# Complexity of Restricted Star Colouring\*

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#### Abstract

Restricted star colouring is a variant of star colouring introduced to design heuristic algorithms to estimate sparse Hessian matrices. For  $k \in \mathbb{N}$ , a k-restricted star colouring (k-rs colouring) of a graph G is a function  $f:V(G) \to \{0,1,\ldots,k-1\}$  such that (i)  $f(x) \neq f(y)$  for every edge xy of G, and (ii) there is no bicoloured 3-vertex path  $(P_3)$  in G with the higher colour on its middle vertex. We show that for  $k \geq 3$ , it is NP-complete to test whether a given planar bipartite graph of maximum degree k and arbitrarily large girth admits a k-rs colouring, and thereby answer a problem posed by Shalu and Sandhya (Graphs and Combinatorics, 2016). In addition, it is NP-complete to test whether a 3-star colourable graph admits a 3-rs colouring. We also prove that for all  $\epsilon > 0$ , the optimization problem of restricted star colouring a 2-degenerate bipartite graph with the minimum number of colours is NP-hard to approximate within  $n^{\frac{1}{3}-\epsilon}$ . On the positive side, we design (i) a linear-time algorithm to test 3-rs colourability of chordal graphs.

Keywords — Graph coloring, Star coloring, Restricted star coloring, Unique superior coloring, Vertex ranking, Complexity

# 1 Introduction

Many large scale optimization problems involve a multi-variable function f. The second-order approximation of f using Taylor series expansion requires an estimation of the Hessian matrix of f. Vertex colouring of graphs and its variants have been found immensely useful as models for estimation of sparse Hessian and Jacobian matrices (see [1] for a survey). To compute a compressed form of a given sparse matrix, Curtis et al. [2] partitioned the set of columns of the matrix in such a way that columns that do not share non-zero entries along the same row are grouped together. By exploiting symmetry, Powell and Toint [3] designed a heuristic algorithm for partitioning columns of a sparse Hessian matrix implicitly using restricted star colouring. Restricted star colouring was first studied as a variant of star colouring (see next para. for definitions). It was also studied independently in the guise of unique superior colouring [4, 5], a generalization of ordered colouring (see Section 6 for the definition of ordered colouring). Therefore, restricted star colouring (abbreviated rs colouring) is intermediate in strength between star colouring and ordered colouring. Ordered colouring is also known as vertex ranking, and has several theoretical as well as practical applications (see [6, 7, 8] and [9, Chapter 6]). It is worth mentioning that every distance-two colouring is an rs colouring. Note that restricted star colouring appears unnamed in [1] and under the name independent set star partition in [10].

This paper is an extension of the work in [11]. In this paper, only vertex colourings of finite simple undirected graphs are considered. A k-colouring of a graph G is a function  $f:V(G) \to \{0,1,\ldots,k-1\}$  such that  $f(x) \neq f(y)$  whenever xy is an edge in G. Let us denote the colour class  $f^{-1}(i)$  by  $V_i$ . A k-colouring f of G is a k-star colouring of G if there is no  $P_4$  in G (not necessarily induced) bicoloured by f. In other words, for  $i \neq j$ , every component of  $G[V_i \cup V_j]$  is a star. A k-colouring of G is a k-restricted star colouring if G contains no bicoloured  $P_3$  with the higher colour on its middle vertex (i.e., no path x, y, z with f(y) > f(x) = f(z); see Figure 1a for an example). That is, for i < j, each vertex in  $V_j$  has at most one neighbour in  $V_i$  (see Figure 1b). In other words, every non-trivial component of  $G[V_i \cup V_j]$  is a star with its centre in  $V_i$  for i < j. Hence, every k-restricted star colouring is a k-star colouring; but the converse is not true [10].

Restricted star colouring is studied in the literature mainly for minor-excluded graph families [4], trees [4], q-degenerate graphs [4], and degree-bounded graph families [5]. The rs chromatic number of a graph G, denoted by  $\chi_{rs}(G)$ , is the least integer k such that G is k-rs colourable. For some classes such as planar graphs and trees, only asymptotic bound for the rs chromatic number is known [4]. On the other hand, there are classes for which we can compute the rs chromatic number in polynomial time using a simple formula; for instance,  $\chi_{rs}(Q_d) = d + 1$  for the hypercube  $Q_d$  [5]. It is also known that  $\chi_{rs}(G) \leq 4\alpha(G)$  for every graph G of girth at least five [10]. Heuristic algorithms for restricted star colouring are given in ColPack software suite [12] as well as [1, 13].

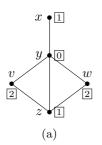
The natural problem of interest is of rs colouring an input graph with the minimum number of colours. Let us call this problem as MIN RS COLOURING. The following are decision versions of this problem (throughout this paper, n denotes the number of vertices unless otherwise specified).

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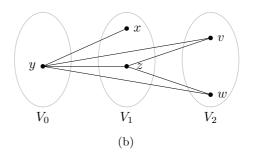


Figure 1: (a) a 3-rs colouring f of the dart graph, (b) colour classes under f

RS COLOURABILITY k-RS COLOURABILITY Instance: A graph G, an integer  $k \le n$ . Instance: A graph G. Question: Is  $\chi_{rs}(G) \le k$ ? Question: Is  $\chi_{rs}(G) \le k$ ?

Note that a graph G is 2-rs colourable if and only if every component of G is a star. Hence, 2-RS COLOURABILITY is in P.

Question [10]: Is k-RS COLOURABILITY NP-complete for  $k \geq 3$ ?

We answer the above question in the affirmative. We prove that for  $k \geq 3$ , k-RS COLOURABILITY is NP-complete for planar bipartite graphs of maximum degree k. We also prove that for all  $\epsilon > 0$ , MIN RS COLOURING is NP-hard to approximate within  $n^{\frac{1}{3}-\epsilon}$  for 2-degenerate bipartite graphs. On the positive side, a linear-time algorithm to test 3-rs colourability of trees is designed. Further, we employ this algorithm to design an  $O(n^3)$ -time algorithm to test 3-rs colourability of chordal graphs. Besides, a few results on rs colouring are obtained using simple observations and known results on star/ordered colouring. Most notable of these is the following: for co-bipartite graphs, k-RS COLOURABILITY is in P for all k, whereas RS COLOURABILITY is NP-complete. Interestingly, this result holds for star colouring as well.

The rest of the paper is organized as follows. Section 2 presents the necessary preliminaries. Section 3 discusses hardness results on bipartite graphs with more focus on planar bipartite graphs. This is followed by Section 4 which mainly acts as preparatory for Section 5. Section 5 presents polynomial-time algorithms to test 3-rs colourability of trees and chordal graphs. Section 6 discusses related colouring variants. Using known results on related colouring variants, complexity results on rs colouring of co-bipartite graphs are obtained.

## 2 Preliminaries

We follow West [14] for graph theory terminology and notation. We write  $v_1, v_2, \ldots, v_n$  for an n-vertex path  $(P_n)$ , and call  $v_1$  and  $v_n$  as the endpoints of the path. We write  $(v_1, v_2, \ldots, v_n)$  for an n-vertex cycle; in particular, a triangle on vertices  $v_1, v_2, v_3$  is  $(v_1, v_2, v_3)$ . A star is a graph isomorphic to  $K_{1,p}$  for some  $p \geq 0$ . The independence number of a graph G is denoted by G(G). If G is a graph and  $G \subseteq V(G)$ , the subgraph of G induced by G(G), denoted by G(G), is  $G = V(G) \setminus S$ . The star chromatic number  $V_{S}(G)$  is the least integer G such that G admits a G-star colouring. The length of a shortest cycle in a graph is called its G-degenerate if there exists an ordering of vertices in G-such that every vertex has at most two neighbours to its left. In this paper, a vertex of degree three or more is called a G-plus vertex.

The next observation follows from the definition.

**Observation 1.** Let G be a graph, and  $f:V(G) \to \{0,1,\ldots,k-1\}$  be a k-rs colouring of G. If v is a vertex of G with f(v)=k-1, then  $\deg(v) < k$ .

We give special attention to 3-restricted star colouring in this paper. The following observations are pivotal to our results on 3-rs colouring.

**Observation 2.** Let G be a graph. If  $f: V(G) \to \{0, 1, 2\}$  is a 3-rs colouring of G, then the following properties hold.

- P1) For every 3-plus vertex u of G, f(u) = 0 or 1 (that is, a binary colour).
- P2) If uv is an edge joining 3-plus vertices u and v in G, then f(v) = 1 f(u).
- P3) Both endpoints of a  $P_3$  in G cannot be coloured 0 by f. So, if u, v, w is a  $P_3$  in G such that f(u) = 0 and w is a 3-plus vertex in G, then f(w) = 1.
- P4) A P4 in G cannot have one endpoint coloured 0 and the other endpoint coloured 1 by f. So, if u, v, w, x is a path in G where u and x are 3-plus vertices in G, then f(x) = f(u).
- P6) Both endpoints of a  $P_6$  in G cannot be coloured 0 by f.

Note: Properties above are numbered as a mnemonic. That is, Property P2 is about path  $P_2$ , Property P3 is about path  $P_3$ , and so on.

*Proof.* One can easily prove Properties P1 to P4 by listing all possible 3-colourings of the path (see supplementary material for detailed proof). To prove Property P6, assume that f is a 3-rs colouring of G and u, v, w, x, y, z is a path in G with f(u) = f(z) = 0. Applying Property P3 on the path u, v, w reveals that  $f(w) \neq 0$ . Similarly,  $f(x) \neq 0$ . Applying Property P4 on the path u, v, w, x reveals that  $f(x) \neq 1$ , and thus f(x) = 2. Similarly,  $f(w) \neq 1$ , and thus f(w) = 2; a contradiction.  $\Box$ 

# 3 Bipartite Graphs

In this section, NP-completeness results on planar bipartite graphs, and an inapproximation result on 2-degenerate bipartite graphs are presented. In fact, our NP-completeness results hold for a much smaller subclass of planar bipartite graphs. To emphasize more interesting parts, the results for the smaller subclass are deferred to the end of the section. Karpas et al. [4] proved that  $\chi_{rs}(G) = O(\log n)$  for every planar graph G. They also proved the following results on 2-degenerate graphs: (i)  $\chi_{rs}(G) = O(\sqrt{n})$  for every 2-degenerate graph G, and (ii) for every integer n, there exists a 2-degenerate graph G on n vertices such that  $\chi_{rs}(G) > n^{\frac{1}{3}}$ .

First, we show that 3-RS COLOURABILITY is NP-complete for subcubic planar bipartite graphs of girth six using a reduction from Cubic Planar Positive 1-in-3 Sat. To describe the latter problem, we introduce necessary terminology assuming that the reader is familiar with satisfiability problems.

A CNF formula  $\mathcal{B}=(X,C)$ , where X is the set of variables and C is the set of clauses, is called a positive CNF formula if no clause contains a negated literal; in other words, the clauses are subsets of X. Let  $\mathcal{B}=(X,C)$  be a positive CNF formula with  $X=\{x_1,x_2,\ldots,x_n\}$  and  $C=\{C_1,C_2,\ldots,C_m\}$ . The graph of formula  $\mathcal{B}$ , denoted by  $G_{\mathcal{B}}$ , is the graph with vertex set  $X\cup C$  and edges  $x_iC_j$  for every variable  $x_i$  in clause  $C_j$   $(i=1,2,\ldots,n;j=1,2,\ldots,m)$ . Figure 2a shows the graph  $G_{\mathcal{B}}$  for the formula  $\mathcal{B}=(X,C)$  where  $X=\{x_1,x_2,x_3,x_4\}$ ,  $C=\{C_1,C_2,C_3,C_4\}$ ,  $C_1=\{x_1,x_2,x_3\}$ ,  $C_2=\{x_1,x_2,x_4\}$ ,  $C_3=\{x_1,x_3,x_4\}$  and  $C_4=\{x_2,x_3,x_4\}$ .

CUBIC PLANAR POSITIVE 1-IN-3 SAT (CPP 1-IN-3 SAT)

Instance: A positive 3-CNF formula  $\mathcal{B} = (X, C)$  such that

 $G_{\mathcal{B}}$  is a cubic planar graph.

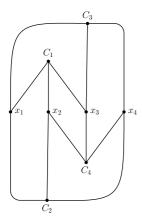
Question: Is there a truth assignment for X such that

every clause in C has exactly one true variable?

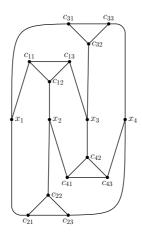
This problem is proved NP-complete by Moore and Robson [15] (Note: in [15], the problem is called Cubic Planar Monotone 1-in-3 Sat. We use 'positive' rather than 'monotone' to be unambiguous; see [16]). Observe that the graph  $G_B$  is cubic if and only if each clause contains three variables and each variable occurs in exactly three clauses. As a result, in a CPP 1-IN-3 SAT instance, the number of variables equals the number of clauses, that is m = n.

**Theorem 1.** 3-RS Colourability is NP-complete for subcubic planar bipartite graphs of girth at least six.

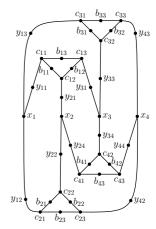
*Proof.* 3-RS COLOURABILITY is in NP because given a 3-colouring f (certificate) of the input graph, we can verify in polynomial time that all bicoloured paths x, y, z satisfy f(y) < f(x).



(a) Graph  $G_{\mathcal{B}}$  for the formula  $(x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_4) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor x_3 \lor x_4)$ 



(b) The intermediate graph obtained from graph  $G_{\mathcal{B}}$  in (a)



(c) The graph G obtained from the intermediate graph in (b)

Figure 2: Construction of graph G from  $G_{\mathcal{B}}$  in Theorem 1

To prove NP-hardness, we transform CPP 1-IN-3 SAT problem to 3-RS COLOURABILITY problem. Let  $\mathcal{B}=(X,C)$  be an instance of CPP 1-IN-3 SAT where  $X=\{x_1,x_2,\ldots,x_m\}$  and  $C=\{C_1,C_2,\ldots,C_m\}$ . Recall that  $\mathcal{B}$  is a positive CNF formula and  $G_{\mathcal{B}}$  is a cubic planar graph. We construct a graph G from  $G_{\mathcal{B}}$  as follows. First, an intermediate graph is constructed. For each clause  $C_j=\{x_{j_1},x_{j_2},x_{j_3}\}$ , replace vertex  $C_j$  in  $G_{\mathcal{B}}$  by a triangle  $(c_{j_1},c_{j_2},c_{j_3})$  and replace edges  $x_{j_1}C_j,\,x_{j_2}C_j,\,x_{j_3}C_j$  in  $G_{\mathcal{B}}$  by edges  $x_{j_1}c_{j_1},\,x_{j_2}c_{j_2},\,x_{j_3}c_{j_3}$  (see Figure 2).

The graph G is obtained by subdividing each edge of this intermediate graph exactly once. Let us call the new vertex introduced upon subdividing the edge  $x_i c_{jk}$  as  $y_{ij}$ , and the new vertex introduced upon subdividing the edge  $c_{jk} c_{jk+1}$  as  $b_{jk}$  (where index k+1 is modulo 3). Each 6-vertex cycle  $(c_{j1}, b_{j1}, c_{j2}, b_{j2}, c_{j3}, b_{j3})$  serves as the gadget for clause  $C_j$ . Since the intermediate graph is a cubic planar graph of girth three (see Figure 2b), G is a subcubic planar bipartite graph of girth six (see Figure 2c).

The graph  $G_{\mathcal{B}}$  can be constructed in O(m) time, and it has 2m vertices and 3m edges. Also, G can be constructed from  $G_{\mathcal{B}}$  in O(m) time since there are only 10m vertices and 12m edges in G. All that remains is to prove that G is 3-rs colourable if and only if  $\mathcal{B}$  is a yes instance of CPP 1-IN-3 SAT. The following claims help to establish this.

- CL1) If f is a 3-rs colouring of G, then for each j, exactly one vertex among  $c_{j1}, c_{j2}, c_{j3}$  is coloured 0 by f.
- CL2) If f is a 3-rs colouring of G, then  $f(c_{jk}) = 1 f(x_i)$  whenever  $x_i, y_{ij}, c_{jk}$  is a path in G.
- CL3) The clause gadget admits a 3-rs colouring scheme that assigns colour 0 on one of the vertices  $c_{j1}, c_{j2}, c_{j3}$  and colour 1 on the other two vertices.

Since  $x_i$ 's and  $c_{jk}$ 's are 3-plus vertices in G, they must receive binary colours by Property P1. Observe that every pair of vertices from  $c_{j1}, c_{j2}, c_{j3}$  is at a distance two from each other. Since Property P3 forbids assigning colour 0 on both endpoints of a  $P_3$ , at most one vertex among  $c_{j1}, c_{j2}, c_{j3}$  is coloured 0 by f. Thus, to prove claim CL1, it suffices to show that at least one of them is coloured 0 by f. On the contrary, assume that  $f(c_{j1}) = f(c_{j2}) = f(c_{j3}) = 1$ . Since  $c_{j1}, b_{j1}, c_{j2}$  is a bicoloured  $P_3$  with colour 1 at the endpoints, its middle vertex must be coloured 0. That is,  $f(b_{j1}) = 0$ . Similarly,  $f(b_{j2}) = 0$ . This means that  $b_{j1}, c_{j2}, b_{j2}$  is a  $P_3$  with  $1 = f(c_{j2}) > f(b_{j1}) = f(b_{j2}) = 0$ ; a contradiction. This proves claim CL1.

Next, we prove claim CL2 for k=1. The proof is similar for other values of k. Let  $x_i, y_{ij}, c_{j1}$  be a path in G where  $i, j \in \{1, 2, ..., m\}$ . Recall that  $f(x_i), f(c_{j1}) \in \{0, 1\}$ . If  $f(x_i) = 0$ , then  $f(c_{j1}) = 1$  due to Property P3. So, it suffices to prove that  $f(x_i) = 1$  implies  $f(c_{j1}) = 0$ . On the contrary, assume that  $f(x_i) = f(c_{j1}) = 1$ . Since  $x_i, y_{ij}, c_{j1}$  is a bicoloured  $P_3$  with colour 1 at its endpoints, its middle vertex  $y_{ij}$  must be coloured 0 (by f). Thus, we have  $f(y_{ij}) = 0$  and  $f(c_{j1}) = 1$ . Therefore, applying Observation 3 on paths  $y_{ij}, c_{j1}, b_{j1}, c_{j2}$  and  $y_{ij}, c_{j1}, b_{j3}, c_{j3}$  gives  $f(b_{j1}) = f(b_{j3}) = 2$  and  $f(c_{j2}) = f(c_{j3}) = 0$  (see Figure 3). The equation  $f(c_{j2}) = f(c_{j3}) = 0$  contradicts Property P3. This completes the proof of claim CL2.

Figure 4 exhibits the colouring scheme guaranteed by claim CL3.

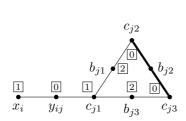


Figure 3:  $f(x_i) = f(c_{i1}) = 1$  leads to a contradiction

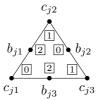


Figure 4: 3-rs colouring scheme for the clause gadget guaranteed by claim CL3 ('rotate' colours if  $c_{j2}$  or  $c_{j3}$  gets colour 0)

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Now, we are ready to prove that G is 3-rs colourable if and only if  $\mathcal{B}$  is a yes instance of CPP 1-IN-3 SAT. Suppose that G has a 3-rs colouring f. Since  $x_i$ 's are 3-plus vertices in G,  $f(x_i) \in \{0,1\}$  for  $1 \le i \le m$ . We define a truth assignment  $\mathcal{A}$  for X by setting variable  $x_i \leftarrow$  true if  $f(x_i) = 1$ , and  $x_i \leftarrow$  false if  $f(x_i) = 0$ . We claim that each clause  $C_j$  has exactly one true variable under  $\mathcal{A}$  ( $1 \le j \le m$ ). Let  $C_j = \{x_p, x_q, x_r\}$ . By claim CL1, exactly one vertex among  $c_{j1}, c_{j2}, c_{j3}$  is coloured 0 under f. Without loss of generality, assume that  $f(c_{j1}) = 0$  and  $f(c_{j2}) = f(c_{j3}) = 1$ . As  $x_p, y_{pj}, c_{j1}$  is a path in G,  $f(x_p) = 1 - f(c_{j1}) = 1$  by claim CL2. Similarly,  $f(x_q) = 1 - f(c_{j2}) = 0$  and  $f(x_r) = 1 - f(c_{j3}) = 0$  (consider the path  $x_q, y_{qj}, c_{j2}$  and the path  $x_r, y_{rj}, c_{j3}$ ). Hence, by definition of  $\mathcal{A}$ ,  $x_p$  is true whereas  $x_q$  and  $x_r$  are false. Therefore,  $\mathcal{A}$  is a truth assignment such that each clause  $C_j$  has exactly one true variable ( $1 \le j \le m$ ). So,  $\mathcal{B}$  is a yes instance of CPP 1-IN-3 SAT.

Conversely, suppose that X has a truth assignment  $\mathcal{A}$  such that each clause has exactly one true variable. We produce a 3-rs colouring f of G as follows. First, colour vertices  $x_i$  by the rule:  $f(x_i) = 1$  if  $x_i$  is true; otherwise,  $f(x_i) = 0$ . Next, vertices  $y_{ij}$  and  $c_{jk}$  are coloured. Assign colour 2 to all  $y_{ij}$ 's. Whenever  $x_i, y_{ij}, c_{jk}$  is a path in G, assign  $f(c_{jk}) = 1 - f(x_i)$ . This ensures that the path  $x_i, y_{ij}, c_{jk}$  is not bicoloured. Finally, extend the partial colouring to clause gadgets using the scheme guaranteed by claim CL3. We claim that f is a 3-rs colouring of G. Obviously, f is a 3-colouring. The bicoloured F3's in G are either entirely within a clause gadget or has a vertex  $y_{ij}$  as an endpoint. Since clause gadgets are coloured by a 3-rs colouring scheme, every bicoloured F3 of the first kind has a lower colour on its middle vertex. Besides, every bicoloured F3 of the second kind has colour 2 at its endpoints because f3's are coloured 2, and thus its middle vertex has a lower colour. Therefore, f3 is a 3-rs colouring of G4.

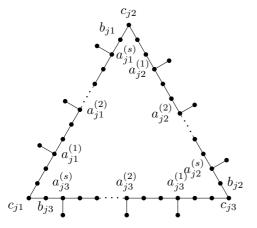
So,  $\mathcal{B}$  is a yes instance of CPP 1-IN-3 SAT if and only if G is 3-rs colourable.

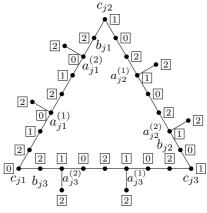
The reduction in Theorem 1 can be modified to give G' arbitrarily large girth. The modification required is to replace the clause gadget by the gadget displayed in Figure 5a.

**Theorem 2.** For  $g \ge 6$ , 3-RS COLOURABILITY is NP-complete for subcubic planar bipartite graphs of girth at least g.

*Proof.* We employ a modified form of the reduction in Theorem 1. Let s be the smallest even number satisfying  $s \ge \lceil \frac{g}{6} \rceil$  (we need s to be even to ensure that the graph to be constructed is bipartite). Replace the old clause gadget by the new one displayed in Figure 5a. For convenience, let us call the graph constructed in Theorem 1 as G and the graph produced by the modified construction as  $G_{new}$ . For the reduction to work, it suffices to show that claims CL1, CL2 and CL3 still hold in  $G_{new}$ .

- CL1) If f is a 3-rs colouring of  $G_{new}$ , then for each j, exactly one vertex among  $c_{j1}, c_{j2}, c_{j3}$  is coloured 0 by f.
- CL2) If f is a 3-rs colouring of  $G_{new}$ , then  $f(c_{jk}) = 1 f(x_i)$  whenever  $x_i, y_{ij}, c_{jk}$  is a path in  $G_{new}$ .
- CL3) The new clause gadget admits a 3-rs colouring scheme that assigns colour 0 on one of the vertices  $c_{j1}$ ,  $c_{j2}$ ,  $c_{j3}$  and colour 1 on the other two vertices.





(a) The new clause gadget. This replaces the old clause gadget  $(c_{j1}, b_{j1}, c_{j2}, b_{j2}, c_{j3}, b_{j3})$ 

(b) a 3-rs colouring scheme for the new clause gadget if s=2 ('rotate' colours if  $c_{i2}$  or  $c_{i3}$  gets colour 0)

Figure 5: (a) The new clause gadget, (b) the new clause gadget for s=2 with a 3-rs colouring

Thanks to Claim 1 below, we can prove claims CL1 and CL2 without difficulty (see supplementary material for complete proof).

Claim 1: For every 3-rs colouring f of the new clause gadget,  $f(a_{jk}^{(t)}) = f(c_{jk})$  for  $1 \le j \le m$ ,  $1 \le k \le 3$ , and  $1 \le t \le s$ .

Claim 1 follows from applying Property P4 to  $c_{jk}$ ,  $a_{jk}^{(1)}$ -path and  $a_{jk}^{(t)}$ ,  $a_{jk}^{(t+1)}$ -paths for  $1 \le t < s$ . Figure 5b exhibits the colouring scheme guaranteed by claim CL3 for s = 2. For higher values of s, a colouring scheme can be produced by following the same pattern.

All that remains is to show that  $G_{new}$  is a subcubic planar bipartite graph of girth at least g. Obviously, it is a subcubic planar graph. Observe that the modified construction can be viewed as replacing each path  $c_{jk}$ ,  $b_{jk}$ ,  $c_{jk+1}$  of G by an even length path and then attaching some pendant vertices (the length of  $c_{jk}$ ,  $c_{jk+1}$ -path in  $G_{new}$  is 3s+2 which is even because s is even). Therefore,  $G_{new}$  is bipartite. Observe that every cycle in G contains at least two paths of the form  $c_{jk}, b_{jk}, c_{jk+1}$ . Similarly, every cycle in  $G_{new}$  contains at least two  $c_{jk}$ ,  $c_{jk+1}$ -paths. Since such paths have length 3s+2, the girth of  $G_{new}$  is at least  $2(3s+2) > 6s \ge g$ . This completes the proof of the theorem.

Theorem 2 can be generalized using a simple operation. For a graph G with  $\Delta(G) = k$ , the graph  $G^+$  is obtained from G by adding enough pendant vertices at every vertex v of G so that  $\deg_{G^+}(v) = k + 1$ . Hence, each vertex in  $G^+$  has degree 1 or k+1. As we are only adding pendant vertices,  $G^+$  preserves the planarity, bipartiteness and girth of G. Moreover, we have  $\Delta(G^+)=k+1$ . Further, this operation is useful to construct graphs of desired rs chromatic number.

**Observation 4.** Let G be a graph with  $\Delta(G) = k$  where  $k \in \mathbb{N}$ . Then, G is k-rs colourable if and only if  $G^+$  is (k+1)-rs colourable.

*Proof.* If G is k-rs colourable, then we can colour the new pendant vertices added to G with a new colour k so that  $G^+$  is (k+1)-rs colourable. Conversely, suppose that  $G^+$  admits a (k+1)-rs colouring f. Recall that for every vertex v of  $G^+$ ,  $\deg_{G^+}(v) = 1$  or k+1. By Observation 1, a vertex of degree k+1 cannot receive colour k under a (k+1)-rs colouring. Hence, no non-pendant vertex in  $G^+$  is coloured k by f. Observe that the set of non-pendant vertices in  $G^+$  is precisely V(G). Since only colours 0 to k-1 are used on non-pendant vertices of  $G^+$  (under f), the restriction of f to V(G) is a k-rs colouring of G.

By mathematical induction, Theorem 2 can be generalized as follows using Observation 4.

**Theorem 3.** For  $k \geq 3$  and  $g \geq 6$ , k-RS Colourability is NP-complete for planar bipartite graphs of maximum degree kand girth at least q.

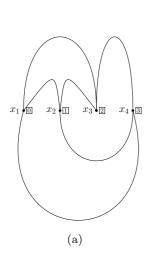
Next, we show that the NP-completeness result presented in Theorem 1 holds for a much smaller subclass.

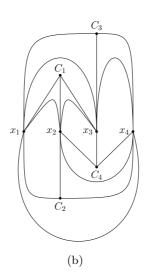
**Theorem 4.** 3-RS Colourability is NP-complete for subcubic planar bipartite graphs G even when G is 2-degenerate,  $girth(G) \geq 6$ ,  $\chi_s(G) = 3$ , and  $\chi_{rs}(G) \leq 4$ .

*Proof.* To prove the theorem, it is enough to show that the graph G constructed in the proof of Theorem 1 is 2-degenerate and satisfies girth $(G) \ge 6$ ,  $\chi_s(G) = 3$ , and  $\chi_{rs}(G) \le 4$ . It is already shown that G has girth at least six. To obtain a 2-degenerate ordering of V(G), list vertices of degree three first, followed by vertices of degree two (note that vertices of degree three form an independent set). Next, we show that G is 3-star colourable and 4-rs colourable.

Claim 1: G admits a 3-star colouring  $\phi$ .

To produce  $\phi$ , an auxiliary graph  $H_{\mathcal{B}}$  is constructed first from the formula  $\mathcal{B} = (X, C)$  of Theorem 1. The vertex set of  $H_{\mathcal{B}}$  is the set of variables X, and two vertices  $x_i$  and  $x_j$  are joined by an edge in  $H_B$  if and only if there is a clause in  $\mathcal{B}$  containing both of them (see Figure 6a for an example).





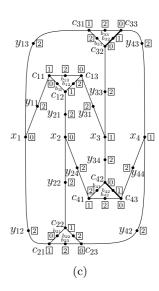
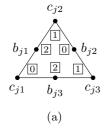
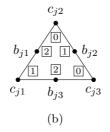


Figure 6: (a) graph  $H_{\mathcal{B}}$  for the formula  $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee x_3 \vee x_4)$  with a 4-colouring h, (b) graph obtained by adding edges in  $H_{\mathcal{B}}$  into  $G_{\mathcal{B}}$ , (c) graph G with a 3-star colouring  $\phi$ 

Recall that the graph  $G_{\mathcal{B}}$  of Theorem 1 is planar. The graph  $H_{\mathcal{B}}$  is planar because adding edges in  $H_{\mathcal{B}}$  into  $G_{\mathcal{B}}$  preserves planarity of  $G_{\mathcal{B}}$  (see Figure 6b; for instance, we can add edge  $x_1x_2$  to  $G_{\mathcal{B}}$  without affecting planarity by drawing it close to the path  $x_1, C_1, x_2$ ). Hence, by 4-colour theorem,  $H_{\mathcal{B}}$  admits a 4-colouring h that uses colours 0,1,2,3 (not necessarily an optimal colouring). Let us partition the set of colours  $\{0, 1, 2, 3\}$  into two sets  $L = \{0, 1\}$  and  $M = \{2, 3\}$ . Observe that for each clause  $C_j = \{x_p, x_q, x_r\}$  in the formula  $\mathcal{B}$ ,  $(x_p, x_q, x_r)$  is a triangle in  $H_{\mathcal{B}}$ . Hence, for each clause  $C_j = \{x_p, x_q, x_r\}$ , either  $|M \cap \{h(x_p), h(x_q), h(x_r)\}| = 1$  or  $|L \cap \{h(x_p), h(x_q), h(x_r)\}| = 1$ . In the former case, we call  $C_j$  as a type-I clause, and in the latter case, a type-II clause.

Construction of  $\phi$  is as follows. First, colour vertices  $x_i$  by assigning  $\phi(x_i) = 0$  if  $h(x_i) \in L$ , and  $\phi(x_i) = 1$  otherwise. Then, all vertices  $y_{ij}$  are coloured 2. Next, for every path  $x_i, y_{ij}, c_{jk}$  in G, vertex  $c_{jk}$  is assigned the opposite binary colour of  $x_i$  (i.e.,  $f(c_{jk}) = 1 - f(x_i)$ ). Note that for each type-I clause  $C_j$  in  $\mathcal{B}$ , exactly one of  $c_{j1}, c_{j2}, c_{j3}$  is coloured 0 by  $\phi$  (and the other two are coloured 1); similarly, for each type-II clause  $C_j$ , exactly one of  $c_{j1}, c_{j2}, c_{j3}$  is coloured 1 by  $\phi$ . For every type-I clause  $C_j$ , extend the colouring to the gadget for  $C_j$  using the scheme in Figure 7a. For every type-II clause  $C_j$ , extend the colouring to the gadget for  $C_j$  using the scheme in Figure 7b. An example is exhibited in Figure 6c. Obviously,  $\phi$  is a 3-colouring ( $y_{ij}$ 's get colour 2 and their neighbours get binary colours, and clause gadgets are coloured by 3-colouring schemes).





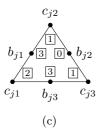


Figure 7: (a) the colouring scheme used by  $\phi$  for the gadget of a type-I clause  $C_j$  ('rotate' the scheme if needed), (b) the colouring scheme used by  $\phi$  for the gadget of a type-II clause  $C_j$  ('rotate' the scheme if needed), (c) colouring of the gadget for a type-II clause  $C_j$  under f (after recolouring  $b_{j2}$ )

Claim 1.1: Under  $\phi$ , if Q is a bicoloured  $P_3$  in G with a higher colour on its middle vertex, then the following hold.

- (1) Q must be a  $c_{jk}, b_{jk}, c_{j\,k+1}$  path coloured 0,1,0 by  $\phi;$  and
- (2) neighbours of  $c_{jk}$  (resp.  $c_{jk+1}$ ) other than  $b_{jk}$  are coloured 2 by  $\phi$  where  $C_j$  is a type-II clause (and index k+1 is modulo 3).

If Q is a bicoloured  $P_3$  in G, then either (i) Q must be within the gadget for a clause  $C_j$ , or (ii) Q has a vertex  $y_{ij}$  as an endpoint. In Case (i), either Q has colour 0 on its middle vertex (when  $C_j$  is a type-I clause), or Q has colour 1 on its middle vertex (when  $C_j$  is a type-II clause). When  $C_j$  is a type-II clause, either Q is a  $b_{jk}, c_{jk+1}, b_{jk+1}$  path coloured 2,1,2 (eg.: path  $b_{j3}, c_{j1}, b_{j1}$  in Figure 7b) or it is a  $c_{jk}, b_{jk}, c_{jk+1}$  path coloured 0,1,0 (eg.: path  $c_{j2}, b_{j2}, c_{j3}$  in Figure 7b); the middle vertex has a higher colour only in the latter case. In Case (ii), Q has a lower colour on its middle vertex because  $y_{ij}$ 's are coloured 2. This proves Point (1) of Claim 1.1. Point (2) of Claim 1.1 is true due to the construction of  $\phi$  (see highlighted  $P_3$ 's in Figure 6c).

We are ready to prove that  $\phi$  is a 3-star colouring. To produce a contradiction, assume that w, x, y, z is a  $P_4$  in G bicoloured by  $\phi$ , say  $\phi(w) = \phi(y) = q$  and  $\phi(x) = \phi(z) = p$  where  $0 \le p < q \le 2$ . Note that x, y, z is a bicoloured  $P_3$  with a higher colour on its middle vertex. Hence, by Point (1) of Claim 1.1, the path x, y, z is a  $c_{jk}, b_{jk}, c_{jk+1}$  path coloured 0,1,0 by  $\phi$  where  $1 \le j \le m$  and  $1 \le k \le 3$ . In particular, p = 0 and q = 1. Note that x has a neighbour w coloured q = 1. That is,  $q_{jk} = 1$  or  $q_{jk+1} = 1$  has a neighbour  $q_{jk} = 1$  and  $q_{jk} = 1$  or  $q_{jk+1} = 1$  has a neighbour  $q_{jk} = 1$  or  $q_{jk+1} = 1$  has a neighbour  $q_{jk} = 1$  or  $q_$ 

colourable. Since G contains cycles, G is not 2-star colourable. Therefore,  $\chi_s(G)=3$ .

Claim 2: G admits a 4-rs colouring f.

We produce f by a simple modification to  $\phi$ . First, define f as  $f(v) = \phi(v) + 1$  for each vertex v of G. Next, for each path  $Q = c_{jk}, b_{jk}, c_{j\,k+1}$  coloured 1,2,1, recolour the middle vertex  $b_{jk}$  with colour 0 (see Figure 7c). We claim that this recolouring makes f a 4-rs colouring of G. Note that if Q is a bicoloured  $P_3$  in G, then either (i) Q is within a clause gadget, or (ii) Q has a vertex  $y_{ij}$  as an endpoint. In Case (i), Q has a lower colour on its middle vertex because f uses an rs colouring scheme on the gadget for each type-I clause (namely the scheme in Figure 7a with colours incremented by one) and each type-II clause (see Figure 7c). In Case (ii), Q has a lower colour on its middle vertex because  $y_{ij}$ 's are coloured 3 by f. Therefore, f is indeed a 4-rs colouring of G. This proves Claim 2. That is,  $\chi_{rs}(G) \leq 4$ . This completes the proof of the theorem.

Corollary 1. MIN RS COLOURING cannot be approximated by a factor less than 4/3 for subcubic planar bipartite graphs G even when  $\chi_s(G) = 3$  and  $girth(G) \ge 6$ .

Proof. By Theorem 4, given a subcubic planar bipartite graph G with  $\chi_s(G)=3$  and  $girth(G)\geq 6$ , it is NP-hard to distinguish between the cases  $\chi_{rs}(G)=3$  and  $\chi_{rs}(G)=4$ . A  $(\frac{4}{3}-\epsilon)$ -approximation algorithm for MIN RS COLOURING for some  $\epsilon>0$  can be used to distinguish between the cases  $\chi_{rs}(G)=3$  and  $\chi_{rs}(G)=4$ . Therefore, MIN RS COLOURING cannot be approximated within a factor less than 4/3.

Similarly, we can strengthen Theorem 3 as follows (proof is omitted; see supplementary material).

**Theorem 5.** For  $k \geq 3$  and  $g \geq 6$ , k-RS COLOURABILITY is NP-complete for planar bipartite graphs G even when G is 2-degenerate,  $\Delta(G) = k$ ,  $girth(G) \geq g$ ,  $\chi_s(G) \leq k$  and  $\chi_{rs}(G) \leq k+1$ .

#### Inapproximability of Min RS Colouring

By an approximation factor preserving reduction from MIN COLOURING to MIN RS COLOURING, we show that for all  $\epsilon > 0$ , MIN RS COLOURING is inapproximable within  $n^{\frac{1}{3}-\epsilon}$  for 2-degenerate bipartite graphs. The construction employed is a slightly modified form of that used by Gebremedhin et al. [17] to prove that MIN STAR COLOURING is inapproximable.

**Theorem 6.** For all  $\epsilon > 0$ , it is NP-hard to approximate MIN RS COLOURING within  $n^{\frac{1}{3}-\epsilon}$  for 2-degenerate bipartite graphs.

Proof. We show that an approximation algorithm for MIN RS COLOURING leads to an approximation algorithm for MIN COLOURING. We assume that for a given  $\epsilon > 0$ , there is an algorithm that takes a graph G' on N vertices as input and approximates  $\chi_{rs}(G')$  within a factor of  $N^{\frac{1}{3}-\epsilon}$ . Then, we show that for a given  $\epsilon > 0$ , there is an algorithm that takes a graph G on n vertices as input and approximates  $\chi(G)$  within a factor of  $n^{1-\epsilon}$ . Given a graph G with n vertices, m edges and maximum degree  $\Delta$ , construct a graph G' from G by replacing each edge e = uv of G by a copy of the complete bipartite graph  $K_{2,\Delta+1}$  with parts  $\{u,v\}$  and  $\{e_1,e_2,\ldots,e_{\Delta+1}\}$  (see Figure 8).

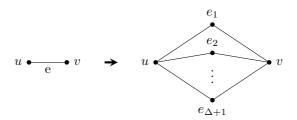


Figure 8: Edge replacement operation to construct G' from G

Note that G' has  $N = n + (\Delta + 1)m < n + (n + 1)\binom{n}{2} \le n^3$  vertices. Clearly, G' is bipartite, and G' can be constructed in time polynomial in m + n. To obtain a 2-degenerate ordering of V(G'), list members of V(G) first, followed by the newly introduced vertices (i.e.,  $e_i$ 's).

First, we prove that G' is (k+1)-rs colourable whenever G is k-colourable. Clearly, each k-colouring f of G can be extended into a (k+1)-colouring f' of G' by assigning the new colour k to all new vertices. A  $P_3$  in G' is either of the form  $u, e_i, v$  where uv is an edge in G, or of the form  $e_i, v, e'_j$ . In the former case, the  $P_3$  is tricoloured by f' because  $f(u) \neq f(v)$ . In the latter case, the  $P_3$  is bicoloured by f' (since  $f'(e_i) = f'(e'_j) = k$ ), but the middle vertex has a lower colour. Hence, f' is a (k+1)-rs colouring of G'. This proves that  $\chi_{rs}(G') \leq \chi(G) + 1$ .

Next, we show that given a (k+1)-rs colouring f' of G', a k-colouring f of G can be obtained in polynomial time. For  $k \geq \Delta + 1$ , a greedy colouring of G can be used as f. Suppose  $k \leq \Delta$ . For each edge e = uv of G, at least two from associated vertices  $e_1, e_2, \ldots, e_{\Delta+1}$  in G' must get the same colour under f' (if not,  $\Delta + 1$  colours are needed for  $e_i$ 's and another colour is needed for u, a contradiction because f' is a (k+1)-colouring and  $k \leq \Delta$ ). W.l.o.g., assume that  $f'(e_1) = f'(e_2)$ . Then, f' must use distinct colours on vertices u and v (if not,  $u, e_1, v, e_2$  is a bicoloured  $P_4$ , and hence f' is not even a star colouring let alone an rs colouring). Moreover, for every vertex  $v \in V(G)$ ,  $deg_{G'}(v) \geq k + 1$  and thus f'(v) < k (if v is coloured k, then  $deg_{G'}(v) \leq k$  because v has at most one neighbour coloured v for  $v \in V(G)$ . Hence, the restriction of v to  $v \in V(G)$  is indeed a v-colouring of v.

Finally, for a given  $\epsilon > 0$ , we produce an algorithm that approximates  $\chi(G)$  within  $n^{1-\epsilon}$ . Let  $n_0$  be a fixed positive integer such that  $n_0^{2\epsilon} \ge 2$ . If  $n < n_0$ , then  $\chi(G)$  can be computed exactly using an exact algorithm. Suppose  $n \ge n_0$ . Now, the assumed approximation algorithm (see the beginning of the proof) gives a (k+1)-rs colouring of G' where  $k+1 \le N^{\frac{1}{3}-\epsilon}\chi_{rs}(G')$ . Using

the method described above, a k-colouring of G can be produced. Since  $\chi_{rs}(G') \leq \chi(G) + 1$ ,  $N < n^3$  and  $2 \leq n_0^{2\epsilon}$ , we have  $k < n^{1-\epsilon}\chi(G)$  by the following equation.

$$k < k + 1 \le N^{\frac{1}{3} - \epsilon} \chi_{rs}(G') < (n^3)^{\frac{1}{3} - \epsilon} (\chi(G) + 1)$$
  
$$\le n^{1 - 3\epsilon} 2\chi(G) \le n^{1 - 3\epsilon} n_0^{2\epsilon} \chi(G) \le n^{1 - 3\epsilon} n^{2\epsilon} \chi(G) = n^{1 - \epsilon} \chi(G)$$

This gives an  $(n^{1-\epsilon})$ -approximation algorithm for MIN COLOURING. So, for a given  $\epsilon > 0$ , if there is an  $(n^{\frac{1}{3}-\epsilon})$ -approximation algorithm for MIN RS COLOURING, then there is an  $(n^{1-\epsilon})$ -approximation algorithm for MIN COLOURING. Since it is NP-hard to approximate MIN COLOURING within  $n^{1-\epsilon}$  for all  $\epsilon > 0$  [18], it is NP-hard to approximate MIN RS COLOURING within  $n^{\frac{1}{3}-\epsilon}$  for all  $\epsilon > 0$ .

Karpas et al. provided a method to produce an rs colouring of a given 2-degenerate graph G in polynomial time using  $O(n^{\frac{1}{2}})$  colours (special case d=2 of Theorem 6.2 in [4]). Hence, the rs chromatic number is approximable within  $O(n^{\frac{1}{2}})$  in 2-degenerate graphs. Along with Theorem 6, this means that in 2-degenerate graphs, MIN RS COLOURING can be approximated within  $O(n^{\frac{1}{2}})$ , yet inapproximable within  $n^{\frac{1}{3}-\epsilon}$  for all  $\epsilon>0$ . The following open problem is of interest in this regard.

**Question:** Is MIN RS COLOURING inapproximable within  $n^p$  for some p in the range  $\frac{1}{3} \le p < \frac{1}{2}$ ?

# 4 Properties of RS Colouring

In this section, some basic observations on rs colouring are presented. From these observations, an easy formula for the rs chromatic number of split graphs is obtained. Also, the material in this section is used in the forthcoming section on trees and chordal graphs.

**Observation 5.** For every graph G,  $\chi_{rs}(G) \leq n - \alpha(G) + 1$ .

Proof. Let  $v_0, v_1, \ldots, v_{n-1}$  be the vertices in G. Let  $\alpha = \alpha(G)$ , and let I be a maximum independent set in G. W.l.o.g., assume that  $I = \{v_{n-\alpha}, \ldots, v_{n-1}\}$ . Define  $f: V(G) \to \{0, 1, \ldots, n-\alpha\}$  as  $f(v_i) = i$  for  $0 \le i \le n-\alpha-1$ , and  $f(v_i) = n-\alpha$  for  $n-\alpha \le i \le n-1$  (i.e., assign colour  $n-\alpha$  on members of I). Since  $f^{-1}(i)$  is a singleton for  $0 \le i \le n-\alpha-1$  and  $f^{-1}(n-\alpha) = I$ , f is an  $(n-\alpha+1)$ -rs colouring of G. Therefore,  $\chi_{rs}(G) \le n-\alpha(G)+1$ .

The next observation is a direct consequence of Observation 1 (proof is available in supplementary material).

**Observation 6.** Let G be a graph, and  $C = \{v_1, v_2, \dots, v_k\}$  be a clique in G. If  $\deg(v_i) \geq k$  for all i, then G is not k-rs colourable.

**Theorem 7.** If G is a split graph, then  $\chi_{rs}(G) = n - \alpha(G) + 1$ .

Proof. Let G be a split graph whose vertex set is partitioned into a clique C and a maximum independent set I (i.e.,  $|I| = \alpha(G)$ ). Since I is a maximum independent set, every vertex in C has a neighbour in I. Hence, each vertex in C has degree at least |C|. Therefore, by Observation 6, G is not |C|-rs colourable. That is,  $\chi_{rs}(G) \ge |C| + 1 = n - \alpha(G) + 1$ . On the other hand, by Observation 5,  $\chi_{rs}(G) \le n - \alpha(G) + 1$ . Therefore,  $\chi_{rs}(G) = n - \alpha(G) + 1$ .

Next, we present a lemma on 3-rs colouring of paths which is used in the design of 3-rs colourability testing algorithm for trees. The aforementioned algorithm for a tree T deals with paths in T whose internal vertices have degree two in T and endpoints are 3-plus vertices in T. Since every 3-rs colouring of a tree must use binary colours on 3-plus vertices, the following question becomes the centre of attention. For  $i, j \in \{0, 1\}$  and  $n \ge 3$ , what are the values of n, i, j such that the path  $P_n$  admits a 3-rs colouring that uses colours i and j at its endpoints? By Properties P3, P4 and P6, such a 3-rs colouring does not exist in the following cases: (i) n = 3, i = j = 0, (ii) n = 4,  $i \ne j$ , and (iii) n = 6, i = j = 0. Soon, we show that such a 3-rs colouring is possible in all other cases. Figure 9 shows that this is the case when 2 < n < 7.

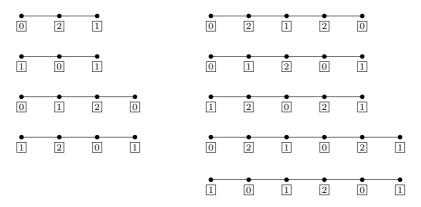


Figure 9: For 2 < n < 7 and  $i, j \in \{0, 1\}$ ,  $P_n$  can be 3-rs coloured with colours i and j at its endpoints except for the cases (i) n = 3, i = j = 0, (ii) n = 4,  $i \neq j$ , and (iii) n = 6, i = j = 0

Lemma 1 below guarantees that such a 3-rs colouring exists whenever  $n \geq 7$ .

**Lemma 1.** If  $n \geq 7$ , for every  $i, j \in \{0, 1\}$ , the path  $P_n$  has a 3-rs colouring with one endpoint coloured i and the other, coloured j.

*Proof.* (by mathematical induction on n)

Base cases:  $n \in \{7, 8\}$ 

For  $n \in \{7, 8\}$  and  $i, j \in \{0, 1\}$ , there exists a 3-rs colouring of  $P_n$  with endpoints coloured i, j as shown in Figure 10.

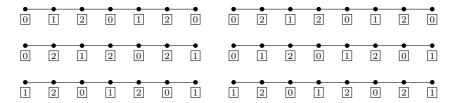


Figure 10: For  $n \in \{7,8\}$ ,  $P_n$  can be 3-rs coloured with colours  $i,j \in \{0,1\}$  at the endpoints

Induction step:  $(n \ge 9)$ 

Let the path be  $v_1, v_2, \ldots, v_n$ . We need to show that the path has a 3-rs colouring f such that  $f(v_1) = i$  and  $f(v_n) = j$ . By the induction hypothesis, the path  $v_3, v_4, \ldots, v_n$  has a 3-rs colouring f' such that  $f'(v_3) = 1 - i$  and  $f'(v_n) = j$ . By assigning  $f'(v_2) = 2$  and  $f'(v_1) = i$ , f' can be extended into a 3-rs colouring of the whole path that uses colours i and j at its endpoints.

The following theorem sums up the picture.

**Theorem 8.** Let  $v_1, v_2, \ldots, v_n$  be a path where  $n \in \mathbb{N}$ , and let  $i, j \in \{0, 1\}$ . The path does not admit a 3-rs colouring f with  $f(v_1) = i$  and  $f(v_n) = j$  in the following cases: (i) n = 2, i = j; (ii) n = 3, i = j = 0; (iii) n = 4,  $i \neq j$ ; (iv) n = 6, i = j = 0. In all other cases, the path admits a 3-rs colouring f such that  $f(v_1) = i$  and  $f(v_n) = j$ .

# 5 Trees and Chordal Graphs

For every tree T,  $\chi_{rs}(T) = O(\log n/\log\log n)$  and the bound is tight [4]. But, the claim [4] that  $\chi_{rs}(T) = \Omega(\log n/\log\log n)$  is false; for instance, all paths are 3-rs colourable no matter how long. Since every tree T admits a distance-two colouring with  $\Delta(T) + 1$  colours [19],  $\chi_{rs}(T) \leq \Delta(T) + 1$  (recall that every distance-two colouring is an rs colouring). This section presents a linear-time algorithm to test 3-rs colourability of trees as well as an  $O(n^3)$ -time algorithm to test 3-rs colourability of chordal graphs.

Note that the decision problem k-RS COLOURABILITY can be expressed in MSO (see supplementary material), and hence can be solved in linear time in graphs of bounded treewidth by Courcelle's theorem [20, 21] (in fact, the problem can be expressed in MSO<sub>1</sub>, MSO without edge set quantification). Unfortunately, algorithms obtained from the MSO expression of problems suffer from extremely large constants hidden in the big-O notation [22]. In contrast, the algorithm for trees provided in this section has a hidden constant of reasonable size (less than 35) making it practically useful.

Note that our algorithm for trees only tests 3-rs colourability of an input tree T with a 3-plus vertex; it does not produce a 3-rs colouring of T in case T is 3-rs colourable (it is possible to extend our algorithm into a 3-rs colouring algorithm for trees; but the extension requires a considerable amount of work).

#### Overview of the algorithm

Consider a rooted tree representation of the input tree T so that we can process it in a bottom-up fashion. The processing part of the algorithm hinges on the following question: What is the 'role' of each connected subgraph of T in deciding 3-rs colourability of T? We categorize relevant connected subgraphs of trees into classes so that members of the same class play the same 'role' (in deciding 3-rs colourability of the tree). As a result, the 3-rs colourability status of a tree is preserved when a connected subgraph is locally replaced by another member of the same class. This notion of local replacement allows us to formalize the classification of connected subgraphs. To this end, it suffices to consider two kinds of connected subgraphs called branches and rooted subtrees (defined below). For convenience, we deal with 3-rs colouring extension (motivation is explained below in detail). So, branches and rooted subtrees may contain coloured vertices, and we are concerned of 3-rs colouring extension status of trees containing them (i.e., whether we can assign colours on the rest of the vertices to produce a 3-rs colouring). We introduce an equivalence relation among branches (resp. rooted subtrees) as  $B_1 \sim B_2$  if replacing  $B_1$  by  $B_2$  in every tree preserves 3-rs colouring extension status. Our algorithm recursively computes the equivalence classes of branches and rooted subtrees of the input tree. The input tree T itself is technically a rooted subtree, and we can determine 3-rs colourability of T based on the equivalence class of that rooted subtree.

#### Ingredients of the algorithm and motivation

There are two basic ideas behind the algorithm. The first idea is that some subgraphs force colours. For example, suppose u, v, w is a path in a tree T where u, v and w are 3-plus vertices in T. If f is a 3-rs colouring of T, then f must use binary colours on u, v and w, and due to Property P3, the only possibility is f(v) = 0 and f(u) = f(w) = 1. In short, the subgraph  $H_1$  displayed in Figure 11 forces colours.

The second idea is more ubiquitous in colouring: we can combine 3-rs colourings of subgraphs to get a 3-rs colouring of the whole graph provided some simple conditions are met. In this context, it takes the following form: we can decompose a tree into two subgraphs with an edge (or a vertex) in common, and combine 3-rs colourings of the subgraphs to produce a 3-rs colouring of the whole tree provided the subgraph colourings agree on common vertices (or the common vertex has colour 0).

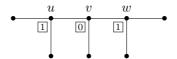


Figure 11: Tree  $H_1$  (with colours forced)

**Lemma 2.** Let T be a tree, and  $e = u_1u_2$  be an edge in T. Let  $U_1$  and  $U_2$  be the vertex sets of components in T - e such that  $u_1 \in U_1$  and  $u_2 \in U_2$ . Suppose  $G_1 = T[U_1 \cup \{u_2\}]$  admits a 3-rs colouring  $f_1$ , and  $G_2 = T[U_2 \cup \{u_1\}]$  admits a 3-rs colouring  $f_2$ . If  $f_1$  and  $f_2$  agree on the common vertices, then they can be combined to give a 3-rs colouring f of f. That is, if  $f_1(u_1) = f_2(u_1)$  and  $f_1(u_2) = f_2(u_2)$ , then the function f defined on V(T) as  $f(w) = f_i(w)$  for  $w \in U_i$ , i = 1, 2 is a 3-rs colouring of f.

*Proof.* Note that every 3-vertex path in T is either entirely in  $G_1$ , or entirely in  $G_2$ . Since f restricted to  $V(G_i)$  is a 3-rs colouring of  $G_i$  for i = 1, 2 (namely  $f_i$ ), f is indeed a 3-rs colouring of T.

**Lemma 3.** Let T be a tree, and v be a vertex of T. Let  $U_1$  be the vertex set of a component in T-v, and let  $U_2 = V(T-v) \setminus U_1$ . Suppose that  $G_i = T[U_i \cup \{v\}]$  admits a 3-rs colouring  $f_i$  such that  $f_i(v) = 0$  for i = 1, 2. Then,  $f_1$  and  $f_2$  can be combined to produce a 3-rs colouring f of T (i.e.,  $f(w) = f_i(w) \ \forall w \in V(G_i)$ , i = 1, 2).

*Proof.* Clearly, for each 3-vertex path Q in T, either (i) Q is entirely in  $G_i$  for some  $i \in \{1, 2\}$ , or (ii) Q has v as its middle vertex. Since  $f_1$  and  $f_2$  are 3-rs colourings, and v is coloured 0, Q has a lower colour on its middle vertex. Since Q is arbitrary, under f, there is no bicoloured  $P_3$  in T with a higher colour on its middle vertex.

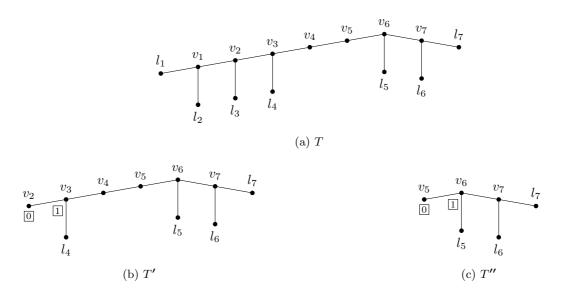


Figure 12: An example of testing 3-rs colourability of a tree

To give a flavour of the problem, we present an example for testing 3-rs colourability of trees (without using the algorithm). If the rooted tree T displayed in Figure 12a admits a 3-rs colouring f, then  $f(v_2) = 0$  and  $f(v_1) = f(v_3) = 1$  due to the presence of the subgraph  $H_1$  (see Figure 11). Therefore, if T admits a 3-rs colouring f, then f' displayed in Figure 12b admits a 3-rs colouring extension (by restricting f to V(T')). Also, any 3-rs colouring extension f' of T' can be extended into a 3-rs colouring of T by assigning colour 1 at  $v_1$  and colour 2 at  $l_1, l_2$  and  $l_3$ . Therefore, T admits a 3-rs colouring if and only if T' admits a 3-rs colouring extension. Note that a 3-rs colouring extension of T' must assign colour 0 at  $v_5$  by Observation 3 (applied on path  $v_2, v_3, v_4, v_5$ ), and colour 1 at  $v_6$  (because  $v_6$  is a 3-plus vertex). Therefore, T' admits a 3-rs colouring extension if and only if T'' displayed in Figure 12c admits a 3-rs colouring extension. So, T admits a 3-rs colouring if and only if T'' admits a 3-rs colouring extension. But, T'' does not admit a 3-rs colouring extension ( $v_7$  must be coloured 0, and thus  $v_5, v_6, v_7$  is a bicoloured  $P_3$  with a higher colour on its middle vertex). Therefore, T doesn't admit a 3-rs colouring.

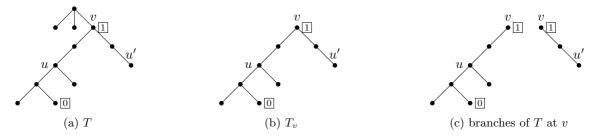


Figure 13: (a) a partially coloured rooted tree T, (b) the rooted subtree of T at v (denoted by  $T_v$ ), (c) branches of T at v

As seen in the above example, dealing with 3-rs colouring extension is helpful. For simplicity, the input tree T (not a path) is viewed as a rooted tree with a 3-plus vertex as the root. The following definitions help to present the algorithm. It is assumed that the reader is familiar with rooted tree terminology such as parent, child, descendant, and so on. Let T be a partially coloured rooted tree. If v is a vertex of T with  $\deg_T(v) \neq 2$ , the rooted subtree of T at v, denoted by  $T_v$ , is the subgraph of T induced by v and its descendants in T along with parent-child relations and colours inherited from T (see Figure 13b). If we 'split'  $T_v$  at v, each resulting piece is called a branch of T at v (see Figure 13; formally, each branch of T at v is  $T_v[U_i \cup \{v\}]$  with inherited parent-child relations and colours where  $U_i$  is the vertex set of some component of  $T_v - v$ ). By our definition, the root of a branch in T is always a 3-plus vertex in T (if  $\deg_T(v) = 1$ , then  $T_v \cong K_1$  and hence there is no branch at v).

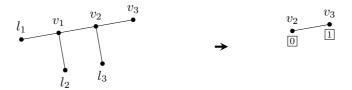


Figure 14: A branch replacement operation. Applying this operation on the left branch at  $v_3$  transforms T into T' (T and T' are displayed in Figure 12)

As far as 3-rs colouring extension is concerned, some branches can be replaced by other branches. For instance, the transformation from T to T' in the first example (see Figure 12) can be viewed as a local replacement operation, namely replacement of the left branch of T at  $v_3$  by another branch (the replacement operation is displayed in Figure 14). A replacement operation that produces a partially coloured tree T' from a partially coloured tree T is said to preserve 3-rs colouring extension status if both T and T' admit a 3-rs colouring extension, or neither does. The replacement operations considered in this paper are either (i) replacement of branches by branches, or (ii) replacement of rooted subtrees by rooted subtrees.

We define an equivalence relation on the set of all branches (of partially coloured rooted trees) as follows:  $B_1 \sim B_2$  if the operation of replacing branch  $B_1$  by branch  $B_2$  in every partially coloured rooted tree preserves 3-rs colouring extension status. It is left as an exercise to the reader to verify that this is indeed an equivalence relation (see supplementary material).

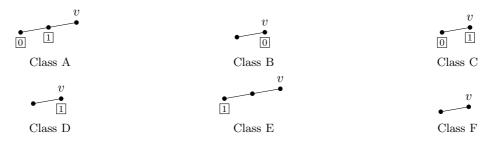


Figure 15: The representatives of Classes A to F

Every uncoloured branch (i.e., branch without any colour) belongs to one of six equivalence classes under this equivalence relation (proved later in Theorem 9). We call these equivalence classes as Class A, Class B, ..., Class F; and a representative of each class is shown in Figure 15 (representatives may contain colours). They are referred to as the representatives of respective classes for the rest of the paper. It is left as an exercise to the reader to verify that branches in Figure 15 belong to different equivalence classes (see supplementary material).

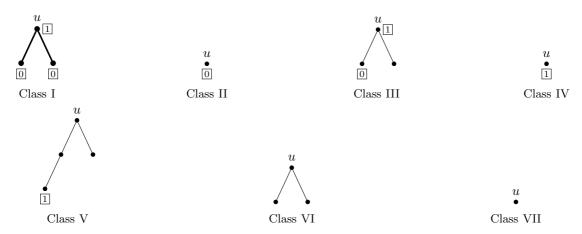


Figure 16: The representatives of Classes I to VII

Similarly, we define an equivalence relation on the set of all rooted subtrees (of partially coloured rooted trees) as follows:  $R_1 \sim R_2$  if replacing rooted subtree  $R_1$  by rooted subtree  $R_2$  in every partially coloured rooted tree preserves 3-rs colouring extension status. An uncoloured rooted subtree belongs to one of seven equivalence classes under this equivalence relation

(proved later in Theorem 9). They are named Class I, Class II, ..., Class VII; and a representative of each class is shown in Figure 16. They are referred to as the representatives of respective classes for the rest of the paper. It is left as an exercise to the reader to verify that rooted subtrees in Figure 16 belong to different equivalence classes (see supplementary material).

Observe that this division of branches (resp. rooted subtrees) into equivalence classes reflect the conditions branches (resp. rooted subtrees) impose on trees containing them to admit a 3-rs colouring extension. Let T be a partially coloured rooted tree with a 3-plus vertex as its root. If T contains a branch  $B^*$  isomorphic to the representative of Class A, then T does not admit a 3-rs colouring extension because (i) the root of branch  $B^*$  must be coloured 2 by Observation 3, and (ii) the root of a branch is always a 3-plus vertex and hence cannot be coloured 2. Thus, by definition of the equivalence relation, T does not admit a 3-rs colouring extension if T contains a Class A branch. Similarly, if T contains a rooted subtree T is a bicoloured T with a higher colour on its middle vertex. Hence, T does not admit a 3-rs colouring extension if T contains a Class I rooted subtree. Since T is arbitrary, we have the following lemma.

**Lemma 4.** A branch B is in Class A if and only if no partially coloured rooted tree (with a 3-plus vertex as root) containing B admits a 3-rs colouring extension. Similarly, a rooted subtree R is in Class I if and only if no partially coloured rooted tree (with a 3-plus vertex as root) containing R admits a 3-rs colouring extension.

The first branch in Figure 13c is composed of the rooted subtree  $T_u$  and a u, v-path. In general, a branch B at a vertex v is composed of a rooted subtree  $T_u$  and a u, v-path (where u is the first descendant of v in B which is not of degree two). The length of the u, v-path is called the up-distance of the branch B. For instance, the first branch in Figure 13c has up-distance two because the length of the u, v-path is two.

Table 1: Equivalence class of branch B in terms of the up-distance of B and the equivalence class of  $T_u$ .

$T_u$	II	III	IV	V	VI	VII
1	С	A	В	С	Е	F
2	D	В	$\mathbf{E}$	$\mathbf{F}$	F	"
3	В	$\mathbf{C}$	D	$\mathbf{E}$	,,	"
4	Е	D	F	$\mathbf{F}$	,,	"
5	D	В	$\mathbf{E}$	,,	,,	"
6	F	E	F	,,	,,	"
7	Е	D	"	,,	"	"
8	F	F	;;	,,	"	"
9	"	Ε	"	,,	"	"
$\geq 10$	"	F	"	,,	"	"

Table 2: Equivalence class of rooted subtree  $T_v$  in terms of the number of branches at v belonging to each class.

b > 0	If $c = d = 0$ , $T_v \in \text{Class II}$ . If $c + d > 0$ , $T_v \in \text{Class I}$ .
b = 0, c + d > 0	If $c + e = 0$ , $T_v \in \text{Class IV}$ . If $c + e = 1$ , $T_v \in \text{Class III}$ . If $c + e \ge 2$ , $T_v \in \text{Class I}$ .
b = c = d = 0	If $e = f = 0$ , $T_v \in \text{Class VII}$ . If $e = 0$ and $f > 0$ , $T_v \in \text{Class VI}$ . If $e = 1$ , $T_v \in \text{Class V}$ . If $e \ge 2$ , $T_v \in \text{Class II}$ .

#### Computation of equivalence classes

Let B be a branch composed of a rooted subtree  $T_u$  and a u,v-path. If  $T_u$  is in Class I, then B is in Class A due to Lemma 4 (note that a tree containing B, in turn, contains  $T_u$ ). Otherwise, we can determine the equivalence class of B from Table 1 based on the equivalence class of  $T_u$  and the up-distance of B. Similarly, a rooted subtree  $T_v$  is composed of branches of T at v. If at least one of those branches in Class A, then  $T_v$  is in Class I by Lemma 4. Otherwise, we can determine the equivalence class of  $T_v$  from Table 2 where b, c, d, e, f denote the number of Class B branches, Class C branches, ..., Class F branches of T at v respectively ( $b, c, d, e, f \ge 0$ ). Subsections 5.1 and 5.2 provide an overview of proofs of Tables 1 and 2, respectively. Complete proofs are available in supplementary material.

One can read the look-up tables as follows. If a branch B is composed of a Class III rooted subtree and a path, and up-distance of B is 5, then B is in Class B (see Table 1; up-distance=5 is the sixth row and  $T_u \in \text{Class III}$  is the third column of the table). If a rooted subtree  $T_v$  is made of one Class C branch and three Class D branches (i.e., b = 0, c = 1, d = 3, and e = f = 0), then  $T_v$  is in Class III (see the case c + e = 1 of the second row of Table 2).

#### Algorithm for testing 3-rs colourability of trees

Assume that the input to the algorithm is a rooted tree T with a 3-plus vertex a as its root (recall that such a representation can be obtained in linear time by BFS). The algorithm determines the equivalence class of the rooted subtree  $T_a$  recursively. To be explicit, the algorithm visits vertices of T in bottom-up order (via post-order traversal) and determines the equivalence classes of all branches and rooted subtrees of T until (i) it comes across a Class A branch or a Class I rooted subtree, or (ii) all vertices of T are visited. In the former case, T is not 3-rs colourable. In the latter case,  $T = T_a$  is in one of the classes Class II, Class III, . . . , Class VII, and thus, T is 3-rs colourable.

To illustrate the algorithm execution, let us revisit the first example. The tree T of Figure 12 is redrawn in Figure 17 (vertex labels are changed for convenience). The equivalence classes of branches and rooted subtrees of T are computed in the

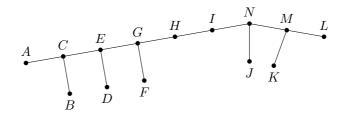


Figure 17: A rooted tree T with root N. A post-order traversal of the tree visits vertices in alphabetical order

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following order using Table 1 as the look-up table for branches, and Table 2, for rooted subtrees.
T_A \in \text{Class VII } (\because b = c = d = e = f = 0; \text{ see Table 2}),
Left branch of T at C \in \text{Class F} (: T_A \in \text{Class VII}, up-dist. = 1; see Table 1),
T_B \in \text{Class VII } (\because b = c = d = e = f = 0),
Right branch of T at C \in \text{Class F} (: T_B \in \text{Class VII}, up-dist. = 1),
T_C \in \text{Class VI } (:: b = c = d = e = 0 \text{ and } f = 2),
Left branch of T at E \in \text{Class E} (:T_C \in \text{Class VI}, up-dist. = 1),
T_D \in \text{Class VII } (:: b = c = d = e = f = 0),
Right branch of T at E \in \text{Class F} (: T_D \in \text{Class VII}, up-dist. = 1),
T_E \in \text{Class V } (\because b = c = d = 0 \text{ and } e = f = 1),
Left branch of T at G \in \text{Class C} (: T_E \in \text{Class V}, up-dist. = 1),
T_F \in \text{Class VII } (\because b = c = d = e = f = 0),
Right branch of T at G \in \text{Class F} (: T_F \in \text{Class VII}, up-dist. = 1),
T_G \in \text{Class III } (\because b = d = e = 0 \text{ and } c = f = 1),
(Next, we consider N, the immediate ancestor of G which is a 3-plus vertex. Note that vertices H and I are of degree two.)
Left branch of T at N \in \text{Class C} (: T_G \in \text{Class III}, up-dist. = 3),
T_J \in \text{Class VII } (\because b = c = d = e = f = 0),
Middle branch of T at N \in \text{Class F}(\cdot; T_J \in \text{Class VII}, \text{up-dist.} = 1),
T_K \in \text{Class VII } (\because b = c = d = e = f = 0),
Left branch of T at M \in \text{Class F} (: T_K \in \text{Class VII}, up-dist. = 1),
T_L \in \text{Class VII } (:: b = c = d = e = f = 0),
Right branch of T at M \in \text{Class F}(:T_L \in \text{Class VII}, \text{up-dist.} = 1),
T_M \in \text{Class VI } (\because b = c = d = e = 0 \text{ and } f = 2),
Right branch of T at N \in \text{Class E} (: T_M \in \text{Class VI}, up-dist. = 1),
T_N \in \text{Class I } (\because b = d = 0 \text{ and } c = e = f = 1).
That is, T_N does not admit a 3-rs colouring extension (by Lemma 4). Therefore, T = T_N is not 3-rs colourable.
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As promised earlier, we next prove that every uncoloured branch belongs to one of the classes Class A, Class B, ..., Class F;

and every uncoloured rooted subtree belongs to one of the classes Class I, Class II, ..., Class VII. Lemmas 5 and 6 below are corollaries of Table 1 and Table 2 respectively.

**Lemma 5.** Let B be an uncoloured branch comprised of a rooted subtree  $T_u$  and a path. If  $T_u$  belongs to one of the classes Class I, Class II, ..., Class VII, then B belongs to one of the classes Class A, Class B, ..., Class F.

**Lemma 6.** Let  $T_u$  be a rooted subtree comprised of branches each of which belongs to one of the classes Class A, Class B, ..., Class F. Then,  $T_u$  belongs to one of the classes Class I, Class II, ..., Class VII.

**Theorem 9.** Every uncoloured branch belongs to one of the classes Class A, Class B, ..., Class F; and every uncoloured rooted subtree belongs to one of the classes Class I, Class II, ..., Class VII.

*Proof.* To produce a contradiction, assume that the theorem is not true. Let X be a counterexample (a branch or a rooted subtree) with the least number of vertices. If X is a branch comprised of a rooted subtree  $T_u$  and a path, then  $T_u$  must belong to one of the classes Class I, Class II, ..., Class VII (if not,  $T_u$  is a counterexample with fewer vertices than X). So, X must belong to one of the classes Class A, Class B, ..., Class F by Lemma 5; a contradiction. If X is a rooted subtree, then X is composed of branches that belong to Classes A to F (if one of the branches is not in Classes A to F, then that branch is a counterexample with fewer vertices than X). So, X must belong to one of the classes Class I, Class II, ..., Class VII by Lemma 6; a contradiction.

The pseudocode and the run-time analysis of the algorithm are presented in Subsection 5.3.

### 5.1 Proof of Table 1 (Overview)



Figure 18: Rooted subtree  $T_u$  and branch B for the first entry of Table 1

The first entry of the table says that a branch B composed of a Class II rooted subtree  $T_u$  and a path of length one (i.e. up-distance(B) = 1) belongs to Class C. Since  $T_u \in \text{Class II}$ , we may assume that  $T_u$  is the representative of Class II by definition of the equivalence relation; i.e.,  $T_u$  is the graph in Figure 18a and B is the graph in Figure 18b. Thus, to prove the entry, it suffices to show that the branch replacement operation displayed in Figure 19 preserves 3-rs colouring extension status (i.e., T' preserves 3-rs colouring extension status of T). Since v is a 3-plus vertex, this branch replacement operation indeed preserves 3-rs colouring extension status. This proves the first entry of Table 1.



Figure 19: A branch replacement operation, and partially coloured rooted trees before and after replacement

Similarly, to prove entries in Table 1, it suffices to show that branch replacement operations displayed in Figure 21 and Figure 22 preserve 3-rs colouring extension status; this is easy to prove with the help of Theorem 8 (see supplementary material for details and proof for each operation). We provide a sample lemma below.

Lemma 7. Branch replacement operation (13) preserves 3-rs colouring extension status (see Figure 20).

$$\begin{array}{c} u_1 & v \\ \ell_1 & & \ell_2 \\ (d=1) \end{array} \longrightarrow \begin{array}{c} w & u_1 & v \\ & & \ell_2 \\ \end{array} \longrightarrow \begin{array}{c} w & u_1 & v \\ & & \ell_1 \\ \end{array}$$

Figure 20: Branch replacement operation (13), and partially coloured rooted trees before and after replacement

*Proof.* Suppose that T admits a 3-rs colouring extension f. Let  $H = T - \{u_1, \ell_1, \ell_2\}$ . If f(v) = 0, then f restricted to V(H) can be extended into a 3-rs colouring extension of T' by assigning colour 2 at  $u_1$  and colour 1 at w. Otherwise, f(v) = 1 and hence  $f(u_1) = 0$  (by Property P2). So, the restriction of f to V(H) can be extended into a 3-rs colouring extension of T' by assigning colour 0 at  $u_1$  and colour 1 at w.

Conversely, suppose that T' admits a 3-rs colouring extension f'. If f'(v) = 0, then f' restricted to V(H) can be extended into a 3-rs colouring extension of T by assigning colour 1 at  $u_1$  and colour 2 at  $\ell_1, \ell_2$ . Otherwise, f'(v) = 1. Since  $w, u_1, v$  is a bicoloured  $P_3$  with endpoints coloured 1, its middle vertex must be coloured 0. That is,  $f'(u_1) = 0$ . Hence, f' restricted to V(H) can be extended into a 3-rs colouring extension of T by assigning colour 0 at  $u_1$  and colour 2 at  $\ell_1, \ell_2$ .

Figure 21: Branch replacement operations (1) to (8)

Figure 22: Branch replacement operations (9) to (15)

## 5.2 Proof of Table 2 (Overview)

Let T be a partially coloured rooted tree with a 3-plus vertex as the root. Let v be a vertex of T with  $\deg_T(v) \neq 2$ . Suppose that the rooted subtree  $R = T_v$  is composed of branches of T at v each of which belong to Classes B to F. Let b, c, d, e, f denote respectively the number of Class B branches, Class B branches, ..., Class B branches (of B at B). Each entry of Table 2 is proved by showing that replacing B by a suitable rooted subtree preserves 3-rs colouring extension status.

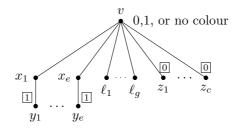


Figure 23: General form of R  $(e \ge 0, g \ge 0, c \ge 0)$ 

By definition of the equivalence relation on branches, each branch  $B^*$  of T at v can be replaced by the representative of the equivalence class of  $B^*$  without affecting 3-rs colouring extension status of T. Unless b>0 and c+d>0, all these replacements can be carried out simultaneously, and thus we may assume that R is the graph in Figure 23 where g=b+d+f and v is coloured 1 if c>0.

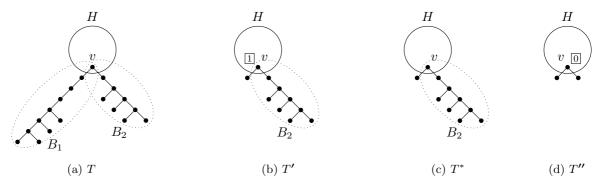


Figure 24: The branch  $B_1 \in \text{Class D. So}$ , T admits a 3-rs colouring extension  $\iff T'$  admits a 3-rs colouring extension  $f^*$  such that  $f^*(v) = 1$ . The branch  $B_2 \in \text{Class B. So}$ ,  $T^*$  admits a 3-rs colouring extension  $f^*$  such that  $f^*(v) = 1 \iff T''$  admits a 3-rs colouring extension f'' such that f''(v) = 1 (impossible obviously). So, T does not admit a 3-rs colouring extension

In the case b > 0 and c + d > 0, T does not admit a 3-rs colouring extension due to a colour conflict at v (see Figure 24 for an example; see supplementary material for the proof of  $B_1 \in \text{Class D}$  and  $B_2 \in \text{Class B}$ ). Thus,  $R \in \text{Class I}$  by Lemma 4. This proves the entry b > 0, c + d > 0 of Table 2. Figure 25 shows an overview of the remaining cases. To prove Table 2, it suffices to show that each rooted subtree replacement operation of Figure 25 (drawn by a thick arrow) preserves 3-rs colouring extension status (proof is available in supplementary material).

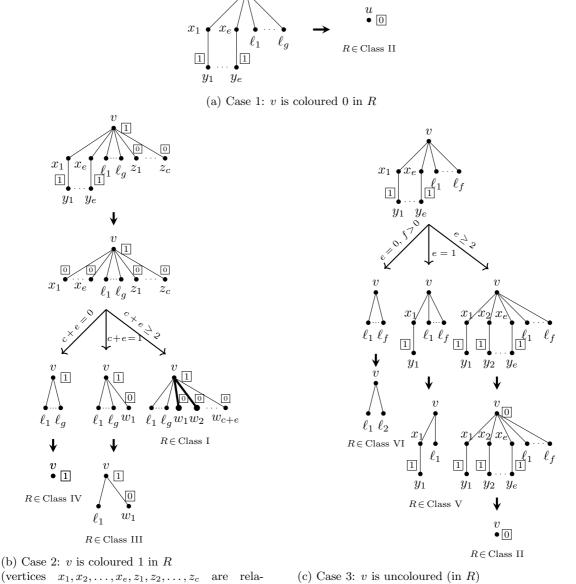


Figure 25: Determining the equivalence class of rooted subtree R in Table 2

(provided e + f > 0)

belled  $w_1, w_2, \ldots, w_{c+e}$ )

#### 5.3 Pseudocode

In this subsection, we provide the pseudocode of 3-rs colourability testing algorithm for trees, and discuss its run-time.

Recall that the input to the algorithm is a rooted tree T with a 3-plus vertex a as the root. Assume that T is stored as an ADT tree. If such a representation is not readily available, it can be obtained by BFS. For simplicity, it is assumed that the number of children of each vertex v is stored as #children[v]. The following global variables are used in the pseudocode: dist, branch\_class, subtree\_class, and arrays colour[v], up\_distance[v], #ClassC\_branches[v] and #ClassE\_branches[v] (where vertex names are used as array indices). The variable dist is used to compute the up-distance of branches. The variables branch\_class and subtree\_class store respectively the equivalence class of the current branch and the current rooted subtree. The variable colour[v] has value 0 or 1 when vertex v is coloured 0 or 1, and value -1 when v is not coloured. Up-distance of a branch of T at v composed of a rooted subtree  $T_u$  and the u, v-path is stored as up\_distance[u] (this is the distance from u to its immediate ancestor which is a 3-plus vertex). The variables #ClassC\_branches[v] and #ClassE\_branches[v] store the number of Class C branches at v and the number of Class E branches at v respectively. To test 3-rs colourability of T, call test3rsColourability (T, a).

The subroutine subtreeClassify(v) computes the equivalence class of the rooted subtree  $T_v$ , and stores it in the variable subtree\_class. If  $T_v \in \text{Class I}$ , then the subroutine prints "T is not 3-rs colourable" and halts the algorithm. Recall that a is the root of T. For each vertex  $u \neq a$  of T, the subroutine branchClassify(u) computes the equivalence class of the branch  $B^*$  composed of the rooted subtree  $T_u$  and the u, v-path where v is the immediate ancestor of u which is a 3-plus vertex; the subroutine also stores it in the variable branch\_class. If  $B^* \in \text{Class A}$ , then the subroutine prints "T is not 3-rs colourable" and halts the algorithm. For u = a, the subroutine branchClassify(u) prints "T is 3-rs colourable" and halts the algorithm. For every vertex v of T with  $\deg_T(v) \neq 2$ , the subroutine traverseSubtree(T, v) performs a post-order traversal of  $T_v$ , and helps to compute the equivalence class of  $T_v$  recursively.

```
Subroutine traverseSubtree(T, v)
   dist \leftarrow dist+1
                                                                         /* to compute up-distance[v] */
   if \#children[v] \ge 2 then
                                                                          /* ie, v is a 3-plus vertex */
      up\_distance[v] \leftarrow dist
      \operatorname{colour}[v] \leftarrow -1
                                                                                       /* initialization */
       \#\text{ClassC\_branches}[v] \leftarrow 0
                                                                                       /* initialization */
       \#\text{ClassE\_branches}[v] \leftarrow 0
                                                                                       /* initialization */
      foreach child w of v do
          \mathrm{dist} \leftarrow 0
                                                                                       /* initialization */
          traverseSubtree(T, w)
          if branch\_class=C then
           else if branch_class=E then
           \#\text{ClassE\_branches}[v] \leftarrow \#\text{ClassE\_branches}[v] + 1
          if branch_class=B then tryToColour(v,0)
          else if branch_class=C or D then tryToColour(v, 1)
          /* (to spot colour conflict; see Figure 24)
                                                                                                             */
      subtreeClassify(v)
                                                                                   /* finds class of T_v */
      branchClassify(v)
                                  /* (for v \neq a) finds class of the branch composed of T_v and the
       v,x-path where x the immediate ancestor of v which is a 3-plus vertex. */
   else if \#children[v]=1 then
      w \leftarrow \text{child of } v
      traverseSubtree(T, w)
   else
                                                                                     /* ie, v is a leaf */
      up\_distance[v] \leftarrow dist
      subtree\_class \leftarrow VII
      /* by the case b=c=d=e=f=0 of Table 2
                                                                                                             */
      branchClassify(v)
```

```
Subroutine subtreeClassify(v)
```

```
/* Determines the equivalence class of T_v assuming T_v \not\cong K_1. The subroutine
   traverseSubtree() deals with the case T_v \cong K_1 (i.e., b=c=d=e=f=0).
                                                                                                        Ιf
   T_v \in \mathtt{Class} I, the subroutine prints "T is not 3-rs colourable", and halts the
   algorithm.
                                                                         /* Case 1 of Table 2: b>0 */
if \operatorname{colour}[v]=0 then
   subtree\_class \leftarrow II
   /* here, c=d=0. The subroutine tryToColour() handles the case b>0, c+d>0.
                                                                                                           */
else if colour[v]=1 then
                                                                 /* Case 2 of Table 2: b = 0, c + d > 0 */
   if \#\text{ClassC\_branches}[v] + \#\text{ClassE\_branches}[v] = 0 then
       /* i.e., Subcase 2.1: c+e=0
      subtree\_class \leftarrow IV
   else if \#\text{ClassC\_branches}[v] + \#\text{ClassE\_branches}[v] = 1 then
       /* i.e., Subcase 2.2: c+e=1
     subtree\_class \leftarrow III
   else
       /* i.e., Subcase 2.3: c+e \ge 2
       subtree\_class \leftarrow I
       Print "T is not 3-rs colourable", and halt
                                                                                          /* by Lemma 4 */
                                                                  /* Case 3 of Table 2: b=c=d=0 */
else
   if \#\text{ClassE\_branches}[v]=0 then
       /* i.e., Subcase 3.1: e=0 and f>0.
       /* (since T_v \not\cong K_1, b=c=d=e=0 \Longrightarrow f>0).
    \bot subtree_class \leftarrow VI
   else if \#\text{ClassE\_branches}[v]=1 then
       /* i.e., Subcase 3.2: e=1
    \_ subtree_class \leftarrow V
   else
       /* i.e., Subcase 3.3: e \ge 2
     \_ subtree_class \leftarrow II
```

### Subroutine branchClassify(u)

```
/* For u \neq a, let v be the immediate ancestor of u which is a 3-plus vertex, and let B^*
   be the branch composed of the rooted subtree T_u and the u,v-path. The subroutine
   determines the equivalence class of B^* if u \neq a. If B^* \in \text{Class A}, it prints "T is
   not 3-rs colourable", and halts the algorithm. Since the algorithm halts whenever a
   Class I rooted subtree is encountered, T_u is not in Class I. For u=a, the
   subroutine reports that T is 3-rs colourable, and halts the algorithm.
if up_distance[u]=0 then
   /* this happens only if u=a (i.e., the root of T)
 \ \ \  Print "T is 3-rs colourable", and halt
else
   Using Table 1, find the equivalence class of B^* based on subtree_class (i.e., class of T_u) and
    up_distance[u]
   (i.e., up-distance of B^*), and store it in branch_class
   if branch_class=A then
                                                                               /* by Lemma 4 */
    Print "T is not 3-rs colourable", and halt
```

It is time to see the run-time analysis of the algorithm. The subroutine traverseSubtree() carries out a post-order traversal of the tree, and performs some processing at each vertex visited; therefore, it is easy to analyze the time complexity of the algorithm by comparing it to post-order traversal. It is easy to verify that the subroutine branchClassify() requires at most 13 steps (see supplementary material for detailed pseudocode of BranchClassify()), and the subroutine subtreeClassify() requires at most 6 steps. In the run-time analysis of the algorithm, the cost of subroutine calls, except for subroutine traverseSubtree(), can be ignored because each call for other subroutines can be replaced by respective code. There is at most one call of subroutines branchClassify() and subtreeClassify() per vertex. Also, other lines of the subroutine traverseSubtree() such as dist  $\leftarrow$  dist+1 and dist  $\leftarrow$  0 are executed at most once per vertex. At most 14 such lines are executed per vertex. Hence, the worst-case run-time of the algorithm is (33 + C)n steps where C is the cost of a subroutine call.

### 5.4 Chordal Graphs

In this subsection, we show that 3-RS COLOURABILITY is polynomial-time decidable for the class of chordal graphs. Note that a chordal graph has no induced cycle of length greater than three. Hence, a connected chordal graph G is either a tree, or it contains a triangle. If G is a tree, then 3-rs colourability of G can be tested using our 3-rs colourability testing algorithm for trees. So, we assume that G has a triangle, say (u, v, w). There are two types of triangles in G namely (i) type-I: u, v and w are 3-plus vertices in G, and (ii) type-II: at least one of the vertices u, v, w is of degree two. Observation 7 deals with a graph (not necessarily chordal) which contains a type-I triangle; and Theorem 10 deals with a graph (not necessarily chordal) which contains a type-II triangle.

**Observation 7.** Let G be a graph, and let (u, v, w) be a triangle in G. If u, v and w are 3-plus vertices in G, then G is not 3-rs colourable.

**Theorem 10.** Let G be a graph, let (u, v, w) be a triangle in G, and let w be a vertex of degree two in G. Let G' be the graph obtained from G - w by attaching two pendant vertices each at u and v (see Figure 26). Then, G is 3-rs colourable if and only if G' is 3-rs colourable.



Figure 26: Graphs G and G'

Proof. Suppose that G admits a 3-rs colouring. Then, G admits a 3-rs colouring f such that f(w)=2 (for instance, if a 3-rs colouring of G assigns colour 2 at v, then  $\deg_G(v)=2$  and hence swapping colours at v and w gives a 3-rs colouring of G that assigns colour 2 at w). Since f(w)=2, we have  $f(u), f(v)\in\{0,1\}$ . Hence, the restriction of f to V(G-w) can be extended into a 3-rs colouring of G' by assigning colour 2 on the newly added pendant vertices. Conversely, suppose that G' admits a 3-rs colouring f'. Since u and v are 3-plus vertices in  $G', f'(u), f'(v) \in \{0,1\}$  by Property P1. Hence, f' restricted to V(G-w) can be extended into a 3-rs colouring of G by assigning colour 2 at w.

**Theorem 11.** 3-RS COLOURABILITY is in P for the class of chordal graphs.

*Proof.* We can test whether a given connected chordal graph G is 3-rs colourable as follows.

Case 1: G is a tree.

Then, 3-rs colourability of G can be tested using our 3-rs colourability testing algorithm for trees.

Case 2: G contains a triangle.

List all triangles in G in  $O(n^3)$  time. If G contains a type-I triangle, then G is not 3-rs colourable by Observation 7. If G contains only type-II triangles, we can get rid of all triangles in G by repeated application of Theorem 10 (see Figure 27 for an example). The resultant graph will be a tree T on at most  $n + 3\binom{n}{3} = O(n^3)$  vertices. Thus, we can test 3-rs colourability of G in  $O(n^3)$  time by testing 3-rs colourability of T.

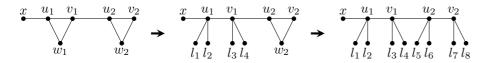


Figure 27: Construction of a tree T from a chordal graph G such that  $\chi_{rs}(T) \leq 3$  if and only if  $\chi_{rs}(G) \leq 3$ 

# 6 Related Colouring Variants

Star colouring and ordered colouring are colouring variants closely related to restricted star colouring. A k-ordered colouring of a graph G is a function  $f: V(G) \to \{0, 1, \dots, k-1\}$  such that (i)  $f(x) \neq f(y)$  for every edge xy of G, and (ii) every non-trivial path in G with the same colour at its endpoints contains a vertex of higher colour (i.e., every path  $P_{q+1}$  with endpoints coloured i contains a vertex of colour j > i) [23]. The ordered chromatic number  $\chi_o(G)$  is the least integer k such that G admits a k-ordered colouring. Analogous to the problem k-RS Colourability (resp. RS Colourability), we can define problems k-Star Colourability and k-Ordered Colourability (resp. Star Colourability and Ordered Colourability).

For every graph G,  $\chi_s(G) \leq \chi_{rs}(G) \leq \chi_o(G)$  [4]. It is known that the three parameters can be arbitrarily apart [4]. On the other hand, all three parameters are equal for cographs because  $\chi_s(G) = \operatorname{tw}(G) + 1 = \chi_o(G)$  [24, 25] where  $\operatorname{tw}(G)$  denotes the treewidth of G. We prove that the star chromatic number equals the ordered chromatic number for co-bipartite graphs as well.

**Theorem 12.** For a co-bipartite graph G,  $\chi_s(G) = \chi_{rs}(G) = \chi_o(G)$ .

Proof. Let G be a co-bipartite graph whose vertex set is partitioned into two cliques A and B. Let  $k=\chi_s(G)$ , and let f be a k-star colouring of G. To prove the theorem, it suffices to show that G admits a k-ordered colouring. Clearly, each colour class under f contains at most one vertex from A and at most one vertex from B. Let  $\{U_0, U_1, \ldots, U_{k-1}\}$  be the set of colour classes under f. Let t be the number of colour classes with cardinality two. We assume that  $t \neq 0$  (if t = 0, then k = n, and f itself is a k-ordered colouring of G). W.l.o.g., we assume that  $|U_i| = 2$  for  $0 \le i \le t-1$ , and  $|U_i| = 1$  for  $t \le i \le k-1$ . Observe that if  $a_i \in U_i \cap A$ ,  $b_i \in U_i \cap B$ ,  $a_j \in U_j \cap A$  and  $b_j \in U_j \cap B$ , then  $a_ib_j \notin E(G)$  and  $a_jb_i \notin E(G)$  (if not, the path  $a_j, a_i, b_j, b_i$  or the path  $a_i, a_j, b_i, b_j$  is a  $P_4$  in G bicoloured by f). Therefore, there are no edges in G between  $A \cap \bigcup_{i=0}^{t-1} U_i$  and  $B \cap \bigcup_{i=0}^{t-1} U_i$ . We claim that  $h: V(G) \to \{0, 1, \ldots, k-1\}$  defined as h(v) = i for all  $v \in U_i$  is a k-ordered colouring of G. Suppose that u and  $v \in B$ . Then,  $v \in A \cap \bigcup_{i=0}^{t-1} U_i$  and  $v \in B \cap \bigcup_{i=0}^{t-1} U_i$ . Hence, every v, v-path v in v must contain a vertex v from v from v and v are arbitrary, v is indeed a v-ordered colouring of v.

Since k-ORDERED COLOURABILITY is in P for every k, and ORDERED COLOURABILITY is NP-complete for co-bipartite graphs [26], we have the following corollary.

Corollary 2. For the class of co-bipartite graphs, problems k-RS Colourability and k-Star Colourability are in P for all k, whereas problems RS Colourability and Star Colourability are NP-complete.

## 7 Conclusion

It is known that deciding whether a planar bipartite graph admits a 3-star colouring is NP-complete [27]. We prove that deciding whether a subcubic planar bipartite graph of arbitrarily large girth admits a 3-restricted star colouring is NP-complete. In addition, we prove that it is NP-complete to test whether a 3-star colourable graph admits a 3-restricted star colouring (see Theorem 4). Karpas et al. [4] produced an  $O(n^{\frac{1}{2}})$  approximation algorithm for the optimization problem of restricted star colouring on a 2-degenerate bipartite graph with the minimum number of colours. We prove that this optimization problem is NP-hard to approximate within  $n^{\frac{1}{3}-\epsilon}$  for all  $\epsilon > 0$ . For the class of co-bipartite graphs, RS Colourability is NP-complete, but k-RS Colourability is in P for all  $k \in \mathbb{N}$ . We present (i) an O(n)-time algorithm to test 3-rs colourability of trees, and (ii) an  $O(n^3)$ -time algorithm to test 3-rs colourability of chordal graphs. The complexity of RS Colourability in the class of chordal graphs remains open.

# Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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# Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.dam.2021.05.015.

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