On the In-Out-Proper Orientations of Graphs

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Abstract

An orientation of a graph G is *in-out-proper* if any two adjacent vertices have different in-out-degrees, where the in-out-degree of each vertex is equal to the in-degree minus the out-degree of that vertex. The *in-out-proper orientation number* of a graph G, denoted by $\overleftrightarrow{\chi}(G)$, is $\min_{D\in\Gamma}\max_{v\in V(G)}|d_D^{\pm}(v)|$, where Γ is the set of in-out-proper orientations of G and $d_D^{\pm}(v)$ is the in-out-degree of the vertex v in the orientation D. Borowiecki *et al.* proved that the in-out-proper orientation number is well-defined for any graph G [Inform. Process. Lett., 112(1-2):1–4, 2012]. So we have $\overleftrightarrow{\chi}(G) \leq \Delta(G)$, where $\Delta(G)$ is the maximum degree of vertices in G. We conjecture that there exists a constant number c such that for every planar graph G, we have $\overleftrightarrow{\chi}(G) \leq c$. Towards this speculation, we show that for every tree T we have $\overleftrightarrow{\chi}(T) \leq 3$ and this bound is sharp. Next, we study the in-out-proper orientation number of subcubic graphs. By using the properties of totally unimodular matrices we show that there is a polynomial time algorithm to determine whether $\overleftrightarrow{\chi}(G) \leq 2$, for a given graph G with maximum degree three. On the other hand, we show that it is NP-complete to decide whether $\overleftrightarrow{\chi}(G) \leq 1$ for a given bipartite graph G with maximum degree three. Finally, we study the in-out-proper orientation number of regular graphs.

Key words: Proper orientation; In-out-proper orientation; In-out-proper orientation number; In-out-degree; Subcubic graphs.

1 Introduction

Let G be a graph and D be an orientation of it. For every vertex v of G, we denote the in-degree (out-degree) of v in the orientation D by $d_D^-(v)$ ($d_D^+(v)$, respectively). An orientation of a graph G is called *proper* if any two adjacent vertices have different in-degrees [1]. The *proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}(G)$, is the minimum of the maximum in-degree taken over all proper orientations of the graph G. A proper orientation D of G can be used to form a proper vertex coloring of G by assigning every vertex v of G the color $d_D^-(v)$ [1]. So, we have

$$\chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G). \tag{1}$$

The proper orientation number of graphs has been studied by several authors, for instance see [1, 2, 3, 4, 5, 6, 8, 10, 11, 13]. In [4], Araujo *et al.* asked whether the proper orientation number of a planar graph is bounded. Toward this question, it was shown that if T is a tree, then $\overrightarrow{\chi}(T) \leq 4$ [4]. Also, it was shown that every cactus admits a proper orientation with maximum in-degree at most 7 [5]. Furthermore, it was proved that every bipartite planar graph with minimum degree at least 3 has proper orientation number at most 3 [13].

Let D be an orientation for a given graph G. The in-out-degree of the vertex v is defined as $d_D^{\pm}(v) = d_D^-(v) - d_D^+(v)$. Note that for a given graph G and orientation D, for each vertex v we have

$$-\Delta(G) \leq d_D^{\pm}(v) \leq \Delta(G). \tag{2}$$

Motivated by the proper orientations of graphs we investigate the in-out-proper orientations. An orientation of a graph G is *in-out-proper* if any two adjacent vertices have different in-out-degrees. The *in-out-proper orientation number* of a graph G, denoted by $\overleftrightarrow{\chi}(G)$, is $\min_{D\in\Gamma} \max_{v\in V(G)} |d_D^{\pm}(v)|$, where Γ is the set of in-out-proper orientations of G and $d_D^{\pm}(v)$ is the in-out-degree of the vertex v in the orientation D. For a given graph G, we say that an in-out-proper orientation D is *optimal* if the maximum of the absolute values of their in-out-degrees is equal to $\overleftrightarrow{\chi}(G)$.

It is interesting to mention that in-out-proper orientation relates to the flow. In more details, an in-out-proper orientation of a graph G can be thought as a 'flow' of G that does not satisfy Kirchhoff's Current Law. Borowiecki *et al.* proved that in-out-proper orientation number is well-defined for any graph G [7].

Theorem 1. *[7] The in-out-proper orientation number is well-defined for any graph G.*

By Theorem 1 and noting that for a given graph G every in-out-proper orientation defines a proper vertex coloring for G, we have

$$\lceil \frac{\chi(G)-1}{2} \rceil \leq \overleftrightarrow{\chi}(G) \leq \Delta(G). \tag{3}$$

Example 1. Let G be a cycle. The degree of each vertex is two, so in each in-out-proper orientation of G, the in-out-degree of each vertex is $-2, +2$, or 0. The graph G has at least two adjacent vertices, so $\overleftrightarrow{\chi}(G) \geq 2$. On the other hand, by Theorem 1, $\overleftrightarrow{\chi}(G) \leq 2$. Consequently, for every cycle C_n we have $\overleftrightarrow{\chi}(G) = 2$.

Araujo *et al.* asked whether the proper orientation number of a planar graph is bounded. We pose the following conjecture for the in-out-proper orientation number of planar graphs.

Conjecture 1. *There is a constant number c such that for every planar graph G, we have $\overleftrightarrow{\chi}(G) \leq c$.*

Towards Conjecture 1, we study the in-out-proper orientation number of trees and show that for every tree T we have $\overleftrightarrow{\chi}(T) \leq 3$.

Theorem 2. *For every tree T we have $\overleftrightarrow{\chi}(T) \leq 3$ and this bound is sharp.*

A graph is called subcubic if it has maximum degree at most three. Let G be a subcubic graph. By Theorem 1, we have $\overleftrightarrow{\chi}(G) \leq 3$. By using the properties of totally unimodular matrices we show that there is a polynomial time algorithm to determine whether $\overleftrightarrow{\chi}(G) \leq 2$.

Theorem 3. *There is a polynomial time algorithm to determine whether $\overleftrightarrow{\chi}(G) \leq 2$, for a given graph G with maximum degree three.*

On the other hand, we show that it is NP-complete to decide whether $\overleftrightarrow{\chi}(G) \leq 1$ for a given bipartite graph G with maximum degree three.

Theorem 4. *It is NP-complete to decide whether $\overleftrightarrow{\chi}(G) \leq 1$ for a given bipartite graph G with maximum degree three.*

Next, we study the computational complexity of determining the the in-out-proper orientation number of 4-regular graphs. Note that for any 4-regular graph G we have $2 \leq \overleftrightarrow{\chi}(G) \leq 4$.

Theorem 5. *It is NP-complete to decide whether* $\overleftrightarrow{\chi}(G) \leq 2$ *for a given 4-regular graph* G.

Let G be a 4-regular graph with $\overleftrightarrow{\chi}(G) \leq 3$ and suppose that D is an optimal in-out-proper orientation. In G the degree of each vertex is four, so the in-out-degree of each vertex is in $\{0, \pm 2\}$. Thus, we have $\overleftrightarrow{\chi}(G) \leq 3$ if and only if $\overleftrightarrow{\chi}(G) \leq 2$. Thus, by Theorem 5, we have the following corollary.

Corollary 1. *It is NP-complete to decide whether* $\overleftrightarrow{\chi}(G) \leq 3$ *for a given 4-regular graph* G.

The organization of the rest of the paper is as follows: In Section 2, we present some definitions and notations. This is followed in Section 3 by some bounds for the in-out-proper orientation number of graphs. Next, in Section 4, we prove that the in-out-proper orientation number of each tree is at most three. In Section 5, we focus on the in-out-proper orientation number of subcubic graphs. Section 6 is devoted to the computational complexity of regular graphs. The paper is concluded with some remarks in Section 7.

2 Definitions

In this work, all graphs are finite and simple (i.e. without loops and multiple edges). We follow [14] for terminology and notation where they are not defined here. If G is a graph, then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For every $v \in V(G)$, $d_G(v)$ denotes the degree of v in the graph G. Also, $\Delta(G)$ denotes the maximum degree of G. The distance between two vertices v and u, denoted by $distance(v, u)$, is the length of a shortest path between them.

An orientation D of a graph G is a digraph obtained from the graph G by replacing each edge by just one of the two possible arcs with the same endvertices. For every vertex v, the in-degree (the out-degree) of v in the orientation D, denoted by $d^-_D(v)$ ($d^+_D(v)$), is the number of arcs with head (tail) v in D. Also, the in-out-degree of v, denoted by $d^{\pm}_D(v)$, is defined as $d^-_D(v) - d^+_D(v)$. An orientation of a graph G is *in-out-proper* if any two adjacent vertices have different in-out-degrees. The *in-out-proper orientation number* of a graph G, denoted by $\overleftrightarrow{\chi}(G)$, is $\min_{D\in\Gamma} \max_{v\in V(G)} |d^{\pm}_D(v)|$, where Γ is the set of in-out-proper orientations of G.

Let G be a graph. A proper vertex t-coloring of G is a function $f : V(G) \longrightarrow \{1, \dots, t\}$ such that if $u, v \in V(G)$ are adjacent, then $f(u)$ and $f(v)$ are different. The smallest integer t such that G has a proper vertex t-coloring is called the chromatic number of G and denoted by $\chi(G)$. Also, a proper edge t-coloring of G is a function $f : E(G) \longrightarrow \{1, \dots, t\}$ such that if $e, e' \in E(G)$ have a same endvertex, then $f(e)$ and $f(e')$ are different. The smallest integer t such that G has a proper edge t-coloring is called the edge chromatic number (or chromatic index) of G and denoted by $\chi'(G)$.

For a graph $G = (V, E)$, the line graph of G is a graph with the set of vertices $E(G)$ and two vertices are adjacent if and only if their corresponding edges share a common endpoint in G.

A matrix A is totally unimodular if every square submatrix of A has determinant 1, 0 or -1. The importance of totally unimodular matrices stems from the fact that when an integer linear program has all-integer coefficients and the matrix of coefficients is totally unimodular, then the optimal solution of its relaxation is integral. Therefore, it can be obtained in polynomial time [12].

3 General bounds

For every graph G we have $\overleftrightarrow{\chi}(G) \leq \Delta(G)$. It is good to mention that the inequality is tight for any complete graph. For any n, each vertex of K_n can have only in-out-degree $n-1, n-3, \dots, -(n-3), -(n-1)$. The number of these values is exactly n. Thus, the in-out-proper orientation number of K_n is at least $n - 1 = \Delta(K_n)$.

Next, we present some observation for the in-out-proper orientation number of graphs.

Lemma 1. *Let G be a graph with at least one edge and assume that D is an in-out-proper orientation of G. Then in the orientation D there is at least one vertex with positive in-out-degree and at least one vertex with negative in-out-degree.*

Proof. Let G be a graph with at least one edge and assume that D is an in-out-proper orientation of G. First, we show that in D there is a vertex with positive in-out-degree. To the contrary assume that the in-out-degree of each vertex is negative or zero. So, we have

$$\sum_{v\in V(G)} d^{\pm}_{D}(v) \leq 0. \tag{4}$$

On the other hand, we have

$$\sum_{v\in V(G)} d^{-}_{D}(v) = \sum_{v\in V(G)} d^{+}_{D}(v). \tag{5}$$

Thus, by (4) and (5), we conclude that for every vertex v we have $d^{\pm}_{D}(v) = 0$. The graph G has at least one edge, but in D the in-out-degrees of all vertices are zero (so, it is not a proper vertex coloring). Thus D is not an in-out-proper orientation for G. This is a contradiction. So, there is a vertex with positive in-out-degree. Similarly, we can show that there is a vertex with negative in-out-degree. □

4 Trees

Next, we study the in-out-proper orientation number of trees and show that for every tree T we have $\overleftrightarrow{\chi}(T) \leq 3$. Also, we show that this bound is sharp.

Proof of Theorem 2. First we show that for each tree T we have $\overleftrightarrow{\chi}(T) \leq 3$. Let T be a tree with n vertices and v be a vertex of T. Sort the vertices of T according to their distance from v and let $v = v_1, v_2, \dots, v_n$ be that sorted set. For each vertex u, the father of u, denoted by $f(u)$, is the unique vertex that is adjacent and closer to the root v. Perform the Algorithm 1 and call the resultant orientation D.

We have the following properties for the orientation D that we obtained from Algorithm 1.

Proposition 1. *Let u be a vertex with $d(u) \geq 3$. If u has an even distance from the root v_1, then $d^{\pm}_{D}(u) \in \{1, 2, 3\}$. Also, if u has an odd distance from the root v_1, then $d^{\pm}_{D}(u) \in \{-1, -2, -3\}$.*

Proof. Let u be a vertex with $d(u) \geq 3$. If u has an even distance from the root v_1, then at Lines 2-3, 5-6, and 8-9, we orient the edges incident with u such that the in-out-degree of u is in $\{1, 2, 3\}$. There is only one other part of the algorithm that we may change the in-out-degree of u. That part is Lines 35-36. In that case the in-out-degree of u is one and by reorienting one of the edges that is incident with u we increase the in-out-degree of u by two. Similarly, if u has an odd distance from the root v, then at Lines 23-24, 26-27, and 17-18, we orient the edges incident with u such that the in-out-degree of u is in $\{-1, -2, -3\}$. ■

Proposition 2. *Let u be a vertex with $d(u) = 2$. If u has an even distance from the root v_1, then $d^{\pm}_{D}(u) \in \{0, 1, 2\}$. Also, if u has an odd distance from the root v_1, then $d^{\pm}_{D}(u) \in \{0, -1, -2\}$.*

Proof. Let u be a vertex with $d(u) = 2$. If u has an even distance from the root v_1, then at Lines 2-3, 5-6, and 10-14, we orient the edges incident with u such that the in-out-degree of u is in $\{0, 1, 2\}$. There is only one other part of the algorithm that we may change the in-out-degree of u. That part is Lines 31-32. In that case the in-out-degree of u is zero and by reorienting one of the edges that is incident with u we increase the

Algorithm 1

for $i = 1$ to n **do**
  if $i = 1$ **then**
    Orient the edges incident with v_1 such that if $d(v_1)$ is an even number then $d_D^{\pm}(v_1) = 2$, and if $d(v_1)$ is an odd number then $d_D^{\pm}(v_1) = 1$.
  else if $distance(v_i, v_1)$ is an even number **then**
    if the edge $v_i f(v_i)$ was oriented from $f(v_i)$ to v_i **then**
      Orient the set of edges $\{v_i v_j | j > i\}$ such that if $d(v_i)$ is an odd number then $d_D^{\pm}(v_i) = 1$, and if $d(v_i)$ is an even number then $d_D^{\pm}(v_i) = 2$.
    else if the edge $v_i f(v_i)$ was oriented from v_i to $f(v_i)$ **then**
      if $d(v_i) \geq 3$ **then**
        Orient the set of edges $\{v_i v_j | j > i\}$ such that $d_D^{\pm}(v_i) \in \{1, 2\}$
      else if $d(v_i) = 2$ **then**
        if $d_D^{\pm}(f(v_i)) \neq 0$ **then**
          Orient the edge $\{v_i v_j | j > i\}$ such that $d_D^{\pm}(v_i) = 0$
        else if $d_D^{\pm}(f(v_i)) = 0$ **then**
          Orient the set of edges incident with v_i such that $d_D^{\pm}(v_i) = 2$ (note that we reorient the edge $v_i f(v_i)$.
        end if
      else if $d(v_i) = 1$ **then**
        if $d_D^{\pm}(f(v_i)) = -1$ **then**
          Reorient the edge $v_i f(v_i)$ from $f(v_i)$ to v_i
        end if
      end if
    end if
  else if $distance(v_i, v_1)$ is an odd number **then**
    if the edge $v_i f(v_i)$ was oriented from v_i to $f(v_i)$ **then**
      Orient the set of edges $\{v_i v_j | j > i\}$ such that if $d(v_i)$ is an odd number then $d_D^{\pm}(v_i) = -1$, and if $d(v_i)$ is an even number then $d_D^{\pm}(v_i) = -2$.
    else if the edge $v_i f(v_i)$ was oriented from $f(v_i)$ to v_i **then**
      if $d(v_i) \geq 3$ **then**
        Orient the set of edges $\{v_i v_j | j > i\}$ such that $d_D^{\pm}(v_i) \in \{-1, -2\}$
      else if $d(v_i) = 2$ **then**
        if $d_D^{\pm}(f(v_i)) \neq 0$ **then**
          Orient the edge $\{v_i v_j | j > i\}$ such that $d_D^{\pm}(v_i) = 0$
        else if $d_D^{\pm}(f(v_i)) = 0$ **then**
          Orient the set of edges incident with v_i such that $d_D^{\pm}(v_i) = -2$ (note that we reorient the edge $v_i f(v_i)$.
        end if
      else if $d(v_i) = 1$ **then**
        if $d_D^{\pm}(f(v_i)) = 1$ **then**
          Reorient the edge $v_i f(v_i)$ from v_i to $f(v_i)$
        end if
      end if
    end if
  end if
end for

in-out-degree of u by two. So, the final in-out-degree of u is in $\{0, 1, 2\}$. Similarly, if u has an odd distance from the root v_1, then at Lines 23-24, 28-32, and 13-14, we orient the edges incident with u such that the in-out-degree of u is in $\{0, -1, -2\}$. ■

Proposition 3. *Let u and u' be two adjacent vertices such that $d(u) = d(u') = 2$. Then $d_D^{\pm}(u) \neq d_D^{\pm}(u')$.*

Proof. By Lines 10-14 and Lines 28-32, the algorithm does not produce any two adjacent vertices u, u' such that $d(u) = d(u') = 2$ and $d_D^{\pm}(u) = d_D^{\pm}(u') = 0$. Thus, by Proposition 2, for any two adjacent vertices u, u' with $d(u) = d(u') = 2$ we have $d_D^{\pm}(u) \neq d_D^{\pm}(u')$. ■

Proposition 4. *Let u be a vertex with $d(u) = 1$. Then $d_D^{\pm}(u) \in \{-1, +1\}$ and $d_D^{\pm}(u) \neq d_D^{\pm}(f(u))$.*

Proof. Let u be a vertex with $d(u) = 1$. If u has an even distance from the root v_1, then at Lines 2-3, 5-6 and 16-19, we orient the edge incident with u such that the in-out-degree of u is in $\{-1, +1\}$ and also if it is -1 then $d_D^{\pm}(u) \neq d_D^{\pm}(f(u))$. By Propositions 1, 2, we also conclude that if the in-out-degree of u is 1, then $d_D^{\pm}(u) \neq d_D^{\pm}(f(u))$. Similarly, if u has an odd distance from the root v_1, then at Lines 23-24, and 34-37, we orient the edge incident with u such that the in-out-degree of u is in $\{-1, +1\}$ and if it is 1, then $d_D^{\pm}(u) \neq d_D^{\pm}(f(u))$. By Propositions 1, 2, we conclude that if the in-out-degree of u is -1, then $d_D^{\pm}(u) \neq d_D^{\pm}(f(u))$. This completes the proof. ■

By Propositions 1, 2, 3 and 4, for every vertex u, we have $d_D^{\pm}(u) \in \{\pm 3, \pm 2, \pm 1, 0\}$ and for every two adjacent vertices u, u', we have $d_D^{\pm}(u) \neq d_D^{\pm}(u')$. Thus D is an in-out-proper orientation such that the maximum of absolute values of their in-out-degrees is at most three.

Finally, we show that there is a tree T such that $\overleftrightarrow{\chi}(T) = 3$. Consider the tree T with the set of vertices v_1, v_2, v_3, v_4 and set of edges v_1v_2, v_1v_3, v_1v_4. We have $d(v_2) = d(v_3) = d(v_4) = 1$, so in any orientation of T, their in-out-degrees are in $\{\pm 1\}$. On the other hand, the degree of v_1 is three, so its in-out-degree is in $\{\pm 1, \pm 3\}$. To the contrary assume that $\overleftrightarrow{\chi}(T) < 3$, and let D be an in-out-proper orientation of T such that $d_D^{\pm}(v_1) \in \{\pm 1\}$. The in-out-degree of at least one of the vertices v_2, v_3, v_4 is 1 (otherwise $d_D^{\pm}(v_1) \notin \{\pm 1\}$) and also the in-out-degree of at least one of the vertices v_2, v_3, v_4 is -1. So, D has two adjacent vertices with the same in-out-degree Thus, it is not an in-out-proper orientation. This is a contradiction. So, we conclude that $\overleftrightarrow{\chi}(T) = 3$. □

5 Subcubic graphs

In this section we focus on subcubic graphs. Let G be a subcubic graph. By Theorem 1, we have $\overleftrightarrow{\chi}(G) \leq 3$. Next, we show that there is a polynomial time algorithm to determine whether $\overleftrightarrow{\chi}(G) \leq 2$. On the other hand, it is NP-complete to decide whether $\overleftrightarrow{\chi}(G) \leq 1$ for a given bipartite graph G with maximum degree three.

Proof of Theorem 3. Let G be a cubic graph. If D is an optimal in-out-proper orientation, then for any vertex v of degree two we have $d_D^{\pm}(v) \in \{0, \pm 2\}$ and also for any vertex v of degree one or three we have $d_D^{\pm}(v) \in \{\pm 1\}$. First, we investigate the subcubic graphs without degree two vertices. Then, we present a polynomial time algorithm for subcubic graphs. Let G be a graph such that the degree of each vertex is one or three and without loss of generality assume that G is connected. Also, suppose that $\overleftrightarrow{\chi}(G) \leq 2$ and D is an optimal in-out-proper orientation of G. Since $d_D^{\pm}(v) \in \{\pm 1\}$ for each vertex v and the in-out-degrees form a proper vertex coloring of G, G should be bipartite.

Proposition 5. *Let G be a graph such that the degree of each vertex is one or three. If $\overleftrightarrow{\chi}(G) \leq 2$, then G is bipartite.*

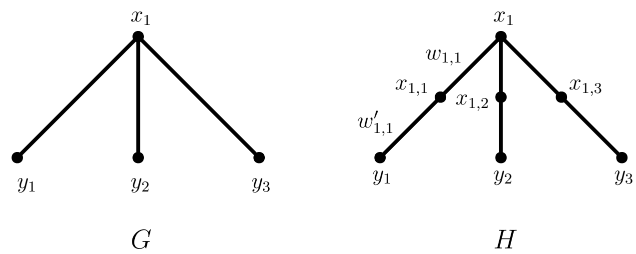


Figure 1: The graph G and its corresponding graph H.

Consequently, at step one we should check that whether G is bipartite. Next, at step two we want to determine whether it is possible to orient the edges of G such that in-out-degrees of vertices of one partite set of G are 1 and in-out-degrees of vertices of the other partite set of G are -1.

Proposition 6. *Let $G = (X \cup Y, E)$ be a bipartite graph such that the degree of each vertex is one or three. If $\overleftrightarrow{\chi}(G) \leq 2$, then there is an orientation for the edges of G such that the in-out-degree of each vertex is 1 or -1, and the in-out-degrees of all vertices in X are the same, and so are those in Y.*

It is well-known that there is a polynomial time algorithm to decide whether a given graph is bipartite [14]. Next, we present a polynomial time algorithm for step two. Let $G = (X \cup Y, E(G))$ be a bipartite graph such that the degree of each vertex is one or three. Without loss of generality assume that $X = x_1, x_2, \dots, x_n$ and $Y = y_1, y_2, \dots, y_{n'}$. From the graph G we construct a bipartite graph H with vertex set $V(H) = (U_X \cup U_Y) \cup U_0$ and edge set $E(H) = W$. Put $U_X = X$, and $U_Y = Y$. Also, for every edge $x_i y_j \in E(G)$, put $x_{i,j}$ in U_0 and the edges $w_{i,j} = x_i x_{i,j}, w'_{i,j} = x_{i,j} y_j$ in W. See Fig. 1.

Consider the following integer linear program for the graph H.

$$
\begin{aligned}
\text{Maximize} \quad & 1 \\
\text{subject to} \quad & \sum_{x_i y_j \in E(G)} w_{i,j} = 1 \quad \forall x_i \in U_X \text{ s.t. } d_G(x_i) = 1 && (6)\\
& \sum_{x_i y_j \in E(G)} w_{i,j} = 2 \quad \forall x_i \in U_X \text{ s.t. } d_G(x_i) = 3 && (7)\\
& \sum_{x_i y_j \in E(G)} w'_{i,j} = 0 \quad \forall y_j \in U_Y \text{ s.t. } d_G(y_j) = 1 && (8)\\
& \sum_{x_i y_j \in E(G)} w'_{i,j} = 1 \quad \forall y_j \in U_Y \text{ s.t. } d_G(y_j) = 3 && (9)\\
& w_{i,j} + w'_{i,j} = 1 \qquad \forall x_{i,j} \in U_0 && (10)\\
& w_{i,j}, w'_{i,j} \in \{0,1\} \qquad \forall x_i y_j \in E(G) && (11)
\end{aligned}
$$

Note that we can write the above integer linear program in the following canonical form:

$$
\begin{aligned}
\text{Maximize} \quad & 1 \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} && (12)\\
& .\mathbf{x} \in \{0,1\}^{|E(H)|}, && (13)
\end{aligned}
$$

where A is the incidence matrix of H, $\mathbf{x}^T = (w_{i_1,j_1}, \ldots, w'_{i_k,j_k})$, and $\mathbf{b} \in \{0,1,2\}^{|V(H)|}$. For instance, for the graph H that was shown in Fig. 1, we have $\mathbf{x}^T = (w_{1,1}, w'_{1,1}, w_{1,2}, w'_{1,2}, w_{1,3}, w'_{1,3})$, $\mathbf{b} = (2,0,0,0,1,1,1)$ and A is

$$\begin{array}{c c}
 & \begin{array}{cccccc} w_{1,1} & w'_{1,1} & w_{1,2} & w'_{1,2} & w_{1,3} & w'_{1,3} \end{array} \\
\begin{array}{c} x_1 \\ y_1 \\ y_2 \\ y_3 \\ x_{1,1} \\ x_{1,2} \\ x_{1,3} \end{array} &
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}$$

For each edge $x_iy_j \in E(G)$ in (11), we consider two variables $w_{i,j}, w'_{i,j}$ such that $w_{i,j}, w'_{i,j} \in \{0,1\}$. On the other hand, in (10), we have $w_{i,j} + w'_{i,j} = 1$, so the value of exactly one of these variables is one and the value of the other variable is zero. We consider the values of $w_{i,j}, w'_{i,j}$ as an orientation for the edge x_iy_j in G such that it is oriented from y_j to x_i if and only if $w_{i,j} = 1$. So, the values of the variables correspond to an orientation for the graph G. Call that orientation D. By (6) and (7), we ensure that in D the in-out-degree of each vertex in X is 1. Also, by (8) and (9), the in-out-degree of each vertex in Y is -1. Consequently, the above integer linear program is feasible if and only if the graph G has an in-out-proper orientation such that the in-out-degree of each vertex in X is 1 and the in-out-degree of each vertex in Y is -1.

When an integer linear program has all-integer coefficients and the matrix of coefficients is totally unimodular, then the optimal solution of its relaxation is integral. Therefore, it can be obtained in polynomial time [12]. On the other hand, it is shown in [[12], Corollary 2.9 in Page 544] that every incidence matrix of a bipartite graph is totally unimodular. So, in our integer linear program the matrix of coefficients is totally unimodular. Consequently, there is a polynomial time algorithm to determine whether the above-mentioned integer linear program is feasible.

Note that there is an orientation of the edges of G such that the in-out-degree of each vertex in X is -1 and the in-out-degree of each vertex in Y is 1 if and only if there is an orientation of the edges of G such that the in-out-degree of each vertex in X is 1 and the in-out-degree of each vertex in Y is -1 (by considering the reverse of the given orientation). Consequently, there is a polynomial time algorithm to determine whether the in-out-proper orientation number of given graph G with degree set $\{1,3\}$ is at most two.

Next, we consider the set of subcubic graphs. Let G be a subcubic graph. If D is an optimal in-out-proper orientation, then for any vertex v of degree two we have $d^{\pm}_D(v) \in \{0, \pm 2\}$ and also for any vertex v of degree one or three we have $d^{\pm}_D(v) \in \{\pm 1\}$.

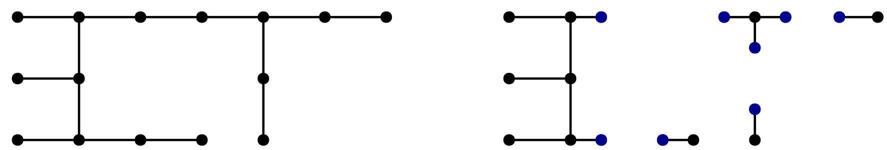
Remove all vertices of degree two from the graph G and call the resultant graph G'. For each vertex v in G' if $d_G(v) \neq d_{G'}(v)$, then put $d_G(v) - d_{G'}(v)$ isolated vertices and join them to v (we call these new vertices dummy vertices). Call the resultant graph G''. Note that in G'' the degree of each vertex is one or three. See Fig. 2.

Assume that $\overleftrightarrow{\chi}(G) \leq 2$. next, we present some necessary conditions for G''.

Proposition 7. *Let $C_1, C_2, \ldots, C_k$ be all the connected components of G''. For any $i \in \{1,2,\ldots,k\}$, (C_i is bipartite and) there exists an orientation D_i of C_i satisfying*
(a) every vertex of C_i has the in-out-degree 1 or -1, and
(b) for any $uv \in E(G) \cap E(G'')$, $d^{\pm}_{D_i}(u) \neq d^{\pm}_{D_i}(v)$.

Proof. By Proposition 5 and Proposition 6 the proof is clear. ■

In order to complete the proof we do the following steps:



G G''

Figure 2: The graph G and its corresponding graph G''. In the graph G'' the degree of each vertex is 1 or 3 and the set of blue vertices are dummy vertices.

Step 1. Proving that the condition in Proposition 7 is a necessary and sufficient one for $\overleftrightarrow{\chi}(G) \leq 2$.
Step 2. Showing that the condition in Proposition 7 can be checked in polynomial time.
Step 3. Concluding that Theorem 3 is true by Step 1 and Step 2.

(Proof of Step 1:) We show that the necessary conditions that are presented in Proposition 7 are also sufficient. In other words, we prove that if each connected component of G'' is bipartite and also if the graph G'' has an orientation such that in each connected component, the in-out-degrees of vertices in different parts (of that bipartite component), except dummy vertices, are different, then we can extend that partial orientation to an in-out-proper orientation of G such that the maximum of absolute values of their in-out-degrees is at most two. To prove that it is enough, we show the following proposition.

Proposition 8. *Each path $P_n = v_1, v_2, \ldots, v_n$ of length at least two (i.e. $n \geq 3$) has the following four kinds of in-out-proper orientations:*
(1) The in-out-proper orientation D_1 such that $d_D^{\pm}(v_1) = d_D^{\pm}(v_n) = 1$.
(2) The in-out-proper orientation D_2 such that $d_D^{\pm}(v_1) = d_D^{\pm}(v_n) = -1$.
(3) The in-out-proper orientation D_3 such that $d_D^{\pm}(v_1) = 1$ and $d_D^{\pm}(v_n) = -1$.
(4) The in-out-proper orientation D_4 such that $d_D^{\pm}(v_1) = -1$ and $d_D^{\pm}(v_n) = 1$.

Proof. Let $P_n = v_1, v_2, \ldots, v_n$ be a path of length at least two and $D \in \{D_1, D_2, D_3, D_4\}$. Orient the edges v_1v_2 and $v_{n-1}v_n$ such that the in-out-degree of v_1 is $d_D^{\pm}(v_1)$ and the in-out-degree of v_n is $d_D^{\pm}(v_n)$. Do Algorithm 2 to orient the remaining edges.

Algorithm 2

```
for i = 2 to n − 2 do
    if d±_D(v_{i−1}) = 0 and v_{i−1}v_i was oriented from v_{i−1} to v_i then
        Orient v_i v_{i+1} from v_{i+1} to v_i
    else
        Orient v_i v_{i+1} from v_i to v_{i+1}
    end if
end for
```

By Algorithm 2, there is no two consecutive vertices with the in-out-degree 0. On the other hand, it not possible to have two consecutive vertices with the in-out-degree 2 or -2. Moreover, the in-out-degree of v_n is in $\{\pm 1\}$ and the in-out-degree of v_{n-1} is in $\{0, \pm 2\}$. So D is an in-out-proper orientation. ■

(Proof of Step 2:) To check whether G'' has such an orientation we can use the previous mentioned integer linear program with some modifications. In fact, for each dummy vertex v we just remove the corresponding condition in (6) or (8), and then we solve the integer linear program.

(Proof of Step 3:) Having Propositions 6, and 8, and noting that there is a polynomial time algorithm to check Proposition 6, we conclude that there is a polynomial time algorithm to decide whether in-out-proper orientation number of a given subcubic graph is at most two. This completes the proof. □

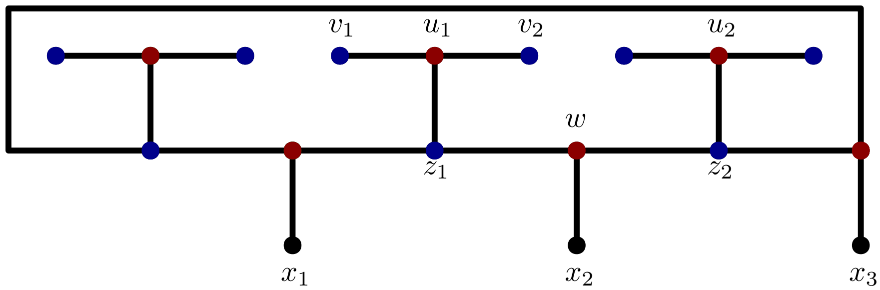


Figure 3: The gadget I_x.

Next, we prove that it is NP-complete to decide whether $\overleftrightarrow{\chi}(G) \leq 1$ for a given bipartite graph G with maximum degree three.

Proof of Theorem 4. It was shown in [9] that the following variant of Not-All-Equal satisfying assignment problem is NP-complete.

Problem: Cubic Monotone Not-All-Equal (2,3)-Sat.
Input: Set X of variables, collection C of clauses over X such that every clause $c \in C$ has $|c| \in \{2, 3\}$, each variable appears in exactly three clauses and there is no negation in the formula.
Question: Is there a truth assignment for X such that every clause in C has at least one true literal and at least one false literal?

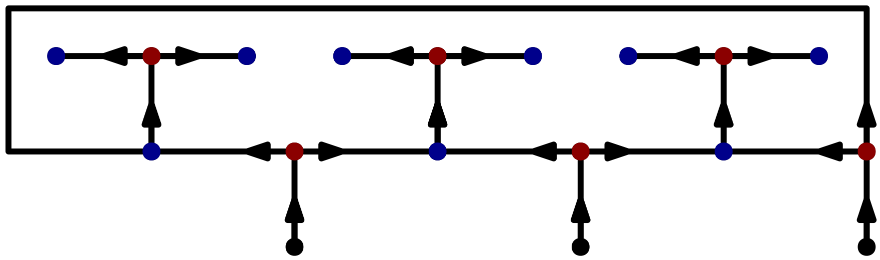
Our proof is a polynomial time reduction from Cubic Monotone Not-All-Equal (2,3)-Sat. Let Φ be an instance with the set of variables X and the set of clauses C. We transform it to a bipartite graph G_Φ with maximum degree three in polynomial time such that $\overleftrightarrow{\chi}(G_\Phi) \leq 1$ if and only if Φ has a Not-All-Equal truth assignment. We use the auxiliary gadget I_x which is shown in Fig. 3. Our construction consists of three steps.
Step 1. For each variable $x \in X$ put a copy of the gadget I_x which is shown in Fig. 3.
Step 2. For each clause $c \in C$ put a vertex c and then for each variable x that appears in the clause c join the vertex c to one of the vertices x_1, x_2, x_3 of I_x such that in the resultant graph for each variable $x \in X$ in the gadget I_x the degrees of the variables x_1, x_2, x_3 are two. Call the resultant graph H_Φ.
Step 3. For each clause $c = (x \vee x') \in C$, without loss of generality assume that $cx_1, cx'_1 \in E(H_\Phi)$. Merge the three vertices c, x_1, x'_1 into a new vertex c'.

Call the resultant graph G_Φ. The degree of every vertex in the graph G_Φ is 1, 2 or 3 and the resultant graph is bipartite. Let us now prove that $\overleftrightarrow{\chi}(G_\Phi) \leq 1$ if and only if Φ has a Not-All-Equal truth assignment.

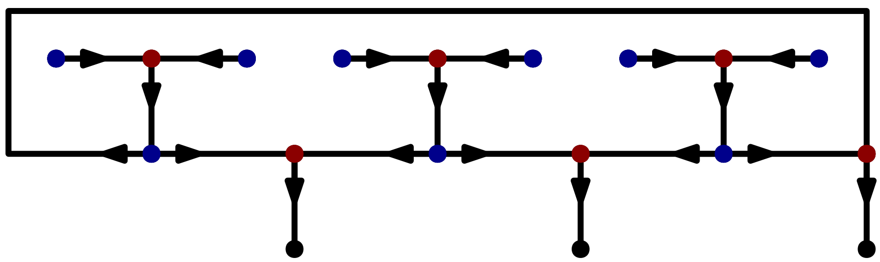
First, assume that $\overleftrightarrow{\chi}(G_\Phi) \leq 1$. We have the following properties.

Proposition 9. *Consider the gadget I_x which is shown in Fig. 3. Let D be an orientation of I_x such that the in-out-degree of each vertex is in $\{0, \pm 1\}$ and the endvertices of any edge in I_x, except three edges incident with the vertices x_1, x_2, x_3, have different in-out-degrees, then $d^{\pm}_D(x_1) = d^{\pm}_D(x_2) = d^{\pm}_D(x_3) = 1$ or $d^{\pm}_D(x_1) = d^{\pm}_D(x_2) = d^{\pm}_D(x_3) = -1$.*

Proof. Note that in the proof of this proposition the notation and colors that we refer are depicted in Fig. 3. In the orientation D the in-out-degree of each vertex is in $\{0, \pm 1\}$. On the other hand, the degree of each vertex is one or three, so the in-out-degree of each vertex is in $\{\pm 1\}$. In orientation D the endvertices of any edge, except three edges incident with the vertices x_1, x_2, x_3, have different in-out-degrees. Thus, the red vertices have the same in-out-degree and also the blue vertices have the same in-out-degree. Now, two cases can be considered.
Case 1. The blue vertices have the in-out-degree 1. Then the red vertices have the in-out-degree -1. We



Type 1



Type 2

Figure 4: The two possible orientations of I_x.

have $d^{\pm}_D(v_1) = d^{\pm}_D(v_2) = 1$, so the edges v_1u_1, v_2u_1 were oriented form u_1 to v_1 and v_2, respectively. The in-out-degree of u_1 is -1. Thus, the edge z_1u_1 was oriented from z_1 to u_1. The in-out-degree of z_1 is 1 and thus the edge $wz_1$was oriented from w to z_1. We have the same situation for z_2. Its in-out-degree is 1 and the edge wz_2 was oriented from w to z_2. The vertex w is a red vertex and its in-out-degree is -1. On the other hand, the edges wz_1, wz_2 were oriented from w to z_1 and z_2. Thus, the edge wx_2 was oriented from w to x_2 and consequently $d^{\pm}_D(x_2) = 1$. We have the same conclusion for x_1 and x_3. Thus, $d^{\pm}_D(x_1) = d^{\pm}_D(x_2) = d^{\pm}_D(x_3) = 1$.
Case 2. The blue vertices have the in-out-degree -1. Then the red vertices have the in-out-degree 1. Similar to Case 1, we can show that $d^{\pm}_D(x_1) = d^{\pm}_D(x_2) = d^{\pm}_D(x_3) = -1$. This completes the proof. ■

Let D be an optimal in-out-proper orientation of G_Φ. Now, we present a Not-All-Equal truth assignment for the formula Φ. Let $\Gamma : X \to \{\mathsf{true}, \mathsf{false}\}$ be the assignment defined by $\Gamma(x_i) = \mathsf{true}$ if the blue vertices in I_x have the in-out-degree 1, and $\Gamma(x_i) = \mathsf{false}$ if the blue vertices in I_x have the in-out-degree -1.

Next, we prove that Γ is a Not-All-Equal truth assignment for Φ. Let $c = (x \vee y \vee r)$ and without loss of generality assume that $cx_1, cy_1, cr_1 \in E(G_\Phi)$. The degree of the vertex c is three, so $d^{\pm}_D(c) \in \{\pm 1\}$. Thus, at least one of the edges incident with c was oriented from c to the other endpoint. Note that the other endpoint is one of the vertices x_1, y_1, r_1. Also, at least one of the edges incident with c was oriented toward c. On the other hand, the degree of vertices x_1, y_1, r_1 are two, so $d^{\pm}_D(x_1) = d^{\pm}_D(y_1) = d^{\pm}_D(r_1) = 0$. Thus, $\mathsf{true}, \mathsf{false} \in \{\Gamma(x), \Gamma(y), \Gamma(r)\}$. Next, assume that $c = (x \vee y)$. The degree of the vertex c' (that corresponds to the clause c in C) is two. So, $d^{\pm}_D(c') = 0$. Thus, $\mathsf{true}, \mathsf{false} \in \{\Gamma(x), \Gamma(y)\}$.

Now, assume that there is a Not-All-Equal assignment $\Gamma : X \to \{\mathsf{true}, \mathsf{false}\}$ for Φ. For each variable $x \in X$ if $\Gamma(x) = \mathsf{true}$ then orient I_x like Type 2 in Fig. 4 and if $\Gamma(x) = \mathsf{false}$ then orient I_x like Type 1 in Fig. 4. Also, for each clause $c = (x \vee y \vee r)$ orient the edges incident with c such that the in-out-degree of each neighbor of c is 0. Call the resultant orientation D. The function Γ is a Not-All-Equal assignment, so D is an in-out-proper orientation such that the maximum of absolute values of their in-out-degrees is one.

This completes the proof.

$\square$

6 Regular graphs

Next, we study the computational complexity of determining the in-out-proper orientation number of 4-regular graphs.

Proof of Theorem 5. It was shown that it is NP-complete to determine whether the edge chromatic number of a given 3-regular graph is three (see [14]). We reduce this problem to our problem in polynomial time. For a given 3-regular graph G we construct a 4-regular graph H such that the edge chromatic number of G is three if and only if $\overleftrightarrow{\chi}(H) \leq 2$.

For a given graph G with the set of edges $e_1, e_2, \ldots, e_n$, let H be the line graph of G with the set of vertices $v_{e_1}, v_{e_2}, \ldots, v_{e_n}$, such that $v_{e_i}v_{e_j} \in E(H)$ if and only if e_i and e_j have a common endvertex. First, assume that the in-out-proper orientation number of H is two and let D be an optimal in-out-proper orientation. The orientation D defines a proper vertex 3-coloring for the vertices of H using three colors $0, \pm 2$. Thus, G has a proper edge 3-coloring.

Next, assume that the edge chromatic number of G is three and let $f : E(G) \to \{1, 2, 3\}$ be a proper edge 3-coloring of G. Define the function $h : V(H) \to \{1, 2, 3\}$ such that $h(e_{v_i}) = k$ if and only if $f(e_i) = k$, for each $k = 1, 2, 3$. Let K be the subset of edges of H such that for each edge $v_{e_i}v_{e_j} \in K$ we have $\{h(v_{e_i}), h(v_{e_j})\} = \{1, 3\}$. In the subgraph $H \setminus K$ the degree of each vertex is even. In fact the degree of each vertex v_{e_i} with $h(v_{e_i}) = 2$ is four and also the degree of each vertex v_{e_i} with $h(v_{e_i}) \in \{1, 3\}$ is two. So we can orient the edges in $H \setminus K$ such that the in-degree of each vertex is equal to its out-degree. Next, for each edge $v_{e_i}v_{e_j} \in K$ orient it from v_{e_i} to v_{e_j} if $h(v_{e_i}) = 1$ and $h(v_{e_j}) = 3$, otherwise orient it from v_{e_j} to v_{e_i}. Consider the union of orientations for $H \setminus K$ and K and call the resultant orientation D. In D the in-out-degree of each vertex v_{e_i} with $h(v_{e_i}) = 1$ ($h(v_{e_i}) = 2$, $h(v_{e_i}) = 3$, respectively) is -2 $(0, 2$, respectively). Thus, D is an in-out-proper orientation such that the maximum of absolute values of their in-out-degree is two. This completes the proof. $\square$

7 Conclusions and future research

In this work we studied the in-out-proper orientation number of graphs. We proved that for any graph G, $\overleftrightarrow{\chi}(G) \leq \Delta(G)$. We conjectured that there exists a constant number c such that for every planar graph G, we have $\overleftrightarrow{\chi}(G) \leq c$. Regarding this conjecture, we showed that for every tree T we have $\overleftrightarrow{\chi}(T) \leq 3$ and this bound is sharp. It is interesting to prove constant bounds for other families of planar graphs.

We also studied the in-out-proper orientation number of subcubic graphs. By using the properties of totally unimodular matrices we proved that there is a polynomial time algorithm to determine whether $\overleftrightarrow{\chi}(G) \leq 2$, for a given graph G with maximum degree three. It is interesting to present a polynomial time algorithm for other families of graphs.

It is also interesting to characterize all graphs G which satisfy $\overrightarrow{\chi}(G) = \overleftrightarrow{\chi}(G)$. It would be interesting to attack this problem for the family of regular graphs.

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