Upper bounds for stabbing simplices by a line

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Abstract

It is known that for every dimension $d \geq 2$ and every k < d there exists a constant $c_{d,k} > 0$ such that for every n-point set $X \subset \mathbb{R}^d$ there exists a k-flat that intersects at least $c_{d,k}n^{d+1-k} - o(n^{d+1-k})$ of the (d-k)-dimensional simplices spanned by X. However, the optimal values of the constants $c_{d,k}$ are mostly unknown. The case k=0 (stabbing by a point) has received a great deal of attention.

In this paper we focus on the case k=1 (stabbing by a line). Specifically, we try to determine the upper bounds yielded by two point sets, known as the *stretched grid* and the *stretched diagonal*. Even though the calculations are independent of n, they are still very complicated, so we resort to analytical and numerical software methods. We provide strong evidence that, surprisingly, for d=4,5,6 the stretched grid yields better bounds than the stretched diagonal (unlike for all cases k=0 and for the case (d,k)=(3,1), in which both point sets yield the same bound). Our experiments indicate that the stretched grid yields $c_{4,1} \leq 0.00457936$, $c_{5,1} \leq 0.000405335$, and $c_{6,1} \leq 0.0000291323$.

1 Introduction

A k-dimensional simplex is the convex hull of k+1 affinely independent points in \mathbb{R}^d , $d \geq k$. The k+1 points are said to span the simplex. The following result was proven for the planar case by Boros and Füredi [9] and for arbitrary dimension by Bárány [5]: For every $d \geq 2$ there exists a constant $c_d > 0$ such that for every n, if X is any n-point set in \mathbb{R}^d in general position, then there exists a point x in \mathbb{R}^d contained in at least $c_d n^{d+1} - o(n^{d+1})$ full-dimensional simplices spanned by X, where $c_d > 0$ is a constant depending only on d. Matoušek [22] called this result the First Selection Lemma. It can be used to construct so-called weak ε -nets (see [22]).

The problem of determining largest possible values of the constants c_d has sparked a lot of interest. For the planar case, Boros and Füredi [9] showed that $c_2 \geq 1/27$. For arbitrary dimension, Bárány [5] proved that $c_d \geq 1/((d+1)!(d+1)^d)$. Wagner [26] subsequently improved this lower bound to $c_d \geq (d^2+1)/((d+1)!(d+1)^{d+1})$. In particular, $c_3 \geq 0.001627$. Basit et al. [7] then improved the bound for c_3 to $c_3 \geq 0.00227$.

Later, Gromov improved the general lower bound to $c_d \geq 2d/((d+1)!^2(d+1))$ [14] (see simpler expositions of this result by Karasev [16] and Jiang [15]). This is an improvement by roughly a factor of e^d over the previous bound. In particular, $c_3 \geq 0.002604$. Matoušek and Wagner [21] then showed that $c_3 \geq 0.00263$. Later, Král et al. [19] slightly improved Gromov's bound for general d, yielding in particular $c_3 \geq (3 - \sqrt{2})/512 \simeq 0.00309$.

Regarding upper bounds, Kárteszi [17] (for d=2) and Bárány [5] (for general d) proved that if X is any point set in general position in \mathbb{R}^d , then no point in \mathbb{R}^d is contained in more than

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 $n^{d+1}/(2^d(d+1)!) + O(n^d)$ simplices spanned by X. Hence, $c_d \leq 1/(2^d(d+1)!)$. This upper bound is "trivial" in the sense that it does not rely any specific construction for X.

Bukh et al. obtained the first "non-trivial" upper bounds, by constructing, for every n and d, a specific point set $X \subset \mathbb{R}^d$ that witnesses $c_d \leq (d+1)^{-(d+1)}$ [10]. These are the best upper bounds currently known. The set X is the so-called *stretched diagonal* (presented below). Another point set, called the *stretched grid* [11] (also presented below) gives the same upper bound.

Hence, $c_2 = 1/27$ is tight, and $c_3 \le 0.0039$. Thus, for $d \ge 3$ the optimal value of c_d is not known, and there is a gap of a factor of roughly d^d between the best lower and upper bounds.

(Some authors prefer to talk about the constant c'_d such that there exists a point in at least $c'_d\binom{n}{d+1} - O(n^d)$ simplices. Then the relation between the two constants is that $c'_d = c_d \cdot (d+1)!$.)

1.1 Generalization of the First Selection Lemma

The First Selection Lemma can be generalized as follows. If $X \subseteq \mathbb{R}^d$ is an n-point set in general position, and k is an integer, $0 \le k < d$, then there exists a k-flat that intersects at least $c_{d,k}n^{d-k+1} - O(n^{d-k})$ of the (d-k)-dimensional simplices spanned by X, for some positive constants $c_{d,k}$ that depend only on d and k. A trivial projection argument yields $c_{d,k} \ge c_{d-k}$. The problem of determining the maximum values of the constants $c_{d,k}$ was raised by Bukh et al. [10].

1.2 Lower and upper bounds for $c_{d,k}$

The case k = d - 1 is trivial: An optimal hyperplane is one that partitions the given point set into two equal parts. Hence, $c_{d,d-1} = 1/4$.

By a simple projection argument, the above-mentioned result of Kártesi and Bárány yields the "trivial" upper bound of $c_{d,k} \leq 1/(2^{d-k}(d-k+1)!)$.

For the case k = d - 2, it was shown in [10] that there exists a (d - 2)-flat that stabs at least $c_{d,d-2}n^3 - O(n^2)$ of the triangles spanned by X, with

$$c_{d,d-2} \ge \frac{1}{24} \Big(1 - \frac{1}{(2d-1)^2} \Big).$$

In particular, for d=3 there always exists a line that stabs at least $n^3/25 - O(n^2)$ triangles.

For the case (d, k) = (3, 1), Bukh claimed without providing details that in the *stretched grid* every line stabs at most $n^3/25 + o(n^3)$ triangles, and therefore $c_{3,1} = 1/25$ is tight (this is mentioned in [23]).

1.3 Related problems

Ashok, Rajgopal and Govindarajan [3] studied variants of the First Selection Lemma for other classes of geometric objects, such as spheres and axis-parallel boxes in \mathbb{R}^d , and quadrants and slabs in the plane. They also considered a *strong* variant of the First Selection Lemma, where the piercing point must come from the point set itself.

The Second Selection Lemma is a generalization of the First Selection Lemma. It states that for every n, if X is an n-point set in \mathbb{R}^d and F is a family of $\alpha\binom{n}{d+1}$ X-simplices, then there exists a point contained in at least $b_d \alpha^{s_d} \binom{n}{d+1}$ simplices of F, for some constants $b_d > 0$ and s_d .

The Second Selection Lemma was conjectured, and proved in the planar case, by Bárány, Füredi and Lovász [6] (see also Matoušek [22]). A proof for the planar case by a different technique, with considerably better quantitative bounds, was given by Aronov et al. [2]. This bound was then slightly improved by Eppstein, Nivasch, and Sharir [13, 24]. The full proof of the Second Selection

Lemma for arbitrary dimension was put together by Bárány et al. [6], Alon et al. [1], and Živaljević and Vrećica [27].

The Second Selection Lemma has been used to bound the number of k-sets in arbitrary dimension, where a k-set of a point set X is a subset of X of size k that can be separated from the rest of X by a hyperplane.

Several variants of the Second Selection Lemma, involving geometric objects other than simplices, were proved by Chazelle et al. [12], Sharir and Smorodinsky [25], and Ashok et al. [3].

A similar problem, of *centerline depth*, has been studied by Magazinov and Pór [20] and Blago-jević, Karasev, and Magazinov [8].

1.4 Our results

In this work we try to determine the upper bounds for the constants $c_{d,1}$ given by the stretched grid and the stretched diagonal.

For d=3, we find that both point sets yield $c_{3,1} \leq 1/25$ according to analytical software methods (as Bukh had already claimed for the stretched grid). Surprisingly, however, for $4 \leq d \leq 6$ we find very strong numerical evidence that the stretched grid yields a better bound than the stretched diagonal: On the one hand, for the stretched diagonal there always exists a line that stabs at least $n^d/(d+2)^{d-1}-o(n^d)$ simplices. On the other hand, the stretched grid likely yields $c_{4,1} \leq 0.00457936$, $c_{5,1} \leq 0.000405335$, and $c_{6,1} \leq 0.0000291323$, according to non-rigorous numerical optimization methods.

Organization of this paper. Section 2 reviews the stretched grid and the stretched diagonal, as well as *stair-convexity*, the framework used to analyze them. Section 3 presents our results regarding the stretched grid. Section 4 presents our results regarding the stretched diagonal. We conclude with some remarks in Section 5.

2 Stair-convexity

Following Bukh et al. [11] we define the *stretched grid* as an axis-parallel grid of points in \mathbb{R}^d where, in each axis direction i, $2 \leq i \leq d$, the spacing between consecutive "layers" increases rapidly, and furthermore, the rate of increase for direction i is much larger than that for direction i-1. To simplify calculations, we make the coordinates increase rapidly also in the first direction. We denote the stretched grid by G_s . Hence, $G_s(n^d) \subseteq \mathbb{R}^d$ is of the form $G_s(n^d) = X_1 \times \ldots \times X_d$ where each $X_i \subseteq \mathbb{R}$ is of the form $X_i = \{x_{i,1}, \ldots, x_{i,n}\}$, where $x_{i,1} < x_{i,2} < \cdots < x_{i,n}$ is the ith axis that contains n points. We define the sets X_i by induction on i, together with relations \ll_i on \mathbb{R} , which describe "at least how fast" the terms in X_i must grow (but we will also use \ll_i for comparing real numbers other than the members of X_i). We start by letting $x \ll_1 y$ mean $K_1 x \leq y$, where $K_1 = 2^d$. Then we choose X_1 so that $x_{1,1} = 1$ and $x_{1,1} \ll_1 x_{1,2} \ll_1 \cdots \ll_1 x_{1,n}$. Having defined X_{i-1} and \ll_{i-1} , we set $K_i = 2^d x_{(i-1),n}$, we define $x \ll_i y$ to mean $K_i x \leq y$, and we choose X_i so that $x_{i,1} = 1$ and $x_{i,1} \ll_i x_{i,2} \ll_i \cdots \ll_i x_{i,n}$.

The *stretched diagonal* is the following subset of the stretched grid:

$$D_s(n) = \{(x_{1,j}, \dots, x_{d,j}) \in \mathbb{R}^d : j = 1, 2, \dots, n\}.$$

In other words, the jth point of the stretched diagonal is built from the jth element of each of X_1, \ldots, X_d .

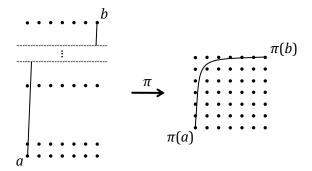


Figure 1: The stretched grid and the mapping π .

Define the uniform grid in the unit cube $[0,1]^d$ by

$$G_u = G_u(n^d) = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-1}{n-1}\right\}^d.$$

Let $BB(G_s) = [1, x_{1,n}] \times [1, x_{2,n}] \times \cdots \times [1, x_{d,n}]$ be the bounding box of G_s , and let $\pi : BB(G_s) \to [0, 1]^d$ be a bijection that maps G_s onto G_u and preserves ordering in each coordinate (that is, we map points of G_s to the corresponding points of G_u and we squeeze the "elementary boxes" of G_s onto the corresponding elementary boxes of G_u). See Figure 1. Let us consider the effect of π on a straight-line segment u = ab connecting two grid points $a, b \in G_s$. Suppose without loss of generality that $b_d \geq a_d$. Since G_s is so much more stretched in the dth direction than in all the previous directions, $\pi(u)$ ascends in the dth direction from $\pi(a)$, reaching almost the height of $\pi(b)$, before moving significantly in any other direction. From there on, we can continue tracing $\pi(u)$ by induction on d. This observation motivates the notion of stair-convexity.

2.1 Stair-convexity

Given a pair of points $a, b \in \mathbb{R}^d$, define the *stair-path* $\sigma(a, b)$ between them as a polygonal path connecting a and b and consisting of at most d closed line segments, each parallel to one of the coordinate axes. The definition goes by induction on d; for d = 1, the stair-path $\sigma(a, b)$ is simply the segment ab. For $d \geq 2$, after possibly interchanging a and b, let us assume $a_d \leq b_d$. We set $a' = (a_1, \ldots, a_d - 1, b_d)$, and we let $\sigma(a, b)$ be the union of the segment aa' and the stair-path $\sigma(a', b)$; for the latter we use induction, ignoring the common last coordinate of a' and b.

We call a set $S \subseteq \mathbb{R}^d$ stair-convex if for every $a, b \in S$ we have $\sigma(a, b) \subseteq S$. We define the stair-convex hull of a set $S \subseteq \mathbb{R}^d$ as the intersection of all stair-convex sets containing S, and we will denote it as stconv(S).

2.2 Intersection of stair-convex hulls of two sets

In stair-convexity we will call the last coordinate of a point its "height". For a real number y, let h(y) denote the horizontal hyperplane $\{x \in \mathbb{R}^d : x_d = y\}$. For a horizontal hyperplane h = h(y), let $h^+ = \{x \in \mathbb{R}^d : x_d \geq y\}$ be the upper closed half-space bounded by h, and similarly let h^- be the lower closed half-space. For a set $S \subseteq \mathbb{R}^d$, let $S(y) = S \cap h(y)$ be the horizontal slice of S at height y. For a point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let $\overline{x} = (x_1, \dots, x_{d-1})$ be the projection of x into \mathbb{R}^{d-1} , and define \overline{S} for $S \subset \mathbb{R}^d$ similarly. For a point $x \in \mathbb{R}^{d-1}$ and a real number x_d , let $x \times x_d = (x_1, \dots, x_{d-1}, x_d)$.

Lemma 1 ([11]). A set $S \subseteq \mathbb{R}^d$ is stair-convex if and only if the following two conditions hold:

- 1. Every horizontal slice $\overline{S(y)}$ is stair-convex.
- 2. For every $y_1 \leq y_2 \leq y_3$ such that $\overline{S(y_3)} \neq \emptyset$ we have $\overline{S(y_1)} \subseteq \overline{S(y_2)}$. (Meaning, the horizontal slice can only grow with increasing height, except that it can end by disappearing abruptly).

Lemma 2 ([11]). The stair-convex hull of a set $X \subseteq \mathbb{R}^d$ can be (recursively) characterized as follows. For every horizontal hyperplane h = h(y) that does not lie entirely above X, let X' stand for the vertical projection of $X \cap h^-$ into h. Then $h \cap \operatorname{stconv}(X) = \operatorname{stconv}(X')$ (where $\operatorname{stconv}(X')$ is a stair-convex hull in dimension d-1).

The following lemma specifies under which conditions the stair-convex hulls of two sets intersect. Recall that in standard geometry, Kirchberger's theorem [18] states that if Y and Z are point sets in \mathbb{R}^d such that $\operatorname{conv}(Y)$ and $\operatorname{conv}(Z)$ intersect, then there exist subsets $Y' \subseteq Y$ and $Z' \subseteq Z$ of total size $|Y| + |Z| \le d + 2$ such that $\operatorname{conv}(Y)$ and $\operatorname{conv}(Z)$ intersect.

Lemma 3 ([11]). Let $Y, Z \subset \mathbb{R}^d$ be two finite point sets that do not share any coordinate, with |Y| = s and |Z| = t. Then:

- 1. If s + t < d + 2, then stconv(Y) and stconv(Z) do not intersect.
- 2. If s+t=d+2 and $\operatorname{stconv}(Y)$, $\operatorname{stconv}(Z)$ intersect, then they do so at a single point. Suppose they do intersect. Then the two highest points of $Y \cup Z$ (in last coordinate) belong one to Y and one to Z. Furthermore, let $y_{\operatorname{top}}, z_{\operatorname{top}}$ be the highest points of Y, Z respectively, and say $y_{\operatorname{top},d} > z_{\operatorname{top},d}$. Then the point of intersection between $\operatorname{stconv}(Y)$, $\operatorname{stconv}(Z)$ is $p=q\times z_{\operatorname{top},d}$, where $q\in\mathbb{R}^{d-1}$ is the point of intersection of $\operatorname{stconv}(\overline{Y}\setminus\{y_{\operatorname{top}}\})$ and $\operatorname{stconv}(\overline{Z})$.
- 3. If s+t>d+2 and $\operatorname{stconv}(Z),\operatorname{stconv}(Y)$ intersect, then there exist subsets $Z'\subseteq Z,Y'\subseteq Y$ of total size |Z'|+|Y'|=d+2, such that $\operatorname{stconv}(Z'),\operatorname{stconv}(Y')$ intersect.

The special case |Z|=1 of Lemma 3 is important enough to be stated separately. Let $a,b \in \mathbb{R}^d$ be two points that do not share any coordinate. We say that b has $type\ 0$ with respect to a if $b_i < a_i$ for every $i=1,2,\ldots,d$. For $j\in\{1,2,\ldots,d\}$ we say that b has $type\ j$ with respect to a if $b_j>a_j$ but $b_i< a_i$ for all $i=j+1,\ldots,d$.

Lemma 4 ([11]). Let $X \subseteq \mathbb{R}^d$ be a point set, and let $a \in \mathbb{R}^d$ be a point. Then $a \in \operatorname{stconv}(X)$ if and only if X contains a point of type j with respect to a for every $j = 0, 1, \ldots, d$.

Transference Lemma. The almost-correspondence between convex hulls and stair-convex hulls in the stretched grid is formalized in the following lemma. Let us say that two points $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in BB(G_s) are far apart if, for every $i = 1, 2, \ldots, d$, we have either $a_i \ll_i b_i$ or $b_i \ll_i a_i$. We also extend this notion to sets: Two sets $Y, Z \subseteq \mathbb{R}^d$ are far apart if each $z \in Z$ is far apart from each $y \in Y$.

Lemma 5 ([11]). Let Y, Z be sets in $BB(G_s)$ that are far apart. Then $stconv(Y) \cap stconv(Z) = \emptyset$ if and only if $conv(Y) \cap conv(Z) = \emptyset$.

If a small set $Y \subset BB(G_s)$ is given, not necessarily from the stretched grid, and we consider all possible fixed-size sets $Z \subset G_s$, then almost all such sets Z will be far apart from Y, except for a negligible fraction of them. What will interest us is whether conv(Z) and conv(Y) intersect. So, according to the lemma above, in the vast majority of cases it is enough to check whether stconv(Z) and stconv(Y) intersect.

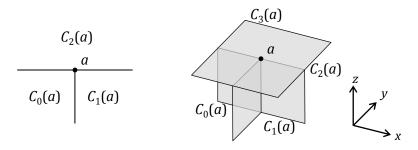


Figure 2: Partition of space into d + 1 parts.

2.3 Warm-up: Upper bounds for the First Selection Lemma

As a warm-up, we recall the proof that both the stretched grid and the stretched diagonal yield the upper bound $c_d \leq (d+1)^{-(d+1)}$ for the First Selection Lemma.

Let X be either the stretched grid or the stretched diagonal, and let $a \in \mathbb{R}^d$ be a point. For each $0 \le j \le d$, let $C_j(a)$ be the set of all points $b \in X$ that have type j with respect to a. By the Transference Lemma and Lemma 4, the number of full-dimensional simplices spanned by X that contain a is very close to the product $\prod_{j=0}^d |C_j(a)|$. By the arithmetic-geometric mean inequality, this expression achieves its maximum when all terms have the same size, namely $|C_j(a)| = |X|/(d+1)$ for each j. The claim follows.

3 Results for the stretched grid

In this section we derive the upper bounds for $c_{d,1}$ yielded by the stretched grid.

3.1 A recursive formula

Let $q, p \in [0, 1]^d$ be two given points with $q = (q_1, \ldots, q_d), p = (p_1, \ldots, p_d)$. Informally, we want to define the probability $\operatorname{RecFSG}_d(q, p)$ that a randomly chosen (d-1)-dimensional stair-simplex from $[0, 1]^d$ intersects the stair-path $\sigma(q, p)$. Formally, let \mathcal{D} be the uniform distribution in $([0, 1]^d)^d$. Every element $A \in ([0, 1]^d)^d$ represents a d-tuple z_1, \ldots, z_d of points in $[0, 1]^d$ which span the stair-simplex $S(A) = \operatorname{stconv}\{z_1, \ldots, z_d\}$. Then define $\operatorname{RecFSG}_d(q, p)$ as the measure

$$\operatorname{RecFSG}_d(q,p) = \mu[A \in \mathcal{D} : S(A) \cap \sigma(q,p) \neq \emptyset].$$

The connection between RecFSG_d and the stretched grid is as follows. Given q, p as above, let $\alpha = \operatorname{RecFSG}_d(q, p)$. Given n, let $G_s(n) = X_1 \times \cdots \times X_d$ be the d-dimensional stretched grid of dimensions $m \times \cdots \times m$ with $m = n^{1/d}$, where $X_i = \{x_{i,1}, \dots, x_{i,m}\}$ for each i. If $q' = (q'_1, \dots, q'_d)$, $p' = (p'_1, \dots, p'_d)$ are points satisfying $x_{i,q_im} \leq q'_i \leq x_{i,q_im+1}$ and $x_{i,p_im} \leq p'_i \leq x_{i,p_im+1}$, then the stair-path $\sigma(q', p')$ intersects an $(\alpha \pm o(1))$ -fraction of the (d-1)-dimensional stair-simplices spanned by $G_s(n)$. Let $Y = \{q, p\}$, and let Z be the set of vertices of a stair-simplex. The fraction of stair-simplices for which Y, Z are not far apart is negligible as $n \to \infty$. Hence, by the Transference Lemma (Lemma 5), the segment qp also intersects an $(\alpha \pm o(1))$ -fraction of the (d-1)-dimensional simplices spanned by $G_s(n)$. Since the number of simplices spanned by $G_s(n)$ is $\binom{n}{d} = n^d/d! - O(n^{d-1})$, the constant that multiplies n^d is $\operatorname{RecFSG}_d(q, p)/d!$.

Types of stair-paths. We define the type T of a stair-path qp by $T = \{j : q_j < p_j\}$ (we can safely ignore cases where $p_j = q_j$ for some j, since they have measure 0). In dimension d there are 2^d possible types of stair-paths, but half of them are equivalent to the other half since p and q just switch positions. This leaves us with 2^{d-1} possible types of stair-paths.

In addition, stair-convexity is symmetric with respect to the first coordinate, and so is the stretched grid. Therefore, without loss of generality we can assume that $1 \notin T$. This leaves us with 2^{d-2} possible types of stair-paths.

Theorem 6. Let $q, p \in [0, 1]^d$ be two points, with $q = (q_1, \ldots, q_d), p = (p_1, \ldots, p_d)$. If $p_d \ge q_d$ let x = p, y = q; otherwise, let x = q, y = p. Then RecFSG_d is given by:

$$\operatorname{RecFSG}_{1}(q, p) = x_{1} - y_{1},$$

$$\operatorname{RecFSG}_{d}(q, p) = d!(x_{d}^{d} - y_{d}^{d}) \prod_{i=1}^{d-1} y_{i}^{i}(1 - y_{i})$$

$$+ d(1 - x_{d})x_{d}^{d-1}\operatorname{RecFSG}_{d-1}(\{\overline{q}, \overline{p}\}), \quad \text{for } d \geq 2.$$

Proof. Let $Y = \{q, p\}$, and let $Z = \{z_1, z_2, \dots, z_d\} \subset [0, 1]^d$ such that Y, Z do not share any coordinate, where $|Y \cup Z| = d + 2$. Hence, $\operatorname{stconv}(Y)$ is a stair-path and $\operatorname{stconv}(Z)$ is a (d-1)-stair-simplex. By Lemma 3 part (2), $\operatorname{stconv}(Y)$ and $\operatorname{stconv}(Z)$ will intersect in at most one point, and if they do intersect, then after projecting the d+1 lower points to dimension d-1, there will also be an intersection point.

Base case: When d=1, the path is of type $T=\emptyset$ which means $q_1 \geq p_1$. The measure of simplices in \mathcal{D} whose single point lies between them is q_1-p_1 .

Recursive case: The recursive function is built out of a two-part addition: a non-recursive part that we get when the highest point belongs to Y, and a recursive part that we get when the highest point belongs to Z. The first part is derived as follows. Let $p = y_{\text{top}}$ be the highest of all points; see Figure 3(a). All the points of Z must be below p_d but not all of them should be below q_d . The measure of simplices in \mathcal{D} with this property is $p_d^d - q_d^d$. Now, when projecting to the lower dimension, d-1, we "discard" the highest point p, and stay with \overline{Z} and the point \overline{q} . So, it remains to calculate the measure of simplices stconv(\overline{Z}) that intersect the point \overline{q} . Since $z_{\text{top}} \in \overline{Z}$ should be above q_{d-1} and the other d-1 points should be below it, this occurs with measure $d(1-q_{d-1})q_{d-1}^{d-1}$. Let us again project to a lower dimension and "discard" the highest point. We are left with a simplex in a lower dimension, stconv $(\overline{Z} \setminus \{z_{\text{top}}\})$, and the point \overline{q} . We continue this way until we reach d=1. Hence, for the first part we get the term

$$(p_d^d - q_d^d)d! \prod_{i=1}^{d-1} q_i^i (1 - q_i).$$

The second part is derived as follows: Let $q_d \leq p_d \leq z_{\text{top},d}$. The point $z_{\text{top}} \in Z$ must be above p_d , while the other d-1 points of Z must be below it; see Figure 3(b, c). This happens with measure $d(1-p_d)p_d^{d-1}$. When we "discard" the highest point, we are left with a stair-path and a stair-simplex one dimension lower. Therefore, we can recursively invoke RecFSG_{d-1} . Hence, for the second part we get the term

$$d(1-p_d)p_d^{d-1}\operatorname{RecFSG}_{d-1}(\overline{q},\overline{p}).$$

If $q_d \geq p_d$, all calculations are the same, except that we interchange q and p.

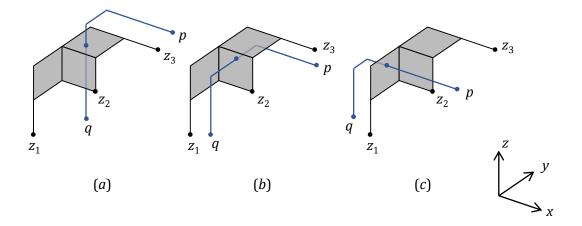


Figure 3: A stair-path stabbing a stair-triangle (3-vertex stair-simplex) in dimension 3.

3.2 Extending the stair-path to the boundary of the cube

Without loss of generality, we can extend a given stair-path qp until the two endpoints touch the boundary of the unit cube. This makes the calculations easier, since for each type T, there are two variables that can be set to 0 or 1.

Without loss of generality assume $d \notin T$, so $p_d \leq q_d$. Then p_d can be extended to 0. For q, the first coordinate that is elevated from q to p, namely $\max\{i: q_i \leq p_i\}$, is the one that can be extended to 0. If, on the other hand, $p_i \leq q_i$ for all i, then q_1 can be extended to 1.

Hence, given the type T of the stair-path qp, we proceed as follows: Say $d \notin T$ (otherwise, switch p and q and let T be $\{1,\ldots,d\}\setminus T$). Then we let $p_d=0$. In addition, if $T\neq\emptyset$ then we let $q_{\max T}=0$, while if $T=\emptyset$ then we let $q_1=1$. (As noted before, for the case of the stretched grid we can assume without loss of generality that $1\notin T$.)

3.3 Maximum for the stretched grid

Using the recursive function of Theorem 6 for the stretched grid, we get 2^{d-2} polynomial expressions in the coordinates $q_1, \ldots, q_d, p_1, \ldots, p_d$. We need to find the maximum for each expression.

3.3.1 Results for dimension 3

For d=3 there are two fundamentally different types: $T=\emptyset$ and $T=\{2\}$.

Type $T = \emptyset$. For this case, we can let $q_1 = 1, p_3 = 0$, so in this case the function we want to maximize is

$$F = \text{RecFSG}_3(q, p)/3! = (p_1 - 1)q_3^2 \cdot \left(p_1(p_2^2(1 - 2q_3) + q_2^2(q_3 - 1) + p_2^3q_3) + q_2(q_2 - 1 + q_3 - q_2q_3)\right)$$

in the domain

$$U = \{(p_1, p_2, q_2, q_3) \in \mathbb{R}^4 : 0 \le p_1 \le 1, 0 \le p_2 \le q_2 \le 1, 0 \le q_3 \le 1\}.$$

In order to find analytically the maximum of F in U, we have to examine the interior of U and its faces. U is a 4-dimensional polytope, which according to the software "polymake" [4] has 62 faces of various dimensions.

Alternatively, we could try using the function Maximize of Mathematica 11, which finds the absolute maximum of a given function in a given range. Unfortunately, when given F and U, Maximize does not terminate in a reasonable amount of time. Hence, we employ a hybrid approach, dealing with the interior of U by hand, and using Maximize for the seven facets of U.

Maximum in the interior. If the maximum is attained by a point (p_1, p_2, q_2, q_3) in the interior of U, then it must satisfy the following four equations:

$$\frac{dF}{dq_2} = (p_1 - 1) \underbrace{(1 + 2(p_1 - 1)q_2)(q_3 - 1)q_3^2}_{\text{II}} = 0,$$

$$\frac{dF}{dq_3} = (p_1 - 1)q_3$$

$$\underbrace{\cdot \left(-(q_2 - 1)q_2(3q_3 - 2) + p_1(p_2^2(2 - 6q_3) + 3p_2^3q_3 + q_2^2(3q_3 - 2)) \right)}_{\text{II}} = 0,$$

$$\frac{dF}{dp_1} = q_3^2$$

$$\underbrace{\cdot \left(q_2(1 + 2(p_1 - 1)q_2)(q_3 - 1) + (2p_1 - 1)p_2^3q_3 + p_2^2(-1 + p_1(2 - 4q_3) + 2q_3) \right)}_{\text{IV}} = 0,$$

$$\frac{dF}{dp_2} = (p_1 - 1)p_1p_2q_3^2 \underbrace{(2 + (3p_2 - 4)q_3)}_{\text{IV}} = 0.$$

The solutions that satisfy $\{q_3 = 0\}$, or $\{p_1 = 1\}$, or both $\{q_3 = 1\}$ and $\{p_1 = 0\}$, or both $\{q_3 = 1\}$ and $\{p_2 = 0\}$, are irrelevant, since they give F = 0. Therefore, any local maximum should satisfy the equations II = 0 and III = 0. One possibility is to satisfy $q_3 = 1$ and IV = 0. The other three possibilities are to satisfy the equation I = 0, as well as one of IV = 0, $\{p_1 = 0\}$, $\{p_2 = 0\}$. These are the relevant solutions to the system:

$$\{p_1 = (2q_2 - 1)/(2q_2), p_2 = 0, q_3 = 2/3\},\$$

$$\{p_1 = 0, p_2 = 0, q_2 = 1/2, q_3 = 2/3\},\$$

$$\{p_1 = 0, p_2 = 1/2, q_2 = 1/2, q_3 = 2/3\},\$$

$$\{p_1 = 1/2, p_2 = 1/2, q_2 = 1, q_3 = 4/5\},\$$

$$\{p_1 = 1/2, p_2 = 2/3, q_2 = 2/3, q_3 = 1\}.$$

After checking all these solutions, we get the maximum, 1/25 by the solution $\{p_1 = 1/2, p_2 = 1/2, q_2 = 1, q_3 = 4/5\}$.

Maximum on the facets. In order to find maximum on the facets of U, we used Maximize. The results can be seen in Table 1.

Type $T = \{2\}$. Here one can proceed similarly. Here the maximum is also 1/25, this time given by $qp = \{(2/3, 0, 4/5), (1/3, 3/4, 0)\}$. (Note that in this case, the maximum is in the interior of the domain, whereas in the case $T = \emptyset$ the maximum was on one of its facets.) The calculations can be found in the ancillary files of the arXiv version of this paper.

```
in: F=(p1-1)q3^2*(p1(p2^2(1-2q3)+q2^2(q3-1)+p2^3q3)
      +q2(q2-1+q3-q2 q3))
in: F1=F/.q2->1;
     Maximize [\{F1,0<q3<1,0<p1<1,0<p2<1\},\{q3,p1,p2\}]
out: {1/25,{q3->4/5,p1->1/2,p2->1/2}}
in: F2=F/.q2->p2;
     Maximize [\{F2,0<q3<1,0<p1<1,0<p2<1\},\{q3,p1,p2\}]
out: \{1/27, \{q3->64/81, p1->10/37, p2->37/64\}\}
in: F3=F/.p2->0;
     Maximize [\{F3,0<q3<1,0<p1<1,0<q2<1\},\{q3,p1,q2\}]
out: \{1/27, \{q3->2/3, p1->1/3, q2->3/4\}\}
in: F4=F/.p1->1;
     Maximize [{F4,0<q3<1,0<p2<q2<1},{q3,q2,p2}]
out: \{0, \{q3->1/2, q2->3/4, p2->1/4\}\}
in: F5=F/.p1->0;
     Maximize [\{F5,0<q3<1,0<p2<q2<1\},\{q3,q2,p2\}]
out: \{1/27, \{q3->2/3, q2->1/2, p2->1/4\}\}
in: F6=F/.q3->1;
     Maximize [{F6,0<p1<1,0<p2<q2<1},{p1,q2,p2}]
out: \{1/27, \{p1->1/2, q2->27/32, p2->2/3\}\}
in: F7=F/.q3->0;
     Maximize [\{F7,0<p1<1,0<p2<q2<1\},\{p1,q2,p2\}]
out: \{0, \{p1->1/2, q2->3/4, p2->1/4\}\}
```

Table 1: Maximum on faces, $T = \emptyset$

T	qp	maximum
Ø	$\{(1, 0.99973, 0.841676, 0.824961), (0.499854, 0.57138, 0.590885, 0)\}$	0.00456416
{2}	$\{(0.70017, 0, 0.841749, 0.824908), (0.400339, 0.714113, 0.590808, 0)\}$	0.00456416
{3}	$\{(0.666089, 0.777112, 0, 0.827549), (0.395765, 0.49188, 0.794824, 0)\}$	0.00457936
$\{2, 3\}$	$\{(0.604237, 0.491879, 0, 0.830208), \\ (0.333913, 0.77711, 0.792279, 0)\}$	0.00457936

Table 2: Results for dimension 4, stretched grid.

3.3.2 Results for dimensions 4, 5, and 6

For dimensions $d \geq 4$, the problem turns out to be too complex for the above approach. Therefore, we used the function NMaximize of Mathematica, which searches for the absolute maximum numerically. The function NMaximize provides four different numerical methods, called NelderMead, DifferentialEvolution, SimulatedAnnealing, and RandomSearch. We tried all four of them. In dimensions 4 and 5 they all gave the same results, though not in dimension 6.

In dimension 4 there are four fundamentally different types of stair-paths. Table 2 sums up the numerical results. The maximum among all the types is $0.00457936n^4$.

In dimension 5 there are eight different types of stair-paths. The maximum among all of them is $0.000405335n^5$. See Table 3.

In dimension 6 there are 16 different types of stair-paths. Here, not all maximization methods gave the same result. The method DifferentialEvolution gave the best results in all types. The maximum obtained is $0.0000291323n^6$, for types $T = \{2,3,5\}$ and $T = \{4,5\}$. For type $T = \{2,3,5\}$, the coordinates that give this maximum are

```
qp = \{(0.592993, 0.545248, 0.59284, 0.843717, 0, 0.869422), \\ (0.38511, 0.750149, 0.798446, 0.658605, 0.849763, 0)\}.
```

4 Results for the stretched diagonal

In this section we prove the following:

Theorem 7. For every $d \ge 3$ there exists a stair-path qp that stabs $n^d/(d+2)^{d-1} - o(n^d)$ stair-simplices spanned by the stretched diagonal $D_s(n)$.

Hence, for d = 4, 5, 6 the stretched diagonal yields worse bounds for $c_{d,1}$ than the stretched grid. In this section we also prove that for d = 3 the stretched diagonal yields $c_{3,1} \le 1/25$, just like the stretched grid.

4.1 A recursive formula for a special case

Let \mathcal{D} be the uniform distribution in $[0,1]^d$. Every element $A=(a_1,\ldots,a_d)\in[0,1]^d$ represents a d-tuple of points $\vec{a}_1,\ldots,\vec{a}_d$ where $\vec{a}_i=(a_i,\ldots,a_i)\in[0,1]^d$ for each i. These points span the stair-simplex $S(A)=\operatorname{stconv}\{\vec{a}_1,\ldots,\vec{a}_d\}$.

T	qp	Maximum
Ø	$\{(1, 0.999998, 0.863413, 0.850444, 0.848693),\\ (0.499999, 0.604701, 0.650764, 0.657374, 0)\}$	0.000402464
{2}	$\{(0.71664, 0, 0.863421, 0.850413, 0.848695), \\ (0.433377, 0.697668, 0.650744, 0.657406, 0)\}$	0.000402464
{3}	$\{(0.675465, 0.796913, 0, 0.850715, 0.848819), \\ (0.428888, 0.554061, 0.78136, 0.657527, 0)\}$	0.00040419
{4}	$\{(0.661946, 0.786912, 0.815446, 0, 0.850046), \\ (0.425712, 0.554894, 0.590175, 0.827888, 0)\}$	0.000404818
$\{2, 3, 4\}$	$\{(0.574368, 0.554597, 0.589951, 0, 0.853691),\\ (0.337538, 0.786894, 0.815268, 0.824237, 0)\}$	0.000404818
$\{2, 3\}$	$\{(0.630424, 0.478229, 0, 0.852643, 0.849618),\\ (0.387534, 0.738946, 0.779659, 0.658423, 0)\}$	0.000404815
$\{2,4\}$	$\{(0.622544, 0.495106, 0.817108, 0, 0.850792),\\ (0.388108, 0.740553, 0.590829, 0.826241, 0)\}$	0.000405335
${3,4}$	$\{(0.612094, 0.740651, 0.590451, 0, 0.852007),\\ (0.377364, 0.494832, 0.816924, 0.824642, 0)\}$	0.000405335

Table 3: Results for dimension 5, stretched grid.

Given two points $q, p \in [0, 1]^d$ with $q = (q_1, \ldots, q_d)$, $p = (p_1, \ldots, p_d)$, let FSD(q, p) be the measure of all the d-tuples $A = (a_1, \ldots, a_d) \in \mathcal{D}$ that satisfy the following two conditions:

- 1. $a_1 < \cdots < a_d$,
- 2. $\operatorname{stconv}\{q,p\} \cap \operatorname{stconv}S(A) \neq \emptyset$.

The connection between FSD and the stretched diagonal is as follows: Let n be large enough, and let $D_s(n)$ be the n-point stretched diagonal, and let q', p' be defined from q, p as before. Then the probability that a random stair-simplex spanned by $D_s(n)$ intersects the stair-path q'p' is very close to d!FSD(q, p), and hence, by the Transference Lemma, the number of simplices spanned by $D_s(n)$ that intersect the segment q'p' is FSD $(q, p)n^d$ plus lower-order terms.

We prove Theorem 7 by calculating FSD for a certain sub-type of stair-path that belongs to the type $T = \emptyset$. Specifically, we will calculate FSD(q, p) for the special case where the points $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ satisfy the following conditions:

$$p_1 \le p_2 \le \dots \le p_d$$
, $q_2 \le q_3 \le \dots \le q_d \le q_1 = 1$, $p_i \le q_i$ for all i . (1)

Let $p, q \in [0, 1]^d$ satisfy conditions (1), and let $Y = \{q, p\}$. We define $\text{RecFSD}_d(q, p)$ as the measure of (d-1)-tuples $a_1, \ldots, a_{d-1} \in [0, 1]$ satisfying the following two conditions:

- 1. $a_1 < a_2 < \cdots < a_{d-1} < q_d$
- 2. $\operatorname{stconv}(\overline{Y}) \cap \operatorname{stconv}(\overline{Z}) \neq \emptyset$, where $Z = \{\vec{a}_1, \dots, \vec{a}_{d-1}\}$.

(For p, q not satisfying conditions (1), $\operatorname{RecFSD}_d(q, p)$ is undefined.) Note that if p, q satisfy (1), then so do $\overline{p}, \overline{q}$.

Observation 8. Let $p, q \in [0, 1]^d$ satisfy conditions (1). Then,

$$FSD_d(q, p) = RecFSD_{d+1}(q \times 1, p \times 1).$$

Lemma 9. For $p, q \in [0, 1]^d$ satisfying conditions (1), RecFSD_d(q, p) is given recursively as follows:

$$\operatorname{RecFSD}_{2}(q, p) = q_{2} - p_{1},$$

$$\operatorname{RecFSD}_{d+1}(q, p) = (q_{d} - p_{d})p_{1} \prod_{i=2}^{d-1} (p_{i} - p_{i-1})$$

$$+ (q_{d+1} - q_{d})\operatorname{RecFSD}_{d}(\overline{q}, \overline{p}), \quad \text{for } d \geq 2.$$

For example, putting together Observation 8 and Lemma 9, we get that for p, q satisfying (1) we have

$$FSD_1(q, p) = 1 - p_1,$$

$$FSD_2(q, p) = (q_2 - p_2)p_1 + (1 - q_2)(q_2 - p_1),$$

$$FSD_3(q, p) = (q_3 - p_3)p_1(p_2 - p_1) + (1 - q_3)((q_2 - p_2)p_1 + (q_3 - q_2)(q_2 - p_1)).$$

Proof of Lemma 9. By induction on d.

Base case: When d=2, we need $a_1 < q_2$ and $p_1 \le a_1 \le q_1 = 1$. Therefore, we need $p_1 \le a_1 \le q_2$. The measure of numbers $a_1 \in [0,1]$ satisfying this condition is $q_2 - p_1$.

Recursive case: Suppose we are in dimension d+1, and let $Y=\{q,p\}$ with $q,p\in[0,1]^{d+1}$ and $Z=\{\vec{a}_1,\ldots,\vec{a}_d\}$ where $\vec{a}_i=(a_i,\ldots,a_i)\in[0,1]^{d+1}$. We need the numbers a_i to satisfy the following two separate conditions:

- 1. $a_d < q_{d+1}$.
- 2. $\operatorname{stconv}(\overline{Y}) \cap \operatorname{stconv}(\overline{Z}) \neq \emptyset$ with $a_1 < a_2 < \dots < a_d$.

Let us calculate the measure of tuples satisfying the second condition. As in the recursive formula for the stretched grid, here there are two possibilities, according to whether the highest point in dimension d is \overline{q} or $\overline{d_d}$. In the first case, we must have $a_d < q_d$ (which automatically implies $a_d < q_{d+1}$). In addition, by Lemma 3, we need to have $p_d < a_d$, and after "discarding" point q and projecting down to dimension d-1, we need to have $p' = \overline{\overline{p}} \in \operatorname{stconv}(\overline{\overline{Z}})$. For this, we apply Lemma 4. The set $\overline{\overline{Z}}$ must contain a point of type j with respect to p' for every $j=0,\ldots,d-1$. We also need the coordinates a_i to be in increasing order. Therefore, for type d-1, the (d-1)-st coordinate of \overline{a}_d should be higher than p_{d-1} , that is $p_{d-1} < a_d$ but since we demand $p_d < a_d$ this condition is irrelevant. For type d-2, the (d-2)-nd coordinate of \overline{a}_{d-1} should be higher than p_{d-2} , that is $p_{d-2} < a_{d-1}$. In addition, \overline{a}_{d-1} should be lower than p in the higher coordinates, and therefore $p_{d-2} < a_{d-1} < p_{d-1}$. And so on. In general, for every type $j=1,\ldots,d-2$ we must have $p_j < a_{j+1} < p_{j+1}$. For type 0, the lowest point \overline{a}_1 must satisfy $a_1 < p_1$. To sum up, the measure in the first case is

$$(q_d - p_d)p_1 \prod_{i=2}^{d-1} (p_i - p_{i-1}).$$

In the second case we must have $a_d > q_d$. Together with the condition $a_d < q_{d+1}$, this implies $q_d < a_d < q_{d+1}$. In addition, by Lemma 3, we need to have $a_{d-1} < q_d$, and after "discarding" point \vec{a}_d and projecting down to dimension d-1, we need to have an intersection between stcony $\{\overline{p}, \overline{q}\}$

and stconv $\{\overline{\overline{a_1}}, \dots, \overline{\overline{a_{d-1}}}\}$. But these are exactly conditions 1 and 2 above, one dimension lower. Therefore, the measure in the second case is

$$(q_{d+1} - q_d) \operatorname{RecFSD}_{d-1}(\overline{q}, \overline{p}).$$

Proof of Theorem 7. Let us take the following stair-path:

$$qp = \left\{ \left(1, \frac{3}{d+2}, \frac{4}{d+2}, \dots, \frac{d+1}{d+2}\right), \left(\frac{1}{d+2}, \frac{2}{d+2}, \dots, \frac{d-1}{d+2}, \frac{d-1}{d+2}\right) \right\}.$$

The points q, p satisfy conditions (1). Hence,

$$FSD_d(q, p) = RecFSD_{d+1}(q \times 1, p \times 1) = \frac{2}{(d+2)^d} + \frac{1}{d+2} \cdot RecFSD_d(q, p)$$

It follows by induction on d that, if $q', p' \in \mathbb{R}^d$ have the form $q' = (1, 3/c, 4/c, \dots, (d+1)/c)$ and $p' = (1/c, 2/c, \dots, (d-1)/c, k)$ for some c, k, then $\text{RecFSD}_d(q', p') = d/c^{d-1}$. Therefore, in our case, $\text{RecFSD}_d(q, p) = 1/(d+2)^{d-1}$. The claim follows.

4.2 Dimension 3

In order to find the maximum for the stretched diagonal in d=3, we examine all different possible types of stair-paths, each one having its own expression F and domain U. See Table 4. The expressions F can be derived from Lemma 3, as in previous sections, or they can be derived more directly as follows: Let $\vec{a}=(a,a,a)$, $\vec{b}=(b,b,b)$, $\vec{c}=(c,c,c)$, with 0 < a < b < c < 1, and consider the stair-simplex $S=\operatorname{stconv}(\vec{a},\vec{b},\vec{c})$. S contains three axis-parallel rectangles with different orientations: rectangle R_1 with opposite corners (a,a,b), (a,b,c), rectangle R_2 with opposite corners (a,b,b), (b,b,c), and rectangle R_3 with opposite corners (a,b,c), (b,c,c). Furthermore, the stair-path qp is composed of three axis-parallel segments with three different orientations. In order for the stair-path qp to intersect S, one of the former's segments must intersect one of the latter's rectangles. Hence, the numbers a,b,c must satisfy some inequalities depending on the coordinates q,p, which are not hard to work out.

We calculated the maximum in each case using Maximize. In contrast to the stretched grid, where the degree of F was 8, here the degree of F is only 3, so Maximize had no problem finding the maximum quickly. The maximum is $(1/25)n^3$, see Table 5.

5 Discussion and future work

Since in dimension d=3, the stretched grid and the stretched diagonal yield the same upper bound of $n^3/25$ (which is known to be tight), we were expecting the same to happen in higher dimensions. We were surprised to find this not to be the case. Also surprising is the fact that the bounds obtained for $d \ge 4$ do not seem to be rational. Running NMaximize with higher precision, we find the bound for d=4 to be 0.004579364805943860006...

In order to gain more confidence in our numerical results, we re-ran the stretched-grid numerical maximization experiments in a newer version of Mathematica (12.3) as well as in Python using SciPy's differential_evolution function. We got the same results. See the ancillary files of the arXiv version.

```
T = \emptyset, 0 < p_1 < p_2 < q_2 < q_3 < 1,
F_1 = p_1(q_2 - p_2)(1 - q_3) + p_1(p_2 - p_1)(q_3 - p_2) + (q_2 - p_1)(1 - q_3)(q_3 - q_2)
T = \emptyset, 0 < p_1 < p_2 < q_3 < q_2 < 1,
F_2 = p_1(p_2 - p_1)(q_3 - p_2) + p_1(1 - q_3)(q_3 - p_2)
T = \emptyset, 0 < p_2 < p_1 < q_3 < q_2 < 1,
F_3 = p_1(1-q_3)(q_3-p_1)
T = \emptyset, 0 < p_2 < p_1 < q_2 < q_3 < 1,
F_4 = p_1(q_2 - p_1)(1 - q_3) + (q_2 - p_1)(1 - q_3)(q_3 - q_2)
T = \{1\}, 0 < p_1 < p_2 < q_2 < q_3 < 1,
F_5 = p_1(q_2 - p_2)(1 - q_3) + p_1(p_2 - p_1)(q_3 - p_2) + p_1(1 - q_3)(q_3 - q_2)
T = \{1\}, 0 < p_1 < p_2 < q_3 < q_2 < 1,
F_6 = p_1(p_2 - p_1)(q_3 - p_2) + p_1(1 - q_3)(q_3 - p_2)
T = \{1\}, 0 < p_2 < p_1 < q_3 < q_2 < 1,
F_7 = p_1(1-q_3)(q_3-p_1)
T = \{1\}, 0 < p_2 < p_1 < q_2 < q_3 < 1,
F_8 = p_1(q_2 - p_1)(1 - q_3) + p_1(1 - q_3)(q_3 - q_2)
T = \{1\}, 0 < p_2 < q_2 < p_1 < 1, 0 < q_2 < q_3 < 1,
F_9 = q_2(1 - q_3)(q_3 - q_2)
T = \{2\}, 0 < p_1 < q_1 < p_2 < q_3 < 1,
F_{10} = (p_2 - q_1)q_1(1 - q_3) + p_1(p_2 - p_1)(q_3 - p_2) + (q_1 - p_1)(1 - q_3)(q_3 - p_2)
T = \{2\}, 0 < p_1 < p_2 < q_1 < 1, 0 < p_2 < q_3 < 1,
F_{11} = p_1(p_2 - p_1)(q_3 - p_2) + (p_2 - p_1)(1 - q_3)(q_3 - p_2)
T = \{2\}, 0 < p_1 < q_1 < q_3 < p_2 < 1,
F_{12} = q_1(1-q_3)(q_3-q_1)
T = \{1, 2\}, 0 < q_1 < p_1 < p_2 < q_3 < 1,
F_{13} = (p_2 - q_1)q_1(1 - q_3) + p_1(p_2 - p_1)(q_3 - p_2) + (p_1 - q_1)(1 - q_3)(q_3 - p_2)
T = \{1, 2\}, 0 < q_1 < p_2 < p_1 < 1, 0 < p_2 < q_3 < 1,
F_{14} = (p_2 - q_1)q_1(1 - q_3) + (p_2 - q_1)(1 - q_3)(q_3 - p_2)
T = \{1, 2\}, 0 < q_1 < p_1 < 1, 0 < q_1 < q_3 < p_2 < 1,
F_{15} = q_1(1 - q_3)(q_3 - q_1)
```

Table 4: Different functions for the stretched diagonal in dimension 3.

F	qp	maximum
F_1	(1,3/5,4/5),(1/5,2/5,0)	1/25
F_2	(1,59/64,27/32),(1/3,49/96,0)	1/27
F_3	(1, 27/32, 2/3), (1/3, 5/32, 0)	1/27
F_4	(1, 1/2, 2/3), (1/6, 5/64, 0)	1/27
F_5	(0, 11/16, 27/32), (1/3, 49/96, 0)	1/27
F_6	(0,59/64,27/32),(1/3,49/96,0)	1/27
F_7	(0, 27/32, 2/3), (1/3, 5/32, 0)	1/27
F_8	(0, 1/2, 2/3), (1/3, 5/32, 0)	1/27
F_9	(0, 1/3, 2/3), (11/16, 5/32, 0)	1/27
F_{10}	(2/5, 0, 4/5), (1/5, 3/5, 0)	$\mathbf{1/25}$
F_{11}	(11/16, 0, 3/4), (1/12, 5/12, 0)	1/27
F_{12}	(1/3, 0, 2/3), (5/32, 27/32, 0)	1/27
F_{13}	(1/5, 0, 4/5), (2/5, 3/5, 0)	$\mathbf{1/25}$
F_{14}	(5/32, 0, 2/3), (3/4, 47/96, 0)	1/27
F_{15}	(1/3, 0, 2/3), (11/16, 27/32, 0)	1/27

Table 5: Maximum of each function in Table 4.

The main open problem is to find the exact value of the constants $c_{d,1}$. Since in dimension 4, the stretched grid and the stretched diagonal do not give the same value, we are not sure that the value given by the stretched grid is tight.

Another interesting problem is to study the corresponding variant of the Second Selection Lemma, in which we look for a line that stabs many simplices from a given subset of X-simplices. One could also study variants in which a line stabs geometric objects other than simplices.

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