

# ANTI-RAMSEY THRESHOLD OF CYCLES

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ABSTRACT. For graphs  $G$  and  $H$ , let  $G \xrightarrow{\text{rb}} H$  denote the property that for every *proper* edge colouring of  $G$  there is a *rainbow* copy of  $H$  in  $G$ . Extending a result of Nenadov, Person, Škorić and Steger (2017), we determine the threshold for  $G(n, p) \xrightarrow{\text{rb}} C_\ell$  for cycles  $C_\ell$  of any given length  $\ell \geq 4$ .

## §1. INTRODUCTION

In this paper we investigate an *anti-Ramsey* property of random graphs. Given graphs  $G$  and  $H$ , we denote by  $G \xrightarrow{\text{rb}} H$  the following anti-Ramsey property: for every *proper* edge colouring of  $G$  there is a *rainbow* copy of  $H$  in  $G$ , i.e. a subgraph of  $G$  isomorphic to  $H$  in which all edges have distinct colours.

In 1992, Rödl and Tuza [12] proved the following result, which answered affirmatively a question raised by Spencer (see [4, p. 29]) asking whether there are graphs of arbitrarily large girth containing a rainbow cycle in every proper edge colouring.

**Theorem 1** ([12]). *For every positive integer  $t$  and every positive  $\delta$  with  $\delta < 1/(2t + 1)$  there exists  $n_0$  such that for every  $n \geq n_0$  there exists an  $n$ -vertex graph  $G$  with girth at least  $t + 2$  having the property  $G \xrightarrow{\text{rb}} C_\ell$ , for  $2t + 1 \leq \ell \leq n^\delta$ , where  $C_\ell$  is an  $\ell$ -vertex cycle.*

In their proof, Rödl and Tuza showed that  $G(n, p) \xrightarrow{\text{rb}} C_\ell$  holds a.a.s.<sup>1</sup> for a small  $p$ . Note that since  $G \xrightarrow{\text{rb}} H$  is an increasing property, there exists a threshold<sup>2</sup>  $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$  for any fixed graph  $H$  (see [2]). In [6], Kohayakawa, Konstadinidis and Mota obtained an upper bound for the threshold  $p_H^{\text{rb}}$  for any fixed graph  $H$  in terms of the maximum 2-density  $m_2(H) = \max \{(e(J) - 1)/(v(J) - 2) : J \subseteq H, v(J) \geq 3\}$ .

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<sup>1</sup>A property in  $G(n, p)$  holds asymptotically almost surely (a.a.s.) if the probability tends to one as  $n$  tends to infinity.

<sup>2</sup>The threshold for a property is a function  $\hat{p} = \hat{p}(n)$  such that  $G(n, p)$  a.a.s. has this property if  $p \gg \hat{p}$  and a.a.s. does not have it if  $p \ll \hat{p}$ .

**Theorem 2** ([6]). *Let  $H$  be a fixed graph. Then there exists a constant  $C > 0$  such that a.a.s.  $G(n, p) \xrightarrow{\text{rb}} H$  whenever  $p = p(n) \geq Cn^{-1/m_2(H)}$ . In particular,  $p_H^{\text{rb}} \leq n^{-1/m_2(H)}$ .*

A classical result in Ramsey Theory obtained by Rödl and Ruciński [11] implies that  $n^{-1/m_2(H)}$  is the threshold for the following Ramsey property, as long as  $H$  contains a cycle: every colouring of  $E(G(n, p))$  with  $r$  colours contains a monochromatic copy of  $H$ . In view of this result, it is plausible to conjecture that  $n^{-1/m_2(H)}$  is also the threshold for the anti-Ramsey property, for any fixed graph  $H$ . However, as proved in [7], there are infinitely many graphs  $H$  for which the threshold  $p_H^{\text{rb}}$  is asymptotically smaller than  $n^{-1/m_2(H)}$ . Recently, this result was extended to a larger family of graphs (see [1]). On the other hand, Nenadov, Person, Škorić and Steger [10] proved that at least for sufficiently large cycles and complete graphs  $H$  the lower bound for  $p_H^{\text{rb}}$  matches the upper bound  $n^{-1/m_2(H)}$  of Theorem 2.

**Theorem 3** ([10]). *If  $H$  is a cycle on at least 7 vertices or a complete graph on at least 19 vertices, then  $p_H^{\text{rb}} = n^{-1/m_2(H)}$ .*

In [8], Kohayakawa, Mota, Parczyk and Schnitzer extended Theorem 3, by showing that for all complete graphs  $K_\ell$  with  $\ell \geq 5$  the threshold  $p_{K_\ell}^{\text{rb}}$  is in fact  $n^{-1/m_2(K_\ell)}$ , and for  $K_4$  we have  $p_{K_4}^{\text{rb}} = n^{-7/15} \ll n^{-1/m_2(K_4)}$ . Our result determines the threshold  $p_{C_\ell}^{\text{rb}}$  for every cycle  $C_\ell$  on  $\ell \geq 4$  vertices.

**Theorem 4.** *Let  $\ell \geq 5$  be an integer. Then  $p_{C_\ell}^{\text{rb}} = n^{-1/m_2(H)}$ . Furthermore,  $p_{C_4}^{\text{rb}} = n^{-3/4}$ .*

In Section 2 we prove Theorem 4 for cycles with at least 5 vertices. Similarly to what happens with complete graphs, the situation for  $C_4$  is different: For  $p = n^{-3/4} \ll n^{-1/m_2(C_4)}$  the random graph  $G(n, p)$  a.a.s. contains a small graph  $F$  such that  $F \xrightarrow{\text{rb}} C_4$ . In Section 3 we prove that  $p_{C_4}^{\text{rb}} = n^{-3/4}$  gives the threshold for  $C_4$ <sup>3</sup> and we finish with some concluding remarks in Section 4. We use standard notation and terminology (see e.g. [3] and [5]). In particular, given a subgraph  $H$  of a graph  $G$ , we write  $G - H$  for the graph obtained from  $G$  by removing all vertices that belong to  $H$  and all edges incident with these vertices.

## §2. CYCLES ON AT LEAST FIVE VERTICES

In [10], Nenadov, Person, Škorić, and Steger provide a general framework that reduces some Ramsey problems into deterministic problems for graphs with bounded maximum density, where the *maximum density* of a graph  $G$  is denoted by

$$m(G) = \max \left\{ \frac{e(J)}{v(J)} : J \subseteq G, v(J) \geq 1 \right\}.$$

The proof of Theorem 3 for cycles relies on the following lemma (see [10, Lemma 24]).

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<sup>3</sup>We remark that a sketch of the proof for  $C_4$  was given in a short abstract of the fourth author [9].

**Lemma 5** ([10]). *Let  $\ell \geq 7$  be an integer and  $G$  be a graph such that  $m(G) < m_2(C_\ell)$ . Then  $G \xrightarrow{\text{rb}} C_\ell$ .*

In fact they prove a slightly stronger statement for which they need a non-strict inequality relating the densities [10, Corollary 13]. The condition  $\ell \geq 7$  in Theorem 3 is simply a consequence of the restriction on the cycle length imposed in Lemma 5, as observed by the authors [10]. We extend Lemma 5, proving the following result, where we note that  $m_2(C_\ell) = (\ell - 1)/(\ell - 2)$ .

**Lemma 6.** *Let  $\ell \geq 5$  be an integer and  $G$  be a graph such that  $m(G) < (\ell - 1)/(\ell - 2)$ . Then,  $G \xrightarrow{\text{rb}} C_\ell$*

Theorem 4 thus follows immediately by replacing Lemma 5 with our Lemma 6 in the proof of Theorem 3 in [10]. We remark that the proof of Lemma 6 considers all the cycle lengths in the range  $\ell \geq 5$ , i.e. it is not a proof only for the cases  $\ell = 5$  and  $\ell = 6$ .

Throughout this section let  $\ell \geq 5$  be an integer and  $G$  be a graph with  $m(G) < (\ell - 1)/(\ell - 2)$ . We use the term *k-path* to refer to a path with  $k$  vertices. For the proof of Lemma 6, we will define a *partial* proper edge colouring of  $G$  such that every  $\ell$ -cycle has two non-adjacent edges with the same colour. Clearly, having defined such a partial edge colouring, we can extend it to a proper edge colouring (for instance, the uncoloured edges may be assigned distinct colours).

**2.1. Cycle components.** Let  $\mathcal{C}_\ell(G)$  be the set of all  $\ell$ -cycles of  $G$ . We start by defining key concepts that we use throughout our proof. The *edge intersection graph* of  $\mathcal{C}_\ell(G)$  is the graph whose vertex set is  $\mathcal{C}_\ell(G)$  and whose edges correspond to pairs  $\{C, C'\}$  such that  $C \neq C'$  and  $E(C) \cap E(C') \neq \emptyset$ . A subgraph  $H \subseteq G$  is a  *$C_\ell$ -component* of  $G$  if it is the union of all  $\ell$ -cycles corresponding to the vertices of some component of the edge intersection graph of  $\mathcal{C}_\ell(G)$ .

Let  $H_1$  be an  $\ell$ -cycle in  $G$ . A  $C_\ell$ -component  $H$  of  $G$  containing  $H_1$  can be constructed from  $H_1$  as follows. Suppose we have defined  $H_1 \subseteq \dots \subseteq H_i$  for  $i \geq 1$ . If there is an  $\ell$ -cycle  $C$  in  $G$  such that  $C \not\subseteq H_i$  and  $E(C) \cap E(H_i) \neq \emptyset$ , then we put  $H_{i+1} = H_i \cup C$ ; otherwise we terminate the construction and set  $H = H_i$ . Let  $t$  be such that  $H = H_t$ . We call  $(H_1, \dots, H_t)$  a *construction sequence* of  $H$ . For brevity, sometimes we will identify a  $C_\ell$ -component with a construction sequence of it; for example, we will write “a  $C_\ell$ -component  $(H_1, \dots, H_t)$ ”.

Note that there can be multiple new  $\ell$ -cycles appearing in  $H_{i+1}$  that were not present in  $H_i$  before; this will be the main problem to deal with when constructing the partial colouring. Also note that the process just described allows us to reconstruct a  $C_\ell$ -component starting from any  $\ell$ -cycle of it. Also note that two  $\ell$ -cycles belonging to distinct  $C_\ell$ -components may share vertices (obviously they do not share edges).

We start the colouring procedure in some  $C_\ell$ -component  $H$  of  $G$ . Once we have coloured the edges of  $H$  avoiding a rainbow  $C_\ell$ , we proceed to assign colours different from those

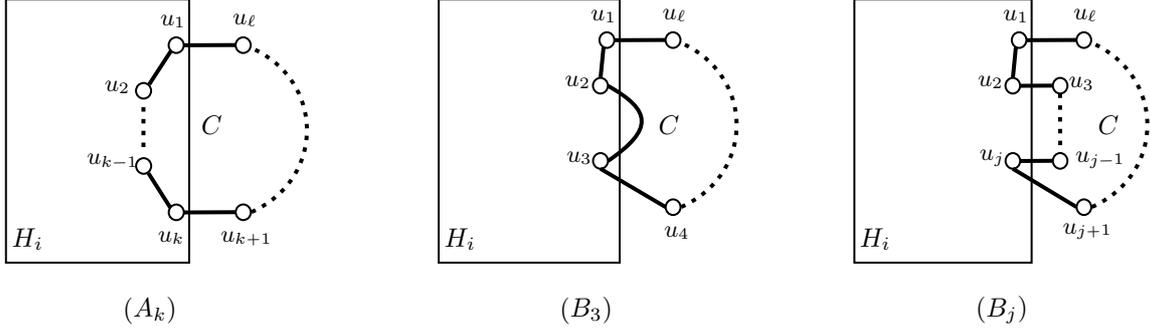


FIGURE 1. Possible configurations of a  $C_\ell$  added to  $H_i$  to form  $H_{i+1}$ .

used in  $H$  to edges of a  $C_\ell$ -component of  $G - E(H)$ , using the same procedure. We continue colouring edges in this manner (taking an uncoloured  $C_\ell$ -component, colouring it and removing its edges) until we have considered all the  $\ell$ -cycles of  $G$ . Thus, our aim is to describe the colouring procedure of an arbitrary  $C_\ell$ -component  $H$  of  $G$ .

Let  $(H_1, \dots, H_t)$  be a  $C_\ell$ -component of  $G$ . Since producing a colouring which avoids rainbow  $C_\ell$  is a trivial task if the  $C_\ell$ -component has only one cycle, we may assume  $t \geq 2$ . The following proposition is crucial in our proof and, given a  $C_\ell$ -component  $(H_1, \dots, H_t)$ , describes for any  $1 \leq i \leq t-1$  the possible structure of an  $\ell$ -cycle  $C$  which is added to  $H_i$  to form  $H_{i+1}$ , i.e.  $C \subseteq H_{i+1}$ , but  $C \not\subseteq H_i$  and  $E(C) \cap E(H_i) \neq \emptyset$ . (see Figure 1).

**Proposition 7.** *Let  $\ell \geq 5$  be an integer,  $G$  be a graph with  $m(G) < (\ell - 1)/(\ell - 2)$  and  $(H_1, \dots, H_t)$  be a  $C_\ell$ -component of  $G$ . Then, the following holds for every  $1 \leq i \leq t-1$ .*

*If  $C$  is an  $\ell$ -cycle added to  $H_i$  to form  $H_{i+1}$ , then there exists a labelling  $C = u_1 u_2 \dots u_\ell u_1$  such that exactly one of the following occurs, where  $2 \leq k \leq \ell$  and  $3 \leq j \leq \ell - 1$ :*

- (A<sub>k</sub>)  $u_1 u_2 \dots u_k$  is a  $k$ -path in  $H_i$  and  $u_{k+1}, \dots, u_\ell \notin V(H_i)$ ;
- (B<sub>j</sub>)  $u_1 u_2 \in E(H_i)$ ,  $u_2 u_3 \notin E(H_i)$ ,  $\{u_3, \dots, u_\ell\} \setminus \{u_j\} \subseteq V(H_{i+1}) \setminus V(H_i)$ ,  $u_j \in V(H_i)$ .

We refer to each of (A<sub>k</sub>) and (B<sub>j</sub>) as a *configuration* of  $H_{i+1}$ . Before proving Proposition 7, let us discuss some ideas used for this purpose. To show that some of the configurations are not possible or do not happen often during the construction of  $(H_1, \dots, H_t)$ , we heavily use the fact that  $m(G) < (\ell - 1)/(\ell - 2)$ .

For any  $1 \leq j \leq i$ , define parameters  $e_j$ ,  $v_j$  and  $c_j$  as follows:  $e_j$  is the number of edges in  $E(H_{j+1}) \setminus E(H_j)$ , while  $v_j$  stands for the number of vertices in  $V(H_{j+1}) \setminus V(H_j)$ . Lastly, let  $c_j$  be the number of components of  $H_{j+1} - H_j$ . Note that if  $v_j = 0$ , then  $e_j \geq 1$ , and if  $v_j \geq 1$ , then the components of  $H_{j+1} - H_j$  are paths and we get  $e_j \geq v_j + c_j \geq v_j + 1$ . Therefore, we conclude that, for  $1 \leq j \leq i$  we have  $e_j \geq v_j + 1$ . Also, since any  $\ell$ -cycle added to  $H_j$  to form  $H_{j+1}$  contains at least one edge of  $H_j$ , for  $1 \leq j \leq i$ , we have  $v_j \leq \ell - 2$ . Note that we have

$$\frac{\ell - 1}{\ell - 2} > m(G) \geq \frac{e(H_{i+1})}{v(H_{i+1})} = \frac{\ell + \sum_{j=1}^i e_j}{\ell + \sum_{j=1}^i v_j}. \quad (1)$$

Using the bounds  $e_j \geq v_j + 1$  and  $v_j \leq \ell - 2$ , we obtain

$$\frac{\ell - 1}{\ell - 2} > \frac{\ell + e_i + \sum_{j=1}^{i-1} (v_j + 1)}{\ell + v_i + \sum_{j=1}^{i-1} v_j} \geq \frac{\ell + e_i + (i - 1)(\ell - 1)}{\ell + v_i + (i - 1)(\ell - 2)}, \quad (2)$$

which implies

$$e_i < \frac{(\ell - 1)v_i + \ell}{\ell - 2}. \quad (3)$$

We are ready to prove Proposition 7.

*Proof of Proposition 7.* We will prove the result for all possible values of  $v_i$  (i.e.,  $0 \leq v_i \leq \ell - 2$ ). If  $v_i = \ell - 2$ , then we have configuration  $(A_2)$ .

Now let  $v_i = \ell - 3$ , which means that there are exactly three vertices of  $C$  in  $H_i$ . If these vertices form a path, then we have configuration  $(A_3)$ . On the other hand, let  $u_1, u_2$  and  $w$  be the vertices of  $C$  in  $H_i$  and let  $u_1u_2$  be an edge of  $H_i$ . If there is an edge of  $C$  between  $w$  and  $\{u_1, u_2\}$ , then let w.l.o.g.  $u_2w$  be this edge. Then, we have configuration  $(B_3)$ , where  $u_3 = w$ . If there is no edge of  $C$  between  $w$  and  $\{u_1, u_2\}$ , then w.l.o.g.  $C$  contains a path  $P_1 = u_2, u_3, \dots, u_{j-1}, w$  (with at least two edges) between  $u_2$  and  $w$  with all edges outside  $H_i$ , and a path  $P_2 = w, u_{j+1}, \dots, u_\ell, u_1$  between  $w$  and  $u_1$  with all edges outside  $H_i$ , such that  $w$  is the only common vertex of  $P_1$  and  $P_2$ . Then, we have configuration  $(B_j)$ , where  $u_j = w$  and  $4 \leq j \leq \ell - 1$  (as  $u_3$  and  $u_\ell$  are vertices outside  $H_i$ ).

Finally, let  $0 \leq v_i \leq \ell - 4$ . From (3) we have  $e_i \leq v_i + 1$ . Then,  $H_{i+1} - H_i$  has only one component, which implies that the vertices of  $C$  in  $H_i$  form a path of length  $\ell - v_i$ , where we have  $4 \leq \ell - v_i \leq \ell$ . Therefore, we have configuration  $(A_k)$  with  $4 \leq k \leq \ell$ .  $\square$

**2.2. Proof of Lemma 6.** Given a  $C_\ell$ -component  $H$  described by a construction sequence  $(H_1, \dots, H_t)$ , we will colour the edges of  $H_1, H_2$  and so on iteratively, avoiding rainbow  $\ell$ -cycles. For configurations  $(A_k)$  with  $2 \leq k \leq \ell - 2$  we are always able to assign a new colour  $i$  to two non-adjacent new edges. All other configurations may appear at most twice in  $(H_1, \dots, H_t)$ , and in these cases we will colour all previous configurations carefully so that we are able to proceed.

Arguments involving calculations similar to those we did on (1) and (2) will be referred to as *density arguments*. For example, when  $H_i$  has configuration  $(A_\ell)$ , we have  $v_i = 0$  and  $e_i = 1$ , which following the calculations in (1) and (2) implies that there cannot be another occurrence of  $(A_\ell)$ , as this would imply

$$\frac{\ell - 1}{\ell - 2} > m(G) \geq \frac{e(H_{i+1})}{v(H_{i+1})} = \frac{\ell + \sum_{j=1}^i e_j}{\ell + \sum_{j=1}^i v_j} \geq \frac{\ell + 2 + (i - 3)(\ell - 1)}{\ell + (i - 3)(\ell - 2)},$$

which gives the following contradiction, as  $\ell \geq 5$ :

$$\ell(\ell - 1) > (\ell - 2)(\ell + 2).$$

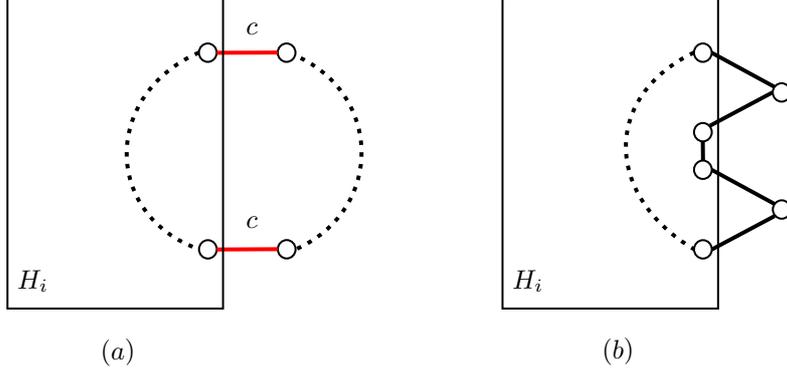


FIGURE 2. Examples of cases to consider when colouring  $E(H_{i+1}) \setminus E(H_i)$ .

Similarly, one can show that configuration  $(A_{\ell-1})$ , where  $v_i = 1$  and  $e_i = 2$ , appears at most twice and any  $(B_j)$ , where  $v_i = \ell - 3$  and  $e_i = \ell - 1$ , at most once. Furthermore, when one of these configurations appears, the occurrence of  $(A_k)$  with  $3 \leq k \leq \ell - 2$  is restricted, while only  $(A_2)$  can appear arbitrarily often. We will refer to these estimates as the *density argument*.

*Proof of Lemma 6.* Let  $\ell \geq 5$  be an integer and  $G$  be a graph such that  $m(G) < (\ell - 1)/(\ell - 2)$ . Choose an arbitrary  $\ell$ -cycle  $H_1$  in  $G$  and assign a colour  $c_1$  to a pair of non-adjacent edges of  $H_1$ . Let  $H = H_t$ , with  $t \geq 2$ , be the  $C_\ell$ -component of  $G$  obtained from a construction sequence  $(H_1, \dots, H_t)$ .

Now we consider a few cases according to which configurations given by Proposition 7 occur in  $(H_1, \dots, H_t)$ . For each  $1 \leq i \leq t - 1$ , note that there can be many cycles in  $H_{i+1}$  that are not in  $H_i$ . We will assign colours to the edges of  $E(H_{i+1}) \setminus E(H_i)$  such that in  $H_{i+1}$  any  $\ell$ -cycle has two edges coloured with the same colour.

Since the connected components of  $H_{i+1} - H_i$  are paths, in case each of these paths contains two vertices, we can give a new colour  $c$  to two non-adjacent edges of  $E(H_{i+1}) \setminus E(H_i)$ . Then, any  $\ell$ -cycle of  $H_{i+1}$  that contains these paths becomes non-rainbow (see Figure 2-(a)). If  $H_{i+1}$  has configuration  $(A_k)$  with  $2 \leq k \leq \ell - 2$ , this is how we proceed, unless stated otherwise. But it may be the case that  $H_{i+1}$  contains an  $\ell$ -cycle that is not in  $H_i$  and it does not contain such paths (it can be formed with edges between vertices of  $H_i$  and components of  $H_{i+1} - H_i$  of only one vertex (see Figure 2-(b)) and we have to be more careful colouring these edges.

Recall that by the density argument preceding this proof, configuration  $(A_\ell)$  appears at most once,  $(A_{\ell-1})$  at most twice, and any  $(B_j)$  at most once. As observed above, if for every  $1 \leq i \leq t - 1$ , the graph  $H_{i+1}$  has configuration  $(A_k)$  with  $2 \leq k \leq \ell - 2$ , we can easily avoid a rainbow  $C_\ell$  by assigning, for each  $1 \leq i \leq t - 1$ , a new colour  $c_{i+1}$  to two non-adjacent edges of  $E(H_{i+1}) \setminus E(H_i)$ . Thus, from now on we assume that there exists at least one  $H_{i+1}$  ( $1 \leq i \leq t - 1$ ) with configuration  $(A_{\ell-1})$ ,  $(A_\ell)$ , or  $(B_j)$  for some  $3 \leq j \leq \ell$ . We split our proof into a few cases, depending on the occurrence of these configurations.

**Case 1.** *There is an index  $1 \leq i_1 \leq t - 1$  such that  $H_{i_1+1}$  has configuration  $(A_\ell)$ .*

In this case, for all  $i \neq i_1$ ,  $H_{i+1}$  has configuration  $(A_2)$  or  $(A_3)$ , by the density argument. Moreover, at most one  $H_{i+1}$  (for some  $1 \leq i \leq t - 1$ ) has configuration  $(A_3)$ .

Let  $C = u_1 u_2 \cdots u_\ell u_1$  be an  $\ell$ -cycle added to  $H_{i_1}$  to form  $H_{i_1+1}$ , where  $P = u_1 u_2 \cdots u_\ell$  is an  $\ell$ -path in  $H_{i_1}$  and  $u_\ell u_1 \notin E(H_{i_1})$ . The number of  $\ell$ -cycles in  $H_{i_1+1}$  which are not in  $H_{i_1}$  is exactly the number of  $\ell$ -paths in  $H_{i_1}$  with endpoints  $u_1$  and  $u_\ell$ .

First suppose that  $P$  is the only  $\ell$ -path between  $u_1$  and  $u_\ell$  in  $H_{i_1}$ . Let  $C'$  be an  $\ell$ -cycle in  $H_{i_1}$  that contains the edge  $u_2 u_3$ . W.l.o.g. we may assume that  $H_1 = C'$ . Then, give colour  $c_1$  to two non-adjacent edges of  $C'$  that are not  $u_2 u_3$ . For every  $H_{i+1}$  with  $1 \leq i \leq i_1 - 1$  we assign a new colour  $c_{i+1}$  to two non-adjacent edges in  $E(H_{i+1}) \setminus E(H_i)$  (different from  $u_2 u_3$ ). Therefore, in step  $H_{i_1+1}$ , we can give a new colour  $c_{i_1+1}$  to  $u_1 u_\ell$  and  $u_2 u_3$ . Note that this partial colouring of  $H_{i_1+1}$  gives two edges of the same colour in each  $C_\ell$ .

Suppose that  $H_{i_1}$  contains more than one  $\ell$ -path between  $u_1$  and  $u_\ell$ . Let  $P' = u_1 x_2 \cdots x_{\ell-1} u_\ell$  with  $P' \neq P$  be one of these paths. Since there is no other configuration  $(A_k)$  with  $k \geq 4$ , one can see that  $P \cup P'$  contains cycle of length  $2\ell - 2$ ,  $2\ell - 4$ , or  $\ell$ . One can check that if  $P \cup P'$  contains an  $\ell$ -cycle  $C'$ , then  $\ell$  must be even, and  $P \cap C'$  has length  $\ell/2$ .

If  $P \cup P'$  forms a  $(2\ell - 2)$ -cycle  $C'$ , then  $C'$  appears in  $H_{i_1}$  with configuration  $(A_2)$ . W.l.o.g. we assume that  $i_1 = 2$ . Then, we colour alternately the edges of  $C'$  with a colour  $c_1$ , which implies that each of  $E(P)$  and  $E(P')$  contains at least two non-adjacent edges with the same colour. Note that  $H_2$  may contain at most one other  $\ell$ -path  $P''$  between  $u_1$  and  $u_\ell$ , in which case  $\ell$  must be even (and so  $\ell \geq 6$ ). But such  $P''$  contains at least two consecutive edges of  $P$  and two consecutive edges of  $P'$  and then it must contain two edges with colour  $c_1$ . Therefore, every  $\ell$ -cycle in  $H_{i_1+1}$  is non-rainbow.

Suppose now that  $P \cup P'$  contains a  $(2\ell - 4)$ -cycle  $C'$ . Then,  $C'$  appears in  $H_{i_1}$  with configuration  $(A_3)$  (with two  $\ell$ -cycles having exactly a 3-path in common). We may assume w.l.o.g. that  $x_2 = u_2$ ,  $H_2$  has configuration  $(A_3)$  and  $(P \cup P') - u_1 \subseteq H_2$  (note that  $C'$  lies in  $(P \cup P') - u_1$ ). We colour the edges of  $C'$  alternately with two colours  $c_1$  and  $c_2$ . If  $\ell$  is even, then there may be another  $(\ell - 1)$ -path  $P''$  between  $u_2$  and  $u_\ell$  in  $H_2$  (other than  $P - u_1$  and  $P' - u_1$ ). One can easily check that  $P''$  must contain two edges with the same colour ( $c_1$  or  $c_2$ ), by observing the colours given to the edges of  $C'$  which are adjacent to the endpoints of the 3-path  $xyz$ , where  $x, z \in C'$  and  $y$  is the unique vertex in  $((P \cup P') - u_1) - C'$ .

Now consider that  $P \cup P'$  contains an  $\ell$ -cycle  $C'$ . W.l.o.g.  $H_1 = C'$ . Thus, we just colour the edges of  $C'$  alternately with two colours  $c_1$  and  $c_2$ . Since  $\ell \geq 6$ , this implies that both paths  $P$  and  $P'$  have two non-adjacent edges with the same colour.

**Case 2.** *There are  $1 \leq i_1 < i_2 \leq t - 1$  such that  $H_{i_1+1}$  and  $H_{i_2+1}$  have configuration  $(A_{\ell-1})$ .*

By the density argument, this case occurs only if  $\ell = 5$  and every  $H_{i+1}$  has configuration  $(A_2)$  for  $i \neq i_1, i_2$ . Let  $C = u_1u_2u_3u_4u_5u_1$  and  $C' = v_1v_2v_3v_4v_5v_1$  be cycles where  $C$  is in  $H_{i_1+1}$  but not in  $H_{i_1}$  and  $C'$  is in  $H_{i_2+1}$  but not in  $H_{i_2}$ . W.l.o.g. let  $P = u_1u_2u_3u_4$  and  $P' = v_1v_2v_3v_4$  in  $H_{i_2}$  be 4-paths, and  $u_5 \notin V(H_{i_1})$  and  $v_5 \notin V(H_{i_2})$ .

Note that  $P$  is the only 4-path between  $u_1$  and  $u_4$  in  $H_{i_1}$  and thus  $C$  is the only 5-cycle added to  $H_{i_1}$  to form  $H_{i_1+1}$ . However, it is possible that besides  $P'$  there exists one other 4-path  $P''$  in  $H_{i_2}$  between  $v_1$  and  $v_4$ . If this is the case, then  $P' \cup P''$  contains either a 4-cycle or a 6-cycle. This information will be useful in what follows.

We divide this proof into three parts depending on the structure of  $P$  in  $H_{i_1}$ : (a) the three edges of  $P$  lie in the same 5-cycle, (b) exactly two consecutive edges of  $P$  lie in the same 5-cycle, or (c) any 5-cycle in  $H_{i_1}$  contains at most one edge of  $P$ .

*(a) the three edges of  $P$  lie in the same 5-cycle.*

W.l.o.g. assume that all the edges of  $P$  lie in  $H_1$  and  $i_1 = 1$ . Hence  $H_1$  is of the form  $H_1 = u_1u_2u_3u_4x_5u_1$  for some  $x_5$ . Note that  $C'' = u_1x_5u_4u_5u_1$  is a 4-cycle in  $H_2$ .

Suppose all the edges of  $P'$  lie in  $H_2$ . Then, w.l.o.g., we may assume  $i_2 = 2$ . If the endpoints of  $P'$  are  $u_1$  and  $u_3$ , then there is another 4-path  $P''$  between  $u_1$  and  $u_3$  in  $H_1$ , say w.l.o.g.  $P' = u_1u_5u_4u_3$  and  $P'' = u_1x_5u_4u_3$ . We assign a colour  $c_1$  to  $u_4x_5$ ,  $u_1u_2$  and  $u_3v_5$ , and a colour  $c_2$  to  $u_2u_3$ ,  $u_4u_5$  and  $v_5u_1$ . In this way we make all 5-cycles in  $H_3$  non-rainbow. The case in which the ends of  $P'$  are  $u_4$  and  $u_2$  is symmetric.

For all the remainder possibilities for the endpoints of  $P'$ , we assign a colour  $c_1$  to  $u_1u_2$  and  $u_4u_3$ . If the endpoints of  $P'$  are two adjacent vertices in  $V(C'')$ , then we colour two non-adjacent edges of  $C''$  with a new colour  $c_2$ . If the ends of  $P'$  are  $u_1$  and  $u_4$ , then the colouring we gave to  $u_1u_2$  and  $u_4u_3$  already makes every 5-cycle in  $H_3$  non-rainbow. If the endpoints of  $P'$  are  $x_5$  and a vertex in  $\{u_2, u_3\}$ , then we assign a new colour  $c_2$  to  $v_5x_5$  and  $u_2u_3$ . The case in which the ends of  $P'$  are  $u_5$  and a vertex in  $\{u_2, u_3\}$  is symmetric. Thus, we assume that there is no 4-path with endpoints  $v_1$  and  $v_4$  and all edges in  $H_2$ .

If at most two edges of  $P'$  are in  $H_2$ , then for any 4-path with endpoints  $v_1$  and  $v_4$  its edges in  $H_2$  must be consecutive. Hence we may assume w.l.o.g. that, for  $P'$ , the edge  $v_3v_4$  is not in  $E(H_2)$ . Because there is no triangle in  $H_{i_2}$  there can be no 6-cycle in  $H_{i_2}$ . As the unique 4-cycle in  $H_{i_2}$  has its edges in  $H_2$ , the 4-path  $P''$  between  $v_1$  and  $v_4$  ( $P'' \neq P'$ ), if it exists, contains the edge  $v_3v_4$ . Note that we can colour two non-adjacent edges of any  $H_i$  with configuration  $(A_2)$  avoiding colouring the edge  $v_3v_4$ . Thus, we assign a colour  $c_1$  to  $u_1u_2$  and  $u_3u_4$ , and a new colour  $c_2$  to  $v_3v_4$  and  $v_5v_1$ .

*(b) exactly two consecutive edges of  $P$  lie in the same 5-cycle.*

W.l.o.g.  $H_1$  contains the edges  $u_1u_2$  and  $u_2u_3$  but does not contain  $u_3u_4$ . Thus  $H_1$  is of the form  $H_1 = u_1u_2u_3x_4x_5u_1$  for some  $x_4$  and  $x_5$ , and  $C'' = u_1x_5x_4u_3u_4u_5u_1$  is a 6-cycle in  $H_{i_1+1}$ . Note that  $C''$  is the only 6-cycle in  $H_{i_2}$ , and  $H_{i_2}$  contains no 4-cycle. Hence, there are at most two 4-paths between  $v_1$  and  $v_4$ . If there are two such paths, they correspond to two internally disjoint paths in  $C''$ . Suppose that  $E(P') \subseteq$

$E(C'')$ . In this case, alternately colour the edges of  $C''$  with colours  $c_1$  and  $c_2$  and, for  $1 \leq i \leq t-1$ , with  $i \neq i_1, i_2$ , assign a new colour  $c_{i+2}$  to two non-adjacent edges in  $E(H_{i+1}) \setminus (E(H_i) \cup \{u_3u_4\})$ . Now we assume that  $E(P') \not\subseteq E(C'')$ . Thus  $P'$  is the only 4-path between  $v_1$  and  $v_4$  in  $H_{i_2}$ . If  $E(P') \subseteq E(H_1)$  then  $E(P') \cap \{u_1u_2, u_2u_3\} \neq \emptyset$  (since  $E(P') \not\subseteq E(C'')$ , the path  $P'$  cannot be  $u_1x_5x_4u_3$ ), and we colour  $u_4u_5$  and the two non-adjacent edges in  $E(P')$  with  $c_1$ . Assign a new colour  $c_{i+1}$  to two non-adjacent edges in  $E(H_{i+1}) \setminus E(H_i)$ , for  $1 \leq i \leq t-1$ ,  $i \neq i_1, i_2$ . Now we assume that  $E(P') \not\subseteq E(H_1)$  (possibly  $P' = P$ ). Therefore,  $P'$  has an edge  $v_jv_{j+1}$  with  $1 \leq j \leq 3$  which does not belong to  $E(H_1)$ . Colour  $u_2u_3, x_4x_5$  and an edge in  $\{u_5u_1, u_5u_4\} \setminus \{v_jv_{j+1}\}$  with  $c_1$ , and give a new colour  $c_{i_2+1}$  to  $v_jv_{j+1}$  and to some edge in  $\{v_5v_1, v_5v_4\}$  not incident with  $v_j$  nor with  $v_{j+1}$ .

(c) any 5-cycle in  $H_{i_1}$  contains at most one edge of  $P$ .

In  $H_{i_2}$  there are neither 4-cycles nor 6-cycles, and therefore  $P'$  is the only 4-path between  $v_1$  and  $v_4$ . We may assume w.l.o.g. that  $H_1$  contains  $u_2u_3$ . If  $P' = P$ , then we assign a colour  $c_1$  to the edges  $u_2u_3, u_5u_1$  and  $v_5u_4$ , and assign a new colour  $c_{i+1}$  to two non-adjacent edges in  $E(H_{i+1}) \setminus E(H_i)$ , for  $1 \leq i \leq t-1$ ,  $i \neq i_1, i_2$ . Now we assume that  $P' \neq P$ . Since  $P'$  is the only 4-path in  $H_{i_2}$  between  $v_1$  and  $v_4$ , we know that  $P'$  and  $P$  cannot have both endpoints in common. Therefore, w.l.o.g., we may assume that  $v_1 \notin \{u_1, u_2, u_4\}$ . We assign a new colour  $c_1$  to the edges  $u_2u_3$  and  $u_5u_1$ . If  $v_2v_3 \in \{u_2u_3, u_5u_1\}$  then colour  $v_5v_1$  with  $c_1$ , otherwise, colour  $v_2v_3$  and  $v_5v_1$  with a new colour  $c_2$ . Then, we assign a new colour  $c_{i+2}$  to two non-adjacent edges in  $E(H_{i+1}) \setminus (E(H_i) \cup \{v_2v_3\})$ , for  $1 \leq i \leq t-1$ ,  $i \neq i_1, i_2$ .

**Case 3.** There is exactly one  $1 \leq i_1 \leq t-1$  such that  $H_{i_1+1}$  has Configuration  $(A_{\ell-1})$ .

By the density argument,  $H_{i+1}$  has Configuration  $(A_k)$  with  $2 \leq k \leq 4$  for all  $i \neq i_1$ . Let  $C = u_1u_2 \cdots u_\ell u_1$  be a cycle where  $C$  is in  $H_{i_1+1}$  but not in  $H_{i_1}$  and let  $P = u_1 \cdots u_{\ell-1}$  be an  $(\ell-1)$ -path in  $H_{i_1}$ . The number of  $\ell$ -cycles in  $H_{i_1+1}$  which are not in  $H_{i_1}$  is exactly the number of  $(\ell-1)$ -paths in  $H_{i_1}$  with endpoints  $u_1$  and  $u_{\ell-1}$ . The remainder of the proof of Case 3 is similar to the proof of Case 1, but we include it here for completeness.

First, suppose that  $P$  is the only  $(\ell-1)$ -path between  $u_1$  and  $u_{\ell-1}$  in  $H_{i_1}$ . Let  $C'$  be an  $\ell$ -cycle in  $H_{i_1}$  that contains the edge  $u_2u_3$ . W.l.o.g.  $H_1 = C'$ . Then, give colour  $c_1$  to two non-adjacent edges of  $C'$  that are not  $u_2u_3$ . For every  $H_{i+1}$  with  $1 \leq i \leq i_1-1$  we assign a new colour  $c_{i+1}$  to two non-adjacent edges in  $E(H_{i+1}) \setminus E(H_i)$  different from  $u_2u_3$ . Therefore, in step  $H_{i_1+1}$ , we give a new colour  $c_{i_1+1}$  to  $u_1u_\ell$  and  $u_2u_3$ . Note that in this partial colouring of  $H_{i_1+1}$  every copy of  $C_\ell$  has two non-adjacent edges of the same colour.

Suppose that  $H_{i_1}$  contains more than one  $(\ell-1)$ -path between  $u_1$  and  $u_{\ell-1}$ . Let  $P' = u_1x_2 \cdots x_{\ell-2}u_{\ell-1}$  with  $P' \neq P$  be one of these paths. Since there is no configuration

$(A_k)$  with  $k \geq 5$ , one can see that  $P \cup P'$  contains an even cycle of length  $2\ell - 4$ ,  $2\ell - 6$ , or  $\ell$ .

If  $P' \cup P$  forms a  $(2\ell - 4)$ -cycle  $C'$  in  $H_{i_1}$  ( $P'$  and  $P$  are internally disjoint), then we may assume w.l.o.g. that  $i_1 = 2$ , that  $H_2$  has configuration  $(A_3)$ , and  $P \cup P' \subseteq H_2$ . Then, we assign alternately colours  $c_1$  and  $c_2$  to the edges of  $C'$ . Note that if  $\ell$  is even then  $H_2$  may contain another  $(\ell - 1)$ -path  $P''$  between  $u_1$  and  $u_{\ell-1}$ . But then it is not hard to see that  $P''$  contains two edges of  $C'$  with the same colour. So assume that there is no  $(2\ell - 4)$ -cycle containing  $P$ .

Suppose now that  $P \cup P'$  contains a  $(2\ell - 6)$ -cycle  $C'$ . Since there is no  $(2\ell - 4)$ -cycle containing  $P$ , we may assume w.l.o.g. that  $x_2 = u_2$ ,  $H_2$  has configuration  $(A_4)$ , and  $(P \cup P') - u_1 \subseteq H_2$ . We colour the edges of  $C'$  alternately with two colours  $c_1$  and  $c_2$ , and colour the two non-adjacent edges of  $C' \cap H_1$  with a new colour  $c_3$ . If  $\ell$  is even, then there may be another path  $P''$  between  $u_2$  and  $u_{\ell-1}$  in  $H_2$ . Such path contains the edges of  $C' \cap H_1$ , and therefore have two edges with the same colour.

Now consider that  $P \cup P'$  contains an  $\ell$ -cycle  $C'$  (of course, we have that  $\ell$  is even). We assume that there is no  $(\ell - 1)$ -path  $P''$  in  $H_{i_1}$  between  $u_1$  and  $u_{\ell-1}$  such that  $P'' \cup P$  or  $P'' \cup P'$  contains a cycle with length  $2\ell - 6$  or  $2\ell - 4$ . W.l.o.g.  $H_1 = C'$ . Thus, we just colour the edges of  $C'$  alternately with two colours  $c_1$  and  $c_2$ , and we assign a new colour  $c_{i+2}$  to two non-adjacent edges in  $E(H_{i+1}) \setminus E(H_i)$  for  $1 \leq i \leq t - 1$ ,  $i \neq i_1$ .

**Case 4.** *There is  $1 \leq i_1 \leq t - 1$  such that  $H_{i_1+1}$  has configuration  $(B_j)$  for some  $3 \leq j \leq \ell$ .*

By the density argument,  $H_{i_1+1}$  has configuration  $(A_2)$  for all  $i \neq i_1$ . Let  $C = u_1 u_2 \cdots u_\ell u_1$  be an  $\ell$ -cycle added to  $H_{i_1}$  to form  $H_{i_1+1}$ , where  $P = u_1 u_2 \cdots u_j$  is a  $j$ -path for some  $3 \leq j \leq \ell$ , and  $(\{u_3, \dots, u_\ell\} \setminus \{u_j\}) \subseteq V(H_{i_1+1}) \setminus V(H_{i_1})$ . If there is a path  $P'$  in  $H_{i_1}$  between  $u_1$  and  $u_j$  such that  $V(P') \cup \{u_{j+1}, \dots, u_\ell\}$  induces an  $\ell$ -cycle in  $H_{i_1+1}$  or there is a path  $P''$  in  $H_{i_1}$  between  $u_2$  and  $u_j$  such that  $V(P'') \cup \{u_3, \dots, u_{j-1}\}$  induces an  $\ell$ -cycle in  $H_{i_1+1}$ , then  $H_{i_1+1}$  can be constructed with a construction sequence in which the last two steps has configuration  $(A_{\ell-j+3})$  and  $(A_j)$ , respectively, and therefore we have a construction sequence that we already know how to colour (see Cases 1, 2, and 3). So we may suppose that  $H_{i_1}$  contains none of these paths, and thus we assign a new colour  $c_{i_1+1}$  to  $u_2 u_3$  and  $u_\ell u_1$ .  $\square$

### §3. CYCLE ON FOUR VERTICES

In this section we prove that  $p_{C_4}^{\text{rb}} = n^{-3/4}$ . By a classical result of Bollobás (see [5]), we know that if  $p \gg n^{-3/4}$ , then a.a.s.  $G(n, p)$  contains a copy of  $K_{2,4}$ . It is not hard to see that in any proper colouring of the edges of  $K_{2,4}$  there is a rainbow copy of  $C_4$ , which implies that  $p_{C_4}^{\text{rb}} \leq n^{-3/4} = n^{-m(K_{2,4})}$ .

Let  $G = G(n, p)$  where  $p \ll n^{-3/4}$ . To prove that a.a.s.  $p_{C_4}^{\text{rb}} \geq n^{-3/4}$ , we define a sequence  $F = C_4^1, \dots, C_4^\ell$  of copies of  $C_4$  in  $G$  as a  $C_4$ -chain if for any  $2 \leq i \leq \ell$  we have  $E(C_4^i) \cap (\bigcup_{j=1}^{i-1} E(C_4^j)) \neq \emptyset$ .

We want to show that a.a.s. there exists a proper colouring of  $G$  that contains no rainbow copy of  $C_4$ . For that, consider maximal  $C_4$ -chains with respect to the number of  $C_4$ 's. First, we colour the edges of the maximal  $C_4$ -chains avoiding in a way that all the  $C_4$ 's in such chains are non-rainbow. Then, it is enough to give new colours for each of the remaining edges (those that do not belong to the  $C_4$ -chains).

To colour the edges in the  $C_4$ -chains, from Markov's inequality and the union bound, we know that a.a.s.  $G$  does not contain any graph  $H$  with  $m(H) \geq 4/3$  and  $|V(H)| \leq 12$ . Let  $F = C_4^1, \dots, C_4^\ell$  be an arbitrary  $C_4$ -chain in  $G$  with  $m(F) \geq 4/3$ . Let  $2 \leq i \leq \ell$  be the smallest index such that  $F' = C_4^1, \dots, C_4^i$  has density  $m(F') \geq 4/3$ . Then, since  $F'' = C_4^1, \dots, C_4^{i-1}$  has density  $m(F'') < 4/3$ , it is not hard to explore the structure of  $G$  to conclude that  $|V(F'')| \leq 10$ , which implies  $|V(F')| \leq 12$ , as  $|V(F') \setminus V(F'')| \leq 2$ , a contradiction. Therefore, a.a.s. every  $C_4$ -chain  $F$  in  $G$  satisfies  $m(F) < 4/3$ .

Let  $F = C_4^1, \dots, C_4^\ell$  be any  $C_4$ -chain in  $G$  (with  $m(F) < 4/3$ ). If we have  $|V(C_4^i) \setminus V(C_4^{i-1})| = 2$  for every  $2 \leq i \leq \ell$ , then it is easy to give a new colour to two non-adjacent edges of  $E(C_4^i) - E(C_4^{i-1})$ , avoiding a rainbow copy of  $C_4$ . Note that  $F$  can have at most one  $C_4^i$  such that  $|V(C_4^i) \setminus V(C_4^{i-1})| = 1$ , as otherwise  $m(F) \geq 4/3$ . But in this case, since  $m(F) < 4/3$ , we have  $\ell \leq 4$ , which makes easy to colour  $F$  with no rainbow copies of  $C_4$ .

#### §4. CONCLUDING REMARKS

The problem of determining the threshold  $p_H^{\text{rb}}$  for the anti-Ramsey property  $G(n, p) \xrightarrow{\text{rb}} H$  for graphs  $H$  is far from being completely solved. We believe that an adaptation of the framework developed in [10] and the ideas described in this paper could be useful to prove that  $n^{-1/m_2(H)}$  is in fact the threshold for other classes of graphs, for example, not so small bipartite graphs  $H$  (note that this is not the case for  $C_4$ ). One of the main direction for future research is to solve the following problem.

**Problem 8.** *Determine all graphs  $H$  such that  $p_H^{\text{rb}} = n^{-1/m_2(H)}$ .*

We remark that the only graphs  $H$  for which the threshold is known and it is not  $n^{-1/m_2(H)}$  are cycles and complete graphs on four vertices. Thus, to determine the threshold for a large family of graphs for which it is not given by the maximum 2-density is also an interesting problem.

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