# Thinness of product graphs 

Flavia Bonomo-Braberman ${ }^{\text {a,b }}$, Carolina L. Gonzalez ${ }^{\text {b }}$, Fabiano S. Oliveira ${ }^{\text {c }}$, Moysés S. Sampaio Jr. ${ }^{\text {d }}$, Jayme L. Szwarcfiter ${ }^{\text {c,d }}$<br>${ }^{a}$ Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales. Departamento de Computación. Buenos Aires, Argentina.<br>${ }^{b}$ CONICET-Universidad de Buenos Aires. Instituto de Investigación en Ciencias de la Computación (ICC). Buenos Aires, Argentina.<br>${ }^{c}$ Universidade do Estado do Rio de Janeiro, Brazil.<br>${ }^{d}$ Universidade Federal do Rio de Janeiro, Brazil.


#### Abstract

The thinness of a graph is a width parameter that generalizes some properties of interval graphs, which are exactly the graphs of thinness one. Many NPcomplete problems can be solved in polynomial time for graphs with bounded thinness, given a suitable representation of the graph. In this paper we study the thinness and its variations of graph products. We show that the thinness behaves "well" in general for products, in the sense that for most of the graph products defined in the literature, the thinness of the product of two graphs is bounded by a function (typically product or sum) of their thinness, or of the thinness of one of them and the size of the other. We also show for some cases the non-existence of such a function.


Keywords: graph operations, product graphs, proper thinness, thinness, independent thinness, complete thinness.

## 1. Introduction

A graph $G=(V, E)$ is $k$-thin if there exist an ordering $v_{1}, \ldots, v_{n}$ of $V$ and a partition of $V$ into $k$ classes $\left(V^{1}, \ldots, V^{k}\right)$ such that, for each triple

[^0]$(r, s, t)$ with $r<s<t$, if $v_{r}, v_{s}$ belong to the same class and $v_{t} v_{r} \in E$, then $v_{t} v_{s} \in E$. The minimum $k$ such that $G$ is $k$-thin is called the thinness of $G$ and denoted by $\operatorname{thin}(G)$.

The thinness is unbounded on the class of all graphs, and graphs with bounded thinness were introduced in [30] as a generalization of interval graphs.

In [5], the concept of proper thinness is defined in order to obtain an analogous generalization of proper interval graphs, and it is proved that the proper thinness is unbounded on the class of interval graphs. A graph $G=(V, E)$ is proper $k$-thin if there exist an ordering $v_{1}, \ldots, v_{n}$ of $V$ and a partition of $V$ into $k$ classes $\left(V^{1}, \ldots, V^{k}\right)$ such that, for each triple $(r, s, t)$ with $r<s<t$, if $v_{r}, v_{s}$ belong to the same class and $v_{t} v_{r} \in E$, then $v_{t} v_{s} \in E$, and if $v_{s}, v_{t}$ belong to the same class and $v_{t} v_{r} \in E$, then $v_{s} v_{r} \in E$. The minimum $k$ such that $G$ is proper $k$-thin is called the proper thinness of $G$ and denoted by pthin $(G)$.

The parameters of thinness and proper thinness represent how far a graph is from being an interval and proper interval graph, respectively. The class of (proper) 1-thin graphs is that of (proper) interval graphs. This is so because, considering a 1-partitioning, a (strongly) consistent ordering is sufficient to characterize (proper) interval graphs [33, 34].

When a representation of the graph as a $k$-thin graph is given, for a constant value $k$, a wide family of NP-complete problems can be solved in polynomial time, and the family of problems can be enlarged when a representation of the graph as a proper $k$-thin graph is given, for a constant value $k$ [5, 7, 30]. Hardness results for relaxed versions of this family of problems are shown in [4], even for classes of graphs with thinness one or two.

Many operations are defined over graphs, and some of them arise in structural characterizations of particular graph families or graph classes, for example union and join in the case of cographs [13]. The operations known as graph products are those for which the graph obtained by the operation of graphs $G_{1}$ and $G_{2}$ has as vertex set the Cartesian product $V\left(G_{1}\right) \times V\left(G_{2}\right)$. Different graph products are determined by the rules that define the edge set of the obtained graph. The main properties of these products are surveyed in [21].

There is a wide literature about the behavior of graph parameters under graph operations, and in particular graph products. For the chromatic number, it includes the famous conjecture of Hedetniemi (1966) [19], that remained open more than fifty years, it was shown to hold for many par-
ticular classes, and was recently disproved by Shitov [36]. Other results on the chromatic number and its variations in product graphs can be found in [1, 3, 66, 8, 11, 14, 16, 20, 21, 22, 23, 24, 25, 26, 27, 28, 35, 37, 39], and on domination in product graphs in [17, 18, 21, 22]. For width parameters, there is a recent paper studying the boxicity and cubicity of product graphs [9].

In [5], the behavior of the thinness and proper thinness under the graph operations union, join, and Cartesian product is studied. These results allow, respectively, to fully characterize $k$-thin graphs by forbidden induced subgraphs within the class of cographs, and to show the polynomiality of the $t$-rainbow domination problem for fixed $t$ on graphs with bounded thinness.

In this paper, we give bounds for the thinness and proper thinness of union and join of graphs, as well as the thinness and proper thinness of the lexicographical, Cartesian, direct, strong, disjunctive, modular, homomorphic and hom-products of graphs in terms of invariants of the component graphs. We also show that in some cases such bounds do not exist. Furthermore, we describe new general lower and upper bounds for the thinness of graphs. Also, we consider the concepts of independent and complete (proper) thinness, corresponding to the situations in which the classes are all independent or complete sets. Several of the results on the bounds of products of graphs are given additionally for these cases.

The organization of the paper is as follows. In Section 2 we state the main definitions and present some basic results on thinness and proper thinness. In Section 3, we determine the (proper) thinness of some graph families, and prove some lower and upper bounds for the parameters. Section 4 contains the main results of the paper, namely bounds of (proper) thinness for different binary operations, in terms of the (proper) thinness of their factors. Some concluding remarks form the last section.

## 2. Definitions and basic results

All graphs in this work are finite, undirected, and have no loops or multiple edges. For all graph-theoretic notions and notation not defined here, we refer to West [38]. Let $G$ be a graph. Denote by $V(G)$ its vertex set, by $E(G)$ its edge set, by $\bar{G}$ its complement, by $\Delta(G)$ (resp. $\delta(G)$ ) the maximum (resp. minimum) degree of a vertex in $G$. A graph is $k$-regular if every vertex has degree $k$.

Denote by $N(v)$ the neighborhood of a vertex $v$ in $G$, and by $N[v]$ the closed neighborhood $N(v) \cup\{v\}$. If $X \subseteq V(G)$, denote by $N(X)$ the set of
vertices of $G$ having at least one neighbor in $X$. A vertex $v$ of $G$ is universal (resp. isolated) if $N[v]=V(G)$ (resp. $N(v)=\emptyset)$.

An homogeneous set is a proper subset $X \subset V(G)$ of at least two vertices such that every vertex not in $X$ is adjacent either to all the vertices in $X$ or to none of them.

Denote by $G[W]$ the subgraph of $G$ induced by $W \subseteq V(G)$, and by $G-W$ or $G \backslash W$ the graph $G[V(G) \backslash W]$. A subgraph $H$ (not necessarily induced) of $G$ is a spanning subgraph if $V(H)=V(G)$.

Denote the size of a set $S$ by $|S|$. A clique or complete set (resp. stable set or independent set) is a set of pairwise adjacent (resp. nonadjacent) vertices. We use maximum to mean maximum-sized, whereas maximal means inclusion-wise maximal. The use of minimum and minimal is analogous. The size of a maximum clique (resp. stable set) in a graph $G$ is denoted by $\omega(G)($ resp. $\alpha(G))$.

A vertex cover is a set $S$ of vertices of a graph $G$ such that each edge of $G$ has at least one endpoint in $S$. Denote by $\tau(G)$ the size of a minimum vertex cover in a graph $G$.

A graph is called trivial if it has only one vertex. A graph is complete if its vertices are pairwise adjacent. Denote by $K_{n}$ the complete graph of size $n$.

Let $H$ be a graph and $t$ a natural number. The disjoint union of $t$ disjoint copies of the graph $H$ is denoted by $t H$. In particular, $\overline{t K_{2}}$ is the complement of a matching of size $t$. Denote by $\operatorname{mim}(G)$ the size of a maximum induced matching of a graph $G$.

Denote by $P_{n}$ the path on $n$ vertices. Given a connected graph $G$, let $\operatorname{lip}(G)$ be the length of the longest induced path of $G$, and $\operatorname{diam}(G)$ its diameter. A graph is a cograph if it contains no induced $P_{4}$.

For a positive integer $r$, the $(r \times r)$-grid $G R_{r}$ is the graph whose vertex set is $\{(i, j): 1 \leq i, j \leq r\}$ and whose edge set is $\{(i, j)(k, l):|i-k|+|j-l|=$ 1 , where $1 \leq i, j, k, l \leq r\}$.

The crown graph $C R_{n}$ (also known as Hiraguchi graph) is the graph on $2 n$ vertices obtained from a complete bipartite graph $K_{n, n}$ by removing a perfect matching.

A dominating set in a graph is a set of vertices such that each vertex outside the set has at least one neighbor in the set.

A coloring of a graph is an assignment of colors to its vertices such that any two adjacent vertices are assigned different colors. The smallest number $t$ such that $G$ admits a coloring with $t$ colors (a $t$-coloring) is called the
chromatic number of $G$ and is denoted by $\chi(G)$. A coloring defines a partition of the vertices of the graph into stable sets, called color classes.

A graph $G(V, E)$ is a comparability graph if there exists an ordering $v_{1}, \ldots, v_{n}$ of $V$ such that, for each triple $(r, s, t)$ with $r<s<t$, if $v_{r} v_{s}$ and $v_{s} v_{t}$ are edges of $G$, then so is $v_{r} v_{t}$. Such an ordering is a comparability ordering. A graph is a co-comparability graph if its complement is a comparability graph.

In the context of thinness, an ordering $v_{1}, \ldots, v_{n}$ of $V(G)$ and a partition of $V(G)$ satisfying that for each triple $(r, s, t)$ with $r<s<t$, if $v_{r}, v_{s}$ belong to the same class and $v_{t} v_{r} \in E(G)$, then $v_{t} v_{s} \in E(G)$, are said to be consistent. If both $v_{1}, \ldots, v_{n}$ and $v_{n}, \ldots, v_{1}$ are consistent with the partition, the partition and the ordering $v_{1}, \ldots, v_{n}$ are said to be strongly consistent. Notice that a graph is (proper) $k$-thin if and only if it admits a vertex ordering and a vertex partition into $k$ classes that are (strongly) consistent.

We will often use the following definitions and results.
Let $G$ be a graph and $<$ an ordering of its vertices. The graph $G_{<}$has $V(G)$ as vertex set, and $E\left(G_{<}\right)$is such that for $v<w$ in the ordering, $v w \in E\left(G_{<}\right)$if and only if there is a vertex $z$ in $G$ such that $w<z$ in the ordering, $z v \in E(G)$ and $z w \notin E(G)$. Similarly, the graph $\tilde{G}_{<}$has $V(G)$ as vertex set, and $E\left(\tilde{G}_{<}\right)$is such that for $v<w$ in the ordering, $v w \in E\left(\tilde{G}_{<}\right)$if and only if either $v w \in E\left(G_{<}\right)$or there is a vertex $x$ in $G$ such that $x<v$ in the ordering, $x w \in E(G)$ and $x v \notin E(G)$. An edge of $G_{<}$ (respectively $\tilde{G}_{<}$) represents that its endpoints cannot belong to the same class in a vertex partition that is consistent (respectively strongly consistent) with the ordering $<$.

Theorem 1. [5, 7] Given a graph $G$ and an ordering $<$ of its vertices, the graphs $G_{<}$and $\tilde{G}_{<}$have the following properties:
(1) the chromatic number of $G_{<}$(resp. $\left.\tilde{G}_{<}\right)$is equal to the minimum integer $k$ such that there is a partition of $V(G)$ into $k$ sets that is consistent (resp. strongly consistent) with the order $<$, and the color classes of a valid coloring of $G_{<}\left(\right.$resp. $\left.\quad \tilde{G}_{<}\right)$form a partition consistent (resp. strongly consistent) with $<$;
(2) $G_{<}$and $\tilde{G}_{<}$are co-comparability graphs;
(3) if $G$ is a co-comparability graph and $<$ a comparability ordering of $\bar{G}$, then $G_{<}$and $\tilde{G}_{<}$are spanning subgraphs of $G$.

Since co-comparability graphs are perfect [31], $\chi\left(G_{<}\right)=\omega\left(G_{<}\right)$and $\chi\left(\tilde{G}_{<}\right)=\omega\left(\tilde{G}_{<}\right)$. We thus have the following.

Corollary 2. Let $G$ be a graph, and $k$ a positive integer. Then $\operatorname{thin}(G) \geq k$ (resp. pthin $(G) \geq k$ ) if and only if, for every ordering $<$ of $V(G)$, the graph $G_{<}\left(\right.$resp. $\left.\tilde{G}_{<}\right)$has a clique of size $k$.

We will define also two new concepts related to (proper) thinness: independent (proper) thinness and complete (proper) thinness. These concepts are involved in some of the bounds of Section 4 .

A graph $G=(V, E)$ is $k$-independent-thin if there exist an ordering of $V$ and a partition of $V$ into $k$ classes, consistent with the ordering, and such that each class is an independent set of the graph. The minimum $k$ such that $G$ is $k$-independent-thin is called the independent thinness of $G$ and is denoted by $\operatorname{thin}_{\text {ind }}(G)$. Similarly, we can define the concept of proper $k$ -independent-thin and independent proper thinness (denoted by pthin ${ }_{\mathrm{ind}}(G)$ ), where the partition has to be consistent with the ordering and its reverse. Exchanging independent set by complete set, we define the concepts of $k$ -complete-thin, complete thinness (denoted by $\operatorname{thin}_{\mathrm{cmp}}(G)$ ), proper $k$-completethin and complete proper thinness (denoted by pthin $\mathrm{cmp}(G))$.

Remark 1. Notice that $\operatorname{pthin}_{\text {ind }}(G) \geq \operatorname{thin}_{\text {ind }}(G) \geq \chi(G)$ and, by Theorem 1, pthin $_{\text {ind }}(G)=\operatorname{thin}_{\text {ind }}(G)=\chi(G)$ when $G$ is a co-comparability graph. Indeed, we can also see the independent (proper) thinness as a coloring problem in a graph whose vertex set is $V(G)$ and whose edge set is $E(G) \cup$ $E\left(G_{<}\right)\left(\right.$resp. $\left.E(G) \cup E\left(\tilde{G}_{<}\right)\right)$. Similarly, $\operatorname{pthin}_{\mathrm{cmp}}(G) \geq \operatorname{thin}_{\mathrm{cmp}}(G) \geq \chi(\bar{G})$ and we can see the complete (proper) thinness as a coloring problem in a graph whose vertex set is $V(G)$ and whose edge set is $E(\bar{G}) \cup E\left(G_{<}\right)$ (resp. $E(\bar{G}) \cup E\left(\tilde{G}_{<}\right)$). Theorem $[3$ and Corollary 6 show that the bounds $\operatorname{thin}_{\mathrm{cmp}}(G) \geq \chi(\bar{G})$ and $\operatorname{thin}_{\mathrm{ind}}(G) \geq \chi(G)$ can be arbitrarily bad. Notice also that, given a (proper) $k$-thin representation of a graph, we can split each class into independent sets and obtain a (proper) $k$-independentthin representation. Thus $(\mathrm{p}) \operatorname{thin}_{\text {ind }}(G) \leq \chi(G)(\mathrm{p}) \operatorname{thin}(G)$. Analogously, (p) thin $_{\text {cmp }} \leq \chi(\bar{G})(\mathrm{p}) \operatorname{thin}(G)$.

## 3. Thinness of some graph families and general bounds

In this section, we determine or give lower bounds for the thinness and proper thinness of families of graphs, as induced matchings, crowns, and


Figure 1: Hasse diagram of the parameters involved (if $\alpha$ precedes $\beta$ in the diagram, then $\alpha \leq \beta$ ). We will state the strongest results, and the consequences for other parameters can be deduced from the diagram.
hypercubes. In addition, we determine both general lower and upper bounds for the thinness and proper thinness of graphs. Also, we relate the thinness to the independence and clique numbers of graphs.

For the complement of an induced matching, the exact value of the thinness is known.

Theorem 3. [10] For every $t \geq 1$, $\operatorname{thin}\left(\overline{t K_{2}}\right)=t$.
The vertex partition used when proving the part "thin $\left(\overline{t K_{2}}\right)=t$ " of the equation of Theorem 3, which is consistent with any vertex ordering, is the one where each class consists of a pair of nonadjacent vertices. It is easy to see that this partition is also strongly consistent with any vertex ordering. So we have the following corollary.

Corollary 4. For every $t \geq 1$, pthin $\left(\overline{t K_{2}}\right)=\operatorname{thin}_{\text {ind }}\left(\overline{t K_{2}}\right)=\operatorname{pthin}_{\text {ind }}\left(\overline{t K_{2}}\right)=$ $t$.

### 3.1. Lower bounds

Let $G_{1}$ and $G_{2}$ be graphs on $n$ vertices, and $f: V\left(G_{1}\right) \rightarrow V\left(G_{2}\right)$ a bijection. Let $G_{1} \boxminus_{f} G_{2}$ with vertex set $V\left(G_{1}\right) \cup V\left(G_{2}\right)$, such that $V\left(G_{i}\right)$
induces $G_{i}$ for $i=1,2$, and the edges from $V\left(G_{1}\right)$ to $V\left(G_{2}\right)$ are exactly $\{v f(v)\}_{v \in V\left(G_{1}\right)}$. When $G_{1}$ or $G_{2}$ is either the complete graph $K_{n}$ or the empty graph $n K_{1}$, we can omit $f$ by symmetry. Notice that $\overline{t K_{2}}=\overline{t K_{1} \boxminus t K_{1}}$, and the crown $C R_{n}=\overline{K_{n} \boxminus K_{n}}$.

Theorem 5. For $G_{1}$ and $G_{2}$ graphs on $n$ vertices and $f: V\left(G_{1}\right) \rightarrow V\left(G_{2}\right)$ a bijection, thin $\left(\overline{G_{1} \boxminus_{f} G_{2}}\right) \geq n / 2$.

Proof. Let $G=\overline{G_{1} \boxminus_{f} G_{2}}$, and $<$ an arbitrary ordering of its vertices. We will show that $\omega\left(G_{<}\right) \geq n / 2$. Let $A=V\left(G_{1}\right)$ and $A^{\prime}=V\left(G_{2}\right)$, and for each vertex $v \in A$, let $v^{\prime}=f(v)$ (the only vertex in $A^{\prime}$ that is not adjacent to $v$ in $G$ ).

By definition of $G_{<}$, if $v<v^{\prime}$ then $v$ is adjacent in $G_{<}$to every vertex $w$ in $A$ such that $w<v$, and to every vertex $w$ in $A$ such that $v<w<w^{\prime}$. Analogously, if $v^{\prime}<v$, then $v^{\prime}$ is adjacent in $G_{<}$to every vertex $w^{\prime}$ in $A^{\prime}$ such that $w^{\prime}<v^{\prime}$, and to every vertex $w^{\prime}$ in $A^{\prime}$ such that $v^{\prime}<w^{\prime}<w$. Therefore, the vertices $v \in A$ such that $v<v^{\prime}$ form a clique in $G_{<}$, and the vertices $v^{\prime} \in A^{\prime}$ such that $v^{\prime}<v$ form a clique in $G_{<}$. Since for each of the $n$ pairs of vertices $v, v^{\prime}$ one of the inequalities holds, by the pigeonhole principle, $G_{<}$ has a clique of size at least $n / 2$.

Corollary 6. For every $n \geq 1$, $\operatorname{thin}\left(C R_{n}\right) \geq n / 2$.
Since a fat spider is the graph $\overline{K_{n} \boxminus n K_{1}}$, the theorem above implies that split graphs have unbounded thinness.

The vertex isoperimetric peak of a graph $G$, denoted as $b_{v}(G)$, is defined as $b_{v}(G)=\max _{s} \min _{X \subset V,|X|=s}|N(X) \cap(V(G) \backslash X)|$, i.e., the maximum over $s$ of the lower bounds for the number of boundary vertices (vertices outside the set with a neighbor in the set) in sets of size $s$.

Theorem 7. [10] For every graph $G$ with at least one edge, $\operatorname{thin}(G) \geq$ $b_{v}(G) / \Delta(G)$.

The thinness of the grid $G R_{r}$ was lower bounded by using Theorem 7 .
Corollary 8. [10] For every $r \geq 2$, $\operatorname{thin}\left(G R_{r}\right) \geq r / 4$.

We will prove next some other lower bounds, which are very useful for bounding the thinness of highly symmetric graphs, as is the case of graph products of highly symmetric graphs.
Theorem 9. Let $G$ be a graph. If $|N(u) \backslash N[v]| \geq k$ for all $u, v \in V(G)$ then $\operatorname{thin}(G) \geq k+1$. Moreover, for every order $<$ of $V(G)$, the first $k+1$ vertices induce a complete graph in $G_{<}$.

Proof. Let $v_{1}, \ldots, v_{n}$ be an ordering of the vertices of $G$. Let $i, j$ be such that $1 \leq i<j \leq k+1$. We know that $\left|N\left(v_{i}\right) \backslash N\left[v_{j}\right]\right| \geq k$. Hence, $\left|\left(N\left(v_{i}\right) \backslash N\left[v_{j}\right]\right) \backslash\left(\left\{v_{1}, \ldots, v_{j-1}\right\} \backslash\left\{v_{i}\right\}\right)\right| \geq 1$. Therefore, there exists a vertex $v_{h}$ with $h>j$ such that $v_{h} \in N\left(v_{i}\right)$ and $v_{h} \notin N\left(v_{j}\right)$, implying that $v_{i}$ and $v_{j}$ are adjacent in $G_{<}$.

So, for every order $<$ of vertices of $G$, we have that the first $k+1$ vertices induce a complete graph in $G_{<}$. By Corollary 2, $\operatorname{thin}(G) \geq k+1$.

Corollary 10. Let $G$ be a graph with $\delta(G) \geq d$ and such that for all $u, v \in$ $V(G),|N(u) \cap N(v)| \leq c<d$. Then $\operatorname{thin}(G) \geq d-c$.

The class of hypercubes $Q_{n}$ consists of the graphs whose vertex sets correspond to all binary strings with fixed size $n$ and two vertices $u$ and $v$ are adjacent if $u$ and $v$ differ exactly in one position. We say that $n$ is the dimension of the hypercube $Q_{n}$. Clearly, $Q_{n}$ is a $n$-regular graph.
Lemma 11. [32] For all $u, v \in V\left(Q_{n}\right),|N(u) \cap N(v)| \leq 2$.
Corollary 12. For every $n \geq 1$, $\operatorname{thin}\left(Q_{n}\right) \geq n-2$.
Theorem 13. Let $G$ be a graph. Let $S \subseteq V(G)$ and $p=|S|$. If $\mid N(u) \backslash$ $N[v] \mid \geq k$ for all $u, v \in S$ and $|V(G)|-p \leq k$ then $\operatorname{thin}(G) \geq 1+k+p-$ $|V(G)|$.

Proof. Let $v_{1}, \ldots, v_{n}$ be an ordering of the vertices of $G$. Let $v_{i}, v_{j} \in S$ be such that $1 \leq i<j \leq k+1$.

We know that $\left|N\left(v_{i}\right) \backslash N\left[v_{j}\right]\right| \geq k$. Hence, $\mid\left(N\left(v_{i}\right) \backslash N\left[v_{j}\right]\right) \backslash\left(\left\{v_{1}, \ldots, v_{j-1}\right\} \backslash\right.$ $\left.\left\{v_{i}\right\}\right) \mid \geq 1$. Therefore, there exists a vertex $v_{h}$ with $h>j$ such that $v_{h} \in$ $N\left(v_{i}\right)$ and $v_{h} \notin N\left(v_{j}\right)$, implying that $v_{i}$ and $v_{j}$ are adjacent in $G_{<}$.

So, for every order $<$ of vertices of $G$, we have that the vertices in $S$ within the first $k+1$ vertices induce a complete graph in $G_{<}$. Since they are at least $k+1-|V(G) \backslash S|=k+1-|V(G)|+p$, by Corollary 2, $\operatorname{thin}(G) \geq k+1-|V(G)|+p$.

### 3.2. Upper Bounds

Two general upper bounds were known for the thinness of a graph.
Theorem 14. [10] Let $G$ be a graph. Then $\operatorname{thin}(G) \leq|V(G)|-\log (|V(G)|) / 4$.
Theorem 15. [10] Let $G$ be a graph. Then $\operatorname{thin}(G) \leq|V(G)|(\Delta(G)+$ $3) /(\Delta(G)+4)$.

We will prove here other general upper bounds.
Lemma 16. Let $S \subseteq V(G)$. Then $\operatorname{thin}(G) \leq|V(G)|-|S|+\operatorname{thin}(G[S])$.
Proof. Consider an order $<$ of vertices of $G$ such that $v<s$ for all $v \in V(G)-S$ and $s \in S$, and such that $\omega\left(G[S]_{<_{S}}\right)=\operatorname{thin}(G[S])$, where $<_{S}$ stands for the order restricted to $S$. Such an order exists by Corollary 2. Notice that, since $v<s$ for all $v \in V(G)-S$ and $s \in S, G_{<}[S]=G[S]_{<_{S}}$. Then $\operatorname{thin}(G) \leq \omega\left(G_{<}\right) \leq \omega\left(G_{<}[S]\right)+\omega\left(G_{<}[V(G) \backslash S]\right) \leq \omega\left(G[S]_{<S}\right)+|V(G)|-$ $|S|=|V(G)|-|S|+\operatorname{thin}(G[S])$.

Corollary 17. Let $S \subseteq V(G)$ be such that $G[S]$ is an interval graph. Then $\operatorname{thin}(G) \leq|V(G)|-|S|+1$.

An interval completion of a graph $G$ is a spanning supergraph of $G$ which is an interval graph.

Lemma 18. Let $G$ be a graph. Let $H$ be an interval completion of $G$. Let $F$ be the subgraph of $H$ whose edges are $E(H)-E(G)$. Then the number of vertices of a maximum induced interval subgraph of $G$ is at least $|V(G)|-$ $\tau(F)$.

Proof. Let $H$ be an interval completion of $G$. Then $H$ has $|V(G)|$ vertices. Let $F$ be the subgraph of $H$ whose edges are $E(H)-E(G)$. If we remove from $H$ the vertices of a vertex cover in $F$, we get an interval graph that is an induced subgraph of $G$.

Corollary 19. Let $G$ be a graph. Let $H$ be an interval completion of $G$. Let $F$ be the subgraph of $H$ whose edges are $E(H)-E(G)$. Then $\operatorname{thin}(G) \leq$ $\tau(F)+1$.

In particular, stable and complete sets induce interval graphs. Moreover, if $\alpha(G)<|V(G)|($ resp. $\omega(G)<|V(G)|)$, we can add one more vertex $v$ and reorder the vertices of the stable or complete set $S$ such that $u<w$ for all $u \in S-N(v)$ and $w \in N(s) \cap S$ and $s<v$ for all $s \in S$, so $G$ has an induced interval graph of size at least $\alpha(G)+1$ (resp. $\omega(G)+1$ ). As corollaries of Corollary 17, we have the following two results.

Corollary 20. If $V(G)$ is not a stable set then $\operatorname{thin}(G) \leq|V(G)|-\alpha(G)$.
Corollary 21. If $G$ is not a complete graph then $\operatorname{thin}(G) \leq|V(G)|-\omega(G)$.
Remark 2. In the same way, one can see that $\operatorname{thin}_{\text {ind }}(G) \leq|V(G)|-\alpha(G)+1$ and $\operatorname{thin}_{\mathrm{cmp}}(G) \leq|V(G)|-\omega(G)+1$ (in this case we cannot add another vertex to the class containing the maximum stable set or maximum clique, respectively).

We have also the following bound for co-comparability graphs. As already noticed in [7], Theorem 1 implies that if $G$ is a co-comparability graph, then thin $(G) \leq \chi(G)$. Recalling that co-comparability graphs are perfect [31], this implies $\operatorname{thin}(G) \leq \omega(G)$.

We prove next a new upper bound for the thinness of co-comparability graphs.

Theorem 22. If $G$ is a non trivial co-comparability graph, then $\operatorname{thin}(G) \leq$ $|V(G)| / 2$.

Proof. If $G$ is complete and non trivial, then $\operatorname{thin}(G)=1 \leq|V(G)| / 2$. If $G$ is not complete, by Corollary [21, $\operatorname{thin}(G) \leq|V(G)|-\omega(G)$. Adding this inequality to the inequality $\operatorname{thin}(G) \leq \omega(G)$ that holds for co-comparability graphs, we have $2 \operatorname{thin}(G) \leq|V(G)|$, thus $\operatorname{thin}(G) \leq|V(G)| / 2$.

The bound is attained, for example, by the family $\overline{t K_{2}}$ (Theorem (3)).

## 4. Thinness and binary graph operations

In this section, we analyze the behavior of the thinness and proper thinness under different binary graph operations. Each one of these operations will be defined over a pair of graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ such that $\left|V_{1}\right|=n_{1},\left|V_{2}\right|=n_{2}$ and $V_{1} \cap V_{2}=\emptyset$. Besides, for some of the following
proofs, we consider an implicit ordering and partition for both $V_{1}$ and $V_{2}$, as defined next.

The ordering of $V_{1}$ will be denoted by $v_{1}, \ldots, v_{n_{1}}$ and that of $V_{2}$ by $w_{1}, \ldots, w_{n_{2}}$. Moreover, if the value $t_{i}$ of some variation of thinness of $G_{i}$ (for $i \in\{1,2\}$ ) is involved in the bound to be proved, the implicit ordering is one consistent, according to the specified variation of thinness, with a partition $\left(V_{i}^{1}, \ldots, V_{1}^{t_{i}}\right)$. If, otherwise, only the cardinality $n_{i}$ of $V_{i}$ is involved in the bound, the implicit ordering is an arbitrary one. For instance, if $G_{1}$ is a proper $t_{1}$-independent-thin graph, and $t_{1}$ is involved in the bound to be proved, it means that the implicit ordering and partition of $V_{1}$ are strongly consistent and all the $t_{1}$ parts of the partition are independent sets.

Although the proofs in this section are not exactly the same, some of them indeed share a common structure in the reasoning. For the sake of conciseness, some of them were omitted and can be found in Appendix A.

### 4.1. Union and join

The union of $G_{1}$ and $G_{2}$ is the graph $G_{1} \cup G_{2}=\left(V_{1} \cup V_{2}, E_{1} \cup E_{2}\right)$, and the join of $G_{1}$ and $G_{2}$ is the graph $G_{1} \vee G_{2}=\left(V_{1} \cup V_{2}, E_{1} \cup E_{2} \cup\left\{v v^{\prime}: v \in\right.\right.$ $\left.V_{1}, v^{\prime} \in V_{2}\right\}$ ) (i.e., $\overline{G_{1} \vee G_{2}}=\overline{G_{1}} \cup \overline{G_{2}}$ ). (The join is sometimes also noted by $G_{1} \otimes G_{2}$, but we follow the notation in [5]).

The class of cographs can be defined as the graphs that can be obtained from trivial graphs by the union and join operations [13]. Aiming to characterize $k$-thin graphs by forbidden induced subgraphs within the class of cographs, the following results were proved.

Theorem 23. [5] Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \cup G_{2}\right)=\max \left\{f\left(G_{1}\right)\right.$, $\left.f\left(G_{2}\right)\right\}$, for $f \in\{$ thin, pthin $\}$.
Theorem 24. [5] Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \vee G_{2}\right) \leq f\left(G_{1}\right)+$ $f\left(G_{2}\right)$, for $f \in\{$ thin, pthin $\}$. Moreover, if $G_{2}$ is complete, then $\operatorname{thin}\left(G_{1} \vee\right.$ $\left.G_{2}\right)=\operatorname{thin}\left(G_{1}\right)$.

Lemma 25. [5] If $G$ is not complete, then $\operatorname{thin}\left(G \vee 2 K_{1}\right)=\operatorname{thin}(G)+1$.
Lemma 25 implies that if there is some constant value $k$ such that recognizing $k$-thin graphs is NP-complete, then for every $k^{\prime}>k$, recognizing $k^{\prime}$-thin graphs is NP-complete. The existence of such $k$ is still not known, and in general the complexity of recognition of $k$-thin and proper $k$-thin graphs, both with $k$ as a parameter and with constant $k$, is open.

Remark 3. By definition of $G_{<}$, every non-smallest vertex of any nontrivial clique has a vertex in $V(G)$ greater than it and non-adjacent to it in $G$.

The following lemma is necessary to prove Theorem 27.
Lemma 26. Let $G=(V, E)$ be a graph with $\operatorname{thin}(G)=k$ and $v_{1}<\cdots<v_{n}$ be an ordering of $V$. If $G$ is not complete, then there exist a clique of size $k$ of $G_{<}, v_{i_{1}}<\cdots<v_{i_{k}}$, and $v_{j}>v_{i_{1}}$, such that $v_{j} v_{i_{1}} \notin E$.

Proof. By Corollary 2, for every order $<$ of the vertices of $G, \omega\left(G_{<}\right) \geq k$. If $\operatorname{thin}(G)=1$, the statement follows because $G$ is not complete. Suppose $\operatorname{thin}(G)>1$, and let $v_{1}<\cdots<v_{n}$ be an ordering of $V$. By definition of $G_{<}$, for every clique of $G_{<}$, all the vertices that are not the smallest one have a vertex in $V(G)$ which is greater than it and non-adjacent to it in $G$. So, if $G_{<}$contains a clique of size greater than $k$, the statement follows. Consider now that $\omega\left(G_{<}\right)=k$. In order to reach a contradiction, suppose that no clique of size $k$ of $G_{<}$satisfies the property, then each vertex in the set $S=\left\{v \in V\left(G_{<}\right): v\right.$ is the first vertex in $<$ of a clique of size $k$ of $\left.G_{<}\right\}$ is adjacent to every vertex greater than it in $G$. So, modifying the order by placing $S$ as the largest vertices produces a graph $G_{<^{\prime}}$ which is a subgraph of $G_{<}$and in which the vertices of $S$ are isolated vertices. In particular, $\omega\left(G_{<^{\prime}}\right)<\omega\left(G_{<}\right)=k$, a contradiction since, by Corollary 2, $\omega\left(G_{<^{\prime}}\right) \geq k$.

We strength the result of Theorem 24 for thinness.
Theorem 27. Let $G_{1}$ and $G_{2}$ be graphs. If $G_{1}$ is complete, then $\operatorname{thin}\left(G_{1} \vee\right.$ $\left.G_{2}\right)=\operatorname{thin}\left(G_{2}\right)$. If neither $G_{1}$ nor $G_{2}$ are complete, then $\operatorname{thin}\left(G_{1} \vee G_{2}\right)=$ $\operatorname{thin}\left(G_{1}\right)+\operatorname{thin}\left(G_{2}\right)$.

Proof. Let $G=G_{1} \vee G_{2}$. If one of them is complete (suppose without loss of generality $G_{1}$ ), then, by Theorem 24, $\operatorname{thin}(G)=\operatorname{thin}\left(G_{2}\right)$. Otherwise, by Theorem [24, thin $\left(G_{1} \vee G_{2}\right) \leq \operatorname{thin}\left(G_{1}\right)+\operatorname{thin}\left(G_{2}\right)$. Let us prove the equality. Let $k_{1}=\operatorname{thin}\left(G_{1}\right), k_{2}=\operatorname{thin}\left(G_{2}\right)$, and $k=\operatorname{thin}\left(G_{1} \vee G_{2}\right)$. Let $<$ be an ordering consistent with a $k$-partition of $V(G)$. Let $G_{1<}$ and $G_{2<}$ be the incompatibility graphs obtained from the order $<$ restricted to $V_{1}$ and $V_{2}$, respectively.

By Lemma [26, there exist a $k_{1}$-clique of $G_{1<}, v_{i_{1}}^{1}<\cdots<v_{i_{k_{1}}}^{1}$, and $v_{j}^{1}>v_{i_{1}}^{1}$, such that $v_{j}^{1} v_{i_{1}}^{1} \notin E_{1}$. As well, there exist a $k_{2}$-clique of $G_{2<}$, $v_{i_{1}}^{2}<\cdots<v_{i_{k_{2}}}^{2}$, and $v_{j}^{2}>v_{i_{1}}^{2}$, such that $v_{j}^{2} v_{i_{1}}^{2} \notin E_{2}$.

Notice also that, by definition of $G_{i<}, i=1,2$, every non-smallest vertex of the clique of $G_{i<}$ has a vertex in $V_{i}$, greater than it and non-adjacent to it in $G_{i}$. Considering this property and the fact that every vertex of $G_{1}$ is adjacent to every vertex of $G_{2}$ in $G$, it follows that every vertex of $\left\{v_{i_{1}}^{1}, \ldots, v_{i_{k_{1}}}^{1}\right\}$ is adjacent to every vertex of $\left\{v_{i_{1}}^{2}, \ldots, v_{i_{2}}^{2}\right\}$ in $G_{<}$, hence $k \geq k_{1}+k_{2}$. This completes the proof of the theorem.

Remark 4. The proper thinness of the join $G_{1} \vee G_{2}$ cannot be expressed as a function whose only parameters are the proper thinness of $G_{1}$ and $G_{2}$ (even excluding simple particular cases, like trivial or complete graphs). The graph $t K_{1}$ has proper thinness 1 for every $t$. By Theorem 24, the proper thinness of the join of two graphs of proper thinness 1 is either 1 or 2, and there are examples for both of the values. The graph $P_{3}=2 K_{1} \vee K_{1}$ has proper thinness 1 but $3 K_{1} \vee K_{1}$, known as claw, or $C_{4}=2 K_{1} \vee 2 K_{1}$ have proper thinness 2 (the claw and $C_{4}$ are not proper interval graphs). Similarly, by Theorem 24, the proper thinness of the join of a graph of proper thinness 2 and a graph of proper thinness 1 is either 2 or 3, and there are examples for both of the values. The graph $\left(\right.$ claw $\left.\cup t K_{1}\right) \vee K_{1}$ has proper thinness 2, but the graph 3 claw $\vee K_{1}$ has proper thinness 3 [5].

Nevertheless, we have a lemma similar to Lemma 25 for proper thinness.
Lemma 28. [5] For every graph $G$, $\operatorname{pthin}\left(3 G \vee K_{1}\right)=\operatorname{pthin}(G)+1$.
Proof. Theorems 23 and 24 imply pthin $\left(3 G \vee K_{1}\right) \leq \operatorname{pthin}(G)+1$. To show pthin $\left(3 G \vee K_{1}\right) \geq \operatorname{pthin}(G)+1$, we will use Corollary 2. Let $u$ be the corresponding vertex of the $K_{1}$ in $H=3 G \vee K_{1}$, and let $<$ be an ordering of the vertices of $H$. Let $w, w^{\prime}$ be the minimum and maximum vertices, respectively, according to $<$ restricted to $H \backslash\{u\}$. Let $S$ be the set of vertices of the copy of $G$ in $H$ which contains neither $w$ nor $w^{\prime}$. We will show that in $\tilde{H}_{<}$, all the vertices of $S$ are adjacent to $u$. Let $v$ be a vertex of $S$. If $u<v$, then $u<v<w^{\prime}, w^{\prime} v \notin V(H)$ and $w^{\prime} u \in V(H)$. If $v<u$, then $w<v<u, w v \notin V(H)$ and $w u \in V(H)$. In either case, by definition of $\tilde{H}_{<}$, $u v \in V\left(\tilde{H}_{<}\right)$. By Corollary 2, $\tilde{H}_{<}[S]$ contains a clique of size pthin $(G)$, and
thus $\tilde{H}_{<}[S \cup\{u\}]$ contains a clique of size pthin $(G)+1$. As the order $<$ was arbitrary, again by Corollary 2, pthin $(H) \geq \operatorname{pthin}(G)+1$.

We will now study the behavior of independent and complete (proper) thinness under the union and join operations.

Theorem 29. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \cup G_{2}\right)=\max \left\{f\left(G_{1}\right)\right.$, $\left.f\left(G_{2}\right)\right\}$, for $f \in\left\{\right.$ thin $_{\text {ind }}$, pthin $\left._{\text {ind }}\right\}$.

Proof. Since both $G_{1}$ and $G_{2}$ are induced subgraphs of $G_{1} \cup G_{2}$, then $\operatorname{thin}_{\text {ind }}\left(G_{1} \cup G_{2}\right) \geq \max \left\{\operatorname{thin}_{\text {ind }}\left(G_{1}\right), \operatorname{thin}_{\text {ind }}\left(G_{2}\right)\right\}$ and the same holds for the independent proper thinness.

Let $G_{1}$ and $G_{2}$ be two graphs with independent thinness (resp. independent proper thinness) $t_{1}$ and $t_{2}$, respectively. Suppose without loss of generality that $t_{1} \leq t_{2}$. For $G=G_{1} \cup G_{2}$, define a partition $\left(V^{1}, \ldots, V^{t_{2}}\right)$ such that $V^{i}=V_{1}^{i} \cup V_{2}^{i}$ for $i=1, \ldots, t_{1}$ and $V^{i}=V_{2}^{i}$ for $i=t_{1}+1, \ldots, t_{2}$, and define $v_{1}, \ldots, v_{n_{1}}, w_{1}, \ldots, w_{n_{2}}$ as an ordering of the vertices. By definition of union of graphs, the sets of the partition are independent and, if three ordered vertices according to the order defined in $V\left(G_{1} \cup G_{2}\right)$ are such that the first and the third are adjacent, either the three vertices belong to $V_{1}$ or the three vertices belong to $V_{2}$. Since the order and the partition restricted to each of $G_{1}$ and $G_{2}$ are the original ones, the properties required for consistency (resp. strong consistency) are satisfied.

Theorem 30. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \cup G_{2}\right)=f\left(G_{1}\right)+f\left(G_{2}\right)$, for $f \in\left\{\right.$ thin $_{\text {cmp }}$, pthin $\left._{\text {cmp }}\right\}$.

Proof. Let $G_{1}$ and $G_{2}$ be two graphs with complete thinness (resp. complete proper thinness) $t_{1}$ and $t_{2}$, respectively. By definition of $G_{1} \cup G_{2}$, any vertex ordering and partition into complete sets of $G_{1} \cup G_{2}$ which are (strongly) consistent are (strongly) consistent when restricted to $V_{1}$ and $V_{2}$. Notice that no complete set of $G_{1} \cup G_{2}$ contains both a vertex of $V_{1}$ and a vertex of $V_{2}$. So, $\operatorname{thin}_{\text {cmp }}\left(G_{1} \cup G_{2}\right)$ (resp. $\left.\operatorname{pthin}_{\text {cmp }}\left(G_{1} \cup G_{2}\right)\right)$ is at least $t_{1}+t_{2}$. On the other hand, consider orderings and partitions of $V_{1}$ and $V_{2}$ into $t_{1}$ and $t_{2}$ complete sets, respectively, which are consistent (resp. strongly consistent). For $G_{1} \cup G_{2}$, define a partition with $t_{1}+t_{2}$ complete sets as the union of the two partitions and define as ordering of the vertices the concatenation of the orderings of $V_{1}$ and $V_{2}$ (i.e., $v<v^{\prime}$ if either $v$ and $v^{\prime}$ belong
to $V_{i}$ and $v<v^{\prime}$, for $i \in\{1,2\}$, or $v \in V_{1}$ and $v^{\prime}$ in $V_{2}$ ). By definition of union of graphs, if three ordered vertices according to the order defined in $V\left(G_{1} \cup G_{2}\right)$ are such that the first and the third are adjacent, either the three vertices belong to $V_{1}$ or the three vertices belong to $V_{2}$. Since the order and the partition restricted to each of $G_{1}$ and $G_{2}$ are the original ones, the properties required for consistency (resp. strong consistency) are satisfied. Thus $\operatorname{thin}_{\text {cmp }}\left(G_{1} \cup G_{2}\right)$ (resp. pthin $\left.\operatorname{cmp}\left(G_{1} \cup G_{2}\right)\right)$ is at most $t_{1}+t_{2}$, which completes the proof.

Theorem 31. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \vee G_{2}\right)=f\left(G_{1}\right)+f\left(G_{2}\right)$, for $f \in\left\{\operatorname{thin}_{\text {ind }}\right.$, pthin $\left._{\text {ind }}\right\}$.

Proof. Let $G_{1}$ and $G_{2}$ be two graphs with independent thinness (resp. independent proper thinness) $t_{1}$ and $t_{2}$, respectively. By definition of $G_{1} \vee G_{2}$, any vertex ordering and partition into independent sets of $G_{1} \vee G_{2}$ which are (strongly) consistent are (strongly) consistent when restricted to $V_{1}$ and $V_{2}$. Notice that no independent set of $G_{1} \vee G_{2}$ contains both a vertex of $V_{1}$ and a vertex of $V_{2}$. So, $\operatorname{thin}_{\text {ind }}\left(G_{1} \vee G_{2}\right)$ (resp. pthin ind $\left.\left(G_{1} \vee G_{2}\right)\right)$ is at least $t_{1}+t_{2}$.

On the other hand, consider orderings and partitions of $V_{1}$ and $V_{2}$ into $t_{1}$ and $t_{2}$ independent sets, respectively, which are consistent (resp. strongly consistent). For $G=G_{1} \vee G_{2}$, define a partition with $t_{1}+t_{2}$ independent sets as the union of the two partitions and define as ordering of the vertices the concatenation of the orderings of $V_{1}$ and $V_{2}$. Let $x, y, z$ be three vertices of $V(G)$ such that $x<y<z, x z \in E(G)$, and $x$ and $y$ are in the same class of the partition of $V(G)$. Then, in particular, $x$ and $y$ both belong either to $V_{1}$ or to $V_{2}$. If $z$ belongs to the same graph, then $y z \in E(G)$ because the ordering and partition restricted to each of $G_{1}$ and $G_{2}$ are consistent. Otherwise, $z$ is also adjacent to $y$ by the definition of join. We have proved that the defined partition and ordering are consistent. The respective proof of the strong consistency, given the strong consistency of the partition and ordering of each of $G_{1}$ and $G_{2}$, is symmetric. Then $\operatorname{thin}_{\mathrm{ind}}\left(G_{1} \vee G_{2}\right)$ (resp. $\left.\operatorname{pthin}_{\text {ind }}\left(G_{1} \vee G_{2}\right)\right)$ is at most $t_{1}+t_{2}$, which completes the proof.

Lemma 28 and Theorems 30 and 31 imply the following corollary.
Corollary 32. If there is some constant value $k$ such that recognizing proper $k$-thin graphs (resp. (proper) $k$-independent-thin and (proper) $k$-complete-
thin graphs) is NP-complete, then for every $k^{\prime}>k$, recognizing proper $k^{\prime}$ thin graphs (resp. (proper) $k^{\prime}$-independent-thin and (proper) $k^{\prime}$-complete-thin graphs) is NP-complete.

Recall that the existence of such $k$ is still not known for all these classes.
Theorem 33. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \vee G_{2}\right) \leq f\left(G_{1}\right)+f\left(G_{2}\right)$, for $f \in\left\{\operatorname{thin}_{\mathrm{cmp}}\right.$, $\left.\mathrm{pthin}_{\mathrm{cmp}}\right\}$. Moreover, if $G_{2}$ is complete, then $\operatorname{thin}_{\mathrm{cmp}}\left(G_{1} \vee\right.$ $\left.G_{2}\right)=\operatorname{thin}_{\mathrm{cmp}}\left(G_{1}\right)$.

### 4.2. Graph composition or lexicographical product

Let $v \in V_{1}$. The lexicographical product of $G_{1}$ and $G_{2}$ over the vertex $v$ is the graph $G_{1} \bullet{ }_{v}$ obtained from $G_{1}$ by replacing vertex $v$ by graph $G_{2}$, i.e., $V\left(G_{1} \bullet G_{2}\right)=V_{2} \cup V_{1} \backslash\{v\}$, and $x, y$ are adjacent if either $x, y \in V_{1} \backslash\{v\}$ and $x y \in E_{1}$, or $x, y \in V_{2}$ and $x y \in E_{2}$, or $x \in V_{1} \backslash\{v\}, y \in V_{2}$, and $x v \in E_{1}$.

Theorem 34. Let $G_{1}$ and $G_{2}$ be two graphs. Then $f\left(G_{1} \bullet{ }_{v} G_{2}\right) \leq f\left(G_{1}\right)+$ $f\left(G_{2}\right)$, for $f \in\{$ thin, pthin, thin cmp , pthin cmp $\}$, and $f\left(G_{1} \bullet{ }_{v} G_{2}\right) \leq f^{\prime}\left(G_{1}\right)+$ $f\left(G_{2}\right)-1$, for $\left(f, f^{\prime}\right) \in\left\{\left(\right.\right.$ thin, $\left.\operatorname{thin}_{\text {ind }}\right),\left(\operatorname{thin}_{\text {ind }}, \operatorname{thin}_{\text {ind }}\right),\left(\mathrm{pthin}, \mathrm{pthin}{ }_{\text {ind }}\right)$, (pthin ${ }_{\text {ind }}$, pthin $\left.\left._{\text {ind }}\right)\right\}$. Moreover, if $G_{2}$ is complete, $f\left(G_{1} \bullet_{v} G_{2}\right)=f\left(G_{1}\right)$, for $f \in\left\{\right.$ thin, pthin, thin $_{\text {cmp }}$, pthin $\left._{\text {cmp }}\right\}$.

Proof. Let $G_{1}$ and $G_{2}$ be two graphs with thinness (resp. proper thinness) $t_{1}$ and $t_{2}$, respectively.

For $G=G_{1} \bullet{ }_{v} G_{2}$, if $v$ is the $i$-th vertex in the ordering of $V_{1}$ and belongs to the class $V_{1}^{j}$, define $v_{1}, \ldots, v_{i-1}, w_{1}, \ldots, w_{n_{2}}, v_{i+1}, \ldots, v_{n_{1}}$ as an ordering of the vertices of $G$, and a partition with at most $t_{1}+t_{2}$ sets as the union of the two partitions, where $V_{1}^{j}$ is replaced by $V_{1}^{j} \backslash\{v\}$ (or eliminated if $v$ is the only vertex in the class, justifying the partition to have at most $t_{1}+t_{2}$ classes).

Let $x, y, z$ be three vertices of $V(G)$ such that $x<y<z, x z \in E(G)$, and $x$ and $y$ are in the same class of the partition of $V(G)$. Then, in particular, $x$ and $y$ both belong either to $V_{1} \backslash\{v\}$ or to $V_{2}$. If $z$ belongs to the same graph, then $y z \in E(G)$ because the ordering and partition restricted to each of $G_{1}$ and $G_{2}$ are consistent.

Otherwise, if $x$ and $y$ belong to $V_{2}$ and $z$ belongs to $V_{1} \backslash\{v\}$, then $z$ is adjacent to $y$ because $V_{2}$ is an homogeneous set in $G$. If $x$ and $y$ belong to $V_{1} \backslash\{v\}$ and $z$ belongs to $V_{2}$, by the definition of the order in $G, y<v$ in the
order of $V_{1}$, so $v$ is adjacent to $y$ in $G_{1}$. By the definition of $G, y$ is adjacent to $z$.

We have proved that the defined partition and ordering are consistent, and thus that thin $\left(G_{1} \bullet{ }_{v} G_{2}\right) \leq \operatorname{thin}\left(G_{1}\right)+\operatorname{thin}\left(G_{2}\right)$. The proof of the strong consistency, given the strong consistency of the partition and ordering of each of $G_{1}$ and $G_{2}$, is symmetric and implies pthin $\left(G_{1} \bullet_{v} G_{2}\right) \leq \operatorname{pthin}\left(G_{1}\right)+$ $\operatorname{pthin}\left(G_{2}\right)$.

Notice that if the partitions of $G_{1}$ and $G_{2}$ are into complete sets (resp. independent sets), so is the defined partition of $G_{1} \bullet{ }_{v} G_{2}$. Moreover, we will define a new partition with $t_{1}+t_{2}-1$ sets as the union of the partitions of $V_{1}$ and $V_{2}$, where $V_{1}^{j}$ and $V_{2}^{1}$ are replaced by $V_{1}^{j} \backslash\{v\} \cup V_{2}^{1}$. Notice that if the partitions of $G_{1}$ and $G_{2}$ are into complete sets (resp. independent sets), so is the new class.

We will prove first that if the partition of $V_{1}$ consists of independent sets (not necessarily the partition of $V_{2}$ ), then the order and the new partition are consistent. For strongly consistence the proof is symmetric.

The only cases that need to be re-considered are the ones in which $x$ and $y$ belong to the new class and, moreover, one of them belongs to $V_{1}$ and the other one belongs to $V_{2}$. If $x$ belongs to $V_{2}$ and $y$ belongs to $V_{1}$, since the vertices of $G_{2}$ are consecutive in the order, this implies that $z$ belongs to $V_{1}$. Since $x z \in E(G), v z \in E_{1}$. By the order definition, $v<y<z$ in $G_{1}$. Since the ordering and partition of $G_{1}$ are consistent, $y z \in E_{1}$ and thus $y z \in E(G)$. If $x$ belongs to $V_{1}$ and $y$ belongs to $V_{2}$, since the partition of $G_{1}$ consists of independent sets, $x v \notin V_{1}$. So, if $x z \in E(G), z$ belongs to $V_{1}$. By the order definition, $x<v<z$ in $G_{1}$. Since the ordering and partition of $G_{1}$ are consistent, $v z \in E_{1}$ and thus $y z \in E(G)$.

Now we will prove that if $G_{2}$ is complete, i.e., $V_{2}=V_{2}^{1}$, then the order and the new partition are consistent. For strongly consistence, the proof is symmetric.

Again, the only cases that need to be re-considered are the ones in which $x$ and $y$ belong to the new class and, moreover, one of them belongs to $V_{1}$ and the other one belongs to $V_{2}$. We will prove them for consistence, and for strongly consistence the proof is symmetric.

If $x$ belongs to $V_{2}$ and $y$ belongs to $V_{1}$, since the vertices of $G_{2}$ are consecutive in the order, this implies that $z$ belongs to $V_{1}$. Since $x z \in E(G)$, $v z \in E_{1}$. By the order definition, $v<y<z$ in $G_{1}$. Since the ordering and partition of $G_{1}$ are consistent, $y z \in E_{1}$ and thus $y z \in E(G)$. If $x$ belongs to $V_{1}$ and $y$ belongs to $V_{2}$, since $G_{2}$ is complete, if $z \in V_{2}, z y \in V(G)$. So,
assume $z$ belongs to $V_{1}$. By the order definition, $x<v<z$ in $G_{1}$. Since the ordering and partition of $G_{1}$ are consistent, $x z \in E_{1}$ implies $v z \in E_{1}$ and thus $y z \in E(G)$.

Remark 5. Notice that if $v$ is isolated in $G_{1}$, then $G_{1} \bullet_{v} G_{2}=G_{1}\left[V\left(G_{1}\right) \backslash\right.$ $\{v\}] \cup G_{2}$, and if $v$ is universal in $G_{1}$, then $G_{1} \bullet v G_{2}=G_{1}\left[V\left(G_{1}\right) \backslash\{v\}\right] \vee G_{2}$, so we can obtain better bounds by using the results of Section 4.1.

An equivalent formulation of Theorem 34 is the following.
Theorem 35. Let $H$ be an homogeneous set of $G$, and $\left.G\right|_{H}$ be the graph obtained by contracting $H$ into a vertex. Then $f(G) \leq f\left(\left.G\right|_{H}\right)+f(H)$, for $f \in$ $\left\{\right.$ thin, pthin, thin $_{\text {cmp }}$, pthin cmp$\}$, and $f(G) \leq f^{\prime}\left(\left.G\right|_{H}\right)+f(H)-1$, for $\left(f, f^{\prime}\right) \in$ $\left\{\left(\right.\right.$ thin, thin $\left._{\text {ind }}\right),\left(\operatorname{thin}_{\mathrm{ind}}\right.$, thin $\left._{\mathrm{ind}}\right),\left(\mathrm{pthin}, \mathrm{pthin}_{\mathrm{ind}}\right),\left(\mathrm{pthin}_{\mathrm{ind}}\right.$, pthin $\left.\left._{\mathrm{ind}}\right)\right\}$. Moreover, if $H$ is complete, $f(G)=f\left(\left.G\right|_{H}\right)$, for $f \in\left\{\right.$ thin, pthin, thin ${ }_{\mathrm{cmp}}$, $\left.\operatorname{pthin}_{\mathrm{cmp}}\right\}$.

The lexicographical product of $G_{1}$ and $G_{2}$ (also known as composition of $G_{1}$ and $G_{2}$ ) is the graph $G_{1} \bullet G_{2}$ (also noted as $\left.G_{1}\left[G_{2}\right]\right)$ whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent in $G_{1} \bullet G_{2}$ if and only if either $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $G_{2}$, or $u_{1}$ is adjacent to $v_{1}$ in $G_{1}$. It is not necessarily commutative.

Theorem 36. Let $G_{1}$ and $G_{2}$ be two graphs. Then, if $G_{2}$ is complete, $f\left(G_{1} \bullet\right.$ $\left.G_{2}\right)=f\left(G_{1}\right)$, for $f \in\left\{\right.$ thin, pthin, thin cmp , pthin $\left.{ }_{\text {cmp }}\right\}$. Also, $f\left(G_{1} \bullet G_{2}\right) \leq$ $f^{\prime}\left(G_{1}\right) f\left(G_{2}\right)$, for $\left(f, f^{\prime}\right) \in\left\{\left(\right.\right.$ thin, thin $\left._{\text {ind }}\right),\left(\right.$ thin $_{\text {ind }}$, thin $\left._{\text {ind }}\right)$, (pthin, pthin $\left._{\text {ind }}\right)$, (pthin ${ }_{\text {ind }}$, pthin $\left.\left._{\text {ind }}\right)\right\}$, and $f\left(G_{1} \bullet G_{2}\right) \leq\left|V_{1}\right| f\left(G_{2}\right)$, for $f \in\left\{\right.$ thin $_{\text {cmp }}$, pthin $\left._{\text {cmp }}\right\}$. If $G_{2}$ is not complete, $\omega\left(G_{1}\right) f\left(G_{2}\right) \leq f\left(G_{1} \bullet G_{2}\right)$, for $f \in\left\{\right.$ thin, thin ${ }_{\text {ind }}$, pthin $\left._{\text {ind }}\right\}$.

Proof. If $G_{2}$ is complete, we can iteratively apply Theorem 34, since $G_{1} \bullet G_{2}=\left(\left(\ldots\left(\left(G_{1} \bullet_{v_{1}} G_{2}\right) \bullet \bullet_{v_{2}} G_{2}\right) \ldots\right) \bullet_{v_{n_{1}}} G_{2}\right)$, with $\left\{v_{1}, \ldots, v_{n_{1}}\right\}=V_{1}$. By induction in $n_{1}, f\left(G_{1} \bullet G_{2}\right)=f\left(G_{1}\right)$, for $f \in\{$ thin, pthin, thin cmp , pthin cmp $\}$.

So, let $G_{1}$ and $G_{2}$ be two graphs, such that $f^{\prime}\left(G_{1}\right)=t_{1}$ and $f\left(G_{2}\right)=t_{2}$, and assume $G_{2}$ is not complete. In $G_{1} \bullet G_{2}$, consider $V_{1} \times V_{2}$ lexicographically ordered with respect to the defined orderings of $V_{1}$ and $V_{2}$.

Consider first $\left(f, f^{\prime}\right) \in\left\{\left(\right.\right.$ thin, thin $\left._{\text {ind }}\right)$, $\left(\operatorname{thin}_{\text {ind }}\right.$, thin $\left._{\text {ind }}\right)$, (pthin, pthin $\left._{\text {ind }}\right)$, $\left(\right.$ pthin $_{\text {ind }}$, pthin $\left.\left._{\text {ind }}\right)\right\}$, and the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq t_{1}, 1 \leq j \leq t_{2}}$ such that $V^{i, j}=$
$\left\{(v, w): v \in V_{1}^{i}, w \in V_{2}^{j}\right\}$ for each $1 \leq i \leq t_{1}, 1 \leq j \leq t_{2}$. Since the partition of $V_{1}$ consists of independent sets, vertices $(v, w)$ and $\left(v^{\prime}, w\right)$ in the same partition are not adjacent for $v \neq v^{\prime}$, and if furthermore the partition of $V_{2}$ consists of independent sets, the same property holds for the defined partition of $V_{1} \times V_{2}$ for $G_{1} \bullet G_{2}$.

We will show now that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent, when $f, f^{\prime}$ are proper). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $p=q=r$. In this case, $i<j<\ell$. Suppose first that $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $w_{i}, w_{j}$ belong to the same class in $G_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$ if and only if $w_{i} w_{\ell} \in E_{2}$. Since the order and partition of $G_{2}$ are consistent, $w_{j} w_{\ell} \in E_{2}$, so $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$, as required. If $f$ and $f^{\prime}$ are proper, the proof for strongly consistence is symmetric.

Case 2: $p=q<r$. Suppose first that $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $w_{i}, w_{j}$ belong to the same class in $G_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$ if and only if $v_{p} v_{r} \in E_{1}$. Since $p=q, v_{q} v_{r} \in E_{1}$, so $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$, as required. Suppose now $f$ and $f^{\prime}$ are proper, and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, and, in particular, $v_{q}, v_{r}$ belong to the same class in $G_{1}$ thus they are not adjacent. Since $p=q,\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are not adjacent in $G_{1} \bullet G_{2}$.

Case 3: $p<q=r$. Suppose first that $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, and, in particular, $v_{p}, v_{q}$ belong to the same class in $G_{1}$. Thus, they are not adjacent. Since $q=r,\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are not adjacent in $G_{1} \bullet G_{2}$. Suppose now $f$ and $f^{\prime}$ are proper, and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $w_{i}, w_{j}$ belong to the same class in $G_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$ if and only if $v_{p} v_{r} \in E_{1}$. Since $q=r, v_{p} v_{q} \in E_{1}$, so $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \bullet G_{2}$, as required.

Case 4: $p<q<r$. Suppose first that $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, and, in particular, $v_{p}, v_{q}$ belong to the same class in $G_{1}$. Since the ordering an the partition of $G_{1}$ are consistent, if $\left(v_{p}, w_{i}\right),\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$, in particular $v_{p}, v_{r}$ are adjacent in $G_{1}$, thus $v_{q}, v_{r}$ are adjacent in $G_{1}$ and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$, as required. If $f$ and $f^{\prime}$ are proper, the proof for strongly consistence is symmetric.

Consider $f \in\left\{\right.$ thin $_{\text {cmp }}$, pthin cmp $\}$, and the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq n_{1}, 1 \leq j \leq t_{2}}$ such that $V^{i, j}=\left\{\left(v_{i}, w\right): w \in V_{2}^{j}\right\}$ for each $1 \leq i \leq n_{1}, 1 \leq j \leq t_{2}$. Since the partition of $V_{2}$ consists of complete sets, the same property holds for the defined partition of $V_{1} \times V_{2}$ for $G_{1} \bullet G_{2}$.

We will show now that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $p=q=r$. In this case, $i<j<\ell$. Suppose first that $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $w_{i}, w_{j}$ belong to the same class in $G_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$ if and only if $w_{i} w_{\ell} \in E_{2}$. Since the order and partition of $G_{2}$ are consistent, $w_{j} w_{\ell} \in E_{2}$, so $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$, as required. If $f=\operatorname{pthin}_{\mathrm{cmp}}$, the proof for strongly consistence is symmetric.

Case 2: $p=q<r$. Suppose first that $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $w_{i}, w_{j}$ belong to the same class in $G_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$ if and only if $v_{p} v_{r} \in E_{1}$. Since $p=q$, $v_{q} v_{r} \in E_{1}$, so $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$, as required. No further restriction has to be satisfied if $f=\operatorname{pthin}_{\mathrm{cmp}}$, since by definition of the classes $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to different classes.

Case 3: $p<q=r$. No restriction has to be satisfied for consistence, as $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to different classes. If $f=\operatorname{pthin}_{\mathrm{cmp}}$, suppose that $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $w_{j}, w_{\ell}$ belong to the same class in $G_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \bullet G_{2}$ if and only if $v_{p} v_{r} \in E_{1}$. Since $q=r, v_{p} v_{q} \in E_{1}$, so $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \bullet G_{2}$, as required.

Case 4: $p<q<r$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

To prove the lower bound when $G_{2}$ is not complete, notice that $K_{r} \bullet G_{2}$ is isomorphic to $\left(\left(\left(G_{2} \vee G_{2}\right) \vee G_{2}\right) \cdots \vee G_{2}\right)(r$ times $)$. By Theorems 27 and 31, $\omega\left(G_{1}\right) f\left(G_{2}\right) \leq f\left(G_{1} \bullet G_{2}\right)$, for $f \in\left\{\right.$ thin, thin $_{\text {ind }}$, pthin $\left._{\text {ind }}\right\}$.

Corollary 37. Let $G_{1}$ and $G_{2}$ be graphs. If $G_{2}$ is not complete, then $\operatorname{thin}\left(G_{1} \bullet\right.$ $\left.G_{2}\right) \geq \omega\left(G_{1}\right)$.

Notice that $K_{n} \bullet 2 K_{1}=\overline{t K_{2}}$. So, we have the following corollary of Theorem 3.

Corollary 38. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \bullet G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right),\left|V\left(G_{2}\right)\right|\right)$ for any pair of graphs $G_{1}, G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

Corollary 39. Let $G_{1}$ be a co-comparability graph. If $G_{2}$ is complete, then $\operatorname{thin}\left(G_{1} \bullet G_{2}\right)=\operatorname{thin}\left(G_{1}\right)$, and if not, then $\operatorname{thin}\left(G_{1} \bullet G_{2}\right)=\omega\left(G_{1}\right) \operatorname{thin}\left(G_{2}\right)$.

Theorem 40. Let $G$ be a graph and $t \geq 3, q \geq 1$. Then, pthin $(G \bullet$ $\left.t K_{1}\right)=\operatorname{pthin}_{\text {ind }}(G)$, and $\operatorname{pthin}\left(\left(G \bullet t K_{1}\right) \vee q K_{1}\right)=\operatorname{pthin}\left(\left(G \bullet t K_{1}\right) \vee K_{q}\right)=$ pthin $_{\text {ind }}(G)+1$.

Proof. The upper bounds are a consequence of Theorems 36 and 24.
For the lower bounds, we will prove the statement for $t=3$ and $q=1$, since $\left(G \bullet 3 K_{1}\right)$ (resp. $\left.\left(G \bullet 3 K_{1}\right) \vee K_{1}\right)$ is an induced subgraph of $\left(G \bullet t K_{1}\right)$ (resp. $\left(G \bullet t K_{1}\right) \vee q K_{1}$ and $\left.\left(G \bullet t K_{1}\right) \vee K_{q}\right)$. Let $G^{\prime}=\left(G \bullet 3 K_{1}\right)$ and $G^{\prime \prime}=G^{\prime} \vee K_{1}$. Let $V\left(G^{\prime}\right)=\left\{v_{i}^{1}<v_{i}^{2}<v_{i}^{3}: v_{i} \in V(G)\right\}$, and $V\left(G^{\prime \prime}\right)=V\left(G^{\prime}\right) \cup\{u\}$. Consider an ordering of the vertices of $G^{\prime \prime}$, and let $<$ be the vertex order of $V(G)$ induced by the order of $\left\{v_{i}^{2}\right\}_{v_{i} \in V(G)}$. We will show the following three statements, that are enough to prove the theorem: if $v w \in E\left(\tilde{G}_{<}\right)$then $v^{2} w^{2} \in E\left(\tilde{G}^{\prime}{ }^{\prime}\right)$; if $v w \in E(G)$ then $v^{2} w^{2} \in E\left(\tilde{G}^{\prime}<\right)$; for any $v \in V(G), v^{2} u \in E\left(\tilde{G}^{\prime \prime}<\right)$.

First, let $v<w$ be adjacent in $\tilde{G}_{<}$. Then either there is a vertex $z$ such that $v<w<z, v z \in E(G)$ and $w z \notin E(G)$, or there is a vertex $z$ such that $z<v<w, z w \in E(G)$ and $z v \notin E(G)$. In either case, the same holds for $v^{2}, w^{2}, z^{2}$, so $v^{2} w^{2} \in E\left(\tilde{G}^{\prime}<\right)$.

Next, let $v<w$ be adjacent in $G$. Then $v^{2}<w^{2}<w^{3}, v^{2} w^{3} \in E\left(G^{\prime}\right)$ and $w^{2} w^{3} \notin E\left(G^{\prime}\right)$, so $v^{2} w^{2} \in E\left(\tilde{G}^{\prime}<\right)$.

Finally, for $v \in V(G)$, if $u<v^{2}$, then $u<v^{2}<v^{3}, u v^{3} \in E\left(G^{\prime}\right)$ and $v^{2} v^{3} \notin E\left(G^{\prime}\right)$, so $u v^{2} \in E\left(\tilde{G}^{\prime}<\right)$. If $v^{2}<u$, then $v^{1}<v^{2}<u, v^{1} u \in E\left(G^{\prime}\right)$ and $v^{1} v^{2} \notin E\left(G^{\prime}\right)$, so $u v^{2} \in E\left(\tilde{G}^{\prime}<\right)$.

By Remark , it holds that $\operatorname{pthin}\left(G^{\prime}\right) \geq \operatorname{pthin}_{\text {ind }}(G)$, and that $\operatorname{pthin}\left(G^{\prime \prime}\right) \geq$ $\operatorname{pthin}_{\text {ind }}(G)+1$.

Theorem 40 implies that if recognizing proper $k$-independent-thin graphs is NP-complete, then recognizing proper $k$-thin graphs is NP-complete (both with $k$ as a parameter and with constant $k$ ).

### 4.3. Cartesian product

The Cartesian product $G_{1} \square G_{2}$ is a graph whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent in $G_{1} \square G_{2}$ if and only if either $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $G_{2}$, or $u_{2}=v_{2}$ and $u_{1}$ is adjacent to $v_{1}$ in $G_{1}$.

The following result was proved in [5]. We include the proof in order to make some remarks about it. Most of the proofs for other graph products are structurally similar to this one.

Theorem 41. [5] Let $G_{1}$ and $G_{2}$ be graphs. Then, for $f \in\{$ thin, pthin\}, $f\left(G_{1} \square G_{2}\right) \leq f\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$.

Proof. Let $G_{1}$ be a $k$-thin (resp. proper $k$-thin) graph. Consider $V_{1} \times V_{2}$ lexicographically ordered with respect to the defined orderings of $V_{1}$ and $V_{2}$. Consider now the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq k, 1 \leq j \leq n_{2}}$ such that $V^{i, j}=\left\{\left(v, w_{j}\right)\right.$ : $\left.v \in V_{1}^{i}\right\}$ for each $1 \leq i \leq k, 1 \leq j \leq n_{2}$. We will show that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $p=q=r$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

Case 2: $p=q<r$. In this case, $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are in different classes. So suppose $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \square G_{2}$ if and only if $i=\ell$ and $v_{p} v_{r} \in E_{1}$. But then $\left(v_{p}, w_{i}\right)=\left(v_{q}, w_{j}\right)$, a contradiction.

Case 3: $p<q=r$. In this case, $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are in different classes. So suppose $G_{1}$ is $k$-thin (resp. proper $k$-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \square G_{2}$ if and only if $i=\ell$ and $v_{p} v_{r} \in E_{1}$. But then $\left(v_{r}, w_{\ell}\right)=\left(v_{q}, w_{j}\right)$, a contradiction.

Case 4: $p<q<r$. Suppose first $G_{1}$ is $k$-thin (resp. proper $k$-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$ and $v_{p}, v_{q}$ belong to the same class in $G_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \square G_{2}$ if and only if $i=\ell$ and $v_{p} v_{r} \in E_{1}$. But then $j=\ell$ and since the ordering and the partition are consistent (resp. strongly consistent) in $G_{1}, v_{r} v_{q} \in E_{1}$ and so $\left(v_{r}, w_{\ell}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \square G_{2}$. Now suppose that $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$. Vertices $\left(v_{p}, w_{i}\right)$ and ( $v_{r}, w_{\ell}$ ) are adjacent in $G_{1} \square G_{2}$ if and only if $i=\ell$ and $v_{p} v_{r} \in E_{1}$. But then $i=j$ and since the ordering and the partition are strongly consistent in $G_{1}, v_{p} v_{q} \in E_{1}$ and so $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \square G_{2}$.

Remark 6. Notice that if the partition of $G_{1}$ consists of independent sets (respectively, complete sets), the partition defined for $G_{1} \square G_{2}$ consists also of independent sets (respectively, complete sets). So, $f\left(G_{1} \square G_{2}\right) \leq f\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$, for $f \in\left\{\right.$ thin, pthin, thin $_{\text {ind }}$, thin $_{\text {cmp }}$, pthin ${ }_{\text {ind }}$, pthin $\left._{\text {cmp }}\right\}$.

These results can be strengthened by replacing $\left|V\left(G_{2}\right)\right|$ by the size of the largest connected component of $G_{2}$, by Theorem 23 and since $G \square\left(H \cup H^{\prime}\right)=$ $(G \square H) \cup\left(G \square H^{\prime}\right)$.

On the negative side, since $P_{r}$ has independent proper thinness 2 (with the order given by the definition of path), but $P_{r} \square P_{r}=G R_{r}$, we have the following corollary of Corollary 8 .

Corollary 42. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \square G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{ind}}\left(G_{1}\right), \operatorname{pthin}_{\mathrm{ind}}\left(G_{2}\right)\right)$ for any pair of graphs $G_{1}, G_{2}$.

Lemma 43. For $n \geq 1$, $\operatorname{thin}\left(K_{n} \square K_{n}\right)=n$.
Proof. For $n \geq 1, K_{n} \square K_{n}$ is ( $2 n-2$ )-regular, and for any pair of vertices $u, v,|N(u) \cap N(v)| \leq n-2$. By Corollary 10, $\operatorname{thin}\left(K_{n} \square K_{n}\right) \geq n$. By Theorem 41, thin $\left(K_{n} \square K_{n}\right) \leq n$.

Corollary 44. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \square G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right)\right.$, pthin $\left.\operatorname{cmp}\left(G_{2}\right)\right)$ for any pair of graphs $G_{1}, G_{2}$.

Lemma 45. For $n \geq 1$, $\operatorname{thin}\left(K_{n} \square K_{n, n}\right) \geq n-1$.
Proof. For $n \geq 1, K_{n} \square K_{n, n}$ is (2n-1)-regular, and for any pair of vertices $u, v,|N(u) \cap N(v)| \leq n$. By Corollary 10, $\operatorname{thin}\left(K_{n} \square K_{n, n}\right) \geq n-1$.

Corollary 46. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \square G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right)\right.$, pthin $\left._{\mathrm{ind}}\left(G_{2}\right)\right)$ for any pair of graphs $G_{1}, G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

Further consequences of the examples above are the following.
Corollary 47. Given two connected graphs $G_{1}$ and $G_{2}$, (min\{diam $\left(G_{1}\right)$, $\left.\left.\operatorname{diam}\left(G_{2}\right)\right\}+1\right) / 4 \leq\left(\min \left\{\operatorname{lip}\left(G_{1}\right), \operatorname{lip}\left(G_{2}\right)\right\}+1\right) / 4 \leq \operatorname{thin}\left(G_{1} \square G_{2}\right)$.

Corollary 48. Given two graphs $G_{1}$ and $G_{2}, \min \left\{\omega\left(G_{1}\right), \omega\left(G_{2}\right)\right\} \leq \operatorname{thin}\left(G_{1} \square G_{2}\right)$.

### 4.4. Tensor or direct or categorical product

The tensor product or direct product or categorical product or Kronecker product $G_{1} \times G_{2}$ is a graph whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent in $G_{1} \times G_{2}$ if and only if $u_{1}$ is adjacent to $v_{1}$ in $G_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $G_{2}$.

Theorem 49. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \times G_{2}\right) \leq f_{\text {ind }}\left(G_{1} \times G_{2}\right) \leq$ $f_{\text {ind }}\left(G_{1}\right)\left|V\left(G_{2}\right)\right| \leq f\left(G_{1}\right) \chi\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$, for $f \in\{$ thin, pthin $\}$.

This result can be strengthened by replacing $\left|V\left(G_{2}\right)\right|$ by the size of the largest connected component of $G_{2}$, by Theorem 29 and since $G \times\left(H \cup H^{\prime}\right)=$ $(G \times H) \cup\left(G \times H^{\prime}\right)$.

Since pthin $\mathrm{cmp}\left(K_{n}\right)=1,\left|K_{2}\right|=2$, and $K_{n} \times K_{2}=C R_{n}$, we have the following consequence of Corollary 6.

Corollary 50. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \times G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right),\left|V\left(G_{2}\right)\right|\right)$ for any pair of graphs $G_{1}, G_{2}$.

The graph $P_{2 r-1} \times P_{2 r-1}$ contains $G R_{r}$ as an induced subgraph. Since $\operatorname{pthin}_{\text {ind }}\left(P_{2 r-1}\right)=2$, we have also the following.

Corollary 51. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \times G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{ind}}\left(G_{1}\right)\right.$, pthin $\left._{\mathrm{ind}}\left(G_{2}\right)\right)$ for any pair of graphs $G_{1}, G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

Further consequences of the examples above are the following.
Corollary 52. Given two graphs $G_{1}$ and $G_{2}$, if $G_{2}$ has at least one edge, then $\omega\left(G_{1}\right) / 2 \leq \operatorname{thin}\left(G_{1} \times G_{2}\right)$.

Corollary 53. Given two connected graphs $G_{1}$ and $G_{2}$, $\left(\min \left\{\operatorname{diam}\left(G_{1}\right)\right.\right.$, $\left.\left.\operatorname{diam}\left(G_{2}\right)\right\}+1\right) / 8 \leq\left(\min \left\{\operatorname{lip}\left(G_{1}\right), \operatorname{lip}\left(G_{2}\right)\right\}+1\right) / 8 \leq \operatorname{thin}\left(G_{1} \times G_{2}\right)$.


Figure 2: The $(r \times r)$-grid $G R_{r}$ as an induced subgraph of $P_{2 r-1} \times P_{2 r-1}$ and of $P_{2 r-1} \boxtimes$ $P_{2 r-1}$.

### 4.5. Strong or normal product

The strong product (also known as normal product) $G_{1} \boxtimes G_{2}$ is a graph whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$ if and only if they are adjacent either in $G_{1} \square G_{2}$ or in $G_{1} \times G_{2}$.

Theorem 54. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \boxtimes G_{2}\right) \leq f\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$ for $f \in\left\{\right.$ thin, pthin, thin $_{\mathrm{cmp}}$, pthin $_{\mathrm{cmp}}$, thin $_{\mathrm{ind}}$, pthin $\left._{\mathrm{ind}}\right\}$. Moreover, if $G_{2}$ is complete, then $f\left(G_{1} \boxtimes G_{2}\right)=f\left(G_{1}\right)$ for $f \in\left\{\right.$ thin, pthin, thin ${ }_{c m p}$, pthin $\left._{\mathrm{cmp}}\right\}$.

This result can be strengthened for $f \in\left\{\right.$ thin, pthin, $\operatorname{thin}_{\text {ind }}$, pthin $\left._{\text {ind }}\right\}$, replacing $\left|V\left(G_{2}\right)\right|$ by the size of the largest connected component of $G_{2}$, by Theorems 23 and 29, and since $G \boxtimes\left(H \cup H^{\prime}\right)=(G \boxtimes H) \cup\left(G \boxtimes H^{\prime}\right)$.

The graph $P_{2 r-1} \boxtimes P_{2 r-1}$ contains $G R_{r}$ as an induced subgraph. Since $\operatorname{pthin}_{\text {ind }}\left(P_{2 r-1}\right)=2$, we have the following corollary of Corollary 8,

Corollary 55. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \boxtimes G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{ind}}\left(G_{1}\right), \operatorname{pthin}_{\mathrm{ind}}\left(G_{2}\right)\right)$ for any pair of graphs $G_{1}, G_{2}$.

Lemma 56. For $n \geq 2$, $\operatorname{thin}\left(\left(K_{n} \square K_{2}\right) \boxtimes\left(K_{n} \square K_{2}\right)\right) \geq n+2$.
Proof. For $n \geq 2,\left(K_{n} \square K_{2}\right) \boxtimes\left(K_{n} \square K_{2}\right)$ is $\left(n^{2}+2 n\right)$-regular, and for any pair of vertices $u, v,|N(u) \cap N(v)| \leq n^{2}+n-2$. By Corollary 10, $\operatorname{thin}\left(\left(K_{n} \square K_{2}\right) \boxtimes\left(K_{n} \square K_{2}\right)\right) \geq n+2$.

Remark 6 implies pthin $\operatorname{cmp}\left(K_{n} \square K_{2}\right)=2$ for $n \geq 2$, since the graph contains an induced cycle of length four, which is not an interval graph. So we have also the following.

Corollary 57. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \boxtimes G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right)\right.$, pthin $\left.\mathrm{cmp}\left(G_{2}\right)\right)$ for any pair of graphs $G_{1}, G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

A further consequence of the example used for Corollary 55 is the following.

Corollary 58. Given two connected graphs $G_{1}$ and $G_{2}$, $\left(\min \left\{\operatorname{diam}\left(G_{1}\right)\right.\right.$, $\left.\left.\operatorname{diam}\left(G_{2}\right)\right\}+1\right) / 8 \leq\left(\min \left\{\operatorname{lip}\left(G_{1}\right), \operatorname{lip}\left(G_{2}\right)\right\}+1\right) / 8 \leq \operatorname{thin}\left(G_{1} \boxtimes G_{2}\right)$.

### 4.6. Co-normal or disjunctive product

The co-normal product or disjunctive product $G_{1} * G_{2}$ is a graph whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent in $G_{1} * G_{2}$ if and only if either $u_{1}$ is adjacent to $v_{1}$ in $G_{1}$ or $u_{2}$ is adjacent to $v_{2}$ in $G_{2}$.

Notice that $\overline{G * H}=\bar{G} \boxtimes \bar{H}$.
Theorem 59. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} * G_{2}\right) \leq f_{\text {ind }}\left(G_{1} * G_{2}\right) \leq$ $f_{\text {ind }}\left(G_{1}\right)\left|V\left(G_{2}\right)\right| \leq f\left(G_{1}\right) \chi\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$, for $f \in\{$ thin, pthin $\}$.

Since pthin $\mathrm{cmp}\left(K_{t}\right)=1,\left|2 K_{1}\right|=2$, and $K_{t} * 2 K_{1}=\overline{t K_{2}}$, we have the following corollary of Theorem 3.

Corollary 60. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} * G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right),\left|V\left(G_{2}\right)\right|\right)$ for any pair of graphs $G_{1}, G_{2}$.

Consider the graph $t K_{2} * t K_{2}$. It is $(4 t-1)$-regular, and for every pair of vertices $u, v$, it holds $|N(u) \cap N(v)| \leq 2 t+1$. By Corollary 10, $\operatorname{thin}\left(t K_{2} *\right.$ $\left.t K_{2}\right) \geq 2 t-2$. Since pthin $\mathrm{ind}\left(t K_{2}\right)=2$, we have also the following corollary.

Corollary 61. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} * G_{2}\right) \leq$ $f\left(\operatorname{pthin}_{\mathrm{ind}}\left(G_{1}\right)\right.$, pthin $\left._{\text {ind }}\left(G_{2}\right)\right)$ for any pair of graphs $G_{1}, G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

Further consequences of the examples above are the following.
Corollary 62. Given two graphs $G_{1}$ and $G_{2}$, if $G_{2}$ is not complete, then $\omega\left(G_{1}\right) \leq \operatorname{thin}\left(G_{1} * G_{2}\right)$.

Corollary 63. Given two graphs $G_{1}$ and $G_{2}, 2 \min \left\{\operatorname{mim}\left(G_{1}\right), \operatorname{mim}\left(G_{2}\right)\right\}-$ $2 \leq \operatorname{thin}\left(G_{1} * G_{2}\right)$.

### 4.7. Modular product

The modular product $G_{1} \diamond G_{2}$ is a graph whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent in $G_{1} \diamond G_{2}$ if and only if either $u_{1}$ is adjacent to $v_{1}$ in $G_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $G_{2}$, or $u_{1}$ is nonadjacent to $v_{1}$ in $G_{1}$ and $u_{2}$ is nonadjacent to $v_{2}$ in $G_{2}$.

Notice that $K_{n} \diamond K_{2}=C R_{n}$ and $t K_{2} \diamond K_{1}=\overline{t K_{2}}$, so we have the following.
Corollary 64. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \diamond G_{2}\right) \leq$ $f\left(h\left(G_{1}\right),\left|G_{2}\right|\right)$, for $h \in\left\{\operatorname{pthin}_{\mathrm{cmp}}\right.$, pthin $\left._{\mathrm{ind}}\right\}$ for any pair of graphs $G_{1}, G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

Further consequences of the examples above are the following.
Corollary 65. Given two graphs $G_{1}$ and $G_{2}$, if $G_{2}$ has at least one edge, then $\omega\left(G_{1}\right) / 2 \leq \operatorname{thin}\left(G_{1} \diamond G_{2}\right)$.

Corollary 66. Given two graphs $G_{1}$ and $G_{2}$, $\operatorname{mim}\left(G_{1}\right) \leq \operatorname{thin}\left(G_{1} \diamond G_{2}\right)$.

### 4.8. Homomorphic product

The homomorphic product $G_{1} \ltimes G_{2}$ [29] is a graph whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent in $G_{1} \ltimes G_{2}$ if and only if either $u_{1}=v_{1}$ or $u_{1}$ is adjacent to $v_{1}$ in $G_{1}$ and $u_{2}$ is nonadjacent to $v_{2}$ in $G_{2}$. It is not necessarily commutative.

Theorem 67. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \ltimes G_{2}\right) \leq f\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$ for $f \in\left\{\right.$ thin, pthin, thin $_{\text {cmp }}$, pthin $_{\text {cmp }}$, thin $_{\text {ind }}$, pthin $\left._{\text {ind }}\right\}$.

Let $G_{1}$ be isomorphic to $K_{2}$, such that $V\left(G_{1}\right)=\left\{v_{1}, v_{2}\right\}$, and $G_{2}$ be isomorphic to $t K_{2}$, for some $t \geq 1$, such that $V\left(G_{2}\right)=\left\{w_{1}, z_{1}, \ldots, w_{t}, z_{t}\right\}$ and $E\left(G_{2}\right)=\left\{w_{i} z_{i}: 1 \leq i \leq t\right\}$. In $G_{1} \ltimes G_{2}$, the vertices $\left\{\left(v_{1}, w_{i}\right): 1 \leq i \leq\right.$ $t\} \cup\left\{\left(v_{2}, z_{i}\right): 1 \leq i \leq t\right\}$ induce the graph $\overline{t K_{2}}$. In other words, $K_{2} \ltimes t K_{2}$ has $\overline{t K_{2}}$ as induced subgraph.

Now let $G_{3}$ be isomorphic to $K_{t} \square K_{2}$, for some $t \geq 1$, such that $V\left(G_{3}\right)=$ $\left\{w_{1}, z_{1}, \ldots, w_{t}, z_{t}\right\},\left\{w_{1}, \ldots, w_{t}\right\}$ is a clique, $\left\{z_{1}, \ldots, z_{t}\right\}$ is a clique, and $w_{i}$ is adjacent to $z_{j}$ if and only if $i=j$. In $G_{1} \ltimes G_{3}$, the vertices $\left\{\left(v_{1}, w_{i}\right)\right.$ : $1 \leq i \leq t\} \cup\left\{\left(v_{2}, z_{i}\right): 1 \leq i \leq t\right\}$ induce the graph $\overline{t K_{2}}$. In other words, $K_{2} \ltimes\left(K_{t} \square K_{2}\right)$ has $\overline{t K_{2}}$ as induced subgraph.

Since pthin ind $^{\text {ind }}\left(t K_{2}\right)=2$ and pthin $\operatorname{cmp}\left(K_{n} \square K_{2}\right)=2$ (Remark 6), we have the following.

Corollary 68. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \ltimes G_{2}\right) \leq$ $f\left(\left|G_{1}\right|, h\left(G_{2}\right)\right)$, for $h \in\left\{\right.$ pthin $_{\text {ind }}$, pthin $\left._{\text {cmp }}\right\}$ for any pair of graphs $G_{1}, G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

Another consequence of the example above is the following.
Corollary 69. Given two graphs $G_{1}$ and $G_{2}$, if $G_{1}$ has at least one edge, then $\operatorname{mim}\left(G_{2}\right) \leq \operatorname{thin}\left(G_{1} \ltimes G_{2}\right)$.

### 4.9. Hom-product

The hom-product $G_{1} \circ G_{2}$ [2] is a graph whose vertex set is the Cartesian product $V_{1} \times V_{2}$, and such that two vertices $\left(u_{1}, u_{2}\right)$ and ( $v_{1}, v_{2}$ ) are adjacent in $G_{1} \circ G_{2}$ if and only if $u_{1} \neq v_{1}$ and either $u_{1}$ is nonadjacent to $v_{1}$ in $G_{1}$ or $u_{2}$ is adjacent to $v_{2}$ in $G_{2}$. It is not necessarily commutative, indeed, $G_{1} \circ G_{2}=\overline{G_{1} \ltimes G_{2}}$.

Theorem 70. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \circ G_{2}\right) \leq f_{\text {ind }}\left(G_{1} \circ G_{2}\right) \leq$ $\left|V\left(G_{1}\right)\right| f_{\text {ind }}\left(G_{2}\right)$, for $f \in\{$ thin, pthin $\}$.

Proof. Let $G_{2}$ be a $k$-independent-thin (resp. proper $k$-independent-thin) graph. Consider $V_{1} \times V_{2}$ lexicographically ordered with respect to the defined orderings of $V_{2}$ and $V_{1}$. Consider now the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq n_{1}, 1 \leq j \leq k}$ such that $V^{i, j}=\left\{\left(v_{i}, w\right): w \in V_{2}^{j}\right\}$ for each $1 \leq i \leq n_{1}, 1 \leq j \leq k$. By definition of the hom-product, each $V^{i, j}$ is an independent set.

We will show now that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $i=j=\ell$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

Case 2: $i=j<\ell$. In this case, $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are in different classes. So, suppose $G_{2}$ is proper $k$-independent-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $q=r$ and $w_{i}=w_{j}$ and $w_{\ell}$ belong to the same class in $G_{2}$. In particular, since the classes are independent sets, $w_{i} w_{\ell} \notin E_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \circ G_{2}$ if and only if $p \neq r$ and either $v_{p} v_{r} \notin E_{1}$ or $w_{i} w_{\ell} \in E_{2}$. So, assume $p \neq r$. Since $q=r, p \neq q$, and since the graph is loopless and $i=j, w_{i} w_{j} \notin E_{2}$. So $\left(v_{p}, w_{i}\right)$ is adjacent to $\left(v_{q}, w_{j}\right)$ in $G_{1} \circ G_{2}$, as required.

Case 3: $i<j=\ell$. In this case, $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are in different classes. So suppose $G_{2}$ is $k$-independent-thin (resp. proper $k$-independentthin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $p=q$ and $w_{i}$ and $w_{j}=w_{\ell}$ belong to the same class in $G_{2}$. In particular, since the classes are independent sets, $w_{i} w_{j} \notin E_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and ( $\left.v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \circ G_{2}$ if and only if $p \neq r$ and either $v_{p} v_{r} \notin E_{1}$ or $w_{i} w_{\ell} \in E_{2}$. So, assume $p \neq r$. Since $p=q, q \neq r$, and since the graph is loopless and $j=\ell$, $w_{j} w_{\ell} \notin E_{2}$. So $\left(v_{q}, w_{j}\right)$ is adjacent to $\left(v_{r}, w_{\ell}\right)$ in $G_{1} \circ G_{2}$, as required.

Case 4: $i<j<\ell$. Suppose first that $G_{2}$ is $k$-independent-thin (resp. proper $k$-independent-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $p=q$ and $w_{i}, w_{j}$ belong to the same class in $G_{2}$. Since the classes are independent, $w_{i}$ and $w_{j}$ are not adjacent. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \circ G_{2}$ if and only if $p \neq r$ and either $v_{p} v_{r} \notin E_{1}$ or $w_{i} w_{\ell} \in E_{2}$. So, assume $p \neq r$, and since $p=q, q \neq r$ too. If $v_{p} v_{r} \notin E_{1}$, then since $p=q, v_{q} v_{r} \notin E_{1}$, and $\left(v_{q}, w_{j}\right)$ is adjacent to $\left(v_{r}, w_{\ell}\right)$ in $G_{1} \circ G_{2}$, as required. If $w_{i} w_{\ell} \in E_{2}$, since $w_{i}, w_{j}$ belong to the same class in $G_{2}$ and the partition of $V_{2}$ is (strongly) consistent with the ordering, $w_{j} w_{\ell} \in E_{2}$, and $\left(v_{q}, w_{j}\right)$ is adjacent to $\left(v_{r}, w_{\ell}\right)$ in $G_{1} \circ G_{2}$, as required.

Now suppose that $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $q=r$ and $w_{j}, w_{\ell}$ belong to the same class in $G_{2}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \circ G_{2}$ if and only if $p \neq r$ and either $v_{p} v_{r} \notin E_{1}$ or $w_{i} w_{\ell} \in E_{2}$. So, assume $p \neq r$, and since $q=r, p \neq q$ too. If $v_{p} v_{r} \notin E_{1}$, then since $q=r, v_{p} v_{q} \notin E_{1}$, and $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \circ G_{2}$, as required. If $w_{i} w_{\ell} \in E_{2}$, since $w_{j}, w_{\ell}$ belong to the same class in $G_{2}$ and the partition of $V_{2}$ is strongly consistent with the order-
ing, $w_{i} w_{j} \in E_{2}$, and $\left(v_{p}, w_{i}\right)$ is adjacent to $\left(v_{q}, w_{j}\right)$ in $G_{1} \circ G_{2}$, as required.
Notice that $K_{2} \circ K_{n}=C R_{n}$. Also, $G \circ K_{1}=\bar{G}$, so, by taking $G=t K_{2}$ and $G=K_{n} \square K_{2}$, we have the following.

Corollary 71. There is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \circ G_{2}\right) \leq$ $f\left(\left|G_{1}\right|\right.$, pthin $\left.\operatorname{cmp}\left(G_{2}\right)\right)$, and there is no function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $\operatorname{thin}\left(G_{1} \circ\right.$ $\left.G_{2}\right) \leq f\left(h\left(G_{1}\right),\left|G_{2}\right|\right)$ for $h \in\left\{\operatorname{pthin}_{\text {ind }}\right.$, pthin $\left._{\text {cmp }}\right\}$ for any pair of graphs $G_{1}$, $G_{2}$.

The non existence of bounds in terms of other parameters can be deduced from diagram in Figure 1 .

Further consequences of the examples above are the following.
Corollary 72. Given two graphs $G_{1}$ and $G_{2}$, if $G_{1}$ has at least one edge, then $\omega\left(G_{2}\right) / 2 \leq \operatorname{thin}\left(G_{1} \circ G_{2}\right)$.

Corollary 73. Given two graphs $G_{1}$ and $G_{2}$, $\operatorname{mim}\left(G_{1}\right) \leq \operatorname{thin}\left(G_{1} \circ G_{2}\right)$.

## 5. Conclusion

In this paper, we give upper bounds for the thinness, complete thinness, independent thinness and their proper versions for the union and join of graphs, as well as the lexicographical, Cartesian, direct, strong, disjunctive, modular, homomorphic and hom-products of graphs. These bounds are given in terms of the parameters (depicted in the Hasse diagram of Figure (1) of the component graphs, and for each of the cases, it is proved that no upper bound in terms of lower parameters (from those in the diagram) of the component graphs exists. The non existence proofs are based on the determination of exact values or lower bounds for the thinness of some families of graphs like complements of matchings, grids, crown graphs, hypercubes, or products of simple graphs like complete graphs, stable sets, induced matchings and induced paths. We summarize the main bounds obtained and the graph families with high thinness in Table 1 .

Furthermore, we describe new general lower and upper bounds for the thinness of graphs, and some lower bounds for the graph operations in terms of other well known graph invariants like clique number, maximum induced matching, longest induced path, or diameter.

Some open problems and possible research directions are:

| Upper bounds | High thinness |
| :---: | :---: |
| (p)thin $\left(G_{1} \cup G_{2}\right)=\max \left\{(\mathrm{p}) \operatorname{thin}\left(G_{1}\right),(\mathrm{p}) \operatorname{thin}\left(G_{2}\right)\right\}$ <br> (p)thin ind $\left(G_{1} \cup G_{2}\right)=\max \left\{(\mathrm{p})\right.$ thin $_{\text {ind }}\left(G_{1}\right),(\mathrm{p})$ thin ind $\left._{\text {ind }}\left(G_{2}\right)\right\}$ <br> $(\mathrm{p}) \operatorname{thin}_{\mathrm{cmp}}\left(G_{1} \cup G_{2}\right)=(\mathrm{p})$ thin $_{\mathrm{cmp}}\left(G_{1}\right)+(\mathrm{p}) \operatorname{thin}_{\mathrm{cmp}}\left(G_{2}\right)$ |  |
|  |  |
|  | $K_{n} \bullet 2 K_{1}$ |
| (p) $\operatorname{thin}\left(G_{1} \square G_{2}\right) \leq(\mathrm{p}) \operatorname{thin}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\|$ <br> (p)thin ind $\left(G_{1} \square G_{2}\right) \leq(\mathrm{p})$ thin $_{\text {ind }}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\|$ <br> (p)thin ${ }_{\text {cmp }}\left(G_{1} \square G_{2}\right) \leq(\mathrm{p})$ thin $_{\text {cmp }}\left(G_{1}\right) \cdot \mid V\left(G_{2}\right)$ | $\begin{aligned} & P_{n} \square P_{n} \\ & K_{n} \square K_{n} \\ & K_{n} \square K_{n, n} \end{aligned}$ |
| $\begin{aligned} & \text { (p)thin }\left(G_{1} \times G_{2}\right) \leq(\mathrm{p}) \operatorname{thininind~}_{\text {ind }}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\| \\ & \text { (p)thin }{ }_{\text {ind }}\left(G_{1} \times G_{2}\right) \leq(\mathrm{p}) \operatorname{thin}_{\text {ind }}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\| \end{aligned}$ | $\begin{aligned} & K_{n} \times K_{2} \\ & P_{n} \times P_{n} \end{aligned}$ |
| $\begin{aligned} & \text { (p)thin }\left(G_{1} \boxtimes G_{2}\right) \leq(\mathrm{p}) \text { thin }\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\| \\ & \text { (p)thin }\left(G_{1} \boxtimes K_{n}\right)=(\mathrm{p}) \text { thin }\left(G_{1}\right) \\ & \text { (p)thin }{ }_{\text {ind }}\left(G_{1} \boxtimes G_{2}\right) \leq(\mathrm{p}) \text { thinind }\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\| \\ & \text { (p)thin }{ }_{c m p}\left(G_{1} \boxtimes G_{2}\right) \leq(\mathrm{p}) \text { thin }{ }_{\text {cmp }}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\| \\ & \text { (p)thin }{ }_{\text {cmp }}\left(G_{1} \boxtimes G_{2}\right)=(\mathrm{p}) \text { thin }{ }_{\mathrm{cmp}}\left(G_{1}\right) \\ & \hline \end{aligned}$ | $\begin{aligned} & P_{n} \boxtimes P_{n} \\ & \left(K_{n} \square K_{2}\right) \boxtimes\left(K_{n} \square K_{2}\right) \end{aligned}$ |
| $\begin{aligned} & \text { (p)thin }\left(G_{1} * G_{2}\right) \leq(\mathrm{p}) \text { thin }{ }_{\text {ind }}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\| \\ & (\mathrm{p}) \text { thin }{ }_{\text {ind }}\left(G_{1} * G_{2}\right) \leq(\mathrm{p}) \text { thin }_{\text {ind }}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\| \\ & \hline \end{aligned}$ | $\begin{aligned} & \hline K_{n} * 2 K_{1} \\ & n K_{2} * n K_{2} \\ & \hline \end{aligned}$ |
|  | $\begin{aligned} & \hline K_{n} \diamond K_{2} \\ & n K_{2} \diamond K_{1} \\ & \hline \end{aligned}$ |
| (p)thin $\left(G_{1} \ltimes G_{2}\right) \leq(\mathrm{p}) \operatorname{thin}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\|$ <br> (p) $\operatorname{thin}_{\text {ind }}\left(G_{1} \ltimes G_{2}\right) \leq(\mathrm{p})$ thin $_{\text {ind }}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\|$ <br> (p) thin $_{\mathrm{cmp}}\left(G_{1} \ltimes G_{2}\right) \leq(\mathrm{p})$ thin $_{\mathrm{cmp}}\left(G_{1}\right) \cdot\left\|V\left(G_{2}\right)\right\|$ | $\begin{aligned} & \hline K_{2} \ltimes n K_{2} \\ & K_{2} \ltimes\left(K_{n} \square K_{2}\right) \end{aligned}$ |
| $\begin{aligned} & (\mathrm{p}) \operatorname{thin}\left(G_{1} \circ G_{2}\right) \leq\left\|V\left(G_{1}\right)\right\| \cdot(\mathrm{p}) \operatorname{thin} \mathrm{ind}\left(G_{2}\right) \\ & (\mathrm{p}) \operatorname{thin}_{\text {ind }}\left(G_{1} \circ G_{2}\right) \leq\left\|V\left(G_{1}\right)\right\| \cdot(\mathrm{p}) \operatorname{thin}_{\text {ind }}\left(G_{2}\right) \end{aligned}$ | $\begin{aligned} & \hline K_{2} \circ K_{n} \\ & n K_{2} \circ K_{1} \\ & \left(K_{n} \square K_{2}\right) \circ K_{1} \\ & \hline \end{aligned}$ |

Table 1: We summarize the upper bounds (when needed, graphs with an asterisk are not complete). We also summarize the families of graphs with bounded parameters whose product have high thinness, used to show the nonexistence of bounds in terms of certain parameters. Recall that the lexicographic, homomorphic, and hom-product are not necessarily commutative.

- It would be interesting to find tighter bounds, in the case in which it is possible.
- It remains as an open problem the computational complexity of computing the independent and complete (proper) thinness for general graphs. For co-comparability graphs, both the independent thinness and independent proper thinness are exactly the chromatic number, which can be computed in polynomial time [15].
- Regarding lower and upper bounds in Section 3, can we have some similar results for proper thinness? Or for the independent and complete versions of (proper) thinness?
- Does there exist a graph $G$ such that $\operatorname{thin}(G)>|V(G)| / 2$ ?
- Does there exist a co-comparability graph $G$ such that pthin $(G)>$ $|V(G)| / 2$ ?


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## References

[1] M. Asté, F. Havet, and C. Linhares Sales. Grundy number and products of graphs. Discrete Mathematics, 310(9):1482-1490, 2010.
[2] R. Bačík. Structure of Graph Homomorphisms. PhD thesis, Simon Fraser University, Vancouver, 1997.
[3] T. Bartnicki, B. Brešar, M. Kovše, Z. Miechowicz, and I. Peterin. Game chromatic number of Cartesian product graphs. The Electronic Journal of Combinatorics, 15(1):R72, 2008.
[4] C. Bentz. Weighted and locally bounded list-colorings in split graphs, cographs, and partial $k$-trees. Theoretical Computer Science, 782:11-29, 2019.
[5] F. Bonomo and D. De Estrada. On the thinness and proper thinness of a graph. Discrete Applied Mathematics, 261:78-92, 2019.
[6] F. Bonomo, I. Koch, P. Torres, and M. Valencia-Pabon. $k$-tuple colorings of the Cartesian product of graphs. Discrete Applied Mathematics, 245:177-182, 2018.
[7] F. Bonomo, S. Mattia, and G. Oriolo. Bounded coloring of cocomparability graphs and the pickup and delivery tour combination problem. Theoretical Computer Science, 412(45):6261-6268, 2011.
[8] V.A. Campos, A. Gyárfás, F. Havet, C. Linhares Sales, and F. Maffray. New bounds on the Grundy number of products of graphs. Journal of Graph Theory, 71(1):78-88, 2012.
[9] S. Chandran, W. Imrich, R. Mathew, and D. Rajendraprasad. Boxicity and cubicity of product graphs. European Journal of Combinatorics, 45:100-109, 2015.
[10] S. Chandran, C. Mannino, and G. Oriolo. The indepedent set problem and the thinness of a graph. Manuscript, 2007.
[11] N.P. Chiang and H.L. Fu. On the achromatic number of the Cartesian product $G_{1} \times G_{2}$. Australasian Journal of Combinatorics, 6:111-117, 1992.
[12] J. Chvátalová. Optimal labelling of a product of two paths. Discrete Mathematics, 11(3):249-253, 1975.
[13] D. Corneil, H. Lerchs, and L. Stewart Burlingham. Complement reducible graphs. Discrete Applied Mathematics, 3(3):163-174, 1981.
[14] D. Geller and S. Stahl. The chromatic number and other functions of the lexicographic product. Journal of Combinatorial Theory. Series B, 19(1):87-95, 1975.
[15] M.C. Golumbic. The complexity of comparability graph recognition and coloring. Computing, 18:199-208, 1977.
[16] F. Harary. Graph Theory. Addison-Wesley, Reading, MA, 1994.
[17] B. Hartnell and D. Rall. Domination in Cartesian products: Vizing's conjecture. In T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, editors, Domination in Graphs-Advanced Topics, pages 163-189. Dekker, New York, 1998.
[18] J.T. Hedetniemi. Problems in Domination and Graph Products. PhD thesis, Clemson University, 2016.
[19] S. Hedetniemi. Homomorphisms of graphs and automata. Technical Report 03105-44-T, University of Michigan, 1966.
[20] P. Hell and D. Miller. Achromatic numbers and graph operations. Discrete Mathematics, 108(1-3):297-305, 1992.
[21] W. Imrich and S. Klavžar. Product Graphs: Structure and Recognition. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
[22] W. Imrich, S. Klavžar, and D.F. Rall. Graphs and Their Cartesian Product. Topics in Graph Theory. A K Peters, Wellesley, MA, 2008.
[23] M. Jakovac and I. Peterin. On the b-chromatic number of some graph products. Studia Scientiarum Mathematicarum Hungarica, 49(2):156169, 2012.
[24] T. Jensen and B. Toft. Graph coloring problems. John Wiley \& Sons, New York, 1995.
[25] S. Klavžar. Coloring graph products - a survey. Discrete Mathematics, 155:135-145, 1996.
[26] I. Koch and I. Peterin. The b-chromatic index of direct product of graphs. Discrete Applied Mathematics, 190-191:109-117, 2015.
[27] M. Kouider and M. Mahéo. Some bounds for the b-chromatic number of a graph. Discrete Mathematics, 256(1-2):267-277, 2002.
[28] M. Kouider and M. Mahéo. The b-chromatic number of the Cartesian product of two graphs. Studia Scientiarum Mathematicarum Hungarica, 44(1):49-55, 2007.
[29] L. Mančinska and D. Roberson. Graph homomorphisms for quantum players. Journal of Combinatorial Theory. Series B, 118:228-267, 2012.
[30] C. Mannino, G. Oriolo, F. Ricci, and S. Chandran. The stable set problem and the thinness of a graph. Operations Research Letters, 35:19, 2007.
[31] H. Meyniel. A new property of critical imperfect graphs and some consequences. European Journal of Combinatorics, 8:313-316, 1987.
[32] H.M. Mulder. Interval-regular graphs. Discrete Mathematics, 41:253269, 1982.
[33] S. Olariu. An optimal greedy heuristic to color interval graphs. Information Processing Letters, 37:21-25, 1991.
[34] F.S. Roberts. Indifference graphs. In F. Harary, editor, Proof Techniques in Graph Theory, pages 139-146. Academic Press, New York, 1969.
[35] G. Sabidussi. Vertex-transitive graphs. Monatshefte für Mathematik, 68:426-438, 1964.
[36] Y. Shitov. Counterexamples to Hedetniemi's conjecture. Annals of Mathematics, 190(2):663-667, 2019.
[37] M. Valencia-Pabon and J. Vera. Independence and coloring properties of direct product of some vertex-transitive graphs. Discrete Mathematics, 306:2275-2281, 2006.
[38] D. West. Introduction to Graph Theory. Prentice Hall, 2nd edition, 2000.
[39] X. Zhu. The fractional version of Hedetniemi's conjecture is true. European Journal of Combinatorics, 32:1168-1175, 2011.

## Appendix A. Omitted Proofs of Section 4

The operations involved in these theorems are defined over a pair of graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ such that $\left|V_{1}\right|=n_{1},\left|V_{2}\right|=n_{2}$ and $V_{1} \cap V_{2}=\emptyset$. Besides, for some of the following proofs, we consider an implicit ordering and partition for both $V_{1}$ and $V_{2}$, as defined next. The ordering of $V_{1}$ will be denoted by $v_{1}, \ldots, v_{n_{1}}$ and that of $V_{2}$ by $w_{1}, \ldots, w_{n_{2}}$. Moreover, if the value $t_{i}$ of some variation of thinness of $G_{i}$ (for $i \in\{1,2\}$ ) is involved in the bound to be proved, the implicit ordering is one consistent, according to the specified variation of thinness, with a partition $\left(V_{i}^{1}, \ldots, V_{i}^{t_{i}}\right)$. If, otherwise, only the cardinality $n_{i}$ of $V_{i}$ is involved in the bound, the implicit ordering is an arbitrary one. For instance, if $G_{1}$ is a proper $t_{1}$-independent-thin graph, and $t_{1}$ is involved in the bound to be proved, it means that the implicit ordering and partition of $V_{1}$ are strongly consistent and all the $t_{1}$ parts of the partition are independent sets.

Theorem 33. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \vee G_{2}\right) \leq f\left(G_{1}\right)+$ $f\left(G_{2}\right)$, for $f \in\left\{\right.$ thin $_{\mathrm{cmp}}$, pthin $\left._{\mathrm{cmp}}\right\}$. Moreover, if $G_{2}$ is complete, then $\operatorname{thin}_{\mathrm{cmp}}\left(G_{1} \vee G_{2}\right)=\operatorname{thin}_{\mathrm{cmp}}\left(G_{1}\right)$.

Proof. Let $G_{1}$ and $G_{2}$ be two graphs with complete thinness (resp. complete proper thinness) $t_{1}$ and $t_{2}$, respectively.

For $G=G_{1} \vee G_{2}$, define a partition with $t_{1}+t_{2}$ complete sets as the union of the two partitions, and $v_{1}, \ldots, v_{n_{1}}, w_{1}, \ldots, w_{n_{2}}$ as an ordering of $V(G)$.

Let $x, y, z$ be three vertices of $V(G)$ such that $x<y<z, x z \in E(G)$, and $x$ and $y$ are in the same class of the partition of $V(G)$. Then, in particular, $x$ and $y$ both belong either to $V_{1}$ or to $V_{2}$. If $z$ belongs to the same graph, then $y z \in E(G)$ because the ordering and partition restricted to each of $G_{1}$ and $G_{2}$ are consistent. Otherwise, $z$ is also adjacent to $y$ by the definition of join.

We have proved that the defined partition and ordering are consistent, and thus that $\operatorname{thin}_{\mathrm{cmp}}\left(G_{1} \vee G_{2}\right) \leq \operatorname{thin}_{\mathrm{cmp}}\left(G_{1}\right)+\operatorname{thin}_{\mathrm{cmp}}\left(G_{2}\right)$. The proof of the strong consistency, given the strong consistency of the partition and ordering of each of $G_{1}$ and $G_{2}$, is symmetric and implies pthin $\operatorname{cmp}\left(G_{1} \vee G_{2}\right) \leq$ $\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right)+\operatorname{pthin}_{\mathrm{cmp}}\left(G_{2}\right)$.

Suppose now that $G_{2}$ is complete (in particular, $t_{2}=1$ ). Since $G_{1}$ is an induced subgraph of $G_{1} \vee G_{2}$, then $\operatorname{thin}_{\mathrm{cmp}}\left(G_{1} \vee G_{2}\right) \geq \operatorname{thin}_{\mathrm{cmp}}\left(G_{1}\right)$. For $G=G_{1} \vee G_{2}$, define a partition $V^{1}, \ldots, V^{t_{1}}$ such that $V^{1}=V_{1}^{1} \cup V_{2}^{1}$ and
$V^{i}=V_{1}^{i}$ for $i=2, \ldots, t_{1}$, and define $v_{1}, \ldots, v_{n_{1}}, w_{1}, \ldots, w_{n_{2}}$ as an ordering of the vertices. By definition of join, $V^{1}$ is a complete set of $G$, as well as $V^{2}, \ldots, V^{t_{1}}$.

Let $x, y, z$ be three vertices of $V(G)$ such that $x<y<z, x z \in E(G)$, and $x$ and $y$ are in the same class of the partition of $V(G)$. If $z$ belongs to $V_{2}$, then $z$ is also adjacent to $y$, because it is adjacent to every vertex in $G-z$. If $z$ belongs to $V_{1}$, then $x, y$, and $z$, belong to $V_{1}$ due to the definition of the order of the vertices, and thus $y z \in E(G)$ because the ordering and partition restricted to $G_{1}$ are consistent. This proves thin $\mathrm{cmp}\left(G_{1} \vee G_{2}\right) \leq \operatorname{thin}_{\mathrm{cmp}}\left(G_{1}\right)$, thus in this case thin $\operatorname{cmp}\left(G_{1} \vee G_{2}\right)=\operatorname{thin}_{\mathrm{cmp}}\left(G_{1}\right)$.

Theorem 49. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \times G_{2}\right) \leq f_{\text {ind }}\left(G_{1} \times\right.$ $\left.G_{2}\right) \leq f_{\text {ind }}\left(G_{1}\right)\left|V\left(G_{2}\right)\right| \leq f\left(G_{1}\right) \chi\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$, for $f \in\{$ thin, pthin $\}$.

Proof. Let $G_{1}$ be a $k$-independent-thin (resp. proper $k$-independent-thin) graph.

Consider $V_{1} \times V_{2}$ lexicographically ordered with respect to the defined orderings of $V_{1}$ and $V_{2}$. Consider now the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq k, 1 \leq j \leq n_{2}}$ such that $V^{i, j}=\left\{\left(v, w_{j}\right): v \in V_{1}^{i}\right\}$ for each $1 \leq i \leq k, 1 \leq j \leq n_{2}$. Since the graphs considered here are loopless, each $V^{i, j}$ is an independent set. We will show now that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $p=q=r$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

Case 2: $p=q<r$. In this case, $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are in different classes. So, suppose $G_{1}$ is proper $k$-independent-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$ and $v_{q}=v_{p}$ and $v_{r}$ belong to the same class in $G_{1}$. In particular, since the classes are independent sets, $v_{p} v_{r} \notin E_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \times G_{2}$ if and only if $w_{i} w_{\ell} \in E_{2}$ and $v_{p} v_{r} \in E_{1}$, a contradiction.

Case 3: $p<q=r$. In this case, $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are in different classes. So suppose $G_{1}$ is $k$-independent-thin (resp. proper $k$-independentthin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$ and $v_{p}$ and $v_{q}=v_{r}$ belong to the same class in $G_{1}$. In particular, since the classes are independent sets, $v_{p} v_{r} \notin E_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \times G_{2}$ if and only if $w_{i} w_{\ell} \in E_{2}$ and $v_{p} v_{r} \in E_{1}$, a contradiction.

Case 4: $p<q<r$. Suppose first $G_{1}$ is $k$-independent-thin (resp. proper
$k$-independent-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$ and $v_{p}, v_{q}$ belong to the same class in $G_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \times G_{2}$ if and only if $w_{i} w_{\ell} \in E_{2}$ and $v_{p} v_{r} \in E_{1}$. Since the ordering and the partition are consistent (resp. strongly consistent) in $G_{1}$, $v_{r} v_{q} \in E_{1}$ and so ( $v_{r}, w_{\ell}$ ) and ( $v_{q}, w_{j}$ ) are adjacent in $G_{1} \times G_{2}$, as required. Now suppose that $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$ and $v_{q}, v_{r}$ belong to the same class in $G_{1}$. Since the ordering and the partition are strongly consistent in $G_{1}$, if $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \times G_{2}$ then $v_{p} v_{q} \in E_{1}$ and so $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \times G_{2}$, as required.

The last inequality is a consequence of Remark [1.
Theorem54, Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \boxtimes G_{2}\right) \leq f\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$ for $f \in\left\{\right.$ thin, pthin, thin $_{\text {cmp }}$, pthin $_{\text {cmp }}$, thin $_{\mathrm{ind}}$, pthin $\left._{\mathrm{ind}}\right\}$. Moreover, if $G_{2}$ is complete, then $f\left(G_{1} \boxtimes G_{2}\right)=f\left(G_{1}\right)$ for $f \in\left\{\right.$ thin, pthin, thin ${ }_{\text {cmp }}$, pthin $\left._{\text {cmp }}\right\}$.

Proof. If $G_{2}$ is complete then $G_{1} \boxtimes G_{2}=G_{1} \bullet G_{2}$, so by Theorem 36, $\operatorname{thin}\left(G_{1} \boxtimes G_{2}\right)=\operatorname{thin}\left(G_{1}\right), \operatorname{pthin}\left(G_{1} \boxtimes G_{2}\right)=\operatorname{pthin}\left(G_{1}\right), \operatorname{thin}_{\mathrm{cmp}}\left(G_{1} \boxtimes G_{2}\right)=$ $\operatorname{thin}_{\mathrm{cmp}}\left(G_{1}\right)$, and pthin $\mathrm{cmp}\left(G_{1} \boxtimes G_{2}\right)=\operatorname{pthin}_{\mathrm{cmp}}\left(G_{1}\right)$.

Let $G_{1}$ be a $k$-thin (resp. proper $k$-thin) graph. Consider $V_{1} \times V_{2}$ lexicographically ordered with respect to the defined orderings of $V_{1}$ and $V_{2}$. Consider now the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq k, 1 \leq j \leq n_{2}}$ such that $V^{i, j}=\left\{\left(v, w_{j}\right): v \in V_{1}^{i}\right\}$ for each $1 \leq i \leq k, 1 \leq j \leq n_{2}$.

If the sets in the partition of $V_{1}$ are furthermore complete or independent, so are the sets of the partition of $V_{1} \times V_{2}$.

We will show now that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $p=q=r$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

Case 2: $p=q<r$. In this case, $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are in different classes. So suppose $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$ if and only if either $i=\ell$ and $v_{p} v_{r} \in E_{1}$, or $i \neq \ell, w_{i} w_{\ell} \in E_{2}$, and $v_{p} v_{r} \in E_{1}$. In the first case, $\left(v_{p}, w_{i}\right)=\left(v_{q}, w_{j}\right)$, a contradiction. In the second case, since $p=q$ and $j=\ell,\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent, as required.

Case 3: $p<q=r$. In this case, $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are in different classes. So suppose $G_{1}$ is $k$-thin (resp. proper $k$-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$
belong to the same class, i.e., $i=j$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$ if and only if either $i=\ell$ and $v_{p} v_{r} \in E_{1}$, or $i \neq \ell, w_{i} w_{\ell} \in E_{2}$, and $v_{p} v_{r} \in E_{1}$. In the first case, $\left(v_{r}, w_{\ell}\right)=\left(v_{q}, w_{j}\right)$, a contradiction. In the second case, since $q=r$ and $i=j,\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent, as required.

Case 4: $p<q<r$. Suppose first that $G_{1}$ is $k$-thin (resp. proper $k$-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$ and $v_{p}, v_{q}$ belong to the same class in $G_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$ if and only if either $i=\ell$ and $v_{p} v_{r} \in E_{1}$, or $i \neq \ell, w_{i} w_{\ell} \in E_{2}$, and $v_{p} v_{r} \in E_{1}$. In the first case, $j=\ell$ and since the ordering and the partition are consistent (resp. strongly consistent) in $G_{1}, v_{r} v_{q} \in E_{1}$ and so $\left(v_{r}, w_{\ell}\right)$ and ( $v_{q}, w_{j}$ ) are adjacent in $G_{1} \boxtimes G_{2}$. In the second case, since $v_{p}, v_{q}$ belong to the same class in $G_{1}$ and the order and the partition are consistent, $v_{q} v_{r} \in E_{1}$. Since $i=j$, $w_{j} w_{\ell} \in E_{2}$. So $\left(v_{r}, w_{\ell}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$, as required.

Now suppose that $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$ and $v_{q}$, $v_{r}$ belong to the same class in $G_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$ if and only if either $i=\ell$ and $v_{p} v_{r} \in E_{1}$, or $i \neq \ell, w_{i} w_{\ell} \in E_{2}$, and $v_{p} v_{r} \in E_{1}$. In the first case, $i=j$ and since the ordering and the partition are strongly consistent in $G_{1}, v_{p} v_{q} \in E_{1}$ and so $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$. In the second case, since $v_{q}, v_{r}$ belong to the same class in $G_{1}$ and the order and the partition are strongly consistent, $v_{p} v_{q} \in E_{1}$. Since $j=\ell, w_{i} w_{j} \in E_{2}$. So ( $v_{p}, w_{i}$ ) and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \boxtimes G_{2}$, as required.

Theorem 59. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} * G_{2}\right) \leq f_{\text {ind }}\left(G_{1} *\right.$ $\left.G_{2}\right) \leq f_{\text {ind }}\left(G_{1}\right)\left|V\left(G_{2}\right)\right| \leq f\left(G_{1}\right) \chi\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$, for $f \in\{$ thin, pthin $\}$.

Proof. Let $G_{1}$ be a $k$-independent-thin (resp. proper $k$-independent-thin) graph. Consider $V_{1} \times V_{2}$ lexicographically ordered with respect to the defined orderings of $V_{1}$ and $V_{2}$. Consider now the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq k, 1 \leq j \leq n_{2}}$ such that $V^{i, j}=\left\{\left(v, w_{j}\right): v \in V_{1}^{i}\right\}$ for each $1 \leq i \leq k, 1 \leq j \leq n_{2}$. Since the graphs considered here are loopless, each $V^{i, j}$ is an independent set.

We will show now that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $p=q=r$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

Case 2: $p=q<r$. In this case, $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are in different classes. So suppose $G_{1}$ is proper $k$-independent-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$
belong to the same class, i.e., $j=\ell$ and $v_{q}=v_{p}$ and $v_{r}$ belong to the same class in $G_{1}$. In particular, since the classes are independent sets, $v_{p} v_{r} \notin E_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and ( $v_{r}, w_{\ell}$ ) are adjacent in $G_{1} * G_{2}$ if and only if either $v_{p} v_{r} \in E_{1}$ or $w_{i} w_{\ell} \in E_{2}$, so, in this case, if and only if $w_{i} w_{\ell} \in E_{2}$. Since $j=\ell,\left(v_{p}, w_{i}\right)$ is adjacent to $\left(v_{q}, w_{j}\right)$ in $G_{1} * G_{2}$, as required.

Case 3: $p<q=r$. In this case, $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are in different classes. So suppose $G_{1}$ is $k$-independent-thin (resp. proper $k$-independentthin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$ and $v_{p}$ and $v_{q}=v_{r}$ belong to the same class in $G_{1}$. In particular, since the classes are independent sets, $v_{p} v_{r} \notin E_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} * G_{2}$ if and only if either $v_{p} v_{r} \in E_{1}$ or $w_{i} w_{\ell} \in E_{2}$, so, in this case, if and only if $w_{i} w_{\ell} \in E_{2}$. Since $i=j,\left(v_{r}, w_{\ell}\right)$ is adjacent to $\left(v_{q}, w_{j}\right)$ in $G_{1} * G_{2}$, as required.

Case 4: $p<q<r$. Suppose first that $G_{1}$ is $k$-independent-thin (resp. proper $k$-independent-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$ and $v_{p}, v_{q}$ belong to the same class in $G_{1}$. Vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} * G_{2}$ if and only if either $v_{p} v_{r} \in E_{1}$ or $w_{i} w_{\ell} \in E_{2}$. In the first case, since the ordering and the partition are consistent (resp. strongly consistent) in $G_{1}, v_{r} v_{q} \in E_{1}$ and so ( $v_{r}, w_{\ell}$ ) and ( $v_{q}, w_{j}$ ) are adjacent in $G_{1} * G_{2}$, as required. In the second case, since $i=j,\left(v_{r}, w_{\ell}\right)$ is adjacent to $\left(v_{q}, w_{j}\right)$ in $G_{1} * G_{2}$, as required.

Now suppose that $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$ and $v_{q}, v_{r}$ belong to the same class in $G_{1}$.

Vertices $\left(v_{p}, w_{i}\right)$ and ( $v_{r}, w_{\ell}$ ) are adjacent in $G_{1} * G_{2}$ if and only if either $v_{p} v_{r} \in E_{1}$ or $w_{i} w_{\ell} \in E_{2}$. In the first case, since the ordering and the partition are strongly consistent in $G_{1}, v_{p} v_{q} \in E_{1}$ and so $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} * G_{2}$, as required. In the second case, since $j=\ell,\left(v_{p}, w_{i}\right)$ is adjacent to $\left(v_{q}, w_{j}\right)$ in $G_{1} * G_{2}$, as required.

The last inequality is a consequence of Remark 1.
Theorem67. Let $G_{1}$ and $G_{2}$ be graphs. Then $f\left(G_{1} \ltimes G_{2}\right) \leq f\left(G_{1}\right)\left|V\left(G_{2}\right)\right|$ for $f \in\left\{\right.$ thin, pthin, thin $_{\text {cmp }}$, pthin $_{\text {cmp }}$, thin $_{\text {ind }}$, pthin $\left._{\text {ind }}\right\}$.

Proof. Let $G_{1}$ be a $k$-thin (resp. proper $k$-thin) graph. Consider $V_{1} \times V_{2}$ lexicographically ordered with respect to the defined orderings of $V_{1}$ and $V_{2}$. Consider now the partition $\left\{V^{i, j}\right\}_{1 \leq i \leq k, 1 \leq j \leq n_{2}}$ such that $V^{i, j}=\left\{\left(v, w_{j}\right): v \in\right.$ $\left.V_{1}^{i}\right\}$ for each $1 \leq i \leq k, 1 \leq j \leq n_{2}$.

If the sets in the partition of $V_{1}$ are furthermore complete or independent,
the sets of the partition of $V_{1} \times V_{2}$ are so, because $G_{2}$ is loopless.
We will show now that this ordering and partition of $V_{1} \times V_{2}$ are consistent (resp. strongly consistent). Let $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ be three vertices appearing in that ordering in $V_{1} \times V_{2}$.

Case 1: $p=q=r$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

Case 2: $p=q<r$. In this case, $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are in different classes, and they are adjacent. So the conditions both for thinness and proper thinness are satisfied.

Case 3: $p<q=r$. In this case, $\left(v_{q}, w_{j}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are in different classes, and they are adjacent. So the conditions both for thinness and proper thinness are satisfied.

Case 4: $p<q<r$. Suppose first that $G_{1}$ is $k$-thin (resp. proper $k$-thin) and $\left(v_{p}, w_{i}\right),\left(v_{q}, w_{j}\right)$ belong to the same class, i.e., $i=j$ and $v_{p}, v_{q}$ belong to the same class in $G_{1}$. Since $p<r$, vertices $\left(v_{p}, w_{i}\right)$ and ( $v_{r}, w_{\ell}$ ) are adjacent in $G_{1} \ltimes G_{2}$ if and only if $v_{p} v_{r} \in E_{1}$ and $w_{i} w_{\ell} \notin E_{2}$. Since $v_{p}, v_{q}$ belong to the same class in $G_{1}$ and the order and the partition are consistent, $v_{q} v_{r} \in E_{1}$. Since $i=j, w_{j} w_{\ell} \notin E_{2}$. So $\left(v_{r}, w_{\ell}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \ltimes G_{2}$, as required.

Now suppose that $G_{1}$ is proper $k$-thin and $\left(v_{q}, w_{j}\right),\left(v_{r}, w_{\ell}\right)$ belong to the same class, i.e., $j=\ell$ and $v_{q}, v_{r}$ belong to the same class in $G_{1}$. Since $p<r$, vertices $\left(v_{p}, w_{i}\right)$ and $\left(v_{r}, w_{\ell}\right)$ are adjacent in $G_{1} \ltimes G_{2}$ if and only if $v_{p} v_{r} \in E_{1}$ and $w_{i} w_{\ell} \notin E_{2}$. Since $v_{q}, v_{r}$ belong to the same class in $G_{1}$ and the order and the partition are strongly consistent, $v_{p} v_{q} \in E_{1}$. Since $j=\ell, w_{i} w_{j} \notin E_{2}$. So $\left(v_{p}, w_{i}\right)$ and $\left(v_{q}, w_{j}\right)$ are adjacent in $G_{1} \ltimes G_{2}$, as required.

## Appendix B. Proofs from the manuscript by Chandran, Mannino and Oriolo

Theorem 3. [10] For every $t \geq 1$, thin $\left(\overline{t K_{2}}\right)=t$.
Proof. Let $G=\overline{t K_{2}}, t \geq 1$. Let $V(G)=\left\{x_{1}, y_{1}, \ldots, x_{t}, y_{t}\right\}$ and suppose that $\left(x_{i}, y_{i}\right)$, for $1 \leq i \leq t$, are the only pairs of non-adjacent vertices. If we define, for $1 \leq i \leq t, V^{i}=\left\{x_{i}, y_{i}\right\}$, then every total order on the vertices of $V(G)$ is consistent with this partition. We now show that $G$ is not $(t-1)$ thin. Suppose the contrary, that is, there exist an ordering $<$ on $V(G)$ and a partition of $V(G)$ into $t-1$ classes $\left(V^{1}, \ldots, V^{t-1}\right)$ which are consistent.

For every class, denote by $f\left(V^{h}\right)$ the smallest element of $V^{h}$ with respect to the ordering $<$. Clearly, there exists at least one pair $\left\{x_{i}, y_{i}\right\}, 1 \leq i \leq t$, such that $\bigcup_{h}\left\{f\left(V^{h}\right)\right\} \cap\left\{x_{i}, y_{i}\right\}=\emptyset$. Without loss of generality, assume that $x_{i}<y_{i}$. Let $V^{q}$ be the class which $x_{i}$ belongs to. It follows that $y_{i}$ is adjacent to $f\left(V^{q}\right)$ and nonadjacent to $x_{i}$; moreover, $y_{i}>x_{i}>f\left(V^{q}\right)$. But this is a contradiction.

Recall that the vertex isoperimetric peak of a graph $G$, denoted as $b_{v}(G)$, is defined as $b_{v}(G)=\max _{s} \min _{X \subset V,|X|=s}|N(X) \cap(V(G) \backslash X)|$, i.e., the maximum over $s$ of the lower bounds for the number of boundary vertices (vertices outside the set with a neighbor in the set) in sets of size $s$.

Theorem 7. [10] For every graph $G$ with at least one edge, $\operatorname{thin}(G) \geq$ $b_{v}(G) / \Delta(G)$.

Proof. Let $t$ be the thinness of $G, \Delta=\Delta(G), k=b_{v}(G)$ and $s$ realizing the vertex isoperimetric peak of $G$. There exist a partition $V^{1}, \ldots, V^{t}$ and an ordering $v_{1}, \ldots, v_{n}$ of $V(G)$ that are consistent. Let $S=\left\{v_{n-s+1}, \ldots, v_{n}\right\}$, that is, $S$ is the set of the $s$ greatest vertices of $G$. Let $N_{S}=N(S) \cap(V(G) \backslash S)$. Note that, since every node in $N_{S}$ is outside $S$, each of them is smaller than any vertex in $S$.

We claim that for each $i, 1 \leq i \leq t,\left|V^{i} \cap N_{S}\right| \leq \Delta$. (That is, none of the classes can contain more than $\Delta$ vertices from $N_{S}$.) Suppose $V^{i}$ contains at least $\Delta+1$ vertices from $N_{S}$. Let $x$ be the smallest vertex in $V^{i} \cap N_{S}$. Clearly, $x$ is adjacent to some vertex $y \in S$, since $x \in N_{S}$. Then, $y$ has to be adjacent to all the vertices in $V^{i} \cap N_{S}$, since all of them are smaller than $y$. Thus, the degree of $y$ has to be at least $\Delta+1$, a contradiction. So, each class contains only at most $\Delta$ vertices of $N_{S}$. It follows that there are at least $\left|N_{S}\right| / \Delta \geq k / \Delta$ classes.

The following lower bound for the vertex isoperimetric peak of the grid $G_{r}$ was proved by Chvátalova.
Lemma 74. [12] For every $r, b_{v}\left(G_{r}\right) \geq r$.
Now, combining the above results, we get
Corollary 8. [10] For every $r \geq 2$, $\operatorname{thin}\left(G R_{r}\right) \geq r / 4$.

We close with a couple of general upper bounds on the thinness of a graph. First, we need to introduce some notation and a lemma whose proof is straightforward.

Let $G$ be a graph. A partition $V^{1}, \ldots, V^{k}$ of $V(G)$ is valid if there exists an ordering which is consistent with the partition. A class $V^{i}$ is called a singleton class if $\left|V^{i}\right|=1$, otherwise $V^{i}$ is a non-singleton class.

Lemma 75. [10] Let $G$ be a graph with a valid partition $V^{1}, \ldots, V^{k}$ of $V(G)$. Let $X=\{v \in V: v$ belongs to a singleton class $\}$. Then there exists an ordering $v_{1}, \ldots v_{n}$ of $V(G)$ which is consistent with the partition such that the vertices of $X$ are the smallest ones in the order.

Theorem 14. [10] Let $G$ be a graph. Then $\operatorname{thin}(G) \leq|V(G)|-\log (|V(G)|) / 4$.
Proof. Let $t$ be the thinness of $G$, and $n=|V(G)|$. Then there exists a valid partition of $V(G)$ using $t$ classes. We can assume that there are at least 2 singleton classes in this valid partition. Otherwise, if every class (except possibly one), contains at least 2 vertices, then clearly $t \leq(n+1) / 2 \leq$ $n-\log (n) / 4$, as required. Let $X=\{u \in V: u$ belongs to a singleton class $\}$. Let $|X|=h$ and $|V(G) \backslash X|=q$ (clearly, $h+q=n$ ).
Claim. $h \leq 2^{q}$.
Let $V \backslash X=\left\{y_{1}, y_{2}, \ldots, y_{q}\right\}$. We define a function $f: X \rightarrow\{0,1\}^{q}$ as follows: for $x \in X, f(x)=\left(a_{1}, a_{2}, \ldots, a_{q}\right)$, where $a_{i}=1$ if $x$ is adjacent to $y_{i}$, and $a_{i}=0$ otherwise. Assume for contradiction that $h>2^{q}$. Then clearly there exist two vertices $r, s \in X$ such that $f(r)=f(s)\left(\right.$ since $\left.\left|\{0,1\}^{q}\right|=2^{q}\right)$. Now, we claim that even if we merge the two classes containing $r$ and $s$ into one class, it will remain a valid partition of $V(G)$. This will provide the required contradiction, since we have assumed that $t$ is the thinness of $G$. By Lemma 755, there exists an ordering $\left\{v_{1}, \ldots, v_{n}\right\}$ which is consistent with the given partition, such that the vertices of the singleton classes are the smallest, i.e. $X=\left\{v_{1}, \ldots, v_{h}\right\}$. In fact, it is possible to further assume (without violating the validity of the partition) that $r=v_{h-1}$ and $s=v_{h}$. Now, consider merging the two classes containing $r$ and $s$ into one class. It is not difficult to verify that the resulting partition is still valid since the previous ordering is still consistent: there is no conflict due to vertices of $X \backslash\{r, s\}$, since they are all smaller than $r$. Also, since $f(r)=f(s)$, it is easy to see (from the definition of the function $f$ ) that for any vertex $y \in V(G) \backslash X, y$ is adjacent to $r$ if and only if it is adjacent to $s$. Therefore,
there cannot be any conflict due to the vertices of $V(G) \backslash X$. Thus, we have a valid partition of $V(G)$ using only $t-1$ classes, contradicting the assumption that the thinness of $G$ is $t$. We infer that $h \leq 2^{q}$. $\diamond$

Now, suppose $h>n-\log (n) / 2$. Then, from $2^{q} \geq h$, we get $q \geq \log (n-$ $\log (n) / 2)$. But this leads to a contradiction since $n=|X|+|V \backslash X|=h+q \geq$ $n-\log (n) / 2+\log (n-\log (n) / 2)>n$. So, we infer that $h \leq n-\log (n) / 2$. Then, $q=n-h \geq \log (n) / 2$. Note that $t \leq h+q / 2$, since every non-singleton class contains at least 2 vertices, and there are only a total of $q$ vertices in non-singleton classes. Thus, $t \leq h+q-q / 2=n-q / 2 \leq n-\log (n) / 4$, as required.

We can also get a bound in terms of the maximum degree of the graph by modifying the above proof.

Theorem [15. [10] Let $G$ be a graph. Then $\operatorname{thin}(G) \leq|V(G)|(\Delta(G)+$ $3) /(\Delta(G)+4)$.

Proof. Let $n=|V(G)|$ and $\Delta=\Delta(G)$. We modify the proof of Theorem 14 as follows.
Claim. $h \leq q(\Delta+2) / 2$.
For $x \in X$, we denote by $|f(x)|$ the number of 1's in the $q$-tuple $f(x)$. Clearly, $\sum_{x \in X}|f(x)| \leq \sum_{y \in V(G) \backslash X}$ degree $(y) \leq q \Delta$. Now partition $X$ into two classes as follows: Let $X_{1}=\{x \in X:|f(x)|=1\}$ and $X_{2}=X \backslash X_{1}$. Note that $\left|X_{1}\right| \leq q$, since otherwise, if $\left|X_{1}\right|>q$, there will be two vertices $r, s \in X_{1}$ such that $f(r)=f(s)$, leading to a contradiction, as described in the proof of Theorem [14. We can assume that there is no vertex $x \in X_{2}$ with $|f(x)|=0$, since if such a vertex existed we could have merged the class of this vertex with that of the greatest vertex in $X$, contradicting the minimality of the size of the partition. Thus, for every vertex $x \in X_{2}$, $|f(x)| \geq 2$. Since $\sum_{x \in X_{2}}|f(x)| \leq \sum_{x \in X}|f(x)| \leq q \Delta$, we have $\left|X_{2}\right| \leq q \Delta / 2$. Thus, $h=|X|=\left|X_{1}\right|+\left|X_{2}\right| \leq q(\Delta+2) / 2 . \diamond$

It follows that $q=n-h \geq n-q(\Delta+2) / 2$. Rearranging, we get $q \geq 2 n /(\Delta+4)$. Now, $t \leq h+q / 2=n-q / 2 \leq n(\Delta+3) /(\Delta+4)$.


[^0]:    Email addresses: fbonomo@dc.uba.ar (Flavia Bonomo-Braberman), cgonzalez@dc.uba.ar (Carolina L. Gonzalez), fabiano.oliveira@ime.uerj.br (Fabiano S. Oliveira), moysessj@cos.ufrj.br (Moysés S. Sampaio Jr.), jayme@nce.ufrj.br (Jayme L. Szwarcfiter)

