A note on the orientation covering number

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Abstract

Given a graph G, its orientation covering number $\sigma(G)$ is the smallest non-negative integer k with the property that we can choose k orientations of G such that whenever x, y, z are vertices of G with $xy, xz \in E(G)$ then there is a chosen orientation in which both xy and xz are oriented away from x. Esperet, Gimbel and King showed that $\sigma(G) \leq \sigma(K_{\chi(G)})$, where $\chi(G)$ is the chromatic number of G, and asked whether we always have equality. In this note we prove that it is indeed always the case that $\sigma(G) = \sigma(K_{\chi(G)})$. We also determine the exact value of $\sigma(K_n)$ explicitly for 'most' values of n.

1 Introduction

Given a non-empty graph G and k orientations $\vec{G}_1, \ldots, \vec{G}_k$ of G, we say that $\vec{G}_1, \ldots, \vec{G}_k$ is an orientation covering of G if whenever $x, y, z \in V(G)$ with $xy, xz \in E(G)$ then there is an orientation in which both xy and xz are oriented away from x (i.e., there is some i such that $(x, y), (x, z) \in E(\vec{G}_i)$). The orientation covering number $\sigma(G)$ of G is the smallest positive integer k such that there is a list of k orientations forming an orientation covering of G. Orientation coverings were introduced by Esperet, Gimbel and King [2], who used them to study the minimal number of equivalence subgraphs needed to cover a given graph.

Esperet, Gimbel and King [2] showed that $\sigma(G) \leq \sigma(K_{\chi(G)})$ for any graph G, where χ denotes the chromatic number. They asked whether we always have $\sigma(G) = \sigma(K_{\chi(G)})$. In this note we answer this question in the positive.

Theorem 1. For any non-empty graph G, we have $\sigma(G) = \sigma(K_{\chi(G)})$.

The value of $\sigma(K_n)$ has been investigated by Esperet, Gimbel and King [2], who determined its order of magnitude and the exact values for small values of n. An observation of Gyárfás (see [2]) shows that we have $\chi(DS_n) \leq \sigma(K_n) \leq \chi(DS_n) + 2$, where DS_n is the double-shift graph on n vertices. Using the results of Füredi, Hajnal, Rödl and Trotter [3] on the chromatic number of DS_n , this gives $\sigma(K_n) = \log \log n + \frac{1}{2} \log \log \log n + O(1)$. (All logarithms in this paper are base 2.) In this note we will also determine the value of $\sigma(K_n)$ exactly in terms of a certain sequence of positive integers sometimes called the Hoşten–Morris numbers. As a corollary, we get the following improved estimate.

Theorem 2. We have $\sigma(K_n) = \lceil \log \log n + \frac{1}{2} \log \log \log n + \frac{1}{2} (\log \pi + 1) + o(1) \rceil$ as $n \to \infty$.

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Given a positive integer k, let [k] denote $\{1, \ldots, k\}$, as usual. Given a family $\mathcal{A} \subseteq \mathcal{P}([k])$ of subsets of [k], we say that \mathcal{A} is *intersecting* if whenever $S, T \in \mathcal{A}$ then $S \cap T \neq \emptyset$. We say that \mathcal{A} is *maximal intersecting* if \mathcal{A} is intersecting and whenever $\mathcal{B} \supseteq \mathcal{A}$ and \mathcal{B} is intersecting then $\mathcal{B} = \mathcal{A}$. (Equivalently, if \mathcal{A} is intersecting and $|\mathcal{A}| = 2^{k-1}$.) The following characterisation of $\sigma(G)$ is the key to our results.

Theorem 3. For any non-empty graph G, $\sigma(G)$ is the smallest positive integer k such that there are at least $\chi(G)$ maximal intersecting families over [k].

Clearly, Theorem 3 implies Theorem 1. Let $\lambda(k)$ denote the number of maximal intersecting families over [k]. The numbers $\lambda(k)$ are sometimes called Hoşten–Morris numbers, after a paper of Hoşten and Morris [4] in which they showed that the order dimension of K_n is the smallest positive integer k with $\lambda(k) \geq n$. An equivalent formulation of their result is that the minimal number of linear orders on [n] with the property that the induced orientations of K_n form an orientation covering is the smallest positive integer k with $\lambda(k) \geq n$. Note that by Theorem 3 this number is the same as the orientation covering number of K_n .

Although no exact or asymptotic formula is known for $\lambda(k)$, it was shown by Brouwer, Mills, Mills and Verbeek [1] that

$$\log \lambda(k) \sim \frac{2^k}{\sqrt{2\pi k}}.$$
(1)

Furthermore, the exact values of $\lambda(k)$ are known [1] for k up to 9, with $\lambda(9) \approx 4 \times 10^{20}$.

Theorem 2 follows from Theorem 3 and (1). Indeed, taking logarithms in (1) shows that $\sigma(K_n)$ is the smallest positive integer k with $\log \log n \le k - \frac{1}{2}(\log \pi + 1) - \frac{1}{2}\log k + o(1)$, which gives $\sigma(K_n) = \lceil \log \log n + \frac{1}{2} \log \log \log n + \frac{1}{2}(\log \pi + 1) + o(1) \rceil$.

2 Proof of Theorem 3

The proof is based on the following observation.

Lemma 4. For any non-empty graph G, $\sigma(G)$ is the smallest positive integer k with the property that there is a collection $(\mathcal{A}_v)_{v \in V(G)}$ of subsets of $\mathcal{P}([k])$ (i.e., $\mathcal{A}_v \subseteq \mathcal{P}([k])$ for all v) such that the following two conditions hold.

- 1. If $uv \in E(G)$, then there exists $S \in \mathcal{A}_u$ and $T \in \mathcal{A}_v$ such that $S \cap T = \emptyset$.
- 2. For all $v \in V(G)$ and $S, T \in \mathcal{A}_v$, we have $S \cap T \neq \emptyset$. (I.e., \mathcal{A}_v is intersecting.)

Proof. First assume that $\sigma(G) = k$ and $\vec{G}_1, \ldots, \vec{G}_k$ form an orientation cover of G. For each directed edge (x, y) of G, let $S_{(x,y)} = \{i \in [k] : (x, y) \in E(\vec{G}_i)\}$. Let $\mathcal{A}_v = \{S_{(v,w)} : vw \in E(G)\}$. Clearly $S_{(v,w)} \cap S_{(w,v)} = \emptyset$, so Condition 1 holds. Also, we have $S_{(v,w)} \cap S_{(v,w')} \neq \emptyset$ whenever $vw, vw' \in E(G)$, since by assumption there is an i such that $(v, w), (v, w') \in E(\vec{G}_i)$. So Condition 2 holds as well.

Conversely, suppose that we have such a collection $(\mathcal{A}_v)_{v \in V(G)}$ with $\mathcal{A}_v \subseteq \mathcal{P}([k])$ for all v. For each $uv \in E(G)$, pick $S_{(u,v)} \in \mathcal{A}_u$ and $S_{(v,u)} \in \mathcal{A}_v$ such that $S_{(u,v)} \cap S_{(v,u)} = \emptyset$. Define the orientations $\vec{G}_1, \ldots, \vec{G}_k$ of G by orienting the edge uv from u to v in \vec{G}_i if $i \in S_{(u,v)}$, from v to u if $i \in S_{(v,u)}$, and arbitrarily otherwise. This is clearly well-defined, and whenever $uv, uw \in E(G)$, then $S_{(u,v)} \cap S_{(u,w)} \neq \emptyset$ (by Condition 2). This gives $\sigma(G) \leq k$, as claimed. **Proof of Theorem 3.** We first show the lower bound for $\sigma(G)$. Let G be any non-empty graph, and let $(\mathcal{A}_v)_{v \in V(G)}$ be as in Lemma 4 for $k = \sigma(G)$. For each $v \in V(G)$, let \mathcal{B}_v be a maximal intersecting family with $\mathcal{B}_v \supseteq \mathcal{A}_v$. Note that the families $(\mathcal{B}_v)_{v \in V(G)}$ still satisfy both conditions in Lemma 4. Furthermore, $v \mapsto \mathcal{B}_v$ is a proper vertex-colouring (since each \mathcal{B}_v is intersecting but $\mathcal{B}_v \cup \mathcal{B}_w$ is not whenever $vw \in E(G)$). It follows that the number of maximal intersecting families over [k] is at least $\chi(G)$.

Conversely, assume that k is a positive integer such that there are at least $\chi(G)$ distinct maximal intersecting families $\mathcal{B}_1, \ldots, \mathcal{B}_k$ over [k]. Let $c : V(G) \mapsto [\chi(G)]$ be a proper vertexcolouring of G, and set $\mathcal{A}_v = \mathcal{B}_{c(v)}$ for each v. Certainly each \mathcal{A}_v is intersecting. Furthermore, by maximality, no $\mathcal{A}_v \cup \mathcal{A}_w$ can be intersecting when $c(v) \neq c(w)$, and hence $\mathcal{A}_v \cup \mathcal{A}_w$ is not intersecting when $vw \in E(G)$. It follows that $(\mathcal{A}_v)_{v \in V(G)}$ satisfies both conditions in Lemma 4 and so $\sigma(G) \leq k$.

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