# A note on the orientation covering number 

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#### Abstract

Given a graph $G$, its orientation covering number $\sigma(G)$ is the smallest non-negative integer $k$ with the property that we can choose $k$ orientations of $G$ such that whenever $x, y, z$ are vertices of $G$ with $x y, x z \in E(G)$ then there is a chosen orientation in which both $x y$ and $x z$ are oriented away from $x$. Esperet, Gimbel and King showed that $\sigma(G) \leq \sigma\left(K_{\chi(G)}\right)$, where $\chi(G)$ is the chromatic number of $G$, and asked whether we always have equality. In this note we prove that it is indeed always the case that $\sigma(G)=\sigma\left(K_{\chi(G)}\right)$. We also determine the exact value of $\sigma\left(K_{n}\right)$ explicitly for 'most' values of $n$.


## 1 Introduction

Given a non-empty graph $G$ and $k$ orientations $\vec{G}_{1}, \ldots, \vec{G}_{k}$ of $G$, we say that $\vec{G}_{1}, \ldots, \vec{G}_{k}$ is an orientation covering of $G$ if whenever $x, y, z \in V(G)$ with $x y, x z \in E(G)$ then there is an orientation in which both $x y$ and $x z$ are oriented away from $x$ (i.e., there is some $i$ such that $\left.(x, y),(x, z) \in E\left(\vec{G}_{i}\right)\right)$. The orientation covering number $\sigma(G)$ of $G$ is the smallest positive integer $k$ such that there is a list of $k$ orientations forming an orientation covering of $G$. Orientation coverings were introduced by Esperet, Gimbel and King [2], who used them to study the minimal number of equivalence subgraphs needed to cover a given graph.

Esperet, Gimbel and King [2] showed that $\sigma(G) \leq \sigma\left(K_{\chi(G)}\right)$ for any graph $G$, where $\chi$ denotes the chromatic number. They asked whether we always have $\sigma(G)=\sigma\left(K_{\chi(G)}\right)$. In this note we answer this question in the positive.

Theorem 1. For any non-empty graph $G$, we have $\sigma(G)=\sigma\left(K_{\chi(G)}\right)$.
The value of $\sigma\left(K_{n}\right)$ has been investigated by Esperet, Gimbel and King [2], who determined its order of magnitude and the exact values for small values of $n$. An observation of Gyárfás (see [2]) shows that we have $\chi\left(D S_{n}\right) \leq \sigma\left(K_{n}\right) \leq \chi\left(D S_{n}\right)+2$, where $D S_{n}$ is the double-shift graph on $n$ vertices. Using the results of Füredi, Hajnal, Rödl and Trotter [3] on the chromatic number of $D S_{n}$, this gives $\sigma\left(K_{n}\right)=\log \log n+\frac{1}{2} \log \log \log n+O(1)$. (All logarithms in this paper are base 2.) In this note we will also determine the value of $\sigma\left(K_{n}\right)$ exactly in terms of a certain sequence of positive integers sometimes called the Hoşten-Morris numbers. As a corollary, we get the following improved estimate.

Theorem 2. We have $\sigma\left(K_{n}\right)=\left\lceil\log \log n+\frac{1}{2} \log \log \log n+\frac{1}{2}(\log \pi+1)+o(1)\right\rceil$ as $n \rightarrow \infty$.

[^0]Given a positive integer $k$, let $[k]$ denote $\{1, \ldots, k\}$, as usual. Given a family $\mathcal{A} \subseteq \mathcal{P}([k])$ of subsets of $[k]$, we say that $\mathcal{A}$ is intersecting if whenever $S, T \in \mathcal{A}$ then $S \cap T \neq \emptyset$. We say that $\mathcal{A}$ is maximal intersecting if $\mathcal{A}$ is intersecting and whenever $\mathcal{B} \supseteq \mathcal{A}$ and $\mathcal{B}$ is intersecting then $\mathcal{B}=\mathcal{A}$. (Equivalently, if $\mathcal{A}$ is intersecting and $|\mathcal{A}|=2^{k-1}$.) The following characterisation of $\sigma(G)$ is the key to our results.

Theorem 3. For any non-empty graph $G, \sigma(G)$ is the smallest positive integer $k$ such that there are at least $\chi(G)$ maximal intersecting families over $[k]$.

Clearly, Theorem 3 implies Theorem 1 . Let $\lambda(k)$ denote the number of maximal intersecting families over $[k]$. The numbers $\lambda(k)$ are sometimes called Hoşten-Morris numbers, after a paper of Hoşten and Morris [4] in which they showed that the order dimension of $K_{n}$ is the smallest positive integer $k$ with $\lambda(k) \geq n$. An equivalent formulation of their result is that the minimal number of linear orders on $[n]$ with the property that the induced orientations of $K_{n}$ form an orientation covering is the smallest positive integer $k$ with $\lambda(k) \geq n$. Note that by Theorem 3 this number is the same as the orientation covering number of $K_{n}$.

Although no exact or asymptotic formula is known for $\lambda(k)$, it was shown by Brouwer, Mills, Mills and Verbeek [1] that

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\begin{equation*}
\log \lambda(k) \sim \frac{2^{k}}{\sqrt{2 \pi k}} \tag{1}
\end{equation*}
$$

Furthermore, the exact values of $\lambda(k)$ are known [1] for $k$ up to 9 , with $\lambda(9) \approx 4 \times 10^{20}$.
Theorem 2 follows from Theorem 3 and (1). Indeed, taking logarithms in (1) shows that $\sigma\left(K_{n}\right)$ is the smallest positive integer $k$ with $\log \log n \leq k-\frac{1}{2}(\log \pi+1)-\frac{1}{2} \log k+o(1)$, which gives $\sigma\left(K_{n}\right)=\left\lceil\log \log n+\frac{1}{2} \log \log \log n+\frac{1}{2}(\log \pi+1)+o(1)\right\rceil$.

## 2 Proof of Theorem 3

The proof is based on the following observation.
Lemma 4. For any non-empty graph $G, \sigma(G)$ is the smallest positive integer $k$ with the property that there is a collection $\left(\mathcal{A}_{v}\right)_{v \in V(G)}$ of subsets of $\mathcal{P}([k])$ (i.e., $\mathcal{A}_{v} \subseteq \mathcal{P}([k])$ for all $v$ ) such that the following two conditions hold.

1. If $u v \in E(G)$, then there exists $S \in \mathcal{A}_{u}$ and $T \in \mathcal{A}_{v}$ such that $S \cap T=\emptyset$.
2. For all $v \in V(G)$ and $S, T \in \mathcal{A}_{v}$, we have $S \cap T \neq \emptyset$. (I.e., $\mathcal{A}_{v}$ is intersecting.)

Proof. First assume that $\sigma(G)=k$ and $\vec{G}_{1}, \ldots, \vec{G}_{k}$ form an orientation cover of $G$. For each directed edge $(x, y)$ of $G$, let $S_{(x, y)}=\left\{i \in[k]:(x, y) \in E\left(\vec{G}_{i}\right)\right\}$. Let $\mathcal{A}_{v}=\left\{S_{(v, w)}: v w \in E(G)\right\}$. Clearly $S_{(v, w)} \cap S_{(w, v)}=\emptyset$, so Condition 1 holds. Also, we have $S_{(v, w)} \cap S_{\left(v, w^{\prime}\right)} \neq \emptyset$ whenever $v w, v w^{\prime} \in E(G)$, since by assumption there is an $i$ such that $(v, w),\left(v, w^{\prime}\right) \in E\left(\vec{G}_{i}\right)$. So Condition 2 holds as well.

Conversely, suppose that we have such a collection $\left(\mathcal{A}_{v}\right)_{v \in V(G)}$ with $\mathcal{A}_{v} \subseteq \mathcal{P}([k])$ for all $v$. For each $u v \in E(G)$, pick $S_{(u, v)} \in \mathcal{A}_{u}$ and $S_{(v, u)} \in \mathcal{A}_{v}$ such that $S_{(u, v)} \cap S_{(v, u)}=\emptyset$. Define the orientations $\vec{G}_{1}, \ldots, \vec{G}_{k}$ of $G$ by orienting the edge $u v$ from $u$ to $v$ in $\vec{G}_{i}$ if $i \in S_{(u, v)}$, from $v$ to $u$ if $i \in S_{(v, u)}$, and arbitrarily otherwise. This is clearly well-defined, and whenever $u v, u w \in E(G)$, then $S_{(u, v)} \cap S_{(u, w)} \neq \emptyset$ (by Condition 2). This gives $\sigma(G) \leq k$, as claimed.

Proof of Theorem 3. We first show the lower bound for $\sigma(G)$. Let $G$ be any non-empty graph, and let $\left(\mathcal{A}_{v}\right)_{v \in V(G)}$ be as in Lemma 4 for $k=\sigma(G)$. For each $v \in V(G)$, let $\mathcal{B}_{v}$ be a maximal intersecting family with $\mathcal{B}_{v} \supseteq \mathcal{A}_{v}$. Note that the families $\left(\mathcal{B}_{v}\right)_{v \in V(G)}$ still satisfy both conditions in Lemma 4. Furthermore, $v \mapsto \mathcal{B}_{v}$ is a proper vertex-colouring (since each $\mathcal{B}_{v}$ is intersecting but $\mathcal{B}_{v} \cup \mathcal{B}_{w}$ is not whenever $\left.v w \in E(G)\right)$. It follows that the number of maximal intersecting families over $[k]$ is at least $\chi(G)$.

Conversely, assume that $k$ is a positive integer such that there are at least $\chi(G)$ distinct maximal intersecting families $\mathcal{B}_{1}, \ldots, \mathcal{B}_{k}$ over $[k]$. Let $c: V(G) \mapsto[\chi(G)]$ be a proper vertexcolouring of $G$, and set $\mathcal{A}_{v}=\mathcal{B}_{c(v)}$ for each $v$. Certainly each $\mathcal{A}_{v}$ is intersecting. Furthermore, by maximality, no $\mathcal{A}_{v} \cup \mathcal{A}_{w}$ can be intersecting when $c(v) \neq c(w)$, and hence $\mathcal{A}_{v} \cup \mathcal{A}_{w}$ is not intersecting when $v w \in E(G)$. It follows that $\left(\mathcal{A}_{v}\right)_{v \in V(G)}$ satisfies both conditions in Lemma 4 and so $\sigma(G) \leq k$.

## References

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