Uniform and Monotone Line Sum Optimization

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Abstract

The line sum optimization problem asks for a (0,1)-matrix minimizing the sum of given functions evaluated at its row and column sums. We show that the *uniform* problem, with identical row functions and identical column functions, and the *monotone* problem, over matrices with nonincreasing row and column sums, are polynomial time solvable.

Keywords: majorization, column sum, row sum, matrix, degree sequence, graph

MSC: 05A, 15A, 51M, 52A, 52B, 52C, 62H, 68Q, 68R, 68U, 68W, 90B, 90C

1 Introduction

For a positive integer n let $[n] := \{1, 2, ..., n\}$. For an $m \times n$ matrix A let $r_i(A) := \sum_{j=1}^n A_{i,j}$ for $i \in [m]$ be its row sums and let $c_j(A) := \sum_{i=1}^m A_{i,j}$ for $j \in [n]$ be its column sums.

We consider here the following algorithmic problem.

Line Sum Optimization. Given m, n and functions $f_i : \{0, 1, ..., n\} \to \mathbb{Z}$ for $i \in [m]$ and $g_j : \{0, 1, ..., m\} \to \mathbb{Z}$ for $j \in [n]$, find an $m \times n$ (0, 1)-matrix, $A \in \{0, 1\}^{m \times n}$, which minimizes

$$\sum_{i=1}^{m} f_i(r_i(A)) + \sum_{j=1}^{n} g_j(c_j(A)) .$$

For instance, for m=n=4, and functions $f_i(x)=(x-1)^2(x-3)^2$ for $i \in [4]$ and $g_j(x)=(x-2)^2(x-3)^2$ for $j \in [4]$, an optimal solution is the following matrix, with row sums $(r_1,r_2,r_3,r_4)=(3,3,3,1)$ and column sums $(c_1,c_2,c_3,c_4)=(3,3,2,2)$ and objective value 0,

$$A = \left(\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \ .$$

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In this article we consider the *uniform* case, where all f_i are the same and all g_j are the same, and show that the problem can then be solved in polynomial time.

Theorem 1.1 The uniform line sum optimization problem, where for some given functions f, g we have $f_i = f$ for all $i \in [m]$ and $g_j = g$ for all $j \in [n]$, can be solved in polynomial time.

We call a matrix A monotone if it has nonincreasing row and column sums, that is, $r_1 \ge \cdots \ge r_m$ and $c_1 \ge \cdots \ge c_n$. We also solve the line sum problem over monotone matrices.

Theorem 1.2 Given $m, n, f_i : \{0, 1, ..., n\} \to \mathbb{Z}$, and $g_j : \{0, 1, ..., m\} \to \mathbb{Z}$, a monotone $A \in \{0, 1\}^{m \times n}$ minimizing $\sum_{i=1}^m f_i(r_i(A)) + \sum_{j=1}^n g_j(c_j(A))$ is polynomial time computable.

Theorem 1.2 clearly implies Theorem 1.1: if for some given functions f, g we have $f_i = f$ for all $i \in [m]$ and $g_j = g$ for all $j \in [n]$, then the objective value of any matrix is invariant under row and column permutations, and hence an optimal solution to the monotone problem is an optimal solution to the uniform problem as well. So we need only prove Theorem 1.2.

The uniform column sum problem, where the row sums r_1, \ldots, r_m are specified, and the objective is to minimize $\sum_{j=1}^{n} g(c_j(A))$, recently solved in [7], is a special case of Theorem 1.2, obtained by assuming $r_1 \ge \cdots \ge r_m$ and taking $f_i(x) = a(x-r_i)^2$ for all i and sufficiently large a. The line sum problem is a special case of the degree sequence optimization problem, where, given a graph H = (V, E) and functions $f_v : \{0, 1, \dots, d_v(H)\} \to \mathbb{Z}$ for $v \in V$, with $d_v(H)$ the degree of v in H, we need to find a subgraph $G = (V, F) \subseteq H$ minimizing $\sum_{v \in V} f_v(d_v(G))$. Indeed, identifying matrices $A \in \{0,1\}^{m \times n}$ with bipartite graphs G = (V,F) where V = $\{u_1,\ldots,u_m\} \uplus \{w_1,\ldots,w_n\}$ and $F=\{\{u_i,w_j\}:A_{i,j}=1\}$, the line sum problem reduces to the degree sequence problem with $H = K_{m,n}$ the complete bipartite graph. In the case of $H = K_n$ the complete graph, the uniform problem, where all functions are the same, $f_v = f$ for all $v \in V$, was recently shown in [3] to be polynomial time solvable, using the characterization of degree sequences by Erdős and Gallai [5]. For general graphs H, the problem was shown in [1] to be NP-hard already when $f_v(x) = -x^2$ for all $v \in V$, but is polynomial time solvable if the functions are convex [1, 4]. We conjecture that the degree sequence problem over $H = K_{m,n}$, which is the line sum problem, as well as over $H = K_n$, is polynomial time solvable for arbitrary functions at the vertices, not necessarily identical.

2 Proof

For an $m \times n$ matrix A let $r(A) = (r_1(A), \ldots, r_m(A))$ and $c(A) = (c_1(A), \ldots, c_n(A))$ be the tuples of row and column sums, and let $f(r(A)) = \sum_{i=1}^m f_i(r_i(A))$ and $g(c(A)) = \sum_{j=1}^n g_j(c_j(A))$. We need the following terminology. A nonincreasing $r = (r_1, \ldots, r_m)$ is majorized by a nonincreasing $s = (s_1, \ldots, s_m)$ if $\sum_{i=1}^h r_i \leq \sum_{i=1}^h s_i$ for $h \in [m]$ and $\sum_{i=1}^m r_i = \sum_{i=1}^m s_i$. (See [6] for more details on the theory and applications of majorization.) The conjugate of a nonincreasing tuple $c = (c_1, \ldots, c_n)$ with $c_1 \leq m$ is the nonincreasing tuple $s = (s_1, \ldots, s_m)$ where $s_i = |\{j : c_j \geq i\}|$ for $i \in [m]$. Note that $s_1 \leq n$ and $\sum_{i=1}^m s_i = \sum_{j=1}^n c_j$. We make use of the following characterization due to Ryser [8].

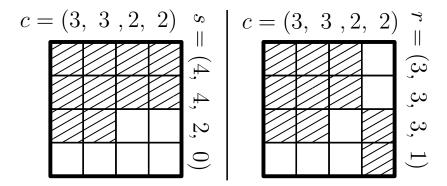


Figure 1: A nonincreasing tuple c=(3,3,2,2), its conjugate s=(4,4,2,0), a nonincreasing tuple r=(3,3,3,1), which is majorized by s, and a monotone matrix $A\in\{0,1\}^{4\times 4}$ with columns sums c and row sums r.

Proposition 2.1 A monotone $A \in \{0,1\}^{m \times n}$ with row and column sums $r = (r_1, \ldots, r_m)$ and $c = (c_1, \ldots, c_n)$ exists if and only if r is majorized by the conjugate $s = (s_1, \ldots, s_m)$ of c.

For instance, if m = n = 4 and c = (3, 3, 2, 2), then s = (4, 4, 2, 0), and r = (3, 3, 3, 1) is majorized by s so there is a monotone matrix $A \in \{0, 1\}^{4 \times 4}$ with row sums r and column sums c, see Figure 1.

Next we note that a matrix with given row and column sums can be efficiently obtained.

Lemma 2.2 Given $r = (r_1, ..., r_m)$ and $c = (c_1, ..., c_n)$ which satisfy the Ryser criterion, a matrix A in $\{0,1\}^{m \times n}$ with row sums r and column sums c is polynomial time computable.

Proof. The problem is solvable either by the efficient simple Gale-Ryser algorithm, see [2, Chapter 3], or by network flows as follows. Define a directed graph with capacities on the edges as follows. There are vertices s, t, u_1, \ldots, u_m , and w_1, \ldots, w_n . There are edges $[s, u_i]$ for $i \in [m]$ with capacity r_i , edges $[u_i, w_j]$ for $i \in [m]$ and $j \in [n]$ with capacity 1, and edges $[w_j, t]$ for $j \in [n]$ with capacity c_j . Then, as is well known, a maximum nonnegative integer flow from s to t can be computed in polynomial time, see e.g. [9]. Then A is read off from the maximum flow by taking $A_{i,j}$ to be the flow on edge $[u_i, w_j]$ for all $i \in [m]$ and $j \in [n]$. \square

A key idea facilitating our algorithm is a different way to view or encode the nonincreasing tuples $c = (c_1, \ldots, c_n)$ with $c_1 \leq m$ and their conjugates $s = (s_1, \ldots, s_m)$ with $s_1 \leq n$. Viewing a nonincreasing tuple c as a series of left-aligned and top-aligned "strips" (see Figure 1). In this view, c describes the length of each strip. An alternative description is by specifying the strip lengths which occur together with their multiplicities. As we will see, this view is much more amenable to designing a dynamic program. Let us now describe it in detail.

The type of a nonincreasing tuple is the number of distinct nonzero values among its components, i.e., the number of occurring strip lengths. It is easy to see that if s is the

conjugate of c then s and c have the same type $0 \le k \le \min\{m, n\}$. For a tuple c of type k, we define numbers $n = t_0 \ge t_1 > \cdots > t_k > t_{k+1} = 0$ and $0 = d_0 < d_1 < \cdots < d_k \le d_{k+1} = m$ such that, for $h = 0, 1, \ldots, k$, we have that c has $t_h - t_{h+1}$ components equal to d_h . Clearly each c has such an encoding using vectors $t = (t_0, \ldots, t_{k+1})$ and $d = (d_0, \ldots, d_{k+1})$, and also each choice of vectors t and d as previously mentioned corresponds to exactly one c. Moreover, the conjugate s of c encoded using t and d has $d_{h+1} - d_h$ components equal to t_{h+1} for $h = 0, 1, \ldots, k$. A particularly neat way to describe c, s using t, d is the abridged form:

$$c = (c_1, \dots, c_n) = (d_k^{t_k - t_{k+1}}, \dots, d_0^{t_0 - t_1}), \quad s = (s_1, \dots, s_m) = (t_1^{d_1 - d_0}, \dots, t_{k+1}^{d_{k+1} - d_k}). \quad (1)$$

Continuing with the example before we have $c = (3, 3, 2, 2) = (3^2, 2^2)$ and $s = (4, 4, 2, 0) = (4^2, 2^1, 0^1)$. See also Figure 3 for a larger example which will be treated in detail in Example 2.3. Note that any c and its conjugate s of type k arise that way from some such t, d.

We can now solve the monotone and hence also the uniform line sum problems.

Theorem 1.2 Given $m, n, f_i : \{0, 1, ..., n\} \to \mathbb{Z}$, and $g_j : \{0, 1, ..., m\} \to \mathbb{Z}$, a monotone $A \in \{0, 1\}^{m \times n}$ minimizing $\sum_{i=1}^m f_i(r_i(A)) + \sum_{j=1}^n g_j(c_j(A))$ is polynomial time computable.

Proof. By Proposition 2.1 and Lemma 2.2 it suffices to find nonincreasing tuples $c = (c_1, \ldots, c_n)$ with $c_1 \leq m$ and $r = (r_1, \ldots, r_m)$ majorized by the conjugate $s = (s_1, \ldots, s_m)$ of c that minimize $f(r) + g(c) = \sum_{i=1}^m f_i(r_i) + \sum_{j=1}^n g_j(c_j)$. For this, we use the encoding of tuples c of type k and their conjugates s discussed above. For type k = 0 we trivially have $c = (0^n)$ and $r = s = (0^m)$ with value $\sum f_i(0) + \sum g_j(0)$. For each type $k = 1, \ldots, \min\{m, n\}$ we provide a construction which reduces the problem of finding the best c, r where c has type k to that of finding a shortest directed path in a suitable directed graph D_k with lengths on the edges. An alternative perspective is that this is a dynamic programming algorithm where we gradually solve larger and larger subproblems; we choose the shortest path encoding to focus attention on the information we keep in each state in order to compute the next one.

We now describe D_k . There are two special vertices u, v. The remaining vertices are labeled by septuples of integers $(h, t_h, d_h, i, r_i, S_i, R_i)$, where $h \in [k+1]$, $0 \le t_h, r_i \le n$, $d_h, i \in [m]$, $S_i, R_i \in [mn]$. We always define the "boundary" values $t_0 = n$, $d_{k+1} = m$, and $d_0 = t_{k+1} = 0$. Our goal is to encode each column vector c of type k, implicitly its conjugate s, and each r majorized by c, in a directed u - v path of length f(r) + g(c).

Before formally describing the edges and their lengths, we explain how such c, r give a path. The reader is referred to Example 2.3 below for a specific demonstration. Consider any choice of numbers $n = t_0 \ge t_1 > \dots > t_k > t_{k+1} = 0$ and $0 = d_0 < d_1 < \dots < d_k \le d_{k+1} = m$. These numbers define the tuple $c = (c_1, \dots, c_n)$ and its conjugate $s = (s_1, \dots, s_m)$ as in (1), where c has $t_h - t_{h+1}$ components equal to d_h and s has $d_{h+1} - d_h$ components equal to t_{h+1} for $h = 0, 1, \dots, k$. Now consider any choice of a nonincreasing tuple $r = (r_1, \dots, r_m)$ majorized by s. Let $S_i = \sum_{j=1}^i s_i$ and $R_i = \sum_{j=1}^i r_i$. For r to be nonincreasing we need $r_{i+1} \le r_i$ for $1 \le i < m$. For r to be majorized by s we need $r_i \le t$ for $r_i \in [m]$ and $r_i = t$ for $r_i \in [m]$ for $r_i \in [m]$ and $r_i = t$ for $r_i \in [m]$ and $r_i = t$ for $r_i \in [m]$ for

starting at u, going through m vertices $(h, t_h, d_h, i, r_i, S_i, R_i)$ with $i = 1, \ldots, m$ and r_1, \ldots, r_m the components of the chosen tuple r, and ending at v.

More specifically, we start with vertex u and go to vertex

$$(h = 1, t_1, d_1, i = d_0 + 1 = 1, r_1 \le t_1, S_1 = t_1, R_1 = r_1 \le S_1)$$

along an edge of length $\sum \{g_j(0): t_1 < j \le t_0\}$ accounting for the contribution of the $(t_0-t_1)=n-t_1$ components $d_0=0$ of c if any. We proceed on a path where $h=1,t_1,d_1$ remain fixed, while we increment i from $d_0+1=1$ to d_1 , where the components s_{d_0+1},\ldots,s_{d_1} of s are all equal to t_1 so that we set $S_{i+1}=S_i+t_1$ for their sum. The components r_{d_0+1},\ldots,r_{d_1} of r are as chosen and we set $R_{i+1}=R_i+r_{i+1}$ for their sum. If $d_1>d_0+1$ then the length of the edge from the vertex with index $i\ge d_0+1$ to $i+1\le d_1$ is $f_i(r_i)$ accounting for the contribution of r_i . When i reaches d_1 , we increment h and proceed to vertex

$$(h = 2, t_2, d_2, i + 1 = d_1 + 1, r_{i+1} \le r_i, S_{i+1} = S_i + t_2, R_{i+1} = R_i + r_{i+1} \le S_{i+1})$$

along an edge of length $f_{d_1}(r_{d_1}) + \sum \{g_j(d_1) : t_2 < j \le t_1\}$ accounting for the contribution of r_{d_1} and the $(t_1 - t_2)$ components d_1 of c. Now we fix $h = 2, t_2, d_2$ and continue on a path where we increment i from $d_1 + 1$ to d_2 , where the components $s_{d_1+1}, \ldots, s_{d_2}$ of s are all equal to t_2 so that we set $S_{i+1} = S_i + t_2$ for their sum. The components $r_{d_1+1}, \ldots, r_{d_2}$ of r are as chosen and we set $R_{i+1} = R_i + r_{i+1}$ for their sum. We continue this way till we arrive at the vertex $(h = k, t_k, d_k, i = d_k, r_{d_k}, S_{d_k}, R_{d_k} \le S_{d_k})$. If $i = d_k = m$ and $R_m = S_m$ then we move to v along an edge of length $f_m(r_m) + \sum \{g_j(m) : t_{k+1} < j \le t_k\}$ accounting for the contribution of r_m and the $(t_k - t_{k+1}) = t_k$ components $d_k = m$ of c. If $i = d_k < m$ then we move to

$$(h = k + 1, t_{k+1} = 0, d_{k+1} = m, i + 1 = d_k + 1, r_{i+1} \le r_i, S_{i+1} = S_i, R_{i+1} = R_i + r_{i+1} \le S_{i+1})$$

along an edge of length $f_{d_k}(r_{d_k})$ accounting for the contribution of r_{d_k} . We proceed on a path where $h = k+1, t_{k+1} = 0, d_{k+1} = m$ remain fixed, while we increment i from d_k+1 to m, where the components s_{d_k+1}, \ldots, s_m of s are all equal to $t_{k+1} = 0$ so that $S_{i+1} = S_i$. The components r_{d_k+1}, \ldots, r_m of r are as chosen and we set $R_{i+1} = R_i + r_{i+1}$ for their sum. If $m = d_{k+1} > d_k + 1$ then the length of the edge from the vertex with index $i \geq d_k + 1$ to $i+1 \leq m$ is $f_i(r_i)$ accounting for the contribution of r_i . Finally, we arrive at the vertex $(h = k+1, 0, m, i = m, r_m, S_m, R_m)$, and if $R_m = S_m$ then we move to v along an edge of length $f_m(r_m)$ accounting for the contribution of r_m . Let us now work through an example with $d_k < m$. After the example we will complete the formal description of D_k .

Example 2.3 We now demonstrate the construction of the directed graph D_k . Consult also Figures 2 and 3. Let m=7, n=9, c=(5,5,3,3,3,1,1,0,0), and r=(6,5,4,3,2,1,0). The conjugate of c is s=(7,5,5,2,2,0,0) which majorizes r. The type of c and s is k=3. The tuples $t=(t_0,\ldots,t_{k+1})=(9,7,5,2,0)$ and $d=(d_0,\ldots,d_{k+1})=(0,1,3,5,7)$ define

 $(5^2, 3^3, 1^2, 0^2) = c$ and $(7, 5^2, 2^2, 0^2) = s$ as in (1). The directed u-v path in D_3 corresponding to c and r, with edge lengths indicated (see also Figure 2), is:

u

$$\downarrow \sum \{g_j(d_0): t_1 < j \le t_0\} = g_8(0) + g_9(0)$$

$$(h = 1, t_1 = 7, d_1 = 1, i = 1 = d_1, r_1 = 6 \le t_1, S_1 = t_1 = 7, R_1 = r_1 = 6 \le S_1)$$

$$\downarrow f_1(r_1) + \sum \{g_j(d_1): t_2 < j \le t_1\} = f_1(6) + g_6(1) + g_7(1)$$

$$(h = 2, t_2 = 5, d_2 = 3, i = 2 < d_2, r_2 = 5 \le r_1, S_2 = S_1 + t_2 = 12, R_2 = R_1 + r_2 = 11 \le S_2)$$

$$\downarrow f_2(r_2) = f_2(5)$$

$$(h = 2, t_2 = 5, d_2 = 3, i = 3 = d_2, r_3 = 4 \le r_2, S_3 = S_2 + t_2 = 17, R_3 = R_2 + r_3 = 15 \le S_3)$$

$$\downarrow f_3(r_3) + \sum \{g_j(d_2): t_3 < j \le t_2\} = f_3(4) + g_3(3) + g_4(3) + g_5(3)$$

$$(h = 3, t_3 = 2, d_3 = 5, i = 4 < d_3, r_4 = 3 \le r_3, S_4 = S_3 + t_3 = 19, R_4 = R_3 + r_4 = 18 \le S_4)$$

$$\downarrow f_4(r_4) = f_4(3)$$

$$(h = 3, t_3 = 2, d_3 = 5, i = 5 = d_3, r_5 = 2 \le r_4, S_5 = S_4 + t_3 = 21, R_5 = R_4 + r_5 = 20 \le S_5)$$

$$\downarrow f_5(r_5) + \sum \{g_j(d_3): t_4 < j \le t_3\} = f_5(2) + g_1(5) + g_2(5)$$

$$(h = 4, t_4 = 0, d_4 = 7, i = 6 < d_4, r_6 = 1 \le r_5, S_6 = S_5 + t_4 = 21, R_6 = R_5 + r_6 = 21 \le S_5)$$

$$\downarrow f_6(r_6) = f_6(1)$$

$$(h = 4, t_4 = 0, d_4 = 7, i = 7 = d_4, r_7 = 0 \le r_6, S_7 = S_6 + t_4 = 21, R_7 = R_6 + r_6 = 21 = S_5)$$

$$\downarrow f_7(r_7) = f_7(0)$$

So the total length of this path is indeed equal to the objective value corresponding to r and c,

$$(f_1(6) + f_2(5) + f_3(4) + f_4(3) + f_5(2) + f_6(1) + f_7(0))$$
+ $(g_1(5) + g_2(5) + g_3(3) + g_4(3) + g_5(3) + g_6(1) + g_7(1) + g_8(0) + g_9(0)) = f(r) + g(c)$.

Note that by Proposition 2.1 a matrix with sums r, c exists and can be found by Lemma 2.2,

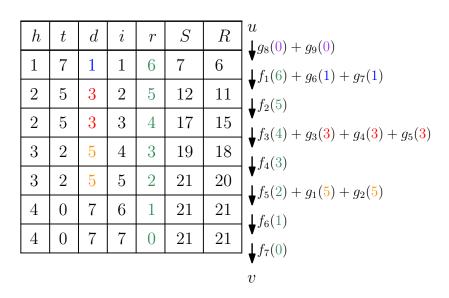


Figure 2: The path of Example 2.3. Each row is one internal vertex of the path; the values on the right of the table are the lengths of the edges between the corresponding consecutive vertices. Notice that t, d are such that they encode some nonincreasing c; S are prefix sums of the conjugate s of c, and because in each row we have $S \ge R$, R are prefix sums of some nondecreasing r which is majorized by s.

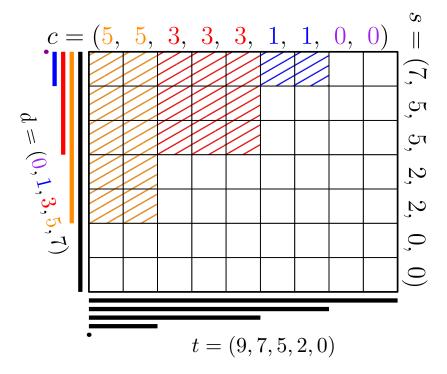


Figure 3: The column sums tuple c and its conjugate s constructed in the path of Example 2.3.

Having given an intuitive explanation of our approach together with a worked example, we now formally describe the edges and their lengths. We include edges from the vertex u to the vertices $(1, t_1, d_1, 1, r_1, S_1 = t_1, R_1 = r_1)$ for $t_1 \in [n]$, $d_1 \in [m]$, $1 \le r_1 \le t_1$, of length $\sum \{g_j(0) : t_1 < j \le t_0\}$. Consider now any $h \in [k]$. For each i with $d_{h-1} + 1 \le i \le d_h - 1$ if any we include the following edges of length $f_i(r_i)$ provided $1 \le r_{i+1} \le r_i$ and $R_{i+1} \le S_{i+1}$,

$$(h, t_h, d_h, i, r_i, S_i, R_i) \longrightarrow (h, t_h, d_h, i+1, r_{i+1}, S_{i+1} = S_i + t_h, R_{i+1} = R_i + r_{i+1})$$
.

In addition, for every $h \in [k-1]$ and $i = d_h$ we include the following edges of length $f_{d_h}(r_{d_h}) + \sum \{g_j(d_h) : t_{h+1} < j \le t_h\}$ provided $1 \le r_{i+1} \le r_i$ and $R_{i+1} \le S_{i+1}$,

$$(h, t_h, d_h, i, r_i, S_i, R_i) \longrightarrow (h+1, t_{h+1}, d_{h+1}, i+1, r_{i+1}, S_{i+1} = S_i + t_{h+1}, R_{i+1} = R_i + r_{i+1})$$
.

Now consider h = k and $i = d_h = d_k$. If i = m then, provided $R_m = S_m$, we include the edges $(k, t_k, m, m, r_m, S_m, R_m) \to v$ of length $f_{d_h}(r_{d_h}) + \sum \{g_j(m) : 0 = t_{k+1} < j \le t_k\}$.

Suppose now h = k but $i = d_h = d_k < m$. We include the following edges of length $f_{d_k}(r_{d_k}) + \sum \{g_j(d_k) : 0 = t_{k+1} < j \le t_k\}$ provided $1 \le r_{i+1} \le r_i$ and $R_{i+1} \le S_{i+1}$,

$$(k, t_k, d_k, i, r_i, S_i, R_i) \longrightarrow (k+1, t_{k+1} = 0, d_{k+1} = m, i+1, r_{i+1}, S_{i+1} = S_i, R_{i+1} = R_i + r_{i+1})$$
.

Also, for each i with $d_k + 1 \le i \le d_k - 1 = m - 1$ if any we include the following edges of length $f_i(r_i)$ provided $1 \le r_{i+1} \le r_i$ and $R_{i+1} \le S_{i+1}$,

$$(k+1,0,m,i,r_i,S_i,R_i) \longrightarrow (k+1,0,m,i+1,r_{i+1},S_{i+1}=S_i,R_{i+1}=R_i+r_{i+1})$$
.

Finally, if $R_m = S_m$, we include the edges $(k+1,0,m,m,r_m,S_m,R_m) \to v$ of length $f_m(r_m)$.

Now, as explained above, it is clear that each r, c with c of type k and r majorized by the conjugate s of c give a u-v path of length f(r)+g(c) in D_k . Conversely, it is clear that every u-v path in D_k visits m intermediate vertices with $i=1,\ldots,m$ and we can read off from this path $r=(r_1,\ldots,r_m)$ directly and $c=(c_1,\ldots,c_n)$ and its conjugate $s=(s_1,\ldots,s_m)$ of type k as in (1) with r majorized by s, and f(r)+g(c) equals the length of the path. So a shortest directed u-v path in D_k gives a pair r,c with c of type k minimizing f(r)+g(c).

Now, the number of vertices of D_k is $O(kn^4m^4)$ and hence is polynomial in m, n. So a shortest directed u-v path in D_k can be obtained in polynomial time, see e.g. [9].

Now for $k = 1, ..., \min\{m, n\}$ we find the shortest path in D_k , read off r, c with minimum f(r) + g(c) among those with c of type k, compare to $c = (0^n)$ of type k = 0 and $r = (0^m)$, and let r, c be the best over all. We now use Lemma 1 to obtain a monotone matrix $A \in \{0, 1\}^{m \times n}$ which has row and column sums r, c, which is an optimal solution to our problem. \square

Acknowledgments

The first author was partially supported by Charles University project UNCE/SCI/004 and by the project 19-27871X of GA ČR. The second author was partially supported by a grant from the Israel Science Foundation and by the Dresner chair at the Technion.

References

- [1] Apollonio, N., Sebő, A.: Minconvex factors of prescribed size in graphs. SIAM Journal on Discrete Mathematics 23:1297–1310 (2009)
- [2] Brualdi, R.A.: Combinatorial Matrix Classes. Cambridge (2006)
- [3] Deza, A., Levin, A., Meesum, S.M., Onn, S.: Optimization over degree sequences. SIAM Journal on Discrete Mathematics 32:2067–2079 (2018)
- [4] Deza, G., Onn S.: Optimization over degree sequences of graphs. Discrete Applied Mathematics (2019). http://doi.org/10.1016/j.dam.2019.12.016
- [5] Erdős, P., Gallai, T.: Graphs with prescribed degrees of vertices (in Hungarian). Matematikai Lopak 11:264–274 (1960)
- [6] Marshall, A.W., Olkin, I., Arnold, B.C.: Inequalities: Theory of Majorization and its Applications. Springer (2011)
- [7] Onn, S.: On line sum optimization. Linear Algebra and its Applications 610:474–479 (2021)
- [8] Ryser, H.J.: Combinatorial properties of matrices of zeroes and ones. Canadian Journal of Mathematics 9:371–377 (1957)
- [9] Schrijver A.: Combinatorial Optimization. Springer (2003)