

On the characterization of some algebraically defined bipartite graphs of girth eight*

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Abstract: For any field \mathbb{F} and polynomials $f_2, f_3 \in \mathbb{F}[x, y]$, let $\Gamma_{\mathbb{F}}(f_2, f_3)$ denote the bipartite graph with vertex partition $P \cup L$, where P and L are two copies of \mathbb{F}^3 , and $(p_1, p_2, p_3) \in P$ is adjacent to $[l_1, l_2, l_3] \in L$ if and only if $p_2 + l_2 = f_2(p_1, l_1)$ and $p_3 + l_3 = f_3(p_1, l_1)$. The graph $\Gamma_3(\mathbb{F}) = \Gamma_{\mathbb{F}}(xy, xy^2)$ is known to be of girth eight. When $\mathbb{F} = \mathbb{F}_q$ is a finite field of odd size q or $\mathbb{F} = \mathbb{F}_{\infty}$ is an algebraically closed field of characteristic zero, the graph $\Gamma_3(\mathbb{F})$ is conjectured to be the unique one with girth at least eight among those $\Gamma_{\mathbb{F}}(f_2, f_3)$ up to isomorphism. This conjecture has been confirmed for the case that both f_2, f_3 are monomials over \mathbb{F}_q , and for the case that at least one of f_2, f_3 is a monomial over \mathbb{F}_{∞} . If one of $f_2, f_3 \in \mathbb{F}_q[x, y]$ is a monomial, it has also been proved the existence of a positive integer M such that $G = \Gamma_{\mathbb{F}_{q^M}}(f_2, f_3)$ is isomorphic to $\Gamma_3(\mathbb{F}_{q^M})$ provided G has girth at least eight. In this paper, these results are shown to be valid when the restriction on the polynomials f_2, f_3 is relaxed further to that one of them is the product of two univariate polynomials. Furthermore, all of such polynomials f_2, f_3 are characterized completely.

Keywords: Bipartite graph; Cycle; Girth; Generalized quadrangle; Isomorphism

1 Introduction

All graphs considered in this paper are undirected, without loops and multiple edges. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$, where each edge in E is a two-element subset of V . Two vertices

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$v, v' \in V$ are said adjacent to each other, and written $v \sim v'$, if $\{v, v'\} \in E$ is an edge. The order of G is the number of vertices in V . The degree of a vertex $v \in V$ is the number of vertices adjacent to v . If every $v \in V$ has degree t , then G is called a t -regular graph. A sequence $(v_1, v_2, \dots, v_k) \in V^k$ of vertices is called a k -*cycle of G if $k \geq 3$, $v_1 \sim v_2 \sim \dots \sim v_k \sim v_1$ and $v_i \neq v_{i+2}$, $i = 1, 2, \dots, k$, where $v_{k+1} = v_1$, $v_{k+2} = v_2$. A k -*cycle (v_1, v_2, \dots, v_k) of G is called a k -cycle of G if the vertices v_1, v_2, \dots, v_k differ from each other. It is clear that any k -*cycle is a k -cycle if $3 \leq k \leq 5$. The graph G is called bipartite if its vertex set can be divided into two parts as $V = P \cup L$ such that each edge in E consists of a vertex in P and a vertex in L . It is not difficult to show that, in a bipartite graph, there is no $(2k+1)$ -*cycle, any 6-*cycle is a 6-cycle and any 8-*cycle is either an 8-cycle or the concatenation of two 4-cycles. In particular, in a bipartite graph with no 4-cycle, any 8-*cycle is an 8-cycle. If G has some cycles, its girth is defined as the largest integer k such that G contains no i -cycles for any k with $3 \leq i < k$. Other standard graph theory definitions can be found in [1].

For $k \geq 2$, let $g_k(n)$ be the greatest number of edges in a graph of order n with girth at least $2k+1$. It is well known that for sufficiently large n ,

$$c'_k n^{1+\frac{2}{3k-3+\epsilon}} \leq g_k(n) \leq c_k n^{1+\frac{1}{k}},$$

where c'_k and c_k are positive constants depending only on k , and $\epsilon = 0$ if k is odd and $\epsilon = 1$ if k is even (see [2]). The upper bound comes from [3] and the lower bound from an explicit construction of [4]. The upper bound is known to be sharp in magnitude $n^{1+\frac{1}{k}}$ only for $k = 2, 3, 5$. In this paper, we concentrate on the case of $k = 3$. A known example which provides such extremal magnitude is the bipartite graph $\Gamma_3(\mathbb{F}_q)$ with vertex partition $P \cup L$, where \mathbb{F}_q is the finite field of q elements, P and L are two copies of \mathbb{F}_q^3 , and $(p_1, p_2, p_3) \in P$ is adjacent to $[l_1, l_2, l_3] \in L$ if and only if $p_2 + l_2 = p_1 l_1$ and $p_3 + l_3 = p_1^2 l_1$. It can be shown easily that $\Gamma_3(\mathbb{F}_q)$ is a q -regular bipartite graph of order $2q^3$ and girth eight. When q is an odd prime power, the graph $\Gamma_3(\mathbb{F}_q)$ is isomorphic to an induced subgraph of the point-line incidence graph of the classical generalized quadrangle $W(q)$ of order q (see [5, 6, 7, 8, 9]).

From now on, we focus on a generalization of the graph $\Gamma_3(\mathbb{F}_q)$. Let \mathbb{F} be an arbitrary field, for polynomials $f_2, f_3 \in \mathbb{F}[x, y]$, the graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ is a bipartite graph with vertex partition $P \cup L$, where P and L are two copies of \mathbb{F}^3 , and $(p_1, p_2, p_3) \in P$ is adjacent to $[l_1, l_2, l_3] \in L$ if and only if

$$p_2 + l_2 = f_2(p_1, l_1) \text{ and } p_3 + l_3 = f_3(p_1, l_1).$$

When $\mathbb{F} = \mathbb{F}_q$, we simplify the notation $\Gamma_{\mathbb{F}_q}(f_2, f_3)$ to $\Gamma_q(f_2, f_3)$ further. For odd prime power q , it is of interest to find a graph $\Gamma_q(f_2, f_3)$ of girth eight that is not isomorphic to $\Gamma_3(\mathbb{F}_q)$, since a new generalized quadrangle could be constructed by “attaching” some tree to such graph. However, many attempts towards this aim failed (see [5, 6, 7, 8, 9]). On the contrary, the following uniqueness conjecture was proposed in [8, 9]:

Conjecture 1. *If $\mathbb{F} = \mathbb{F}_q$ is a finite field of odd size or $\mathbb{F} = \mathbb{F}_\infty$ is an algebraically closed field of characteristic zero, then every graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_3(\mathbb{F}) = \Gamma_{\mathbb{F}}(xy, x^2y)$.*

When $\mathbb{F} = \mathbb{F}_q$ is a finite field of odd size and f_2, f_3 are monomials, Conjecture 1 was investigated in [5, 6, 7] and confirmed in [2]. When $\mathbb{F} = \mathbb{F}_\infty$ is an algebraically closed field of characteristic zero and at least one of f_2, f_3 is monomial, Conjecture 1 was confirmed in [8, 9]. When $\mathbb{F} = \mathbb{F}_q$ is a finite field of odd size and one of f_2, f_3 is monomial, the following result was also shown in [9]: If q is a power of some odd prime p and $f \in \mathbb{F}_q[x, y]$ has degree at most $p - 2$ with respect to each of x and y , then, for any integers k, m coprime to p , there exists a positive integer $M = M(k, m, q)$ such that, for all positive integers r , every graph $\Gamma_{q^{Mr}}(x^k y^m, f)$ of girth at least eight is isomorphic to $\Gamma_3(\mathbb{F}_{q^{Mr}})$. In this paper, we will show that the main results of [9] are still true if one of f_2, f_3 is of form $f(x)g(y)$ for univariate polynomials $f, g \in \mathbb{F}[x]$.

For any field \mathbb{F} , let $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$. For positive integer k , let $[1, k]$ denote the set $\{1, 2, \dots, k\}$, $\mathbb{F}[x]_k$ the set of polynomials in $\mathbb{F}[x]$ of degree at most k and $\mathbb{F}[x, y]_k$ the set of polynomials in $\mathbb{F}[x, y]$ of degree at most k with respect to each of x and y , respectively. Through this paper, we assume that q is a power of some prime p and m, n are positive integers such that

$$q > \max\{2mn + 3, mn + 3n + 1, n(n + 1) + 2\}. \quad (1)$$

Let $M = M(mn)$ be the least common multiple of the integers $2, 3, \dots, mn$. Clearly, any polynomials $T(x) \in \mathbb{F}_q[x]_{mn}$ can be decomposed completely in \mathbb{F}_{q^M} . For any $a \in \mathbb{F}_q$, let $\rho_a(x) = x(x - a) \in \mathbb{F}_q[x]$. Let $K_p = \{p^j \mid j \geq 0\}$. For $u, v \in K_p$, let $\Phi_p(u, v) = \{(i, j) \in K_p^2 : iv = ju\}$. Let $f, g \in \mathbb{F}_q[x]_m$ be monic polynomials with $f(0) = g(0) = 0$ and

$$h(x, y) = \sum_{1 \leq i, j \leq n} h_{i,j} x^i y^j \in \mathbb{F}_q[x, y]_n \quad (2)$$

be a nonzero polynomial. For $a \in \mathbb{F}_q$, $u, v \in K_p$ and the polynomial h given in (2), let $\mu_{a,u,v}(h)$, $\nu_{a,u,v}(h)$ and $\pi_{u,v}(h)$ denote the polynomials in $\mathbb{F}_q[x, y]$

defined respectively by

$$\begin{aligned}\mu_{a,u,v}(h)(x,y) &= h(x,y) - \sum_{(i,j) \in \Phi_p(u,v)} h_{2i,j} \rho_a^i(x) y^j, \\ \nu_{a,u,v}(h)(x,y) &= h(x,y) - \sum_{(i,j) \in \Phi_p(v,u)} h_{i,2j} x^i \rho_a^j(y), \\ \pi_{u,v}(h)(x,y) &= h(x,y) - \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j} x^i(x) y^j.\end{aligned}$$

The main result of this paper is as follows.

Theorem 1. *The graph $G = \Gamma_{q^M}(f(x)g(y), h(x,y))$ is isomorphic to $\Gamma_3(\mathbb{F}_{q^M})$ if it has girth at least eight. Furthermore, G has girth at least eight if and only if there are some $a \in \mathbb{F}_q$, $\zeta \in \mathbb{F}_q^*$, $u, v \in K_p \cap [1, m]$ and $s \in K_p \cap [1, n]$ such that one of the following is valid.*

- (i) $f(x) = \rho_a^u(x), g(y) = y^v$ and $\mu_{a,u,v}(h)(x,y) = \zeta x^{su/v} y^s$.
- (ii) $f(x) = x^v, g(y) = \rho_a^u(y)$ and $\nu_{a,u,v}(h)(x,y) = \zeta x^s y^{su/v}$.
- (iii) $f(x) = x^u, g(y) = y^v$ and $\pi_{u,v}(h)(x,y) = \zeta x^{2su/v} y^s$ or $\zeta x^s y^{2sv/u}$.
- (iv) $p = 2$, $a \neq 0$ and either
 - (a) $f(x) = \rho_a^u(x), g(y) = y^{2su}$ and $\mu_{a,u,2su}(h)(x,y) = \zeta x y^s$, or
 - (b) $f(x) = x^{2su}, g(y) = \rho_a^u(y)$ and $\nu_{a,u,2su}(h)(x,y) = \zeta x^s y$.

The following theorem is an analog of Theorem 1 for the case that $\mathbb{F} = \mathbb{F}_\infty$ is an algebraically closed field of characteristic zero.

Theorem 2. *Suppose that $f, g \in \mathbb{F}_\infty[x]$ are monic polynomials with $f(0) = g(0) = 0$ and $h(x,y) = \sum_{i,j \geq 1} h_{i,j} x^i y^j \in \mathbb{F}_\infty[x,y]$ is a nonzero polynomial. The graph $G = \Gamma_{\mathbb{F}_\infty}(f(x)g(y), h(x,y))$ is isomorphic to $\Gamma_3(\mathbb{F}_\infty)$ if G has girth at least eight. Furthermore, G has girth at least eight if and only if there are some $a \in \mathbb{F}_\infty$, $\zeta \in \mathbb{F}_\infty^*$ such that one of the following is valid.*

- (i) $f(x) = \rho_a(x), g(y) = y$ and $h(x,y) = \zeta xy + h_{2,1} \rho_a(x) y$.
- (ii) $f(x) = x, g(y) = \rho_a(y)$ and $h(x,y) = \zeta xy + h_{1,2} x \rho_a(y)$.
- (iii) $f(x) = x, g(y) = y$ and $h(x,y) = \zeta \rho_a(x) y$ or $\zeta x \rho_a(y)$.

This paper is organized as follows. In Section 2 we show some preliminaries, including a necessary and sufficient condition of $2k$ -*cycles in $\Gamma_{\mathbb{F}}(f_2, f_3)$, some isomorphisms of $\Gamma_{\mathbb{F}}(f_2, f_3)$, some simple conclusions on a few monomial graphs, and a simple but useful lemma on the characterization of polynomials according to their roots in some extension field. In Sections 3 to 5, we characterize the monic polynomials $f, g \in \mathbb{F}_q[x]_m$ and nonzero polynomial $h \in \mathbb{F}_q[x, y]_n$ under the conditions $f(0) = g(0) = h(0, y) = h(x, 0) = 0$ and that $G = \Gamma_{q^M}(f(x)g(y), h(x, y))$ has girth at least eight. In Section 6, we complete the proof of Theorem 1, and make some concluding remarks, including a simple illustration for the proof of Theorem 2.

2 Preliminaries

For $k \geq 2$, let Δ_k be the function defined by

$$\Delta_k : \mathbb{F}[x, y] \rightarrow \mathbb{F}[x_1, \dots, x_k; y_1, \dots, y_k],$$

$$f(x, y) \mapsto \sum_{i=1}^k (f(x_i, y_i) - f(x_{i+1}, y_i)),$$

where $x_{k+1} = x_1$. Suppose $k \geq 2$, $S = (a_1, \dots, a_k; r_1, \dots, r_k) \in \mathbb{F}^{2k}$ and $f_2, f_3 \in \mathbb{F}[x, y]$, it is clear that the graph $G = \Gamma_{\mathbb{F}}(f_2, f_3)$ contains a $2k$ -*cycle of form

$$((a_1, b_1, c_1), [r_1, s_1, t_1], \dots, (a_k, b_k, c_k), [r_k, s_k, t_k]) \quad (3)$$

if and only if

$$\begin{cases} \Delta_k(f_2)(S) = \Delta_k(f_3)(S) = 0, \\ a_i \neq a_{i+1}, r_i \neq r_{i+1}, i \in [1, k], \end{cases} \quad (4)$$

where $a_{k+1} = a_1$ and $r_{k+1} = r_1$ (see [6]). If G contains some $2k$ -*cycles of form (3), we also call $S = (a_1, \dots, a_k; r_1, \dots, r_k)$ a $2k$ -*cycle of G for simplicity.

Some useful isomorphisms of the graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$ are integrated in the following lemma.

Lemma 1. *Assume $f_2, f_3 \in \mathbb{F}[x, y]$. Then*

- (i) $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_3, f_2)$.
- (ii) $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(\bar{f}_2, \bar{f}_3)$, where $\bar{f}(x, y) = f(y, x)$.

(iii) For any $\alpha \in \mathbb{F}^*$, $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, \alpha f_3)$.

(iv) For any $\beta \in \mathbb{F}$, $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, f_3 + \beta f_2)$.

(v) For any $t \in \mathbb{F}[x]$, $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2(x, y), f_3(x, y) + t(x))$.

Furthermore, if $\mathbb{F} = \mathbb{F}_q$, then, for any $u \in K_p$,

(vi) $\Gamma_q(f_2, f_3) \cong \Gamma_q(f_2(x^u, y), f_3(x^u, y))$.

(vii) $\Gamma_q(f_2, f_3) \cong \Gamma_q(f_2^u, f_3)$.

Proof. We refer the reader to [8] for the proofs of (i) \sim (v).

For (vi), let v be the least positive integer such that vu is a power of q and π_1 the map from $V(\Gamma_q(f_2, f_3))$ to $V(\Gamma_q(f_2(x^u, y), f_3(x^u, y)))$ defined by $\pi_1 : (p_1, p_2, p_3) \mapsto (p_1^v, p_2, p_3)$ and $\pi_1 : [l_1, l_2, l_3] \mapsto [l_1, l_2, l_3]$. Then, π_1 is a graph isomorphism.

For (vii), let π_2 be the map from $V(\Gamma_q(f_2, f_3))$ to $V(\Gamma_q(f_2^u, f_3))$ defined by $\pi_2 : (p_1, p_2, p_3) \mapsto (p_1, p_2^u, p_3)$ and $\pi_2 : [l_1, l_2, l_3] \mapsto [l_1, l_2^u, l_3]$. Then, π_2 is also a graph isomorphism. \square

If f_2, f_3 are monomials, the graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ is referred to as a monomial graph. Now we show some simple results for a few monomial graphs.

Lemma 2. (i) The girth of $\Gamma_3(\mathbb{F}) = \Gamma_{\mathbb{F}}(xy, x^2y)$ is 8.

(ii) The girth of $\Gamma_{\mathbb{F}}(x^3y, x^2y)$ is 6 if $\mathbb{F} \notin \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5\}$.

(iii) The girth of $\Gamma_{\mathbb{F}}(xy, x^2y^3)$ is 6 if $\mathbb{F} \neq \mathbb{F}_3$ and the characteristic of \mathbb{F} is not equal to 2.

(iv) $\Gamma_5(x^3y, x^2y) \cong \Gamma_3(\mathbb{F}_5)$.

(v) $\Gamma_3(x^3y, x^2y) \cong \Gamma_3(xy, x^2y^3) \cong \Gamma_3(\mathbb{F}_3)$.

Proof. Let (a, b) and (c, d) be two pairs of distinct elements in \mathbb{F} . From $\Delta_2(xy)(a, b; c, d) = (a - b)(c - d) \neq 0$, we see that $\Gamma_3(\mathbb{F})$ and $\Gamma_{\mathbb{F}}(xy, x^2y^3)$ contain no 4-cycle and thus have girth at least 6.

(i) Let $S_0 = (a, b, e; c, d, f)$, where $e \in \mathbb{F} \setminus \{a, b\}$ and $f \in \mathbb{F} \setminus \{c, d\}$. If $\Delta_3(xy)(S_0) = (a - b)(c - d) - (a - e)(f - d)$ is equal to 0, then we have

$$\begin{aligned} \Delta_3(x^2y)(S_0) &= (a^2 - b^2)(c - d) - (a^2 - e^2)(f - d) \\ &= (a - b)(c - d)(e - b) \neq 0. \end{aligned}$$

Hence, $\Gamma_3(\mathbb{F})$ contains no 6-cycle. Furthermore, $\Gamma_3(\mathbb{F})$ has girth 8 since it contains the 8-cycle $(1, 0, 1, 0; 1, 0, -1, 0)$.

(ii) If $\Delta_2(x^2y)(a, b; c, d) = (a^2 - b^2)(c - d)$ is equal to 0, then we have $a^2 = b^2 \neq 0$ and thus

$$\Delta_2(x^3y)(a, b; c, d) = (a^3 - b^3)(c - d) = a^2(a - b)(c - d) \neq 0.$$

Hence, $\Gamma_{\mathbb{F}}(x^3y, x^2y)$ has no 4-cycle and has girth at least 6. Furthermore, if there is some $t \in \mathbb{F} \setminus \{0, 1, -1\}$ such that $0 \notin \{t - 2, 2t - 1\}$, then

$$(t, 1 - t, t(t - 1); t^2(t - 1)^2, t^2, (t - 1)^2)$$

is a 6-cycle of $\Gamma_{\mathbb{F}}(x^3y, x^2y)$. Hence, the girth of $\Gamma_{\mathbb{F}}(x^3y, x^2y)$ is equal to 6 if $\mathbb{F} \notin \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5\}$.

(iii) If $\mathbb{F} \neq \mathbb{F}_3$ and the characteristic of \mathbb{F} is not equal to 2, for any $t \in \mathbb{F} \setminus \{0, 1, -1\}$,

$$(-t, t + 2t^2, t + 2; 1, 0, t)$$

is a 6-cycle of $\Gamma_{\mathbb{F}}(xy, x^2y^3)$ and thus $\Gamma_{\mathbb{F}}(xy, x^2y^3)$ has girth 6.

(iv) Since x^3 is a permutation in \mathbb{F}_5 , from $(x^3)^2 \equiv x^2 \pmod{x^5 - x}$ we have

$$\Gamma_5(x^3y, x^2y) \cong \Gamma_5(x^3y, (x^3)^2y) \cong \Gamma_5(xy, x^2y) = \Gamma_3(\mathbb{F}_5).$$

(v) The desired proof follows simply from $x^3 \equiv x \pmod{x^3 - x}$. \square

To deal with the graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ in general, according to Lemma 1 one can assume, without loss of generality, that f_2 and f_3 consist of only mixed terms, i.e. $f_i(x, 0)$ and $f_i(0, y)$ are zero polynomials for $i = 2, 3$. Hereafter, we assume that the bipartite graph $G = \Gamma_{q^M}(f(x)g(y), h(x, y))$ has girth at least 8, where $f, g \in \mathbb{F}_q[x]_m$ are monic polynomials with $f(0) = g(0) = 0$ and $h(x, y) = \sum_{1 \leq i, j \leq n} h_{i,j} x^i y^j \in \mathbb{F}_q[x, y]_n$ is a nonzero polynomial.

In the end of this section we show a lemma which is useful for the characterization of the polynomials f, g, h .

Lemma 3. *Suppose that $1 \leq D \leq mn$, $1 \leq N < q/2$ and $W \subseteq \mathbb{F}_q^2$ is a nonempty set such that, for any $(a, b) \in W$,*

$$\min\{|\{c \in \mathbb{F}_q : (a, c) \in W\}|, |\{d \in \mathbb{F}_q : (d, b) \in W\}|\} > 2N. \quad (5)$$

Let $\{e_i\}_{1 \leq i \leq D}$ be a family of polynomials in $\mathbb{F}_q[x, y]_N$ such that, for any $(a, b) \in W$, the t -polynomial $\sum_{1 \leq i \leq D} e_i(a, b)t^i \in \mathbb{F}_q[t]_D$ has no root in $\mathbb{F}_{q^M}^$. Then, there is an integer $s \in [1, D]$ such that $e_s(a, b) \neq 0$ for each $(a, b) \in W$ and e_i is the zero polynomial for any $i \neq s$.*

Proof. Let $s \in [1, D]$ be an integer such that f_s is not the zero polynomial. Let $e_s(x, y) = \sum_{0 \leq i, j \leq N} w_{i,j} x^i y^j$. If e_s is always equal to 0 over W , then, for any $(a, b) \in W$, from $|\{c \in \mathbb{F}_q : (a, c) \in W\}| > 2N$ we see $\sum_{0 \leq i \leq N} w_{i,j} a^i = 0$, $0 \leq j \leq N$ and thus from $|\{d \in \mathbb{F}_q : (d, b) \in W\}| > 2N$ we see $w_{i,j} = 0$, $0 \leq i, j \leq N$, contradicts to the assumption. Hence, there is some pair $(a, b) \in W$ such that $e_s(a, b) \neq 0$. For such a pair (a, b) , let

$$A = \{d \in \mathbb{F}_q : (d, b) \in W, e_s(d, b) \neq 0\},$$

$$B = \{c \in \mathbb{F}_q : (a, c) \in W, e_s(a, c) \neq 0\}.$$

From (5) and that the polynomials $e_s(a, y) \in \mathbb{F}_q[y]_N$ and $e_s(x, b) \in \mathbb{F}_q[x]_N$ have degree at most N , we see

$$\min\{|A|, |B|\} > N. \quad (6)$$

Since for any $d \in A$ the t -polynomial $\sum_{1 \leq i \leq D} e_i(d, b) t^i \in \mathbb{F}_q[t]_D$ can be decomposed completely in \mathbb{F}_{q^M} and has no nonzero root, we have $e_i(d, b) = 0$ for any $i \neq s$. Hence, from (6) the x -polynomial $e_i(x, b) \in \mathbb{F}_q[x]_N$ is the zero polynomial for any $i \neq s$. Similarly, one can conclude that the y -polynomial $e_i(a, y) \in \mathbb{F}_q[y]_N$ is the zero polynomial for any $i \neq s$. Therefore, from (6) we see that the polynomial $e_i(x, y) \in \mathbb{F}_q[x, y]$ is the zero polynomial for any $i \neq s$. Clearly, we have $e_s(a, b) \neq 0$ for any $(a, b) \in W$. \square

3 Characterization of f, g

In this section, we consider to characterize the univariate polynomials f, g .

Lemma 4. (i) If $a, b \in \mathbb{F}_{q^M}$ are distinct with $f(a) = f(b)$, then the y -polynomial

$$\theta_{a,b}(y) = h(a, y) - h(b, y) \in \mathbb{F}_{q^M}[y]$$

is injective in \mathbb{F}_{q^M} .

(ii) If $c, d \in \mathbb{F}_{q^M}$ are distinct with $g(c) = g(d)$, then the x -polynomial

$$\phi_{c,d}(x) = h(x, c) - h(x, d) \in \mathbb{F}_{q^M}[x]$$

is injective in \mathbb{F}_{q^M} .

Proof. Since (ii) is symmetrical to (i), we only give proof for (i).

Suppose $a, b \in \mathbb{F}_{q^M}$ are distinct with $f(a) = f(b)$. Let $S_1 = (a, b; c, d)$. Since G has no 4-cycle, from $\Delta_2(f(x)g(y))(S_1) = 0$ we see

$$\Delta_2(h(x, y))(S_1) = \theta_{a,b}(c) - \theta_{a,b}(d) \neq 0$$

for any $c, d \in \mathbb{F}_{q^M}$ with $c \neq d$, and thus $\theta_{a,b}(y)$ must be injective in \mathbb{F}_{q^M} . \square

Lemma 5. *At least one of the polynomials f, g is injective in \mathbb{F}_{q^M} .*

Proof. Suppose that neither f nor g is injective in \mathbb{F}_{q^M} . Let a, b, c, d be elements in \mathbb{F}_{q^M} with $a \neq b$, $c \neq d$ such that $f(a) = f(b)$, $g(c) = g(d)$. According to Lemma 4, the polynomials $\theta_{a,b}, \phi_{c,d} \in \mathbb{F}_{q^M}[x]$ are injective in \mathbb{F}_{q^M} . For $S_2 = (a, b, t'; t, c, d)$, we have $\Delta_3(f(x)g(y))(S_2) = 0$ and

$$\Delta_3(h(x, y))(S_2) = h(b, c) - h(a, d) + \theta_{a,b}(t) - \phi_{c,d}(t').$$

Since $\theta_{a,b}$ is injective in \mathbb{F}_{q^M} , for any $t \in \mathbb{F}_{q^M} \setminus \{c, d\}$ we have

$$h(b, c) - h(a, d) + \theta_{a,b}(t) \notin \{\phi_{c,d}(a), \phi_{c,d}(b)\}.$$

Therefore, from that $\phi_{c,d}$ is injective in \mathbb{F}_{q^M} , for $t \in \mathbb{F}_{q^M} \setminus \{c, d\}$ there exists some $t' \in \mathbb{F}_{q^M} \setminus \{a, b\}$ such that $\Delta_3(h(x, y))(S_2) = 0$ and thus G has a 6-cycle of form S_2 , contradicts to the assumption. \square

Lemma 6. *There are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^M}$ satisfying $f(x_0) = f(x_1) = f(x_2)$ or $g(x_0) = g(x_1) = g(x_2)$.*

Proof. Assume in contrast that distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^M}$ satisfy $f(x_0) = f(x_1) = f(x_2)$. For $S_3 = (x_0, x_1, x_2; y_0, y_1, y_2)$, we have $\Delta_3(f(x)g(y))(S_3) = 0$ and

$$\Delta_3(h(x, y))(S_3) = \theta_{x_0, x_1}(y_0) + \theta_{x_1, x_2}(y_1) + \theta_{x_2, x_0}(y_2),$$

where the polynomials $\theta_{x_0, x_1}, \theta_{x_1, x_2}, \theta_{x_2, x_0}$ are injective in \mathbb{F}_{q^M} according to Lemma 4. Therefore, from $\theta_{x_0, x_1} + \theta_{x_1, x_2} + \theta_{x_2, x_0} = 0$ there are distinct $y_0, y_1, y_2 \in \mathbb{F}_{q^M}$ satisfying $\Delta_3(h(x, y))(S_3) = 0$, and thus G contains a 6-cycle of form S_3 , contradicts to the assumption.

Similarly, one can show that there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^M}$ satisfying $g(x_0) = g(x_1) = g(x_2)$. \square

Lemma 7. *Let $D \in [1, mn]$ and $T(x) \in \mathbb{F}_q[x]_D$ be a monic polynomial with $T(0) = 0$.*

(i) *If $T(x)$ is injective in \mathbb{F}_{q^M} , then there is some $u \in K_p$ such that*

$$T(x) = x^u. \quad (7)$$

(ii) *If $T(x)$ is not injective in \mathbb{F}_{q^M} and there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^M}$ with $T(x_0) = T(x_1) = T(x_2)$, then there are some $v \in K_p$ and $a \in \mathbb{F}_q$ such that*

$$T(x) = \rho_a^v(x). \quad (8)$$

Proof. From $D \in [1, mn]$ we see any polynomial in $\mathbb{F}_q[x]_D$ can be completely decomposed in $\mathbb{F}_{q^M}^*$.

Assume first that the polynomial $T(x) \in \mathbb{F}_q[x]_D$ has no root in $\mathbb{F}_{q^M}^*$. Clearly, we have $T(x) = x^d$, where d is the degree of $T(x)$. Let u be the largest integer in K_p with $u \mid d$. Then the integer $k = d/u \in [1, D]$ is coprime to p . We note that $x^k - 1$ has no repeated roots in $\mathbb{F}_{q^M}^*$ since its derivative kx^{k-1} has no root in $\mathbb{F}_{q^M}^*$. Therefore, the polynomial $x^k - 1 \in \mathbb{F}_q[x]_N$ has k distinct roots in $\mathbb{F}_{q^M}^*$. If $T(x)$ is injective in \mathbb{F}_{q^M} , we have $k = 1$ and thus (7) follows, where we note that x^i is an injective in \mathbb{F}_{q^M} if and only if i is coprime to $q^M - 1$. If $T(x)$ is not injective in \mathbb{F}_{q^M} and there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^M}$ with $T(x_0) = T(x_1) = T(x_2)$, then we must have $k = 2$ and thus (8) is valid for $a = 0$.

Assume now that $T(x) \in \mathbb{F}_q[x]_D$ has roots in $\mathbb{F}_{q^M}^*$ and there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^M}$ with $T(x_0) = T(x_1) = T(x_2)$. Clearly, $T(x)$ has just one root in $\mathbb{F}_{q^M}^*$. Hence, we have $T(x) = x^r(x-a)^s$ for some $a \in \mathbb{F}_{q^M}^*$ and positive integers r, s . Let v be the largest integer in K_p dividing both r and s . Then, the positive integers $k = r/v$ and $l = s/v$ satisfy $2 \leq k+l \leq D$ and that at least one of $k+l$ and k is not divided by p . Hence, the x -polynomial $(k+l)x - ka \in \mathbb{F}_{q^M}[x]$ is not the zero polynomial.

Furthermore, we assume $k+l > 2$ and write $R(x) = x^k(x-a)^l$. From $v \in K_p$ and $R^v(x) = T(x)$, we see that $R(x)$ is also a polynomial in $\mathbb{F}_q[x]_D$ and there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^M}$ with $R(x_0) = R(x_1) = R(x_2)$. Therefore, for any $b \in \mathbb{F}_q \setminus \{0, a\}$ we have $R(b) \in \mathbb{F}_q^*$ and there are some $\alpha(b) \in \mathbb{F}_{q^M} \setminus \{0, a, b\}$ and integers $u(b) \geq 1, v(b) \geq 0$ such that

$$R(x) - R(b) = (x-b)^{u(b)}(x-\alpha(b))^{v(b)}. \quad (9)$$

For any $b \in \mathbb{F}_q \setminus \{0, a\}$ with $(k+l)b - ka \neq 0$, since the derivative of $R(x)$ is

$$R'(x) = x^{k-1}(x-a)^{l-1}((k+l)x - ka),$$

we see $R'(b) \neq 0$ and thus from (9) we have $u(b) = 1, v(b) = k+l-1 \geq 2$ and $(k+l)\alpha(b) - ka = 0$. Hence, $k+l$ is not divided by p and $\alpha = \alpha(b) = ka/(k+l)$ is independent of b . Then, for any $b \in \mathbb{F}_q \setminus \{0, a, \alpha\}$, from (9) we have

$$x^k(x-a)^l - b^k(b-a)^l - (x-b)(x-\alpha)^{k+l-1} = 0. \quad (10)$$

From (1) we have

$$|\mathbb{F}_q \setminus \{0, a, \alpha\}| \geq q - 3 > mn \geq D \geq k+l, \quad (11)$$

and thus from (10) we see that the y -polynomial

$$x^k(x-a)^l - y^k(y-a)^l - (x-y)(x-\alpha)^{k+l-1}$$

is the zero polynomial, which is impossible since its leading term is y^{k+l} . Hence, we must have $k = l = 1, a \in \mathbb{F}_q^*$ and thus (8) follows. \square

From Lemmas 5, 6 and 7, we can determine the forms of $f(x)$ and $g(y)$ as showing in the following corollary.

Corollary 1. *The polynomials f, g can be classified into two cases:*

Case 1. *Just one of f, g is injective in \mathbb{F}_{q^M} and thus there are $u \in K_p \cap [1, m/2], v \in K_p \cap [1, m]$ and $a \in \mathbb{F}_q$ such that $f(x) = \rho_a^u(x), g(y) = y^v$ or $f(x) = x^v, g(y) = \rho_a^u(y)$.*

Case 2. *Both f, g are injective in \mathbb{F}_{q^M} and thus there are $u, v \in K_p \cap [1, m]$ such that $f(x) = x^u, g(y) = y^v$.*

4 Characterization of h for Case 1

In this section, we consider to characterize the polynomial $h(x, y)$ for the first case shown in Corollary 1.

For any $a \in \mathbb{F}_q$, let $Q_q(a)$ denote the set of pairs $(c, d) \in \mathbb{F}_q^2$ such that $\rho_a(c), \rho_a(d)$ are distinct elements in \mathbb{F}_q^* , i.e.

$$Q_q(a) = \{(c, d) \in (\mathbb{F}_q \setminus \{0, a\})^2 : c \neq d, c + d \neq a\}. \quad (12)$$

For $j \in [1, n]$, let $h_j \in \mathbb{F}_q[x]_n$ denote the polynomial defined by

$$h_j(x) = \sum_{1 \leq i \leq n} h_{i,j} x^i. \quad (13)$$

Then, the polynomial $h(x, y)$ can be expressed as $\sum_{1 \leq j \leq n} h_j(x) y^j$. The following lemma gives some conditions on the polynomials h_j for the case $f(x)g(y) = \rho_a^u(x)y^v, a \in \mathbb{F}_q, u, v \in K_p$.

Lemma 8. *Suppose $f(x) = \rho_a^u(x), g(y) = y^v$ for some $a \in \mathbb{F}_q$ and $u, v \in K_p \cap [1, m]$ with $2u \leq m$. Let b be the element in \mathbb{F}_q with $b^v = a$. Then, there exists an $s \in K_p \cap [1, n]$ such that*

$$h_s(d^v)\rho_b^{su}(c) - h_s(c^v)\rho_b^{su}(d) \neq 0, \text{ for any } (c, d) \in Q_{q^M}(b), \quad (14)$$

$$h_s(c) - h_s(a - c) \neq 0, \text{ for any } c \in \mathbb{F}_{q^M} \text{ with } 2c \neq a, \quad (15)$$

$$h(x, y) = h_s(x)y^s + \sum_{(i,j) \in \Phi_p(u,v), j \neq s} h_{2i,j} \rho_a^i(x) y^j. \quad (16)$$

Proof. For $(c, d) \in Q_{q^M}(b)$ and $t \in \mathbb{F}_{q^M}^*$, let

$$S_4 = (0, c^v, d^v; t\rho_b^u(d), 0, t\rho_b^u(c)).$$

Then, from $v \in K_p$ and $b^v = a$ we see

$$\begin{aligned} \Delta_3(f(x)g(y))(S_4) &= f(d^v)g(t\rho_b^u(c)) - f(c^v)g(t\rho_b^u(d)) \\ &= t^v(cd)^{uv}((d^v - a)^u(c - b)^{uv} - (c^v - a)^u(d - b)^{uv}) \end{aligned}$$

is equal to 0 and thus the t -polynomial

$$\begin{aligned} \Delta_3(h(x, y))(S_4) &= h(d^v, t\rho_b^u(c)) - h(c^v, t\rho_b^u(d)) \\ &= \sum_{1 \leq j \leq n} (h_j(d^v)\rho_b^{ju}(c) - h_j(c^v)\rho_b^{ju}(d))t^j \\ &= \sum_{1 \leq j \leq n} H_j(d, c)t^j \end{aligned} \tag{17}$$

has no root in $\mathbb{F}_{q^M}^*$, where

$$H_j(x, y) = h_j(x^v)\rho_b^{ju}(y) - h_j(y^v)\rho_b^{ju}(x). \tag{18}$$

Clearly, for any $j \in [1, n]$, the polynomial $H_j(x, y)$ belongs to $\mathbb{F}_q[x, y]_{mn}$. By applying Lemma 3 for $W = Q_q(b) \subset Q_{q^M}(b)$, $e_j = H_j$, $D = n$ and $N = mn$, according to (1) we see that there is an integer $s \in [1, n]$ such that

$$H_j(x, y) = 0, \text{ for } j \neq s. \tag{19}$$

For any $(c, d) \in Q_{q^M}(b)$, from (17) and (19) we have $\Delta_3(h(x, y))(S_4) = H_s(d, c)t^s$ and thus (14) follows.

If $j \neq s$ and $h_j(x)$ is not the zero polynomial, from (18) and (19) we see

$$h_j(x^v) = \frac{h_j(c^v)}{\rho_b^{ju}(c)}\rho_b^{ju}(x),$$

and then v divides ju and, moreover, from $v \in K_p$ and $b^v = a$ we have

$$h_j(x) = h_{2ju/v, j}\rho_a^{ju/v}(x). \tag{20}$$

Hence, we have

$$h(x, y) = h_s(x)y^s + \sum_{j \neq s, v|ju} h_{2ju/v, j}\rho_a^{ju/v}(x)y^j. \tag{21}$$

For any $c \in \mathbb{F}_{q^M}$ with $2c \neq a$, from $f(c) = \rho_a^u(c) = \rho_a^u(a - c) = f(a - c)$, Lemma 4 and (21) we see that the y -polynomial

$$h(c, y) - h(a - c, y) = \sum_{1 \leq j \leq n} (h_j(c) - h_j(a - c))y^j = (h_s(c) - h_s(a - c))y^s$$

is injective in \mathbb{F}_{q^M} . Therefore, we have (15) and according to Lemma 7 we see $s \in K_p \cap [1, n]$.

Assume now $c \in \mathbb{F}_q \setminus \{0, b\}$. Since for any $j \in [1, n] \setminus K_p$ the y -polynomial $\rho_b^j(c) - \rho_b^j(y) + (\rho_b(y) - \rho_b(c))^j$ has at most $2j$ roots in \mathbb{F}_q , from (1) we see $q - 4 > \sum_{2 \leq j \leq n} 2j = n(n+1) - 2$ and thus there is some $d \in \mathbb{F}_q \setminus \{0, b, c, b - c\}$ such that, for any $j \in [1, n] \setminus K_p$,

$$\rho_b^j(c) - \rho_b^j(d) + (\rho_b(d) - \rho_b(c))^j \neq 0. \quad (22)$$

Clearly, $(c, d) \in Q_q(b)$. For $t \in \mathbb{F}_{q^M}^*$, let

$$S_5 = (0, c^v, d^v; 0, t\rho_b^u(d), t(\rho_b^u(d) - \rho_b^u(c))).$$

Then, from $s \in K_p$ we have

$$\begin{aligned} & \Delta_3(h_s(x)y^s)(S_5) \\ &= (h_s(c^v) - h_s(d^v))(t\rho_b^u(d))^s + h_s(d^v)(t(\rho_b^u(d) - \rho_b^u(c)))^s \\ &= (h_s(c^v)\rho_b^{su}(d) - h_s(d^v)\rho_b^{su}(c))t^s. \end{aligned} \quad (23)$$

For any $j \in [1, n]$ with $v \mid ju$, from $v \in K_p$ and $b^v = a$ we see

$$\rho_b^{ju}(x) = (x(x - b))^{ju} = (x^v(x^v - a))^{ju/v} = \rho_a^{ju/v}(x^v)$$

and thus from $u \in K_p$ we have

$$\begin{aligned} & \Delta_3(\rho_a^{ju/v}(x)y^j)(S_5) \\ &= \left(\rho_a^{ju/v}(c^v) - \rho_a^{ju/v}(d^v) \right) (t\rho_b^u(d))^j + \rho_a^{ju/v}(d^v) (t(\rho_b^u(d) - \rho_b^u(c)))^j \\ &= t^j \rho_b^{ju}(d) \left(\rho_b^j(c) - \rho_b^j(d) + (\rho_b(d) - \rho_b(c))^j \right)^u. \end{aligned} \quad (24)$$

If there is some $j \in [1, n] \setminus K_p$ with $v \mid ju$ such that $h_{2ju/v, j} \neq 0$, from (14) and (21) to (24) we see that the t -polynomial $\Delta_3(h(x, y))(S_5) \in \mathbb{F}_q[t]_n$ has at least two nonzero coefficients, and thus there is some $t \in \mathbb{F}_{q^M}^*$ such that $\Delta_3(h(x, y))(S_5) = 0$. Since $\Delta_3(f(x)g(y))(S_5) = 0$ can also be obtained from (24) by replacing j with v , the graph G has a 6-cycle of form S_5 , contradicts to the assumption.

Hence, we have $h_{2ju/v, j} = 0$ for any $j \in [1, n] \setminus K_p$ with $v \mid ju$, and thus (16) follows from (21). \square

The polynomial h_s in this lemma can be characterized further by the following lemma.

Lemma 9. *Let $z(x) = \sum_{1 \leq i \leq n} z_i x^i \in \mathbb{F}_q[x]_n$ and*

$$R(x, y) = z(x^v) \rho_b^w(y) - z(y^v) \rho_b^w(x), \quad (25)$$

where $b \in \mathbb{F}_q, w \in K_p \cap [1, mn/2]$ and $v \in K_p \cap [1, m]$. Suppose

$$z(c) - z(b^v - c) \neq 0, \text{ for any } c \in \mathbb{F}_q \text{ with } 2c \neq b^v, \quad (26)$$

$$R(d, c) \neq 0, \text{ for any } (c, d) \in Q_{qM}(b). \quad (27)$$

Then one of the following three cases is valid.

1. $b \neq 0, w \geq v, z_{w/v} + z_{2w/v} b^w \neq 0$ and

$$z(x) = (z_{w/v} + z_{2w/v} b^w) x^{w/v} + z_{2w/v} \rho_b^{w/v}(x). \quad (28)$$

2. $b = 0, 2w \geq v$ and there are some $w_1 \in K_p$ and $\sigma \in \{1, -1\}$ with $z_{2w/v+\sigma w_1} \neq 0$ such that

$$z(x) = z_{2w/v+\sigma w_1} x^{2w/v+\sigma w_1} + z_{2w/v} x^{2w/v}. \quad (29)$$

3. $z(x) = z_1 x, z_1 \neq 0$ and

(a) $b = 0, p = 3$ and $v = 3w$, or

(b) $b = 0, p = 2$ and $v = 4w$, or

(c) $b \neq 0, p = 2$ and $v = 2w$.

Proof. Let $[1, n]_v = \{v, 2v, \dots, nv\}$ and $R(x, y) = \sum_{i \geq 1} r_i(y) x^i$, where $r_i(y) \in \mathbb{F}_q[y]$. From (25) we have

$$r_i(y) = \begin{cases} z_j(y^{2v} - b^v y^v)^j + b^w z(y^v), & \text{if } i = w = jv \in [1, n]_v, \\ b^w z(y^v), & \text{if } i = w \notin [1, n]_v, \\ z_{2j}(y^{2v} - b^v y^v)^j - z(y^v), & \text{if } i = 2w = 2jv \in [1, n]_v, \\ -z(y^v), & \text{if } i = 2w \notin [1, n]_v, \\ z_{i/v} \rho_b^w(y), & \text{if } i \in [1, n]_v \setminus \{w, 2w\}, \\ 0, & \text{else.} \end{cases} \quad (30)$$

Let k denote the largest integer with $r_k(y) \neq 0$ and l the least integer with $r_l(y) \neq 0$. Clearly, we have $1 \leq l \leq k \leq mn$ and $R(x, y) = \sum_{l \leq i \leq k} r_i(y)x^i$.

For any $c \in \mathbb{F}_q \setminus \{0, b\}$ with $2c^v \neq b^v$, from (26) we have

$$\begin{aligned} R(b, c) &= z(b^v)\rho_b^w(c) - z(c^v)\rho_b^w(b) = z(b^v)\rho_b^w(c) \neq 0, \text{ if } b \neq 0, \\ R(b - c, c) &= z((b - c)^v)\rho_b^w(c) - z(c^v)\rho_b^w(b - c) \\ &= (z(b^v - c^v) - z(c^v))\rho_b^w(c) \neq 0, \end{aligned}$$

and thus, from (27) and $R(x, c) \in \mathbb{F}_q[x]_{mn}$, we see there are $\alpha(c) \in \mathbb{F}_q^*$ and positive integers $l(c), k(c)$ with $1 \leq l(c) < k(c) \leq mn$ such that

$$R(x, c) = \alpha(c)x^{l(c)}(x - c)^{k(c)-l(c)}. \quad (31)$$

Let

$$\Theta = \{c \in \mathbb{F}_q \setminus \{0, b\} : 2c^v \neq b^v, r_k(c) \neq 0, r_l(c) \neq 0\}.$$

For any $c \in \Theta$, according to the definitions of k and l , we have $l(c) = l$ and $k(c) = k$, and thus we see $1 \leq l < k \leq mn$ and

$$R(x, c) = \alpha(c)x^l(x - c)^{k-l}, \text{ for any } c \in \Theta. \quad (32)$$

For any $c, d \in \Theta$, from (32) we see

$$\alpha(c)d^l(d - c)^{k-l} = R(d, c) = -R(c, d) = -\alpha(d)c^l(c - d)^{k-l},$$

and then, we have $2 \nmid (k - l)$ if $p \neq 2$, and there is some $\alpha \in \mathbb{F}_q^*$ such that $\alpha(c) = \alpha c^l$ holds for any $c \in \Theta$. Therefore, we have

$$R(x, c) = \alpha c^l x^l (x - c)^{k-l}, \text{ for any } c \in \Theta. \quad (33)$$

Since for any j with $1 \leq 2j \leq n$ the polynomial $z_{2j}(t^2 - b^v t)^j - z(t) \in \mathbb{F}_q[t]$ has degree at most n and for any $j' \in [1, n]$ the polynomial $z_{j'}(t^2 - b^v t)^{j'} + b^w z(t) \in \mathbb{F}_q[t]$ has degree at most $2n$, from (1), (30) and that the polynomial $y^v \in \mathbb{F}_q[y]$ is a permutation on \mathbb{F}_q , we see easily

$$|\Theta| \geq q - 1 - 3n > mn \geq k, \quad (34)$$

where $r_i(0) = 0$ for any i has been taken into account in the first inequality. From (33) and (34) we see the polynomial $R(x, y) \in \mathbb{F}_q[x, y]$ is of form

$$R(x, y) = \alpha x^l y^l (x - y)^{k-l}. \quad (35)$$

Assume $b \neq 0$ first. From (25) and (35) we have

$$\alpha x^l b^l (x - b)^{k-l} = R(x, b) = -z(b^v) \rho_b^w(x) = -z(b^v) x^w (x - b)^w$$

and thus we see $z(b^v) = -\alpha b^l$, $l = k - l = w$ and, for $c \in \mathbb{F}_q \setminus \{0, b\}$,

$$\begin{aligned} z(x^v) &= \rho_b^{-w}(c)(z(c^v) \rho_b^w(x) + \alpha c^w x^w (x - c)^w) \\ &= \rho_b^{-w}(c)(z(c^v) + \alpha c^w) x^{2w} - \rho_b^{-w}(c)(z(c^v) b^w + \alpha c^{2w}) x^w. \end{aligned} \quad (36)$$

If $w \geq v$, then from (36) we see (28) and

$$z_{w/v} + z_{2w/v} b^w = \alpha \rho_b^{-w}(c) c^w (b^w - c^w) = -\alpha \neq 0.$$

If $w < v$, then from (36) we see $p = 2$, $v = 2w$, $z(x) = z_1 x$ and $z_1 = -\alpha b^{-w} \neq 0$.

Assume $b = 0$ now. For any $c \in \mathbb{F}_q^*$, from (25) and (35) we have

$$z(x^v) = c^{-2w} (z(c^v) x^{2w} + \alpha c^l x^l (x - c)^{k-l}). \quad (37)$$

Since the left side of (37) is independent of c , we see that the expansion of $(x - c)^{k-l}$ has at most two terms and thus we have $k - l \in K_p$. Furthermore, from (37) we have $2w \in \{k, l\}$ and

$$z(x^v) = \begin{cases} c^{-2w} (z(c^v) + \alpha c^l) x^{2w} - \alpha x^l, & \text{if } l < 2w = k, \\ c^{-2w} (z(c^v) - \alpha c^k) x^{2w} + \alpha x^k, & \text{if } l = 2w < k, \end{cases}$$

namely, there are integers $w_0 \in K_p$ and $\sigma \in \{1, -1\}$ such that

$$z(x^v) = c^{-2w} (z(c^v) - \sigma \alpha c^{2w+\sigma w_0}) x^{2w} + \sigma \alpha x^{2w+\sigma w_0}. \quad (38)$$

From (38) and $\sigma \alpha \neq 0$ we see

$$v \mid (2w + \sigma w_0), \quad (39)$$

$z_{(2w+\sigma w_0)/v} = \sigma \alpha$ and

$$c^{-2w} (z(c^v) - \sigma \alpha c^{2w+\sigma w_0}) = \begin{cases} z_{2w/v}, & \text{if } 2w \geq v, \\ 0, & \text{otherwise.} \end{cases} \quad (40)$$

If $2w \geq v$, from (39) we have $v \leq w_0$ and thus from (38) and (40) we see that (29) is true for $w_1 = w_0/v \in K_p$ and $z_{2w/v+\sigma w_1} = \sigma \alpha \neq 0$.

If $2w < v$, from (39) we have $\sigma = 1$,

$$p = 3, v = 3w, w_0 = w, \text{ or } p = 2, v = 4w, w_0 = 2w,$$

and thus from (38) and (40) we see $z(x) = z_1 x$ and $z_1 = \sigma \alpha \neq 0$.

The proof is completed. \square

It has been shown in Corollary 1 that the polynomials f, g can be classified into two cases, the following theorem characterizes the polynomial $h(x, y)$ further for the first case by using Lemmas 8 and 9.

Theorem 3. *Let $a \in \mathbb{F}_q, u, v \in K_p$ with $u \leq m/2$ and $v \leq m$.*

(i) If $f(x) = \rho_a^u(x)$ and $g(y) = y^v$, then there are $s \in K_p \cap [1, n]$ and $\zeta \in \mathbb{F}_q^$ such that either*

$$v \leq su \text{ and } \mu_{a,u,v}(h)(x, y) = \zeta x^{su/v} y^s, \text{ or} \quad (41)$$

$$p = 2, a \neq 0, v = 2su \text{ and } \mu_{a,u,2su}(h)(x, y) = \zeta xy^s, \text{ or} \quad (42)$$

$$p = 2, a = 0, v \leq 4su \text{ and } \mu_{0,u,v}(h)(x, y) = \zeta x^{4su/v} y^s. \quad (43)$$

(ii) If $f(x) = x^v$ and $g(y) = \rho_a^u(y)$, then there are $s \in K_p \cap [1, n]$ and $\zeta \in \mathbb{F}_q^$ such that either*

$$v \leq su \text{ and } \nu_{a,u,v}(h)(x, y) = \zeta x^s y^{su/v}, \text{ or} \quad (44)$$

$$p = 2, a \neq 0, v = 2su \text{ and } \nu_{a,u,2su}(h)(x, y) = \zeta x^s y, \text{ or} \quad (45)$$

$$p = 2, a = 0, v \leq 4su \text{ and } \nu_{0,u,v}(h)(x, y) = \zeta x^s y^{4su/v}. \quad (46)$$

Proof. Since (ii) is symmetrical to (i), we only give proof for (i).

Assume $f(x) = \rho_a^u(x)$ and $g(y) = y^v$. According to Lemmas 8 and 9, we see that there are $s \in K_p \cap [1, n]$ and $\zeta \in \mathbb{F}_q^*$ such that either (41), or (42), or

$$a = 0, v \leq 2su \text{ and } \mu_{0,u,v}(h)(x, y) = \zeta x^{2su/v + \sigma w_1} y^s \quad (47)$$

for some $w_1 \in K_p \cap [1, n]$ and $\sigma \in \{1, -1\}$, or

$$a = 0, p = 3, v = 3su \text{ and } \mu_{0,u,3su}(h)(x, y) = \zeta xy^s, \text{ or} \quad (48)$$

$$a = 0, p = 2, v = 4su \text{ and } \mu_{0,u,4su}(h)(x, y) = \zeta xy^s. \quad (49)$$

Assume that (47) is valid for some $w_1 \in K_p \cap [1, n]$ and $\sigma \in \{1, -1\}$. From Lemma 1 we see that G is isomorphic to

$$G_1 = \Gamma_{q^M}(x^{2su/v + \sigma w_1} y^s, x^{2u} y^v),$$

and thus G_1 has girth at least 8. If $\sigma = -1$, from Lemmas 6 and 7 we have $w_1 = su/v$ and that (41) is valid for $a = 0$. If $\sigma = 1$, from Lemmas 6 and 7 we have either $p = 2, w_1 = 2su/v$ or $p = 3, w_1 = su/v$. Clearly, the former case implies (43), and the later case is impossible since G_1 is isomorphic to $\Gamma_{q^M}(x^3 y, x^2 y)$ whose girth is 6 according to (ii) of Lemma 2.

Assume that (48) is valid. Then, from Lemma 1 and $p = 3$ we see that G is isomorphic to

$$\Gamma_{q^M}(x^{2u}y^{3su}, xy^s) \cong \Gamma_{q^M}(x^2y^3, xy)$$

whose girth is 6 according to (ii) of Lemma 2, contradicts to that G has girth at least 8.

Clearly, (49) implies (43) for $v = 4su$. \square

5 Characterization of h for Case 2

The following theorem characterizes the polynomial $h(x, y)$ for the second case of Corollary 1.

Theorem 4. *Assume $f(x) = x^u$ and $g(y) = y^v$ for some $u, v \in K_p \cap [1, m]$. Then, either there is some $s \in K_p \cap [1, n]$ with $v \leq su$ and $h_{2su/v, s} \neq 0$ such that*

$$\pi_{u,v}(h)h(x, y) = h_{2su/v, s}x^{2su/v}y^s, \quad (50)$$

or there is some $r \in K_p \cap [1, n]$ with $u \leq rv$ and $h_{r, 2rv/u} \neq 0$ such that

$$\pi_{u,v}(h)(x, y) = h_{r, 2rv/u}x^ry^{2rv/u}. \quad (51)$$

Proof. For $a, b \in \mathbb{F}_{q^M}^*, t \in \mathbb{F}_{q^M} \setminus \{0, 1\}$, let

$$S_6 = (0, at^v, a; 0, b, b(1-t)^u). \quad (52)$$

Then, from $\Delta_3(f(x)g(y))(S_6) = \Delta_3(x^uy^v)(S_6) = 0$ we see that, for any $a \in \mathbb{F}_{q^M}^*$ and $t \in \mathbb{F}_{q^M} \setminus \{0, 1\}$, the b -polynomial

$$\Delta_3(h(x, y))(S_6) = \sum_{1 \leq j \leq n} E_j(a, t)b^j \quad (53)$$

has no root in $\mathbb{F}_{q^M}^*$, where

$$E_j(x, y) = h_j(xy^v) - h_j(x) + h_j(x)(1-y)^{uj}. \quad (54)$$

Since E_j is a polynomial in $\mathbb{F}_q[x, y]_{mn}$, by applying Lemma 3 for $W = \mathbb{F}_q^* \times (\mathbb{F}_q \setminus \{0, 1\})$, $D = n$, $N = mn$ and $e_j = E_j$, according to (1) we see that there is an $s \in [1, n]$ such that

$$E_j(x, y) = 0, \text{ for any } j \neq s. \quad (55)$$

Therefore, we have $\Delta_3(h(x, y))(S_6) = E_s(a, t)b^s$ and thus

$$E_s(a, t) \neq 0, \text{ for any } a \in \mathbb{F}_{q^M}^* \text{ and } t \in \mathbb{F}_{q^M} \setminus \{0, 1\}. \quad (56)$$

For $j \notin K_p \cup \{s\}$, from (54) and (55) we see

$$h_j(xy^v) = h_j(x)(1 - (1 - y)^{uj}) = h_j(x)(1 - (1 - y)^j)^u$$

and thus, for any positive integer k ,

$$h_j(x)(1 - (1 - y^k)^j)^u = h_j(xy^{kv}) = h_j(x)(1 - (1 - y)^j)^{ku}.$$

Hence, we have

$$h_j(x) = 0, \text{ if } j \notin K_p \cup \{s\}. \quad (57)$$

For $j \in K_p \setminus \{s\}$, from (54) and (55) we see $h_j(xy^v) = h_j(x)y^{ju}$ and thus we have $v \leq ju$ and

$$h_j(x) = \begin{cases} h_{ju/v, j}x^{ju/v}, & \text{if } v \leq ju, \\ 0, & \text{otherwise.} \end{cases} \quad (58)$$

By following the proof of Lemma 3, from (56) and

$$E_s(t, y) = \sum_{1 \leq i \leq n} h_{i, s}(y^{iv} - 1 + (1 - y)^{su})t^i,$$

one can show that there is an $r \in [1, n]$ such that $h_{r, s} \neq 0$ and

$$t^{rv} - 1 + (1 - t)^{su} \neq 0, \text{ for any } t \in \mathbb{F}_{q^M} \setminus \{0, 1\}, \quad (59)$$

$$h_{i, s}(y^{iv} - 1 + (1 - y)^{su}) = 0, \text{ for any } i \neq r. \quad (60)$$

From (59), we see $su \neq rv$ if $s \in K_p$. Therefore, from (60) we see

$$h_s(x) = \begin{cases} h_{r, s}x^r + h_{su/v, s}x^{su/v}, & \text{if } s \in K_p \text{ and } v \leq su, \\ h_{r, s}x^r, & \text{otherwise,} \end{cases}$$

and thus from (57) and (58) we have

$$\pi_{u, v}(h)(x, y) = h_{r, s}x^ry^s. \quad (61)$$

According to (61) and Lemma 1, we see that $G_2 = \Gamma_{q^M}(x^ry^s, x^uy^v)$ is isomorphic to G and thus has girth at least 8.

Assume $r, s \in K_p$ first. From $su \neq rv$ we see there is a $w_2 \in K_p \setminus \{1\}$ such that either $su = rvw_2$ or $rv = suw_2$. If the polynomial $t^{w_2-1} - 1 \in \mathbb{F}_q[t]$

has some root t_0 in $\mathbb{F}_{q^M} \setminus \{1\}$, then G_2 contains the 6-cycle $(0, 1, t_0^v; t_0^u, 0, 1)$, contradicts to that G_2 has girth at least 8. Hence, $t^{w_2-1} - 1 \in \mathbb{F}_q[t]$ has no root in $\mathbb{F}_{q^M} \setminus \{1\}$ and thus $w_2 = 2 = p$, and either (50) or (51) is valid.

Assume $r \notin K_p$ now. According to Lemmas 6 and 7, we have $s \in K_p$, $2 \mid r$, $r/2 \in K_p$ and $p \neq 2$. Then, from (59) we see $t^{rv} - t^{su} \neq 0$ for any $t \in \mathbb{F}_{q^M} \setminus \{0, 1\}$, and thus

$$|rv - su| \in K_p. \quad (62)$$

If $rv < su$, then from (62) we have $p = 3$, $3rv = 2su$ and thus from Lemma 1 we see that G_2 is isomorphic to

$$\Gamma_{q^M}(x^{3vr}y^{3vs}, x^u y^v) \cong \Gamma_{q^M}(x^{2su}y^{3vs}, x^u y^v) \cong \Gamma_{q^M}(x^2 y^3, xy)$$

whose girth is of 6, contradicts to that G_2 has girth at least 8. Hence, we have $rv > su$ and thus from (62) we see $rv = 2su$ and (50).

Similarly, one can show that (51) is valid if $s \notin K_p$. \square

6 Proof of the main result and concluding remarks

In this section, we complete the proof of the main result of this paper and give some concluding remarks.

Proof of Theorem 1: One hand, we assume that the bipartite graph $G = \Gamma_{q^M}(f(x)g(y), h(x, y))$ has girth of at least eight. According to Corollary 1, Theorems 3 and 4, we see that the polynomials f, g, h must be of the desired forms, namely, there are some $a \in \mathbb{F}_q$, $\zeta \in \mathbb{F}_q^*$, $u, v \in K_p \cap [1, m]$ and $s \in K_p \cap [1, n]$ such that one of (i) to (iv) is valid, where it should be noticed that (43) and (46) are included in (iii) indeed.

On the other hand, we assume that there are some $a \in \mathbb{F}_q$, $\zeta \in \mathbb{F}_q^*$, $u, v \in K_p \cap [1, m]$ and $s \in K_p \cap [1, n]$ such that one of (i) to (iv) is valid. According to the isomorphisms given in Lemma 1, one can show easily that the graph G must be isomorphic to $\Gamma_3(\mathbb{F}_{q^M})$. For example, if (a) of (iv) is valid, then from $u, s \in K_p$, $p = 2$ and $a \in \mathbb{F}_q^*$ we see G is isomorphic to

$$\begin{aligned} \Gamma_{q^M}(\rho_a^u(x)y^{2su}, xy^s) &\cong \Gamma_{q^M}(\rho_a(x)y^{2s}, xy^s) \cong \Gamma_{q^M}(\rho_a(x)y^2, xy) \\ &\cong \Gamma_{q^M}(\rho_a(x)y^2, x^2y^2) \cong \Gamma_{q^M}(xy^2, x^2y^2) \cong \Gamma_{q^M}(xy^2, xy) \cong \Gamma_3(\mathbb{F}_{q^M}). \end{aligned}$$

The proof is completed. \square

Clearly, according to Theorem 1 and Lemma 1, one can characterize easily all of the polynomials $f, g \in \mathbb{F}_q[x]_m$ and $h \in \mathbb{F}_q[x, y]_n$ for which the graphs $\Gamma_{q^M}(f(x)g(y), h(x, y))$ have girth at least eight, where $M =$

$\text{lcm}\{2, 3, \dots, mn\}$. In particular, for any assemble of such polynomials f, g, h and any positive integer k , the graph $\Gamma_{q^k}(f(x)g(y), h(x, y))$ is isomorphic to $\Gamma_3(\mathbb{F}_{q^k})$.

The integer M is chosen in such a way so as to ensure any polynomial in $\mathbb{F}_q[x]_{mn}$ is decomposable in \mathbb{F}_{q^M} . However, only the decomposability in \mathbb{F}_{q^M} of the polynomials in $\mathbb{F}_q[x]_n$ is demanded excepting the proof of Lemma 9 needs such property for the polynomials of form $R(x, c) = z(x^v)\rho_a^w(c) - z(c^v)\rho_a^w(x) \in \mathbb{F}_q[x]_{mn}$, where $z \in \mathbb{F}_q[x]_n$, $a \in \mathbb{F}_q$, $v \in K_p \cap [1, m]$ and $w \in K_p \cap [1, mn/2]$. Hence, M may be replaced with a smaller integer if alternative proofs of Lemma 9 are available.

Even if the condition (1) is not valid, the conclusions of Theorem 1 are still true provided the integer M is replaced by rM for any positive integer r with

$$q^r > \max\{2mn + 3, mn + 3n + 1, n(n + 1) + 2\},$$

namely, it is sufficient to replace the basic field \mathbb{F}_q with \mathbb{F}_{q^r} in the proof.

At the end of this paper, in stead of giving a detailed proof for Theorem 2, we just point out that it can be proved by simply following the clues of the proof of Theorem 1. In fact, a proof of Theorem 2 can be obtained directly from that of Theorem 1 if we replace the prime powers q, q^M by ∞ , the finite fields $\mathbb{F}_q, \mathbb{F}_{q^M}$ by \mathbb{F}_∞ , the set K_p by $\{1\}$ (the set $\Phi_p(u, v)$ is then equal to $\{(1, 1)\}$), respectively, and the main arguments are still true while some of them can be simplified by skipping the redundant discussions. For example, in the proof of Theorem 1, almost the restrictions on the degree of polynomials over \mathbb{F}_q are designed to ensure their decomposability in the extension field \mathbb{F}_{q^M} , and thus these restrictions can be simply dropped since \mathbb{F}_∞ is an algebraically closed field.

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