4-COP-WIN GRAPHS HAVE AT LEAST 19 VERTICES

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ABSTRACT. We show that the cop number of any graph on 18 or fewer vertices is at most 3. This answers a question posed by Andreae in 1986, as well as more recently by Baird et al. We also find all 3-cop-win graphs on 11 vertices, narrow down the possible 4-cop-win graphs on 19 vertices and make some progress on finding the minimum order of 3-cop-win planar graphs.

1. Introduction

The game of cops and robbers was first defined by Quilliot in [29] and Nowakowski and Winkler in [27]. Playing on the vertices of a connected, undirected and finite graph, a group of cops pursues a robber. On the first turn of the game, each cop selects a vertex as its initial position, followed by the robber. The cops and the robber then alternate turns with the cops going first. On the cops' turn (resp. robber's turn), each cop (resp. the robber) must choose either to remain on its current vertex or go to a vertex in the neighbourhood of its current position. If at some point during the game a cop and the robber are on the same vertex, the robber is said to be caught and the cops win. The robber wins if it has a strategy ensuring it is never caught by the cops. At all times, the positions of the cops and of the robber are known by all. Furthermore, the cops may coordinate their strategies, and are allowed to share vertices. An alternative formulation is to define the game of cops and robbers as a two-player game in which the first player controls some number of pieces named the cops, the second player controls a single piece named the robber, and the graph is the game board (for example as in [23]).

For a connected graph G, we denote by c(G) the minimum number of cops which can always catch the robber on G. Introduced by Aigner and Fromme in [2], c(G) is called the *cop number* of G. If c(G) = k, we say G is k-cop-win.

The cop number has been the main focus of most articles on cops and robbers, but other parameters, such as the capture time, are also studied. See [10] for a quick overview of this field or [13] for a more in-depth introduction. Although in this paper we study the classical version of the game, a multitude of variants of this game have been considered in recent years. For instance, we can allow the robber to move multiple edges at once [19], or loosen the winning condition to the cops being within some distance of the robber [11].

While there has been a significant amount of research on the cop number, often on specific classes of graphs (for instance in [3, 18, 23]), there are still surprisingly many elementary open questions. We consider here the problem of finding the minimum order of a k-cop-win graph, more specifically for k = 4. This question was first posed by Andreae in [3]. It is also raised, seemingly independently, by Baird et al. in [5].

The case of k=3 has already been solved. Andreae claimed without proof in [3] that the Petersen graph (see Figure 1) is the unique smallest 3-cop-win graph. This statement was later proved by Baird et al. in [5].

For a given graph G, we denote by V(G) and E(G), respectively, the set of vertices and of edges of G. We denote by M_k the minimum order of a k-cop-win graph; formally,

$$M_k = \min\{|V(G)| : G \text{ is a connected graph, } c(G) = k\}.$$

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Hosseini proved in [21] that $M_k < M_{k+1}$, confirming the intuition that if one scans all graphs by increasing order, one cannot find a (k+1)-cop-win graph before finding a k-cop-win graph.

Although interesting by itself, the concept of minimum k-cop-win graphs is also useful in regards to Meyniel's conjecture [17], which asks whether the cop number G is $O(\sqrt{n})$, where n is the order of G. Currently the best known upper bound is $n2^{-(1+o(1))}\sqrt{\log_2 n}$, as proved by Lu and Peng [25], Scott and Sudakov [32], and Frieze, Krivelevich and Loh [19]. A deeper survey of Meyniel's conjecture is [6]. It was proved by Baird et al. in [5] that Meyniel's conjecture is equivalent to proving that $M_k \in \Omega(k^2)$.

The specific problem of finding the minimum order of a 4-cop-win graph has received some interest. Hosseini proved in [20] that $M_4 \geq 16$, and that such a minimal graph is 3-connected, provided it does not contain a vertex of degree 2. The problem is also briefly referenced in [10]. It was suggested in [3, 5] that the value of M_4 might be 19. Indeed, the smallest known 4-cop-win graph is the Robertson graph (see Figure 1). This graph was first discovered by Robertson in [31] as the smallest 4-regular graph with girth 5.

Aigner and Fromme proved in [2] that graphs with girth at least 5 have a cop number of a least their minimum degree. This result has since been generalized by Frankl in [17], and more recently by Bradshaw et al. in [14], where they prove stronger lower bounds on the cop number of graphs with high girth. One deduces that the cop number of the Robertson graph is therefore at least 4. It is easily seen in Figure 1 that placing a cop on each of the three exterior vertices (a, b, c) only leaves 4 unprotected vertices, which form independent edges. A last cop may then capture the robber, hence the Robertson graph is 4-cop-win (this argument appears in [8]).

More generally, it was suggested in [5] that the smallest d-cop-win graphs might be the (d, 5)-cages for $d \geq 3$. A (d, g)-cage is a regular graph of degree d and girth g of minimum order. In particular, the Petersen graph is the unique (3, 5)-cage and the Robertson graph is the unique (4, 5)-cage.

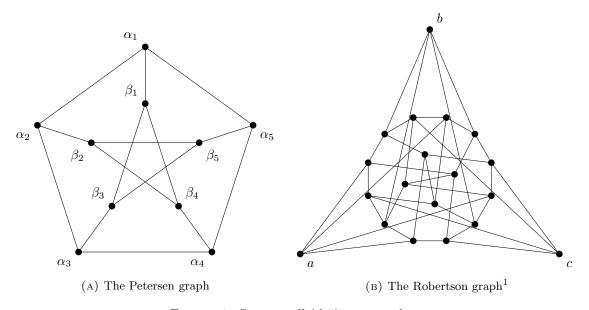


FIGURE 1. Some small (d, 5)-cage graphs

The main result of this article is to confirm that $M_4 = 19$. Although we are not able to prove a complete uniqueness result, we narrow down the possible 4-cop-win graphs on 19 vertices by proving there are no 4-cop-win graphs on 19 vertices with maximum degree at most 4 or at least 7, except for the Robertson graph.

¹Computer-generated drawing [34].

Although our proof is not directly based on those in [20] and [5] (we however use two results from the latter), there are certainly some common elements between those arguments and ours. In particular, we similarly break down the problem by maximum degree and gradually develop properties of potential 4-cop-win graphs by constructing explicit winning strategies for 3 cops if those properties do not hold.

While we are able to obtain many interesting results using structural properties, this article makes extensive use of computational methods to verify the remaining cases. All of the code and data produced in the writing of this article is available online at [33]. This includes not only the final results, but the graphs we generate in the intermediate algorithms, precise counts of the number of graphs we generate at every step in these algorithms, and the time required for almost all computations. All of the computations are split up in small parts to facilitate verification. At various points in this article, we will also discuss possible improvements and alternative computational approaches.

The structure of this article is as follows. In Section 2, we present the previously known results which we will need in our proof. In Section 3, we compute the 3-cop-win graphs on 11 to 14 vertices (with some maximum degree restrictions), which we will use in later sections to construct all graphs which are possibly 4-cop-win. In Section 4, by considering graphs containing the Petersen graph, we show that all graphs on $n \le 18$ vertices with maximum degree n-12 or n-11 have at most 3 vertices. By refining the techniques used for $n \le 18$, we will be able to expand the result to n = 19. In Section 5, we present the results of an exhaustive search which shows there are no 4-cop-win subcubic graphs on $n \le 20$ vertices, and mention an application to a related problem. In Section 6, we introduce the Merging Algorithm in order to tackle the cases with maximum degree $3 < \Delta < n-12$. These are be the last remaining cases, and will allow us to prove in Section 7 that the minimum order of 4-cop-win graphs is 19. A visual aid to how each part will be used in the proof is provided in Table 7.

2. Notation and previous results

In this section, we introduce most of the notation used in the article. We also cite previously known results that will be useful in the following sections.

When considering a graph G, we will respectively denote by n(G), $d_G(u)$, $\delta(G)$ and $\Delta(G)$ the number of vertices of G, the degree of a vertex u in G, the minimum degree of G and the maximum degree of G. If u is a vertex of G, $N_G(u)$ will denote the (open) neighbourhood of u and $N_G[u] = N_G(u) \cup \{u\}$ will denote the closed neighbourhood of u. For these symbols, we will usually omit the G when the choice of graph is easily deduced.

For $S \subseteq V(G)$, S^c will denote the complement (relative to V(G)) of S, $\langle S \rangle$ will denote the subgraph of G induced by S, and G - S will denote $\langle S^c \rangle$. When $S = \{x\}$, we will use the notation G - x instead of G - S. Similarly, if H is a subgraph of G, then G - H will denote G - V(H). When G is isomorphic to a graph G', we will write $G \simeq G'$.

Denote by \mathcal{P}_0 the Petersen graph, as seen in Figure 1. As \mathcal{P}_0 is 3-regular with girth 5, we know that $c(\mathcal{P}_0) \geq 3$, and as it contains a dominating set of size 3, we know that c(G) = 3. As noted in the introduction, the following theorem was stated by Andreae in [3] and proved by Baird et al., first by computer verification, as well as by using structural properties.

Theorem 2.1. [3, 5] Let G be a connected graph.

- (1) If $n \le 9$, then $c(G) \le 2$.
- (2) If n = 10, then $c(G) \le 2$, unless G is the Petersen graph.

In particular, $M_3 = 10$.

The proof of the previous theorem makes use of the following lemma, which will also be useful to us.

Lemma 2.2. [5] Let G be a connected graph. If $\Delta \geq n-5$, then $c(G) \leq 2$.

A simple, visual proof of this lemma is available in [8]. We now introduce the concept of a retract, a tool often used to study the game of cops and robbers (see for example [2, 27]). The following is based on [7].

Definition 2.3. Let G be a graph. If H is an induced subgraph of G, we say H is a *retract* of G if there exists a mapping $f: V(G) \to V(H)$ such that:

- (1) If $xy \in E(G)$, then $f(x)f(y) \in E(H)$ or f(x) = f(y).
- (2) $f|_{V(H)}:V(H)\to V(H)$ is the identity mapping.

Such a mapping f is called a *retraction*.

This definition formalizes the intuitive idea that G can be "folded" onto H, where each edge must either be sent onto an edge or onto a vertex. Those familiar with graph homomorphisms will notice that condition (1) states that f is a homomorphism from G to H if we consider H to be reflexive (that is, if we add a loop at each vertex of H). Reflexivity is necessary as consequence of allowing the cops and the robber to stay on a vertex at their turn, implying a loop on each vertex. The concept of retracts has been central in the study of the game of cops and robbers, for instance also appearing in [27].

If G is disconnected, denote G_1, \ldots, G_t the connected components of G. By extension, we may define the cop number of a disconnected graph by $c(G) = \max_{1 \leq i \leq t} c(G_i)$. These definitions allow us to state the following result of Berarduci and Intriglia, which we will use many times to reduce the number of cases we need to consider.

Theorem 2.4. [7] If G is a connected graph and H is a retract of G, then

$$c(H) \le c(G) \le \max\{c(H), c(G-H)+1\}.$$

A specific case of this theorem is the following reformulation of a corollary in [7], which will often be easier to use.

Corollary 2.5. If G is a connected graph, u is a vertex of G and K is a union of some connected components of G - N[u], then

$$c(G - K) \le c(G) \le \max\{c(G - K), c(K) + 1\}.$$

In particular, if $c(K) \leq k-1$, then $c(G) \leq k$ if and only if $c(G-K) \leq k$.

Proof. It is easy to verify that $f:V(G)\to V(G-K)$ defined by

$$f(x) = \begin{cases} u & \text{if } x \in V(K) \\ x & \text{otherwise} \end{cases}$$

is a retraction. It is only left to apply Theorem 2.4 to H = G - K.

One trivial consequence of this corollary is that if the cop number of every component of G - N[u] is at most k-1, then $c(G) \leq k$. One can also see this directly by leaving a fixed cop on u and playing with k-1 cops on the connected component of G - N[u] in which the robber is located.

We then easily get the following result, which is implicit in [20].

Corollary 2.6. If G is a connected graph and $\Delta > n-11$, then $c(G) \leq 3$.

Proof. If $\Delta > n-11$ and u is a vertex of maximum degree, then |V(G-N[u])| < 10. By Theorem 2.1, every connected component of G-N[u] has cop number at most 2. The last remark yields the result.

Finally, we recall a well known concept in the study of the game of cops and robbers.

Definition 2.7. Let x, u be distinct vertices of a graph G. If $N(x) \subseteq N[u]$, we say x is *cornered* by u or that x is a *corner*.

We note that this is a slight variation on the classical notion of a corner (or irreducible vertex), as it normally requires ux to be an edge, see [27]. We may now get the following well-known result as a further simplification of Corollary 2.5.

Corollary 2.8. Let G be a connected graph and x be a corner of G. If $c(G - x) \ge 2$, then c(G) = c(G - x). If c(G - x) = 1, then $c(G) \in \{1, 2\}$.

Proof. Let u be a vertex cornering x. First define $f:V(G)\to V(G-x)$ by

$$f(y) = \begin{cases} u & \text{if } y = x \\ y & \text{otherwise.} \end{cases}$$

It is easily seen that the condition that x be cornered by u implies that f is a retraction. The statement then follows from Theorem 2.4.

3. Computational results for small 3-cop-win graphs

In this section, we find some 3-cop-win graphs on at most 14 vertices respecting some degree conditions. We will do this by computing the cop number of every graph of the desired orders and degrees. The results will be useful in the following sections.

Graph generation in this section and in Section 5 is done using the geng function provided with the nauty/Traces package (version 26r12) [26]. The algorithm to compute the cop number is similar to that proposed, in particular, in [11, 16, 30], which we have implemented for cop numbers 1, 2, 3 in the Julia language [9, 15]. For a given k (which will be between 1 and 3 in our case), the algorithm determines whether $c(G) \leq k$ or c(G) > k. Hence, if we run the algorithms for $1, \ldots, m$ cops on a graph, we can determine the cop number exactly if it is a most m, or if not then say that $c(G) \geq m+1$.

To test the validity of our implementation, we have compared the results for the cop number of connected graphs up to 10 vertices to those in [5]. Following a small discrepancy between the counts, our tallies of cop-win graphs were also verified to be correct by implementing a dismantling algorithm [27] and by comparing with the implementation at [1]. To test our code for higher cop numbers, it was also run on some cage graphs which we know 3 cops lose. Based on the results of these tests, we are confident in the correctness of our implementation.

We first define a variant of the Petersen graph.

Definition 3.1. We say a connected graph G is a *cornered Petersen graph* if G contains a corner m such that $G - m \simeq \mathcal{P}_0$. There are 6 such graphs up to isomorphism. We denote them \mathcal{P}_i , $i = 1, \ldots, 6$, as seen in Figure 2.

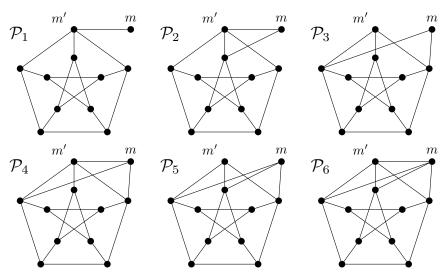


FIGURE 2. The cornered Petersen graphs

We now solve a question raised in [5], classifying the 3-cop-win graphs on 11 vertices, albeit computationally.

Proposition 3.2. If G is a connected graph such that n = 11, then c(G) = 3 if and only if $G \simeq \mathcal{P}_i$ for some $1 \le i \le 6$. Otherwise, $c(G) \le 2$.

Proof. Firstly, it is clear by Theorem 2.1 and Corollary 2.8 that the cornered Petersen graphs are 3-cop-win. We would like to show that these graphs are the only graphs on 11 vertices with cop number 3, and that all other graphs have cop number at most 2.

By Lemma 2.2, we may only consider graphs such that $\Delta \leq n-6=5$. We generate all connected graphs on 11 vertices such that $\Delta \leq 5$ and classify each graph according to its cop number (by running the algorithm for k=1,2). The results are presented in Table 1 (the counts are up to isomorphism). The 6 graphs found are the graphs \mathcal{P}_i for $i=1,\ldots,6$, which concludes the proof.

This is an interesting phenomenon: the unique 3-cop-win graph on 10 vertices is a retract of all the 3-cop-win graphs on 11 vertices. This behaviour does not occur for the 2-cop-win graphs: the minimum 2-cop-win graph is the 4-cycle, on which the 5-cycle does not retract. Although we will not have any answer for this question in this article, it would be interesting to know whether in general (even for 4-cop-win graphs only), the k-cop-win graphs on $M_k + 1$ vertices can be retracted on k-cop-win graph(s) on M_k vertices.

In Section 6, we will also need the following lemma.

Lemma 3.3. There exist

- 80 connected graphs G on 12 vertices with $\Delta \leq 4$,
- 173 connected graphs G on 12 vertices with $\Delta \leq 5$,
- 1105 connected graphs G on 13 vertices with $\Delta \leq 4$, and
- 16523 connected graphs G on 14 vertices with $\Delta \leq 4$

such that c(G) = 3. All other connected graphs G considered with these orders and maximum degrees satisfy $c(G) \leq 2$.

Proof. Firstly, all graphs on at most 14 vertices have cop number at most 3. For cases where $\Delta \geq 4$, this is a direct consequence of Corollary 2.6. For $\Delta = 2$, the graph is either a path or a cycle. For $\Delta = 3$, see the results of Table 4. This is also a direct consequence of knowing that $M_4 \geq 16$, see [20].

We generate, up to isomorphism, all connected graphs on 12 vertices such that $\Delta \leq 5$ and on 13 and 14 vertices such that $\Delta \leq 4$. We classify these graphs according to their cop number (by running the algorithm for k=1,2). Afterwards, we also count which of the graphs on 12 vertices with $\Delta \leq 5$ and $c(G) \geq 3$ are such that $\Delta \leq 4$. The results are in Table 1.

				Cop number	
n	Degree bounds	Number of graphs	1	2	≥ 3
11	$\Delta \leq 5$	21503340	69310	21434024	6
12	$\Delta \leq 4$	-	-	-	80
12	$\Delta \leq 5$	471142472	295377	470846922	173
13	$\Delta \le 4$	68531618	73876	68456637	1105
14	$\Delta \leq 4$	748592936	247022	748329391	16523

Table 1. Cop number breakdown for connected graphs on 11-14 vertices with some degree restrictions

While the counts are presented to summarize the results, the precise 3-cop-win graphs are the focus of our attention as we use them in the following sections.

We now discuss some possible improvements to our approach. This part is not essential to prove the main result, but may be helpful in motivating the methods of Section 6. To achieve these results, we exhaustively computed the cop number of every connected graph that satisfied our maximum degree constraints. Since we proceeded by exhaustion, the run time of these computations is somewhat long

due to the high number of graphs, especially in the case of $\Delta = 14$. We note that a more clever approach might yield faster calculation time.

The first and most obvious improvement would be to only look at graphs with a minimum degree of at least 2, which can reduce the number of graphs to consider by up to around 50%. If we already know the 3-cop-win graphs on one fewer vertices, we can then just consider all possible ways to attach an extra vertex of degree 1 to those graphs. Using this method, we can get all connected 3-cop-win graphs of a given order.

However, this method is still an exhaustive search. A more clever approach would be to consider every (not necessarily connected) 2-cop-win graph G' on $n-\Delta-1$ vertices, add a vertex u with Δ neighbours new neighbours and consider each way of adding edges between N(u) and G' (up to isomorphism), then checking which of these graphs are 3-cop-win. We would recommend the interested reader try this approach.

A more refined approach of this would be to use the algorithm of Section 6 to build candidate 3-cop-win graphs, by merging 2-cop-win graphs on fewer vertices. We will see later that although this method can reduce significantly the computation time, in practice it requires some effort to make sure all the possible cases are considered. For the size of graphs we are considering, this approach does not appear worthwhile.

4. Graphs with high maximum degree

In this section, we consider the cop number of graphs G such that $\Delta = n - 11$ or $\Delta = n - 12$. We start by investigating some properties of the game of cops and robbers on the Petersen graph (denoted by \mathcal{P}_0) and its variants, the cornered Petersen graphs (denoted by \mathcal{P}_i for $1 \leq i \leq 6$). Many of the arguments in this section are extremely simple once visualized. For this reason, we have provided many figures representing parts of the proofs. Of course, we cannot provide figures for every case, so we encourage the reader draw out the graphs while following the proofs, especially regarding the movements of the cops and the robber.

By considering the Petersen graph as the Kneser graph $KG_{5,2}$ [4], one easily gets the following well-known result (although maybe not with this precise formulation), an illustration of the fact that the Petersen graph is highly transitive.

Definition 4.1. We say a set of 3 vertices $\{x, y, z\}$ is a *strong stable set* if it is a stable set and if $N(x) \cap N(y) \cap N(z) = \emptyset$.

Lemma 4.2.

- (a) If $\{x, y, z\}$ and $\{x', y', z'\}$ are strong stable sets of \mathcal{P}_0 , then there exists an automorphism ϕ_1 of \mathcal{P}_0 such that $\phi_1(x) = x'$, $\phi_1(y) = y'$ and $\phi_1(z) = z'$.
- (b) If ab and a'b' are two edges of \mathcal{P}_0 , then there exists an automorphism ϕ_2 of \mathcal{P}_0 such that $\phi_2(a) = a'$, $\phi_2(b) = b'$. This property is known as being arc-transitive.

We use the labels m, m' on the graphs \mathcal{P}_i for $i=1,\ldots,6$, as shown in Figure 2. In particular, for each of these graphs, $\mathcal{P}_i - m \simeq \mathcal{P}_0$. We also see that m' always corners m, which will be very useful. We also note that as $m, m' \notin V(\mathcal{P}_0)$, we can say that $\mathcal{P}_0 - m = \mathcal{P}_0 - m' = \mathcal{P}_0$.

As stated in Theorem 2.1, we know that $c(\mathcal{P}_0) = 3$. In the next two lemmas, we show that although two cops do not have a winning strategy, they have a lot of power as to which positions can be reached. We will later use these lemmas as parts of strategies on graphs containing the Petersen or the cornered Petersen graph. These lemmas would be very easy to establish computationally, but we consider a proof to be worthwhile.

Lemma 4.3. Let $0 \le i \le 6$. If $\{x, y, z\}$ is a strong stable set of $\mathcal{P}_i - m$, then there exists a strategy for 2 cops on \mathcal{P}_i to reach the following situation.

- (1) The robber is on x, except in the case x = m' and $i \in \{5, 6\}$, where the robber is either on m' or m.
- (2) The cops are on y and z.

(3) It is the cops' turn.

Proof. Let us first consider the case of \mathcal{P}_0 . Consider the labelling of \mathcal{P}_0 with α_i and β_i as shown in Figure 1. For any j, j', observe that if two cops are on β_j, β_{j+1} (working in modulo 5), they can directly move to the pair $\beta_{j'}, \beta_{j'+1}$.

Without loss of generality, we may consider that $x = \alpha_1, y = \beta_2, z = \beta_3$, as for all other strong stable sets we can apply the automorphism of Lemma 4.2(a). For some k, we start the game with two cops on β_k and β_{k+1} (modulo 5). The robber must choose one of the α_ℓ as its starting vertex. Notice that if the robber is on α_j , moving the cops to β_j and β_{j+1} forces the robber to move to α_{j-1} . By repeating this strategy, the cops can essentially make the robber turn in circles on the outer 5-cycle of \mathcal{P}_0 . At the end of every cops' turn (except the first), the robber is on α_j and the cops are on β_j and β_{j+1} , for some j. The cops repeat until the robber is on α_1 (unless the robber is already on α_1 on the first turn, in which case the cops make the robber do a full circle around the graph); it is now the cops' turn and the game is in the desired situation. Observe that this strategy works for any initial choice of k. This will be useful later, as for any vertex $w \in \mathcal{P}_0$, we may choose an initial position such that one of the cops is in N[w]. We call this the *chasing strategy* for the Petersen graph. An example is illustrated in Figure 3. Even though this might be a very simple idea, this strategy is critical for the rest of this section as it will enable us to construct more complicated strategies.

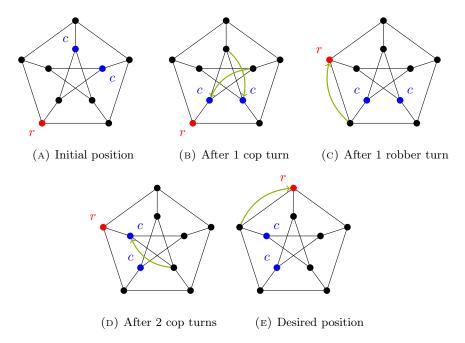


FIGURE 3. Typical application of the chasing strategy on the Petersen graph.

We now consider the cases of \mathcal{P}_5 and \mathcal{P}_6 . In both cases, observe that m and m' are completely indistinguishable: N(m) = N(m'). It is then easily seen that the strategy for 2 cops on \mathcal{P}_5 or \mathcal{P}_6 will be the same as the strategy developed above for \mathcal{P}_0 , except that the robber may choose to go to either m or m'. We apply the strategy for \mathcal{P}_0 by considering the robber to be on m' whenever it is actually on m. This is essentially a simplified version of the well-known argument used to prove, in particular, Theorem 2.4.

Finally, we consider the cases \mathcal{P}_i , $i \in \{1, ..., 4\}$. Our goal is to apply the strategy of \mathcal{P}_0 developed above, with only slight modifications. Using that strategy, we choose initial positions for the cops in $\mathcal{P}_i - m$ such that one of the cops is in N[m'] (it is described above why this is possible). If the robber chooses m as an initial position, this cop may then move to m'. As m' corners m, the robber cannot

move without being captured. The other cop may then, within a few turns, capture the robber. Thus, the robber will choose an initial vertex in $\mathcal{P}_i - m$. Now, as long as the robber is not on m, copy the strategy for \mathcal{P}_0 . Suppose that, at some point, the robber moves to m. In the strategy for \mathcal{P}_0 , the robber is adjacent to a cop before every one of its turns. Thus, this cop can move to a vertex adjacent to m. One easily verifies that in all graphs \mathcal{P}_i for $i \in \{1, \ldots, 4\}$, if one cop is adjacent to m, there is at most one other escape route t for the robber. As $\mathcal{P}_i - m \simeq \mathcal{P}_0$ has diameter 2, the other cop can move to block this escape route by moving to some vertex in N[t]. Thus, while applying this strategy, the robber will never move to m. Hence, the strategy copied from \mathcal{P}_0 yields the desired final position. \square

By weakening the condition that it is the cops' turn at the end of the strategy, we can get more freedom as to where we can place the cops, enabling more strategies.

Lemma 4.4. Let $0 \le i \le 6$. If x, y, z are any three distinct vertices of $\mathcal{P}_i - m$, then there exists a strategy for 2 cops on \mathcal{P}_i to reach the following situation.

- (1) The robber is on x, except in the case x = m' and $i \in \{5, 6\}$, where the robber is either on m' or m.
- (2) The cops are on y and z.
- (3) It is the robber's turn.

Proof. Without loss of generality, we show the statement for \mathcal{P}_0 . For this lemma, generalizing to the cornered Petersen graphs is immediate.

We first consider the case where $xz \in E(\mathcal{P}_0)$. We will enumerate the main cases and conclude by symmetry for the others. We may assume that $x = \alpha_1$ and $z = \beta_1$ (using the labelling from Figure 1), all other possibilities can be solved using the automorphisms of Lemma 4.2 (b). We apply Lemma 4.3 to place the robber on vertex α_1 and the cops on the vertices specified in Table 2 (always forming a strong stable set), and then specify the additional move required to place the cops in the desired final position.

Final position for cops (y, z)	Position after applying Lemma 4.3	Movements
α_2, β_1	β_2, β_3	$\beta_2 \to \alpha_2, \beta_3 \to \beta_1$
eta_2,eta_1	β_2, β_3	$\beta_2 \to \beta_2, \beta_3 \to \beta_1$
β_3, β_1	eta_4,eta_5	$\beta_4 \to \beta_1, \beta_5 \to \beta_3$
$lpha_3,eta_1$	α_3, eta_4	$\alpha_3 \to \alpha_3, \beta_4 \to \beta_1$

TABLE 2. Strategy on \mathcal{P}_0 to bring the robber to α_1 with the cops in the desired final position, when at least one cop needs to be adjacent to robber.

It is easily seen that all other choices of y are analogous by reflection of the graph relative to the vertical axis.

We use a similar approach for the case where $xz \notin E$. We may suppose without loss of generality that $x = \alpha_1$ and $z = \beta_2$: it is easily verified that any two non-adjacent vertices can be expanded into a strong stable set, then apply Lemma 4.2 (a). We must still verify the position can be reached for the different possible y. To further reduce the number of cases, we can also assume that $xy \notin E$ (if $xy \in E$, switching the roles of y and z brings us back to the previous case), as we can see in Table 3.

In the next lemmas, we will consider graphs with the following properties, with the goal of eventually showing that they do not exist. We state these properties now to avoid repetition.

Hypothesis 4.5. Let G be a connected graph such that c(G) > 3 and $u \in V(G)$ such that $G - N[u] \simeq \mathcal{P}_i$, for some $0 \leq i \leq 6$.

In the cases of $1 \le i \le 6$, we may in particular consider that $m, m' \in V(G)$ by fixing the isomorphism. In the cases of i = 5, 6, as the labels m and m' can be switched, we will always suppose m' to be the vertex of the two which has the greatest degree in G (if both have the same degree, then

Final position for cops (y, z)	Position after applying Lemma 4.3	Movements
$lpha_3,eta_2$	$lpha_3,eta_4$	$\alpha_3 \to \alpha_3, \beta_4 \to \beta_2$
α_4, β_2	$lpha_3,eta_4$	$\alpha_3 \to \alpha_4, \beta_4 \to \beta_2$
β_3, β_2	eta_3,eta_2	$\beta_3 \to \beta_3, \beta_2 \to \beta_2$
β_4, β_2	eta_4,eta_5	$\beta_4 \to \beta_4, \beta_5 \to \beta_2$
eta_5,eta_2	eta_4,eta_5	$\beta_4 \to \beta_2, \beta_5 \to \beta_5$

TABLE 3. Strategy on \mathcal{P}_0 to bring the robber to α_1 with the cops in the desired final position, when neither cop needs to be adjacent to robber.

we choose arbitrarily). To simplify notation, we will denote $B_u = V(G - N[u] - m)$. It is easily seen that in all cases $\langle B_u \rangle \simeq \mathcal{P}_0$, that is B_u induces a Petersen graph.

The approach will be to build up a number of structural properties of G by showing that otherwise there exists a winning strategy for 3 cops, yielding a contradiction. We start by proving that all vertices in B_u have a neighbour in N(u).

Lemma 4.6. Consider Hypothesis 4.5. For all $x \in B_u$, $|N(x) \cap N(u)| \ge 1$.

Proof. Let $x \in B_u$. We prove a restricted version of the statement before proving it generally: we first suppose that there exists a neighbour $v \neq m'$ of x in B_u such that $|N(v) \cap N(u)| \geq 1$. In the case of \mathcal{P}_0 , then m' does not exist so v could be any vertex of B_u . Suppose that $|N(x) \cap N(u)| = 0$, with the goal of reaching a contradiction. We note that in the case with x = m', then by our choice of m' we also know that m also has no neighbours in N(u). We show this situation yields a winning strategy for 3 cops.

Let y, z be the other neighbours of x in B_u and let $w \in N(v) \cap N(u)$. The situation is portrayed in Figure 4. We start by placing a cop on u, which will only move if the robber enters N[u]. This cop is commonly referred to as a stationary cop. As long as this cop stays on u, the robber is stuck in $G - N[u] \simeq \mathcal{P}_i$. Thus, the two other cops may apply the strategy from Lemma 4.4 on G - N[u] to place the robber on x, a cop on y and a cop on z. In the special case where $i \in \{5,6\}$ and x = m', the robber might actually be on m.

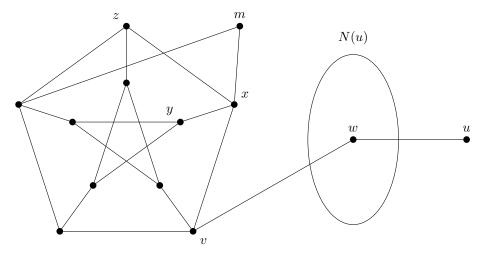


FIGURE 4. Example situation during the proof of Lemma 4.6. Unused or unknown vertices and edges are omitted.

During the last turn of this strategy, the cop on u moves to w. It is now the robber's turn. All of the robber's neighbours in B_u are protected by cops, there is a cop adjacent to the robber, and the robber has no neighbour in N(u). Note that this statement is also true if the cop is on m, in the

special case with i = 5, 6 and x = m', as m and m' have the same neighbours v, y, z, in B_u (and both don't have neighbours in N(u) as stated above).

The robber is caught unless it can move to an unprotected vertex outside B_u , which must necessarily be m. Note that there are some cases in which m is still protected, for instance the special case discussed above. The robber was on x and is now on m, so x is adjacent to m. If one of the cops is adjacent to m, the robber is caught, so we assume otherwise, that m is adjacent to neither y nor z.

If x = m', the cop on w moves back to u and the cop on y moves to m': as m' covers all neighbours of m in B_u and u covers all neighbours in N(u), the robber is trapped and will be caught one turn later.

Now, suppose that $x \neq m'$. As x is adjacent to m, x is also adjacent to m'. Recall that we supposed that $v \neq m'$. Thus, m' is either y or z. The cop on m' stays put and the cop on w moves back to u, trapping the robber on m. The last cop may then capture the robber within a few turns.

In all cases, a contradiction is reached with the hypothesis that c(G) > 3. This proves the specific case in which x has a neighbour $v \in B_u$ (respecting $v \neq m'$) which has at least one neighbour in N(u). We now wish to prove the full statement without this condition.

With what we have proved so far, we know that as soon as a vertex of B_u (other than m') has a neighbour in N(u), we can say the same for its 3 neighbours in B_u (and then we can repeat this argument to their neighbours, etc.). Thus, in order to prove the lemma, it suffices to show that there exists at least one vertex of $B_u \setminus \{m'\}$ that has a neighbour in N(u) (because $\langle B_u \rangle - m'$ is necessarily connected).

Suppose the contrary: that no vertex of $B_u \setminus \{m'\}$ has a neighbour in N(u). If we are in the case of $G - N[u] \simeq \mathcal{P}_0$, then G would be disconnected (as $B_u \setminus \{m'\} = B_u$), which is a contradiction. In the remaining cases, we give a winning strategy for 3 cops. Place a stationary cop on a vertex $t \in N[m] \cap N[m'] \cap B_u$, place one cop on u, and place the third cop anywhere. The robber must choose an initial position in $B_u \setminus N[t]$. As any exit from B_u will go through m or m' (by our hypothesis), the stationary cop on t guarantees that the robber will never leave B_u . The two other cops then have a winning strategy on the component of $\langle B_u \rangle - N[t]$ containing the robber, which has at most 6 vertices.

We will frequently use this idea of applying the chasing strategy on G - N[u] while we leave a cop on u. As long as there is a cop on u, it is as if we were playing on G - N[u]. Moving the cop from u during the last move of this strategy does not affect it, as it happens after the last robber move which is part of that strategy.

Similarly to in the first part of the last proof, many of the arguments in this section will end with the only remaining possible situation being that there is a cop on u, a cop on m', and the robber on m. In this situation, the robber cannot move, as m' corners m in G - N[u], and a third cop may, within a few turns, capture the robber.

We now characterize the neighbourhoods in B_u of vertices in N(u).

Lemma 4.7. Consider Hypothesis 4.5. If $w \in N(u)$, then $N(w) \cap B_u$ does not contain a subset $\{a,b,c\}$ of distinct vertices such that :

- (1) $ab \notin E(G)$;
- (2) $c \notin N(a) \cap N(b)$ (c is not the common neighbour of a and b in $\langle B_u \rangle$);
- (3) $c \notin N(x)$ where $\{x\} = N(a) \cap N(b) \cap B_u$ (c is not adjacent to the common neighbour of a and b in $\langle B_u \rangle$).

Proof. Suppose that $N(w) \cap B_u$ does contains a subset $\{a,b,c\}$ respecting these conditions. We give a winning strategy for 3 cops on G, which will lead to a contradiction. We denote by x the common neighbour of a and b in B_u , which exists by condition (1) as vertices of distance 2 in a Petersen graph have exactly one common neighbour. Furthermore by condition (2), $c \neq x$. Denote by d the neighbour of x in B_u that is neither a or b. Then by condition (3), $c \neq d$.

Let z be a vertex of B_u such that $\{x, c, z\}$ is a strong stable set of $\langle B_u \rangle$ (it is easily seen that any stable set of size 2 in the Petersen graph can be expanded into a strong stable set). The situation

is portrayed in Figure 5. We place a cop on u at the start of the game, and then use Lemma 4.3 to place the other cops on c and z, and the robber on x (or if x = m' and $i \in \{5, 6\}$ possibly on m). As stated in Lemma 4.3, it is now the cops' turn.

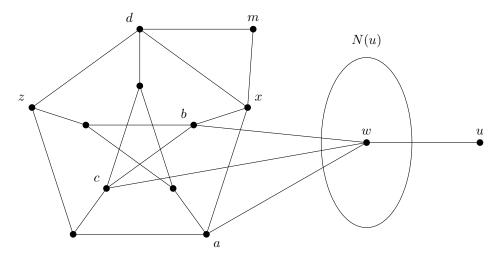


FIGURE 5. Example situation during the proof of Lemma 4.7. Unused or unknown vertices and edges are omitted.

The cop on c moves to w, the cop on z moves to either d or a neighbour of d (this is possible because the Petersen graph has diameter 2), and the cop on u stays still. All neighbours of x in N(u) are covered by the cop on u, the vertices a and b are covered by the cop on w, and d is covered by the 3rd cop, which is either on d or on a neighbour of d. Hence, after the robber's move the robber must either be on x or on m (if m exists, that is $1 \le i \le 6$, and if moving to m from x is possible). We note that in the special case with x = m' and $i \in \{5,6\}$ the robber might have already been on m, but then the same argument as above shows that the robber could not have moved outside of either x = m' or m.

If the robber is still on x, but cannot be immediately captured, the cop which is adjacent to d moves to d. Now, the robber cannot stay put without being captured. Hence, for the rest of the proof we can assume the robber has now moved to m or was already on m (in the latter case, there is not necessarily a cop now on d, as we only do this move if the robber was still on x). This in particular excludes the case i = 0.

If m' is a or b, the cop on w moves to m'. If m' = d, the cop that is adjacent to d or on d moves to (or stays on) d. In both cases, there is now a cop on m', which, together with the cop on u, guarantees that the robber is now stuck on m. The third cop may capture the robber within a few turns.

If m' = x, then, by definition, $N(m) \cap B_u \subseteq \{a, b, d, x\}$. As previously done, move a cop to d (if it is not already there). The pair of cops on d and w cover this set, hence the robber cannot move. At the next cops' turn, the cop on d moves to m'. Then, we are in the same situation as above.

In all cases, there is a contradiction as c(G) > 3.

Although the statement of the previous lemma appears somewhat convoluted, it can essentially be reformulated as the following lemma.

Lemma 4.8. Consider Hypothesis 4.5. If $w \in N(u)$, then there exists a vertex of B_u dominating $N(w) \cap B_u$.

Proof. Recall that $\langle B_u \rangle$ is a Petersen graph. The result is trivial if $|N(w) \cap B_u| \leq 2$ as the diameter of $\langle B_u \rangle$ is 2. If $|N(w) \cap B_u| \geq 3$, suppose the statement is false. As $\langle B_u \rangle$ does not contain a triangle, not all vertices of $N(w) \cap B_u$ can be pairwise adjacent, so we can choose $a, b \in N(w) \cap B_u$ such that

a,b are not adjacent. Denote x the common neighbour of a,b in B_u . By our supposition, x does not dominate $N(w) \cap B_u$, thus we can choose $c \in N(w) \cap B_u$ not in N[x]. The subset $\{a,b,c\}$ then contradicts Lemma 4.7.

In particular, every vertex of N(u) can have at most 4 neighbours in B_u because $\langle B_u \rangle$ is 3-regular. We also note that in some of the cases there is a unique choice for this dominating vertex, in particular when $N(w) \cap B_u$ has 3 or 4 vertices. We are now ready to strengthen Lemma 4.6.

Lemma 4.9. Consider Hypothesis 4.5. For all $x \in B_u$, the following holds.

- (1) If $x \notin N[m']$, then $|N(x) \cap N(u)| \geq 3$.
- (2) If $x \in N[m']$, then $|N(x) \cap N(u)| \ge 2$.

Proof.

(1) Suppose the contrary: there exists $x \in B_u \setminus N[m']$ such that $|N(x) \cap N(u)| \in \{1, 2\}$ (as by Lemma 4.6, $|N(x) \cap N(u)| \ge 1$). We give a winning strategy for 3 cops on G. Denote by w_1, w_2 the neighbours of x in N(u) (if there is only one neighbour, set $w_1 = w_2$) and by y_1, y_2, y_3 the neighbours of x in B_u .

By Lemma 4.8, there exists a vertex of B_u dominating the neighbourhood of w_1 in B_u . As x is in this neighbourhood, we know this dominating vertex (there might be more than one possible choice) is in $\{y_1, y_2, y_3, x\}$. This is also true for w_2 . Thus, we can pick at most 2 elements of $\{y_1, y_2, y_3, x\}$ that dominate all neighbours of w_1, w_2 in B_u .

Without loss of generality (by symmetry of y_1, y_2, y_3 in the Petersen graph), we assume the vertices dominating the neighbourhoods of w_1 and of w_2 in B_u are in $\{y_1, y_2, x\}$. By Lemma 4.6, y_3 must have a neighbour t in N(u). The situation is portrayed in Figure 6.

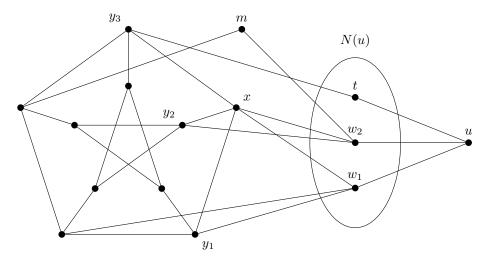


FIGURE 6. Example situation during the proof of Lemma 4.9 (1). Unused or unknown vertices and edges are omitted.

We place one cop on u. We use Lemma 4.4 to place the robber on x and the two other cops on y_1, y_2 . During the last move of this strategy, the cop on u moves to t. It is now the robber's turn. The robber on x cannot move to a neighbour inside of B_u (there are cops on y_1 and y_2 , and y_3 is covered by the cop on t) and there are cops adjacent to the robber. As $x \notin N[m']$, we know that $m \notin N(x)$. Thus, the robber has no choice but to move to either w_1 or w_2 . Let us say the robber moves to w_1 (the strategy for w_2 is analogous).

Denote by a the vertex dominating the neighbours of w_1 in B_u . We recall that a is either y_1, y_2 or x. We now move the cop on t back to u. Of the two cops on y_1 and y_2 , one must be able to move to a, and does so. If $m' \in B_u$ (that is, if we are not in the case of \mathcal{P}_0), the third

cop moves to either m' or a neighbour of m'. After this move, all escapes in N(u) are covered by the cop on u, all escapes in B_u are covered by the cop on a, and the robber cannot stay still as u is adjacent to w_1 . Thus, the robber is caught one move later unless the robber can move to m. In this case, the third cop can now move to m' and trap the robber. Leaving the cops on u and m' fixed, the cop on a can then go capture the robber. This is a contradiction as c(G) > 3.

(2) As the proof will be very similar to the previous case, we outline the main differences. We suppose the statement is false, so that m' has at most one neighbour in N(u) (and so exactly 1, by Lemma 4.6).

If x = m' and $i \in \{5, 6\}$, recall that m' is chosen to be of higher degree (or equal) than m, so m' has exactly one neighbour in N(u), and m has at most one neighbour in N(u). Then, we denote by w_1 and w_2 these neighbours and apply the same strategy as above. Even though the robber will have chosen to go to either m' or m, both cases for its subsequent move will be covered using the strategy above. In particular, we note that the robber will not be able to stay on either x = m' or m, as both are adjacent to y_1, y_2 , so the robber will indeed have to move to either w_1 or w_2 . Thus, either m or m' must have 2 or more neighbours in N(u). As we have selected m' to have the greatest degree of the two, the statement follows for this case.

Consider now that $x \in N[m']$ but $x \neq m'$ or $i \notin \{5,6\}$ (as we covered that case above). Our goal is to prove that x cannot have a unique neighbour in N(u). Since we are supposing the contrary, let w_1 be a unique neighbour of x in N(u). As in the above strategy, one cop's role will be to cover the vertex dominating the neighbourhood of w_1 in B_u , or if this is x, then to be on an adjacent vertex. We will also want a cop to be on m', or on a neighbour of m' if x = m'. In the notation of the original case, this could informally be seen as considering w_2 to be m. If the robber moves to w_1 , we follow a similar strategy to above. If the robber moves to m, then one of the cops will move to m'. Recalling that another robber will return to u, the last cop will be able to go capture the robber.

We are now ready to prove the desired results.

Proposition 4.10. If G is a connected graph such that $\Delta \in \{n-12, n-11\}$ and $n \leq 18$, then $c(G) \leq 3$.

Proof.

(1) We consider $\Delta = n-11$. Let u be a vertex of maximum degree. We know that |V(G-N[u])| = 10. If G-N[u] is disconnected, each one of its connected components has cop number at most 2, as no connected component can contain at least 10 vertices. Applying Corollary 2.5 yields the desired result. Otherwise, G-N[u] must be connected. Suppose that c(G) > 3. Then, c(G-N[u]) > 2, and by Theorem 2.1, G-N[u] must be isomorphic to \mathcal{P}_0 . Then, G and u satisfy the conditions of Hypothesis 4.5.

By Lemma 4.9, every vertex of B_u has at least 3 neighbours in N(u) (recall that m' does not exist in \mathcal{P}_0). As B_u has 10 vertices, this means there are at least 30 edges between N(u) and B_u .

As $n \leq 18$, we have that $\Delta \leq 7$. By Lemma 4.8, every vertex of N(u) has at most 4 neighbours in B_u . Thus, there are at most $4\Delta \leq 28$ edges between N(u) and B_u . This is a contradiction, as we have claimed there are at least 30 but at most 28 edges between N(u) and B_u . Thus, $c(G) \leq 3$.

(2) We consider $\Delta = n-12$. Let u be a vertex of maximum degree. We know that |V(G-N[u])| = 11. Let us first consider the case where G-N[u] is disconnected. If every connected component has cop number at most 2, then, as in the previous case, we are done. By Theorem 2.1, the only other case is if one component is isomorphic to \mathcal{P}_0 and the other is an isolated vertex x.

By applying Corollary 2.5 (or Corollary 2.8), $c(G) \le 3$ if and only if $c(G-x) \le 3$. As G-x satisfies the conditions of the previous case of this proposition, we conclude that $c(G-x) \le 3$.

We may now consider that G - N[u] is connected. Suppose c(G) > 3. By Proposition 3.2, $G - N[u] \simeq \mathcal{P}_i$, for some $1 \le i \le 6$. Then, G and u satisfy the condition of Hypothesis 4.5.

By Lemma 4.9, every vertex of B_u has at least 2 neighbours in N(u), and in fact those not in N[m'] (of which there are at least 6) have at least 3 neighbours in N(u). In total, there are at least 26 edges between N(u) and B_u .

By Lemma 4.8, each vertex of N(u) has at most 4 neighbours in B_u . As $n \leq 18$, we have that $\Delta \leq 6$. Thus, there are at most $4\Delta \leq 24$ edges between N(u) and B_u . This is a contradiction, as we have claimed there are at least 26 but at most 24 edges between N(u) and B_u , hence $c(G) \leq 3$.

These results will be used to prove that $M_4 = 19$ in Section 7. Readers only interested in the main result may skip the remainder of this section, as it concerns reducing the number of possible 4-cop-win graphs on 19 vertices. We first need the following definition.

Definition 4.11. Consider Hypothesis 4.5. Let $w \in N(u)$ such that $|B_u \cap N(w)| = 4$. The vertex $x \in B_u$ dominating $B_u \cap N(w)$ will be called the *projection* of w. If x is the projection of k vertices of N(u), we will call p(x) = k the *projection multiplicity* of x.

By Lemma 4.8, this is well defined and the projection of a vertex is unique.

Observation 4.12. Consider Hypothesis 4.5. If $x \in B_u$, $|N(x) \cap N(u)| \ge \sum_{y \in N[x] \cap B(u)} p(y)$. In particular, if $y \in B_u$ has projection multiplicity k, then each vertex in $N(y) \cap B_u$ has at least k neighbours in N(u).

Proof. We recall that when a vertex y has projection multiplicity k, this means that k vertices of N(u) have for neighbours in B_u exactly $N[y] \cap B_u$, giving each vertex in this set at least k neighbours in N(u).

Noting that the projection of a vertex is unique, we see that the neighbours that x inherits from each projection on x or on its neighbours in B_u are pairwise distinct. The lower bound follows immediately by summing the projection multiplicity for each vertex in N[x].

We now see an interesting property of projections.

Lemma 4.13. Consider Hypothesis 4.5. Let $x \in B_n \setminus N[m']$.

- (1) If $|N(x) \cap N(u)| = 3$, then p(x) = 0.
- (2) More generally, $p(x) \leq |N(x) \cap N(u)| 2$.

Proof.

(1) Supposing the contrary, we give a winning strategy for 3 cops. Let us say that x is the projection of a vertex w of N(u): w is adjacent to x and to each neighbour of x in B_u . As $|N(x) \cap N(u)| = 3$, x has two other neighbours in N(u), which we will denote by t_1 and t_2 . If t_1 has a neighbour in B_u other than x, choose one and denote it r_1 . If not, then choose r_1 to be any neighbour of x in B_u . We choose r_2 similarly. The situation is portrayed in Figure 7.

We start by placing one cop on u. Using Lemma 4.4 (recall that $x \neq m'$, which avoids the exceptional case), we place the robber on x and the two other cops on r_1 and r_2 (if $r_1 = r_2$, then place one cop there and another cop on any vertex in $B_u \setminus \{x, r_1\}$). During the last move of that strategy, move the cop from u to w. It is now the robber's turn. As there is a cop on w, the robber cannot stay in B_u . As $x \notin N[m']$, x is not adjacent to m. If t_1 had a neighbour in B_u other than x, then the cop on r_1 blocks the robber from moving to t_1 , and similarly for t_2

Thus, the only scenario in which the robber does not get captured immediately after moving is if (without loss of generality), t_1 only has one neighbour in B_u , and the robber moves to t_1 . In this case, we chose r_1 to be some neighbour of x. The cop on w moves back to to u,

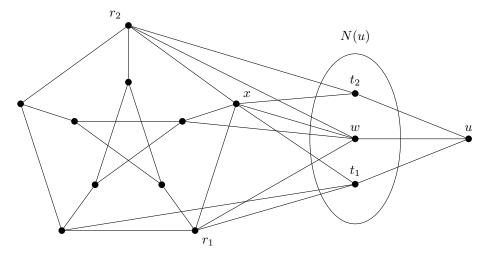


FIGURE 7. Example situation during the proof of Lemma 4.13 (1). Unused or unknown vertices and edges are omitted.

and the cop on r_1 moves to x. The third cop moves to m' or a neighbour of m' (if m' is in the graph). The robber is caught one turn later, as x is the only neighbour of t_1 in B_u and u dominates N(u), unless the robber can move to m. In this case, the third cop can move to m' and trap the robber. The cop on x can capture the robber within a few turns. This contradicts that c(G) > 3.

(2) The strategy is similar to the previous case. Suppose, to the contrary, that x has projection multiplicity at least $|N(x) \cap N(u)| - 1$. Then, there is at most 1 neighbour of x in N(u) which does not project onto x. Choose t_1 to be this vertex (if there is any) and select the corresponding r_1 as above. Choose r_2 to be any other neighbour of x in B_u : r_2 covers all vertices projecting onto x. The rest of the strategy is identical.

We are now ready for the desired result.

Proposition 4.14. If G is a connected graph such that n = 19 and $\Delta \in \{7, 8\}$, then $c(G) \leq 3$.

Proof. Suppose c(G) > 3. Let u be a vertex of maximal degree in G.

(1) We consider $\Delta = 8$. Recall the arguments of the proof of Proposition 4.10. In particular, we can consider that $G - N[u] \simeq \mathcal{P}_0$.

There are at most $4\Delta = 32$ edges between N(u) and B_u , by Lemma 4.8. By Lemma 4.9, each vertex in B_u has at least 3 neighbours in N(u): there are at least 30 edges between B_u and N(u). Thus, there are at most 2 extra edges. By extra edges, we mean that these are edges which, if removed, would leave each vertex in B_u with exactly the lower bound number of neighbours in B_u , as specified in Lemma 4.9. Then, there are at least 8 vertices in B_u which have no extra edges, having exactly 3 neighbours in N(u).

Furthermore, if there are fewer than 6 vertices of N(u) that each have exactly 4 neighbours in B_u , then there cannot be at least 30 edges between N(u) and B_u . Thus, $\sum_{x \in B_u} p(x) \ge 6$.

Recall that Lemma 4.13 states that no vertex in N(u) with exactly 3 neighbours in $B_u = B_u \setminus N[m']$ can be a projection. Hence, only the vertices with extra edges may have non-zero projection multiplicity. The first consequence of this is that it is impossible for all vertices of B_u to have exactly 3 neighbours in N(u). Furthermore, we have seen that there are at most 2 vertices which have extra edges (these are the vertices of B_u that have either 4 or 5 neighbours in N(u)). Denote them by a_1, a_2 (if there is only one vertex, let $a_1 = a_2$). Then, $p(a_1) + p(a_2) \ge 6$ (if $a_1 = a_2$, then simply say $p(a_1) \ge 6$). Let $x \in N[a_1] \cap N[a_2] \cap B_u$ (which

exists as \mathcal{P}_0 has diameter 2). As x is adjacent to all projections, x must be adjacent at least 6 vertices of N(u) (Observation 4.12). As x also has 3 neighbours in B_u , the degree of x is at least 9, which is a contradiction.

(2) We consider $\Delta = 7$. Recall the arguments of the proof of Proposition 4.10. In particular, we can say that $G - N[u] \simeq \mathcal{P}_i$, for some $1 \leq i \leq 6$.

There are at most $4\Delta = 28$ edges between N(u) and B_u , by Lemma 4.8. By Lemma 4.9, each vertex in $B_u \setminus N[m']$ has at least 3 neighbours in N(u) and each vertex in $B_u \cap N[m']$ has at least 2 neighbours in N(u): in total, there are at least 26 edges between B_u and N(u). Using the same argument as above, depending on the number of edges between B_u and N(u), we can find between 5 and 7 vertices in N(u) which have 4 neighbours each in B_u , and thus the total projection multiplicity of B_u is as follows.

- (a) 26 edges: $\sum_{x \in B_u} p(x) \ge 5$; (b) 27 edges: $\sum_{x \in B_u} p(x) \ge 6$; (c) 28 edges: $\sum_{x \in B_u} p(x) = 7$ (as there are exactly 7 vertices in N(u) there cannot be more than 7 projections).

Recall that Lemma 4.13 states that no vertex in $B_u \setminus N[m']$ with 3 neighbours in N(u) can be a projection. Also, if $x \in B_u \setminus N[m']$, $p(x) \leq |N(x) \cap N(u)| - 2$: if x has 4 neighbours in N(u) it can be the projection of at most 2 vertices.

If all vertices in $B_u \setminus N[m']$ have exactly 3 neighbours in N(u), then this implies all projections will be vertices in N[m']: at least 5 vertices project on m' or on a neighbour. Thus, m'will have at least 5 neighbours in N(u). As m' also has at least 3 neighbours in B_u , $d(m') \geq 8$, which is impossible as $\Delta = 7$. This situation includes the case in which there are exactly 26 edges between B_u and N(u).

Suppose there is exactly one vertex x of $B_u \setminus N[m']$ with exactly 4 neighbours in N(u), with all others having exactly 3. This vertex will have projection multiplicity at most 2, so the total projection multiplicity of vertices of N[m'] is at least 4. Thus, m' will have at least 4 neighbours in N(u), which is impossible as this would imply there are 29 edges between B_u and N(u) (x has 4, the other 5 vertices in $B_u \setminus N[m']$ have 3, m' has at least 4, and each of the 3 vertices of $B_u \cap N(m')$ has at least 2).

Suppose now there are 2 vertices x_1, x_2 of $B_u \setminus N[m']$ with 4 neighbours in N(u). These two additional edges bring the total to 28. Thus, the total projection multiplicity is 7. There are at most 2 vertices projecting onto x_1 and 2 vertices projecting on x_2 (and none on the other vertices in $B_u \setminus N[m']$). Thus, at least 3 vertices project onto vertices in N[m']. This a contradiction, as m' must have exactly 2 neighbours in N(u), otherwise there would be more than 28 edges between B_u and N(u). Considering that with $\Delta = 7$, no vertex of B_u can have 5 or more neighbours in N(u), there are no cases left.

In all possible cases, a contradiction was found. Thus, c(G) < 3.

5. Graphs with maximum degree 3

In this section, we consider the cop number of graphs with maximum degree 3. We start with the main result of this section.

Proposition 5.1. If G is a connected graph such that $\Delta \leq 3$ and $n \leq 20$, then $c(G) \leq 3$.

Proof. We first prove the statement for $\delta \geq 2$. For $10 \leq n \leq 20$, we generate all graphs such that $\delta > 2$ and $\Delta < 3$, and then classify each graph according to its cop number. Both steps use the same software and scripts as in Section 3. We present the results in Table 4, which shows that no such graph with cop number at least 4 exists. We have also saved the precise 3-cop-win graphs.

We now considers graphs which contain vertices of degree 1. We know that removing a vertex of degree 1 from a graph does not change the cop number nor the fact that it is connected (as the vertex of degree 1 is cornered by its neighbour, see Corollary 2.8). We successively remove vertices of degree 1 from the graph. We eventually either get to a graph such that $\delta \geq 2$ and $n \geq 10$ (in which case the

			Cop number					
n	$G: \delta \ge 2, \Delta \le 3$	1	2	3	≥ 4			
10	458	7	450	1	0			
11	1353	12	1341	0	0			
12	4566	21	4543	2	0			
13	15530	35	15495	0	0			
14	56973	63	56901	9	0			
15	214763	114	214642	7	0			
16	848895	211	848622	62	0			
17	3454642	388	3454093	161	0			
18	14542574	735	14540858	981	0			
19	62871075	1389	62865352	4334	0			
20	279175376	2664	279147564	25148	0			

Table 4. Cop number breakdown for connected subcubic graphs.

above results can now be applied) or we eventually get to a graph of order at most 9 (in which case we apply Theorem 2.1).

We will use this proposition to prove our main result in Section 7. The remainder of this section is not essential and mostly concerns possible improvements and applications of this proposition.

Notwithstanding the slight improvement of considering $\delta \geq 2$, the approach here is clearly far from optimal. The algorithm described in the Section 6 is an example of a possibly better strategy. However, as we will see, this algorithm would not be the most efficient for maximum degree 3: to compute potential 4-cop-win graphs on 19 vertices, one would still need to compute subcubic 3-cop-win graphs on 15 vertices. A potentially more interesting algorithm for building possible 4-cop-win subcubic graphs would consist in building graphs around long shortest paths (see [2, Lemma 4], which describes how a cop can protect a shortest path) by adding the desired number of other vertices and considering all possible ways to add edges. Nonetheless, our exhaustive testing approach is not without its advantages, as we can use it to gain further knowledge on the cop number of small graphs.

In fact, Hosseini, Mohar and González Hermosillo de la Maza [22] have recently shown that studying the cop number of graphs with $\Delta \leq 3$ is of interest for the study of the cop number at large. More precisely, they show that if it were true that the cop number of subcubic graphs is in $O(\sqrt{n})$, then the cop number of general graphs is in $O(n^{\alpha})$ for all $\alpha > \frac{3}{4}$. In other words, proving Meyniel's conjecture for subcubic graphs would substantially improve the best known upper bound on the cop number. Hence, we consider that getting a distribution of the cop-number of small subcubic graphs might be interesting, even if it is somewhat skewed by adding the condition $\delta \geq 2$. Our computations show that not only there are no 4-cop-win subcubic graphs on at most 20 vertices, but that subcubic 3-cop-win graphs are overwhelmingly rare for these orders.

The exhaustive search approach also gives us progress on a related problem. Arguably the best-known result on the game of cops and robbers is Aigner and Fromme's proof that the cop number of any planar graph is at most 3, see [2]. This yields the analogous question of finding the minimum order of a 3-cop-win planar graph, and an enumeration of such graphs. The smallest known planar 3-cop-win graph is the dodecahedral graph (see Figure 8), which has 20 vertices. It is easy to see that this graph requires 3 cops, as it has girth 5 and is 3-regular. It has been asked, first in [3], as well as in [12], whether the dodecahedral graph is the unique smallest 3-cop-win planar graph.

There are some partial results for this problem. In [20], Hosseini proves that a 3-cop-win planar graph of minimum order must be 2-connected. Furthermore, Pisantechakool and Tan have shown in [28] that any planar graph on 19 or fewer vertices must contain a winning position for 2 cops, although it has not been proved that the cops can bring the game to this winning state. Using the computations in the proof of Proposition 5.1, we are able to get more evidence supporting the conjecture.

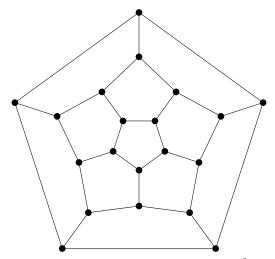


FIGURE 8. The dodecahedral graph²

Corollary 5.2. If G is a connected planar graph such that $\Delta \leq 3$ and $n \leq 20$, then $c(G) \leq 2$, unless G is the dodecahedral graph.

Proof. We simply test the 3-cop-win graphs found in the proof of Proposition 5.1 for planarity [34]. The only graph which was planar was the dodecahedral graph. The arguments relating to the fact that we only computed the graphs such that $\delta \geq 2$ still apply here.

6. Remaining cases

In this section, we consider the few remaining cases needed to prove that $M_4 = 19$, and also work towards reducing the possible 4-cop-win graphs on 19 vertices. More precisely, we consider graphs such that n = 17 with $\Delta = 4$, n = 18 with $\Delta = 4$, 5 and n = 19 with $\Delta = 4$.

As in Section 4, our main tool will be knowing that if a graph G is 4-cop-win, then for each vertex $u, c(G-N[u]) \geq 3$. We know there are relatively few such graphs. In the cases of $\Delta = n-11$ or $\Delta = n-12$, if u is a vertex of maximum degree, we know that these G-N[u] can only be the Petersen graph and cornered Petersen graphs (at least in the connected case). As these are very few and very similar, we were able to build structural properties that allowed us to show that G had cop number at most 3. Having somewhat a large maximum degree, a computational approach would have been difficult due to the fact that there are too many possible edges to consider to effectively construct all possible graphs.

For the cases we will now consider, a computational approach is possible as the maximum degree is not too high. On the other hand, a proof similar to the one of Section 4 for these cases would be difficult, although certainly not impossible given a large amount of time. Indeed, there are too many possible choices for G - N[u], and they are not similar enough. As most graphs found in Lemma 3.3 contain the Petersen graph as an induced subgraph, modifying the strategy to take these vertices into account could seem reasonable. However, we saw that adding even a single corner to the Petersen graphs yields significant complications for the proof. Adding multiple vertices to the Petersen graph would likely be much more complex. Furthermore, some of the graphs found in Lemma 3.3 do not contain the Petersen graph as an induced subgraph at all, and would need to be considered separately. For these reasons, we have mostly investigated the computational approach.

Our goal is to build graphs which are possibly 4-cop-win: graphs G for which we cannot say that $c(G) \leq 3$ simply by looking at G - N[u] for the vertices u of maximum degree, which we will call candidate 4-cop-win graphs.

²Computer-generated drawing [34].

The simplest idea, which we have briefly discussed in Section 3, would be simply to consider a 3-cop-win graph G' on 12 or 13 vertices, add a vertex u of chosen maximum degree and its neighbourhood, and then look at every possible ways of joining N(u) to G' by respecting the maximum degree condition. Then we could compute the cop number of each of these graphs. Even by reducing the number of cases by isomorphism, the number of graphs to consider is massive, especially in the case $\Delta = 5$, so we must be a tad smarter. We present the Merging Algorithm as a way to generate candidate 4-cop-win graphs, which we then test using a standard cop-number algorithm.

We briefly introduce some notation. In general, when considering a graph G and a vertex u, the degree of u will always refer to the degree of G in u. If we want to discuss the degree of u in some induced subgraph H, we will refer to it as the H-degree of u. In general, if we say "there exists a vertex of H-degree r", then we are also implicitly stating that this vertex is in H. Given some function ϕ defined on some set containing T, we will use the notation $\phi(T) = {\phi(t) : t \in T}$.

6.1. Presentation of the Merging Algorithm.

6.1.1. Quick Overview. Our approach to build candidate 4-cop-win graphs G will be the following. Let v_1 and v_2 be non-adjacent vertices, which we will in general choose to be a pair with the highest possible degrees. Denoting $G_1 = G - N[v_2]$ and $G_2 = G - N[v_1]$, we know that if G is to be a candidate 4-cop-win graph, the cop number of G_1 and G_2 must have cop number at least 3 (we will see later that we can consider G_1 and G_2 to be connected).

In Section 3, we computed the 3-cop-win graphs on specific numbers of vertices and with some maximum degree conditions. As we have a few different cases to consider, the possible choices of G_1 and G_2 will vary. For now, we can simply denote by \mathcal{L}_1 and \mathcal{L}_2 some sets of 3-cop-win graphs. Our goal is to determine every possible graph G, with maximum degree Δ , for which $G_1 \in \mathcal{L}_1$ and $G_2 \in \mathcal{L}_2$. We will call the process the Merging Algorithm, which we will now detail.

- 6.1.2. Input of the Algorithm. Integers n, D_1, D_2 and sets of (isomorphism classes of) graphs \mathcal{L}_1 and \mathcal{L}_2 , such that
 - (1) the graphs in \mathcal{L}_1 and \mathcal{L}_2 are connected, 3-cop-win and have maximum degree at most Δ , and
 - (2) the graphs in \mathcal{L}_1 have $n D_2 1$ vertices and the graphs in \mathcal{L}_2 have $n D_1 1$ vertices.
- 6.1.3. Output of the Algorithm. The algorithm returns all connected graphs G on n vertices and maximum degree $\Delta = D_2$ which contain a pair of non-adjacent (and distinct) vertices v_1 and v_2 , with the following 4 properties. Denote $G_1 = G N[v_2]$ and $G_2 = G N[v_1]$. Then,
 - (1) v_1 and v_2 have degree respectively D_1 and D_2 ,
 - (2) $G_1 \in \mathcal{L}_1$ and $G_2 \in \mathcal{L}_2$,
 - (3) for all other vertices u of maximum degree $(d(u) = \Delta)$, $G N[u] \in \mathcal{L}_1$, and
 - (4) if $D_1 < D_2$, then the set of vertices of G of maximum degree forms a clique and v_1 and v_2 have at least 1 common neighbour.

Isomorphic graphs may be omitted from the results, as we are not interested in the precise labellings of the graphs.

6.1.4. Required definitions. We define a few concepts which will be useful in the description of the algorithm.

A partially-constructed graph is a triple $(\widehat{G}, \mathcal{F}, R)$ where \widehat{G} is a graph, \mathcal{F} is a partition of $V(\widehat{G})$, and $R \subseteq V(\widehat{G})$. The graph \widehat{G} is a graph to which we might still add edges in later steps of the algorithm. The set \mathcal{F} will be used to contain some information on how \widehat{G} was constructed. We will use R to denote some set of vertices which must have degree strictly smaller than $D_2 = \Delta$, which will be useful to reduce the number of cases we need to verify.

We say ϕ is a *strong isomorphism* between $(\widehat{G}, \mathcal{F}, R)$ and $(\widehat{G}', \mathcal{F}, R)$ (we suppose both graphs have the same extra structure given to them, in particular $V(\widehat{G}) = V(\widehat{G}')$) if ϕ is an isomorphism between \widehat{G} and \widehat{G}' such that $\phi(S) = S$ for each $S \in \mathcal{F}$ and such that $\phi(R) = R$, i.e. ϕ is an isomorphism that preserves the extra structure we attach to our graphs.

Let $G_1 \in \mathcal{L}_1$. We define an equivalence relation on $V(G_1)$ as follows: $u \sim_{G_1} v$ if there is some automorphism ψ of G_1 such that $\psi(u) = v$. Then, we can define a total ordering \leq_{G_1} on $V(G_1)/\sim_{G_1}$ by comparing the vertices themselves as follows: if u, v are two distinct (equivalence classes of) vertices, we set $u <_{G_1} v$ if $d_{G_1}(u) < d_{G_1}(v)$, and if u, v have the same degree, then we choose the order arbitrarily.

6.1.5. Phase 1 of the Algorithm. First choose some G_1 and G_2 from \mathcal{L}_1 and \mathcal{L}_2 respectively. The rest of the algorithm will be repeated for each possible choice of G_1 and G_2 . Also choose strictly positive integers d_1 and d_2 such that $D_2 - d_2 = D_1 - d_1$, $d_1 \leq D_1$ and $d_2 \leq D_2$. Again, every possible choice will be considered. When $D_1 < D_2$, it will suffice to pick d_1 such that $d_1 < D_1$ (and thus $d_2 < D_2$).

Then consider every possible choice of $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that v_1 has G_1 -degree d_1 and v_2 has G_2 -degree d_2 (of course, v_1 and v_2 can be considered up to automorphism in G_1 and in G_2). For each choice of vertices, consider every possible way of identifying $G_1 - N[v_1]$ and $G_2 - N[v_2]$ by computing every isomorphism between these graphs. If there are none, this branch of the algorithm simply doesn't yield a graph. For each identification, the graphs may be merged by union, keeping the closed neighbourhoods of v_1 and v_2 distinct.

In the case $D_1 = D_2$, let $R = \{u \in V(G_1) : u >_{G_1} v_1\}$ be the set of vertices of G_1 strictly greater than v_1 in the ordering defined above. In the second main case of the algorithm, when $D_1 < D_2$, set $R = V(G_1)$. Note that as R is a subset of the vertices of $V(G_1)$, it is also a subset of the vertices in the graph resulting from the identification above. If the merging process above has created vertices of degree greater than $D_2 = \Delta$, the graph is thrown out: as the rest of the algorithm can only raise the degree again, it would yield graphs we do not want to consider. Similarly, throw out the graphs in which any vertex in R has maximum degree. Finally, add $D_2 - d_2 = D_1 - d_1$ common neighbours to v_1 and v_2 (these are new vertices). Now v_1 has degree D_1 and v_2 has degree D_2 . Note that in the case $D_1 < D_2$, we limited d_1 to $d_1 < D_1$, so v_1 and v_2 have at least 1 common neighbour.

Denote by \widehat{G} some graph resulting from these operations. It is easily seen in Figure 9 that the construction implicitly partitions the vertices into 6 sets (or "types" of vertices):

$$\mathcal{F} = \{ \{v_1\}, N(v_1) \setminus N(v_2), (N[v_1] \cup N[v_2])^c, N(v_1) \cap N(v_2), N(v_2) \setminus N(v_1), \{v_2\} \}$$

where the complement is taken relative to $V(\widehat{G})$. We call the partially-constructed graph $(\widehat{G}, \mathcal{F}, R)$ a base graph. We will sometimes call \widehat{G} itself a base graph, as the remaining structure is usually clear.

Suppose \widehat{G} and \widehat{G}' are generated in Phase 1 of the algorithm with the same properties, that is, they are produced from same algorithm inputs and the same choices of $G_1, G_2, d_1, d_2, v_1, v_2$ but by choosing a different identification in the merging step. In this case, it is clear that \mathcal{F}, R are identical for both. If $(\widehat{G}, \mathcal{F}, R)$ and $(\widehat{G}', \mathcal{F}, R)$ are strongly isomorphic, we can consider these graphs to be duplicates: only one needs to be sent to Phase 2 of the algorithm. We will see later that, up to isomorphism, these strongly isomorphic base graphs necessarily result in isomorphic candidate 4-cop-win graphs.

The result of Phase 1 should then be some (usually large) collection of base graphs, as we have to consider every choice of $G_1, G_2, d_1, d_2, v_1, v_2$ and identification of $G_1 - N[v_1]$ and $G_2 - N[v_2]$.

- 6.1.6. Phase 2 of the Merging Algorithm. The goal of this phase is to complete the construction of the 4-cop-win candidates graphs by adding edges to the base graphs. We will run Phase 2 for every base graph from Phase 1. Suppose we have some base graph $(\widehat{G}, \mathcal{F}, R)$. We know that we do not want to add any edge such that both ends are in G_1 or both ends in G_2 , as we want these to be induced subgraphs of G. Furthermore, we have already created all incident edges to either v_1 or v_2 , by giving them the desired number of neighbours. Thus, we only need to consider adding edges which are either
 - (1) between $N(v_1) \cap N(v_2)$ and $\{v_1, v_2\}^c$, including edges with both ends in $N(v_1) \cap N(v_2)$, or
 - (2) between $N(v_2) \setminus N(v_1)$ and $N(v_1) \setminus N(v_2)$.

We fix an order on the vertices of $N(v_2)$ as follows: first the vertices in $N(v_1) \cap N(v_2)$, then those in $N(v_2) \setminus N(v_1)$. Denote a_i to be *i*-th vertex in this order, which will be considered at the *i*-th step of Phase 2.

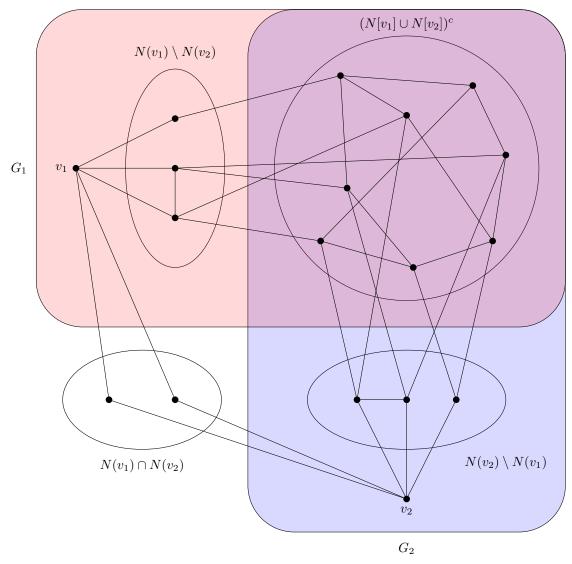


FIGURE 9. Example from Phase 1 of the Merging Algorithm. Here, the base graph was generated using parameters $n=18,\,D_1=D_2=\Delta=5$ and $d_1=d_2=3$.

At the *i*-th step of Phase 2, a new partially-constructed graphs is created for each way of adding edges incident to a_i of the form described above, with the following restrictions:

- (1) not to add an edge if one of its end vertices already has degree Δ ,
- (2) not to add an edge if one of its end vertices is in R and adding this edge would bring that vertex to degree Δ , and
- (3) not to add an edge between $a_i a_j$ if i > j.

Then, for each of these newly created graphs, repeat the process by moving on to a_{i+1} , and so on. Hence restriction (3) comes from the fact that such an edge $a_i a_j$ has already been considered at the j-th step.

In this phase, we will also reduce some cases by strong isomorphism. Let $A_i = \{a_1, \ldots, a_i\}$. If $a_i \in N(v_1) \cap N(v_2)$, let

$$\mathcal{T}_i = \{ N(v_1) \cap N(v_2) \cap A_i, (N(v_1) \cap N(v_2)) \setminus A_i, N(v_2) \setminus N(v_1) \}.$$

Otherwise, if $a_i \in N(v_2) \setminus N(v_1)$, then let

$$\mathcal{T}_i = \{ N(v_1) \cap N(v_2), (N(v_2) \setminus N(v_1)) \cap A_i, (N(v_2) \setminus N(v_1)) \setminus A_i \}.$$

Finally, define

$$\mathcal{F}_i = \{\{v_1\}, N(v_1) \setminus N(v_2), (N[v_1] \cup N[v_2])^c, \{v_2\}\} \cup \mathcal{T}_i.$$

In other words, we refine \mathcal{F} by distinguishing the vertices for which we have already considered adding extra edges.

Hence, we will attach to the graphs produced in *i*-th step of Phase 2 this extra structure, meaning the *i*-th step produces partially-constructed graphs of the form $(\hat{G}', \mathcal{F}_i, R)$. Note that \mathcal{F}_i only depends on the base graph and at the step number in the edge adding procedure, and not the choices of edges added in steps $1, \ldots, i$. Hence, we may compare partially-constructed graphs and remove strongly isomorphic repetitions (we remind the reader that this is only for those constructed from the same base graph). As this procedure is often lengthy, we only remove graphs by strong-isomorphism in some cases.

In the case $D_1 < D_2$, if at any point the (partially-constructed) graph contains non-adjacent vertices both of degree Δ , we can throw out this graph, since these vertices will necessarily remain non-adjacent in final graphs.

After considering every possible way of adding edges (after the $|N(v_2)|$ -th step), all graphs G such that there is some vertex u of maximum degree where G - N[u] is not in \mathcal{L}_1 are thrown out (by construction, we do not need to verify this for v_1 and v_2). From the remaining graphs, we also remove isomorphic graphs. Note that since we have split up the computations in many pieces, we only compare for isomorphism graphs which were generated from the same base graph.

6.1.7. Comments on the Merging Algorithm. We note that in practice, in order to verify strong isomorphism, we do not need to test whether $\phi(R) = R$ holds. Indeed, as ϕ preserves in particular $\{v_1\}$, $N(v_1) \setminus N(v_2)$ and $V(G) \setminus (N[v_1] \cup N[v_2])$, we know that ϕ restricted to the vertices of G_1 is an automorphism of G_1 . Recall that R was constructed using the ordering \leq_{G_1} on $V(G_1)/\sim_{G_1}$ (when $D_1 = D_2$) or was chosen to be $V(G_1)$ (when $D_1 < D_2$). Hence R is always closed under under automorphism of G_1 , so in particular $\phi(R) = R$.

In Phase 1, one particular consequences of having a set R is the following. Suppose $D_1 = D_2$ and G_1 already (before merging) contains multiple vertices of degree $D_1 = D_2 = \Delta$. Let u be some vertex maximal relative to \leq_{G_1} (it is unique up to automorphisms). Then, in Phase 1 when v_1 is not u, no graph will be generated. Indeed, in this case $u \in R$ but u necessarily has degree Δ after merging, a case we have excluded. For this reason, when G_1 contains a vertex of degree Δ , we do not need to try multiple choices of v_1 . In particular, when $\Delta(G_1) = \Delta$, we can only consider the case $d_1 = D_1 = \Delta$.

6.1.8. Validity of the Merging Algorithm. Considering that the algorithm itself is relatively straightforward, we do not present a complete proof of the validity of the algorithm. We however present a few key points.

Consider a graph G respecting the conditions described in the Section 6.1.3. Our goal is to show that G, or a graph isomorphic to G, is constructed by the Merging Algorithm. Choose v_2 to be any vertex of degree D_2 in G such that $V(G) \setminus N[v_2]$ contains at least one vertex respecting the conditions for v_1 in Section 6.1.3. Denote by S this non-empty set of possible choices for v_1 . In the case $D_1 < D_2$, we choose v_1 to be any vertex of S. If $D_1 = D_2$, then choose v_1 to be maximal in S relative to the ordering \leq_{G_1} . Then, write $G_1 = G - N[v_2]$, $G_2 = G - N[v_1]$, $D_1 = d_G(v_1)$, $D_2 = d_G(v_2)$, $d_1 = d_{G_1}(v_1)$

and $d_2 = d_{G_2}(v_2)$. It is easy to verify that

$$\begin{split} d_2 - d_1 &= d_{G_2}(v_2) - d_{G_1}(v_1) \\ &= |N_{G-N[v_1]}(v_2)| - |N_{G-N[v_2]}(v_1)| \\ &= |N(v_2) \setminus (N[v_1] \cap N(v_2))| - |N(v_1) \setminus (N(v_1) \cap N[v_2])| \\ &= |N(v_2) \setminus (N(v_1) \cap N(v_2))| - |N(v_1) \setminus (N(v_1) \cap N(v_2))| \\ &= (|N(v_2)| - |N(v_1) \cap N(v_2)|) - (|N(v_1)| - |N(v_1) \cap N(v_2)|) \\ &= |N(v_2)| - |N(v_1)| \\ &= |N(v_2)| - |N(v_1)| \\ &= D_2 - D_1. \end{split}$$

Thus, the pair of degrees d_1 and d_2 is indeed considered in the Merging Algorithm. It is then easy to see that the Phase 1 of the algorithm will have some branch considering this exact choice of $G_1, G_2, d_1, d_2, v_1, v_2$ and the actual isomorphism between $G_1 - N[v_1]$ and $G_2 - N[v_2]$. Note that in the case $D_1 = D_2$, choosing v_1 as maximal in S relative to \leq_{G_1} guarantees that all vertices of $G - N[v_2]$ which are strictly greater to v_1 in this order do not have maximal degree in G, which is consistent with how we use R in the algorithm.

Then, as in Phase 2 of the algorithm we consider adding every possible edge not totally contained in G_1 or G_2 (precisely the edges between $N(v_1) \cap N(v_2)$ and $\{v_1, v_2\}^c$, or between $N(v_2) \setminus N(v_1)$ and $N(v_1) \setminus N(v_2)$) while still respecting some degree conditions, G must have been constructed.

Other steps in the algorithm, such as requiring that $G - N[u] \in \mathcal{L}_1$ for u of maximum degree, or in the case $D_1 < D_2$ removing graphs with non-adjacent vertices of maximum degree, are consistent with Section 6.1.3.

In fact, what we have stated so far is only true up to the removal of partially-constructed graphs using strong isomorphism. Indeed, although we claim a graph isomorphic to G has necessarily been constructed, the graph G might not have been constructed itself. To see this, suppose that $(\widehat{G}, \mathcal{F}, R)$ was thrown out at some step because it was strongly isomorphic to $(\widehat{G}', \mathcal{F}, R)$ (where the strong isomorphism is $\phi: V(\widehat{G}) \to V(\widehat{G}')$). We wish to prove that every graph that would be constructed from $(\widehat{G}, \mathcal{F}, R)$ (had it not been pruned) can be constructed from $(\widehat{G}', \mathcal{F}, R)$, up to isomorphism.

Suppose that the edges we want to add are $E_1 = \{w_1 z_1, w_2 z_2, \dots, w_k z_k\}$, which we can note as $G = \widehat{G} + E_1$. Then, it is simple to see that ϕ is also an isomorphism between $G = \widehat{G} + E_1$ and $G' = \widehat{G}' + E_2$, where $E_2 = \{\phi(w_1)\phi(z_1), \phi(w_2)\phi(z_2), \dots, \phi(w_k)\phi(z_k)\}$. It remains to see that Phase 2 algorithm allows us to add the edges in E_2 to \widehat{G}' , that is that $\widehat{G}' + E_2$ will actually be constructed. For now, suppose that none of the partially-constructed graph in the intermediate steps between $(\widehat{G}', \mathcal{F}, R)$ and G' are throw out because of some other strong isomorphism.

As ϕ is a strong isomorphism, the sets $\{v_1\}$, $N(v_1)\setminus N(v_2)$, $(N[v_1]\cup N[v_2])^c$, $N(v_1)\cap N(v_2)$, $N(v_2)\setminus N(v_1)$, $\{v_2\}$ are preserved by ϕ (the permutations between vertices are inside these sets). Hence, if w_iz_i is between $N(v_1)\cap N(v_2)$ and $\{v_1,v_2\}^c$, then $\phi(w_i)\phi(z_i)$ is between $N(v_1)\cap N(v_2)$ and $\{v_1,v_2\}^c$ as well. The same goes for the other type of edge we add. In Phase 2, we also required that ϕ preserves the vertices in $N(v_1)\cap N(v_2)$ and in $N(v_2)\setminus N(v_1)$ to which we have already added (possibly zero) edges (recall the refinement \mathcal{F}_i or \mathcal{F}). Hence, if some vertex $a\in N(v_2)$ has not yet been considered in Phase 2 by the time we throw out $(\widehat{G},\mathcal{F},R)$, neither has $\phi(a)$ in $(\widehat{G},\mathcal{F},R)$. This excludes the possibility that because of the strong isomorphism some vertex which had to eventually be considered never is. When adding edges, some degree conditions also had to be respected. In particular, there is a fixed maximum degree, and the vertices in R must have degree strictly smaller than the maximum degree. Let x be a vertex in $V(\widehat{G})$. As ϕ is an isomorphism, the degrees of x and $\phi(x)$ are identical. As ϕ is a strong isomorphism, $x \in R$ if and only if $\phi(x) \in R = \phi(R)$. Hence, the maximum number of edges that we can add to a vertex x is the same as $\phi(x)$. Therefore, as the edge w_iz_i was authorized to be added to \widehat{G} , there will be no degree restrictions forbidding the edge $\phi(w_i)\phi(z_i)$ to be added to \widehat{G}' .

This shows that for every G that would be constructed from $(\widehat{G}, \mathcal{F}, R)$, there is some G' constructed from $(\widehat{G}', \mathcal{F}, R)$ which is isomorphic to G, proving that throwing away $(\widehat{G}, \mathcal{F}, R)$ is valid. However, this supposed that during the steps between $(\widehat{G}', \mathcal{F}, R)$ and G', none of the intermediate partially-constructed graphs were thrown out by some other strong isomorphism. We may now suppose the contrary. Consider that at some point, the intermediary graph $(\widehat{G}' + E_2', \mathcal{F}, R)$ (in particular $E_2' \subset E_2$) was thrown out in favour of some other graph $(\widehat{G}'', \mathcal{F}, R)$. While neither G nor G' was constructed, we wish to show there is some $G'' = G' + E_3$ constructed from $(\widehat{G}'', \mathcal{F}, R)$ which is isomorphic to G' (and to G). We can apply the same argument as above to show that one such G'' exists, assuming that between \widehat{G}'' and G'' no intermediate graph is removed by strong isomorphism. If it is, then by repeating this argument (at most $|N(v_2)|$ times, the number of iterations), we will eventually find some graph constructed that was isomorphic to G.

We note that even if $(\widehat{G}, \mathcal{F}, R)$ and $(\widehat{G}', \mathcal{F}, R)$ are strongly isomorphic graphs, the partially-constructed graphs they lead to one step later are not necessarily strongly isomorphic, in particular as the ordering of the vertices a_i is not preserved by ϕ (not completely, although the information on whether a_i has been considered or not by Phase 2 yet is preserved). Hence, the proof above requires finding a correspondence between the final graphs they produce, and not tracking the constructions step by step.

6.2. **Results.** We will now use the Merging Algorithm to build all possible 4-cop-win graphs. Our implementation of the algorithm is done in the Wolfram language [34].

To lighten the proof, we will use the following notation. Define a property P with the usual definition: a property P is a function from a set to a Boolean value. For instance, if C_3 is the graph property of being 3-cop-win, then $C_3(\mathcal{P}_0)$ is whether the Petersen graph is 3-cop-win (which is true).

Proposition 6.1. Let G be a connected graph such that either

- (1) n = 17 and $\Delta = 4$,
- (2) $n = 18 \text{ and } \Delta \in \{4, 5\}, \text{ or }$
- (3) $n = 19 \text{ and } \Delta = 4.$

If every proper induced connected subgraph H of G respects $c(H) \leq 3$, then $c(G) \leq 3$, unless G is the Robertson graph (see Figure 1).

Proof. Let u be any vertex of G. We know that G - N[u] has at most $n - 2 \le 17$ vertices. If G - N[u] is disconnected, it must contain at least one component K which has at most 8 vertices. Then, Theorem 2.1 implies that $c(K) \le 2$. Furthermore, our hypothesis implies that $c(G - K) \le 3$, as G - K is necessarily a connected induced subgraph of G. By Corollary 2.5, we get that $c(G) \le \max\{c(G - K), c(K) + 1\} \le 3$. Thus, for the remainder of the proof, we assume that for every vertex u, G - N[u] is connected. Likewise, we can assume that G does not contain a corner x. Indeed, G - x is necessarily connected and has cop number at most 3, therefore Corollary 2.8 implies that G also has cop number at most 3. We may also assume that c(G - N[u]) = 3: if $c(G - N[u]) \le 2$, placing a stationary cop on u implies that $c(G) \le 3$.

We can bring together these assumptions by defining the property M(G) as follows: G is a graph respecting the hypotheses of the proposition and such that G - N[u] is a connected 3-cop-win graph for every vertex u of G and such that G does not contain a corner. By the previous discussion, it suffices to show the proposition for graphs respecting M.

We now define property P_1 . A graph G is said to have property P_1 if G contains two non-adjacent vertices of maximum degree Δ . We use the Merging Algorithm to generate all graphs G such that M(G) holds and that respect property P_1 , and then compute their cop numbers. More precisely, we choose n and Δ according to the case we are considering, we set $D_1 = D_2 = \Delta$, and choose $\mathcal{L}_1 = \mathcal{L}_2$ to be the set of 3-cop-win graphs on $n - \Delta - 1$ vertices with maximum degree at most Δ , as computed in Lemma 3.3. We note that the Merging Algorithm computes a somewhat larger class of graphs than we want. In particular, the Merging Algorithm does not exclude graphs which contain corners and

in its last step only tests whether G - N[u] is 3-cop-win (if $G - N[u] \in \mathcal{L}_1$) for vertices of maximum degree.

								Cop number					
Δ	n	Δ_1	G_1	d_1	Base graphs	Final graphs	1	2	3	≥ 4			
	17	4	78	4	123	0	0	0	0	0			
	11	3	2	3	10	0	0	0	0	0			
4	18	4	1105	4	1668	0	0	0	0	0			
	19	4	16514	4	33785	3	0	0	0	3			
		3	9	3	911	0	0	0	0	0			
	18	5	93	5	14232	24416	0	5484	18932	0			
			18 4	4		4	10062	39318	0	7410	31908	0	
5		18			4	1	78	3	534	18645	0	3455	15190
				10	2	111	24238	0	1494	22744	0		
				1	88	698809	0	82882	615927	0			
		3	2	3	22	12778	0	4960	7818	0			

TABLE 5. Results of the first wave of computations using the Merging Algorithm. It presents the counts for the graphs built with the property that they contain 2 non-adjacent vertices of maximum degree. In particular, $d_1 = d_2$ and $\Delta = D_1 = D_2$. Furthermore, G_1 is chosen with maximum degree Δ_1 .

The summary results are presented in Table 5. For more detail, we also split up the graphs relative to the various possible maximum degrees of G_1 , although we of course always merge with all of the possible graphs G_2 , not only the G_2 with the same maximum degree. We note that there are no 3-cop-win graphs with maximum degree 3 on 13 vertices (which can also be seen in Table 4), and that the 3-cop-win graphs with maximum degree 3 on 12 and 14 vertices are 3-regular (and thus the only possible value of d_1 is 3).

We note that the 4-cop-win graphs found on 19 vertices are actually all copies of the Robertson graph. In fact, the 3 copies correspond to 3 different choices of G_1 which can yield the Robertson graph.

With these results, we will then only consider graphs which do not have property P_1 . In other words, the graphs left to consider are those such that the set of vertices of maximum degree of G forms a clique. This is a very restrictive property, and will be very useful.

We see that graphs G such that M(G) holds and for which $\Delta=4$ necessarily respect property P_1 . Indeed, let u be a vertex of maximum degree in G. Consider G'=G-N[u]. If G' contains a vertex of degree 4, $P_1(G)$ is satisfied. Otherwise, we must have $\Delta(G')=3$. If G' is not 3-regular, it is at most 2 cop-win (this holds for orders 12 to 14 as mentioned above) and therefore M(G) does not hold. Therefore, any vertex in G' that was adjacent to a vertex of N(u) is also of degree 4 in G and not adjacent to u. Hence $P_1(G)$ holds. We can therefore suppose $\Delta(G)=5$ for the remainder of the proof. Furthermore, since $P_1(G)$ is false, we can assume that if there exists two vertices of maximum degree, they must be adjacent (hence they must form a clique).

We now define property P_2 . We say a graph G has property P_2 if G contains two non-adjacent vertices v_1 and v_2 such that v_1 has degree either 3 or 4, v_2 has degree 5, v_1 and v_2 have a common neighbour, and $G - N[v_1]$ has maximum degree at most 4. Then, we compute the graphs G such that M(G) holds and $P_2(G)$ holds, but not $P_1(G)$. Precisely, run the Merging Algorithm by setting n = 18, $D_2 = \Delta = 5$, D_1 to respectively either 3 or 4, \mathcal{L}_1 to be the 3-cop-win graphs on 12 vertices with maximum degree at most 4 (if u is a vertex of degree 5 and G - N[u] has maximum degree 5, then G necessarily respect property P_1), and \mathcal{L}_2 to be the 3-cop-win graphs on respectively either 14 or 13 vertices with maximum degree at most 4. We have computed these lists \mathcal{L}_1 and \mathcal{L}_2 in Lemma 3.3. The results of this computation are presented in Table 6.

We note that as the number of possible vertices of maximum degree is generally smaller than before, there are fewer graphs thrown out because of some vertex u of maximum degree such that G - N[u] is not a 3-cop-win graph. Furthermore, we note that as the graphs on 14 vertices with maximum degree 3 are 3-regular, when choosing any of these graphs as G_1 it is impossible for d_1 to be anything other than 3.

								Cop	number					
D_1	G_2	Δ_1	G_1	d_1	Base graphs	Final graphs	1	2	3	≥ 4				
4	1105	4			3	993	41872	0	9299	32573	0			
			78	78	78	78	78	78	2	504	70224	0	4278	65946
4				1	1138	3350712	0	417144	2933568	0				
		3	2	3	153	41006	0	15440	25566	0				
3	16523	23 4	22 4	78	2	2419	83509	0	4187	79322	0			
			10	1	10582	6293171	0	786173	5506998	0				

TABLE 6. Results of the second wave of computations with the Merging Algorithm. It presents the counts for the graphs G built with the property that $G - N[v_1]$ has maximum degree 4, v_1 and v_2 always have a common neighbour (in particular $d_1 < D_1$) and the vertices of maximum degree form a clique. Here, we always have n = 18 and $D_2 = \Delta = 5$, and G_1 is chosen with maximum degree Δ_1 .

We see that none of the graphs are 4-cop-win. We claim that for all graphs G, if M(G) holds then either $P_1(G)$ holds or $P_2(G)$ holds. Supposing the contrary, let G be as graph such that M(G) holds, but for which neither $P_1(G)$ nor $P_2(G)$ holds. We attempt to find a contradiction. As discussed earlier, we may only consider the case where $\Delta = 5$ as P_1 is always respected for $\Delta = 4$.

Let v_2 be a vertex of maximum degree, and denote $G_1 = G - N[v_2]$. As G_1 is 3-cop-win (in order for M(G) to hold), G_1 contains at most 2 vertices of degree 1. Otherwise, removing these vertices of degree 1, which does not change the cop number, would yield a graph with at most 9 vertices and having cop number at most 2 (by Theorem 2.1). By our supposition, we know there cannot be a vertex $v_1 \in N[v_2]^c$ of degree either 3 or 4 such that v_1 and v_2 have a common neighbour and $G - N[v_1]$ has maximum degree at most 4. A sufficient condition for this last requirement is that v_1 has a neighbour a which is adjacent (or equal) to all vertices of maximum degree.

We first consider the case that G contains a unique vertex of degree 5. For any choice of v_1 in $N[v_2]^c$, if v_1 and v_2 have a common neighbour, v_1 trivially has a common neighbour with all vertices of degree 5. So, there are no vertices of $N[v_2]^c$ of degree 3 or 4. Then, no vertex of G_1 -degree 2 or 3 has a neighbour in $N(v_2)$, and no vertex of G_1 -degree 1 has more than 1 neighbour in $N(v_2)$. Therefore, only vertices of G_1 -degree 1 can have a neighbour in $N(v_2)$, and even then they can only receive 1 each, implying there are at most 2 edges between $N(v_2)$ and $N[v_2]^c$. On the other hand, as G does not contain a corner, each vertex in $N(v_2)$ must have at least 1 neighbour in $N[v_2]^c$. Thus, there are at least 5 edges between $N(v_2)$ and the vertices of $N[v_2]^c$. Hence, there is a contradiction in this case.

The other main case is if $N(v_2)$ contains at least one other vertex of degree 5. We recall that these vertices of degree 5 must form a clique as G does not respect property P_1 . Hence, if there is a vertex v_1 in $N[v_2]^c$ which has a neighbour $x \in N(v_2)$ of degree 5 (necessarily in $N(v_2)$) and such that v_1 has degree 3 or 4, then we have a contradiction as x is adjacent (or equal) to all vertices of degree 5. Hence, no vertex of degree 5 in $N(v_2)$ can be adjacent to a vertex of G_1 -degree 2 or 3, and its neighbours (of G_1 -degree 1) can have no other neighbours in $N(v_2)$. As G does not contain a corner, each vertex in $N(v_2)$ must have at least 1 neighbour in $N[v_2]^c$. Hence, as there are at most 2 vertices of G_1 -degree 1, there can be only be at most 2 vertices of degree 5 in $N(v_2)$.

We first consider the case with 2 vertices $x_1, x_2 \in N(v_2)$ of degree 5. Therefore, x_1 and x_2 each have exactly one neighbour in $N[v_2]^c$ (which are different). This means that there are 2 vertices of G_1 -degree 1, one of which is adjacent to x_1 and the other is adjacent to x_2 . Hence both x_1 and x_2 ,

having degree 5, must have 3 neighbours in $N(v_2)$ (possibly including each other). In particular, x_1 and x_2 must have at least one common neighbour in $N(v_2)$, as $N(v_2)$ has 5 vertices. Denote this common neighbour y. We know that y must have a neighbour in $N[v_2]^c$ (otherwise y is cornered by v_2), which will be our choice of v_1 . As both vertices of G_1 -degree 1 cannot have more edges with vertices in $N(v_2)$, v_1 necessarily has degree either 3 or 4. As y is a common neighbour to v_2 , x_1 and x_2 , this choice of v_1 yields a contradiction.

We can similarly consider the case with exactly 1 vertex x of degree 5 in $N(v_2)$. Then, x has at most 2 neighbours (of G_1 -degree 1) in $N[v_2]^c$. Hence, to have degree 5, x has at least 2 neighbours y_1, y_2 in $N(v_2)$. We have already seen that they cannot be adjacent to the neighbours of x in $N[v_2]^c$. If one of them (say y_1) is adjacent to a vertex of G_1 -degree 2 or 3, let v_1 be this last vertex. Then, v_1 is a vertex of degree 3 or 4, and its neighbour y_1 is adjacent to both vertices of maximum degree (v_2 and x), which is a contradiction. Otherwise, y_1 and y_2 are both adjacent to some vertex of G_1 -degree 1 (to which x was necessarily not adjacent). Choose v_1 to be this last vertex of G_1 -degree 1. It has degree at least 3 (one neighbour in G_1 as well as y_1, y_2), and its neighbour y_1 is adjacent to both vertices of maximum degree (v_2 and x). This is a contradiction.

Thus, our claim is verified: all graphs G that satisfy M respect either P_1 or P_2 . We have computed the cop number of graphs such that M(G) and $P_1(G)$ hold, and graphs such that M(G) and $P_2(G)$ hold but not $P_1(G)$. Hence, we have computed the cop number of all graphs such that M(G) holds. This proves the current proposition.

It is interesting that in the first part of computations, for all cases with $\Delta=4$, not only is the Robertson graph the only 4-cop-win graph, but there are no other candidate 4-cop-win graphs. A reasonable explanation for this might be that when merging, too many vertices of high degree are created: either a vertex of degree 5 or more is created (in which case the graph is immediately thrown out) or there are "too many" vertices of degree 4, such that there is always some u of maximum degree for which G-N[u] not 3-cop-win.

6.3. **Possible improvements.** This is only one of many possible computational approaches to solving the problem. We now discuss a few improvements and alternatives that the interested reader may want to apply.

Our approach was based on merging 3-cop-win graphs by looking at non-adjacent vertices v_1, v_2 . It is easy to see that one could also choose v_1 and v_2 to be adjacent. Even if the construction would be somewhat different, the ideas are similar. In particular, after proving that G does not contain non-adjacent vertices of maximum degree, we could have proved that G does not contain any adjacent vertices of maximum degree, instead of considering v_1 of smaller degree. This would then leave only the case with a single vertex of maximum degree to be treated. With some additional heuristics or with a simplification of the methods we used, this case could be dealt with more specifically.

We decided against this approach for few reasons. Although our approach required us to compute more 3-cop-win graphs than otherwise needed, it allowed us to implement only one Merging Algorithm. Furthermore, computing the 3-cop-win graphs on 14 vertices with maximum degree (at most) 4 allowed us to simultaneously handle on the case on n = 18 with $d(v_1) = 3$, and build the candidate 4-cop-win graphs on 19 vertices with maximum degree 4.

Another method would be to not only merge graphs relative to pairs of vertices, but varying sizes of subsets. This approach would certainly reduce the number of intermediate graphs generated by the algorithm: instead of pruning graphs after adding edges, we could build up a larger part of the graph. The difficulty lies in implementing this approach. In particular, the structure of the partially-constructed graphs would not be as simple, as there will be much more than 6 or 7 types of vertices.

Although at the expense of some computation time, we have chosen not to implement these improvements in order to keep the code as simple as possible. Indeed, the simplicity of the code reduces the chances of it being erroneous, as well as making it easier to verify. As the proof is completely dependent on the results of the algorithm, we felt this compromise was justified.

A last idea essentially combines the processes of generating the graphs and testing their cop number. Let G be a connected graph and e be some edge of G. In general, it is unclear whether removing e will help the robber or help the cops, as this depends on many other factors. If we consider a slightly modified ruleset so that the robber can use the edge e but not the cops, we might achieve some results. Denote c' the cop number of this modified game. With these rules, $c(G) \le c'(G)$, as the new restrictions on e can only benefit the robber. Furthermore, $c'(G-e) \le c'(G)$ because removing e can only help the cops, as they were not allowed to use it anyway. Thus, both c(G) and c(G-e) are bounded above by c'(G). If we modify the algorithm which calculates the cop number to take into account the robber-only edge e (a fairly easy modification), we could then determine simultaneously whether both G and G - e have cop number at most 3. This generalizes to larger subsets of edges. Hence, in theory, we can reduce by a significant amount the number of cases to consider by not distinguishing G and G-e. If done for many edges, this idea could decrease the number of graphs to consider exponentially. It is not clear how many such "special edges" we can take in G before the cop number with the special edges diverges from the cop number without those edges. We leave implementing and studying this approach as a problem. Note that modifying slightly the rules of the game to study the cop number has been done many times before. For instance, cop-only edges are an important tool in [18] and allowing the cops to teleport is used in [24].

7. Main results

We are now ready to prove the desired results.

Theorem 7.1. If G is a connected graph such that $n \leq 18$, then $c(G) \leq 3$.

Proof. The statement for all cases except n=17 with $\Delta=4$ and n=18 with $\Delta=4,5$ follows from Corollary 2.6 and Propositions 4.10 and 5.1. Then, apply Proposition 6.1 first for n=17 then for n=18. Indeed, applying our results in this order will allow us to know that every proper induced connected subgraph H of G respects $c(H) \leq 3$ as H necessarily has smaller order.

See Table 7 for a visual guide to which proposition to apply in each case.

			η	$\overline{\imath}$		
Δ	≤ 14	15	16	17	18	19
≤ 3	5.1	5.1	5.1	5.1	5.1	5.1
4	2.6	4.10 (1)	4.10(2)	6.1	6.1	6.1
5	2.6	2.6	4.10(1)	4.10(2)	6.1	
6	2.6	2.6	2.6	4.10 (1)	4.10(2)	
7	2.6	2.6	2.6	2.6	4.10(1)	4.14(2)
8	2.6	2.6	2.6	2.6	2.6	4.14(1)
≥ 9	2.6	2.6	2.6	2.6	2.6	2.6

TABLE 7. Proposition/corollary to apply for each case in Theorems 7.1 and 7.3. For Propositions 4.10 and 4.14 we specify the case using the numbering from the proof.

Considering there exists a known 4-cop-win graph on 19 vertices, the Robertson graph, we get the following corollary.

Corollary 7.2. $M_4 = 19$.

We also want to narrow down the possible 4-cop-win graphs on 19 vertices.

Theorem 7.3. Let G be a connected graph such that n = 19. If $\Delta \leq 3$ or $\Delta \geq 7$, then $c(G) \leq 3$. If $\Delta = 4$, then $c(G) \leq 3$, unless G is the Robertson graph.

Proof. This is a direct consequence of Corollary 2.6 and Propositions 4.14, 5.1 and 6.1 (in the last case using Theorem 7.1 to say that $c(H) \leq 3$ when H is a proper induced connected subgraph of G). See Table 7 for a visual guide to which proposition to apply in each case.

We leave filling the missing cases in this theorem as a conjecture.

Conjecture 7.4. There does not exist a connected graph G such that n = 19, $\Delta \in \{5, 6\}$ and c(G) = 4.

This would show that the Robertson graph is the unique 4-cop-win graph on 19 vertices. With a better implementation of the algorithm, in some low overhead programming language such as C, and with a few additional good ideas, this problem seems within reach. On the other hand, finding M_5 with the methods used in this article is out of reach. A fully non-computational proof $M_4 = 19$ would also be of great interest, and may be more instructive on how to approach M_5 .

It is asked in [5] whether the minimum d-cop-win graphs are (d, 5)-cage graphs for every d. Although we now have further evidence pointing towards this conjecture, any general proof of this statement is still beyond our grasp.

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