Regular homogeneously traceable nonhamiltonian graphs[*]

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Abstract

A graph is called homogeneously traceable if every vertex is an endpoint of a Hamilton path. In 1979 Chartrand, Gould and Kapoor proved that for every integer $n \geq 9$, there exists a homogeneously traceable nonhamiltonian graph of order n. The graphs they constructed are irregular. Thus it is natural to consider the existence problem of regular homogeneously traceable nonhamiltonian graphs. We prove two results: (1) For every even integer $n \geq 10$, there exists a cubic homogeneously traceable nonhamiltonian graph of order n; (2) for every integer $p \geq 18$, there exists a 4-regular homogeneously traceable graph of order p and circumference $p - 4$. Unsolved problems are posed.

Key words. Homogeneously traceable; regular graph; circumference

Mathematics Subject Classification. 05C38, 05C45, 05C76

1 Introduction

We consider finite simple graphs. The *order* of a graph is its number of vertices, and the *size* is its number of edges. We denote by $V(G)$ the vertex set of a graph G. The following concept is introduced by Skupień in 1975 (see [3, p.185], and [4]). Note that the preprint of the 1984 paper [4] was cited by the 1979 paper [2].

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Definition 1. A graph G is said to be *homogeneously traceable* if every vertex of G is an endpoint of a Hamilton path.

Obviously, hamiltonian graphs and hypohamiltonian graphs are homogeneously traceable. Chartrand, Gould and Kapoor [2] proved that for every integer n with $3 \leq n \leq 8$, any homogeneously traceable graph of order n is hamiltonian and that for $n \geq 9$, there exists a homogeneously traceable nonhamiltonian graph of order n. This result was rediscovered in [1] where the term "homogeneously traceable" was called "fully strung". The homogeneously traceable nonhamiltonian graphs constructed in [2] are irregular while the homogeneously traceable nonhamiltonian graphs constructed in [1] are also irregular except the Petersen graph of order 10 which is cubic (i.e., 3-regular). Thus it is natural to consider the existence problem of regular homogeneously traceable nonhamiltonian graphs.

In Section 2 we construct regular homogeneously traceable nonhamiltonian graphs, and in Section 3 we pose two unsolved problems.

2   Regular homogeneously traceable nonhamiltonian graphs

Given a vertex v in a graph, a *v-path* is a path with v as an endpoint. We use K_d to denote the complete graph of order d, and use $N(v)$ to denote the neighborhood of a vertex v. The notation circum(G) means the circumference of a graph G.

Definition 2. Let v be a vertex of degree d in a graph. *Blowing up v into the complete graph K_d* is the operation of replacing v by K_d and adding d edges joining the vertices of K_d to the vertices in $N(v)$ such that the new edges form a matching.

The operation of blowing up a vertex of degree 4 into K_4 is depicted in Figure 1.
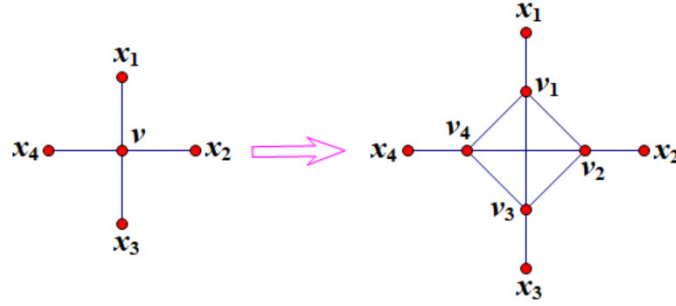
Fig. 1. Blowing up v into K$_4$

Definition 3. A graph G is called *doubly homogeneously traceable* if for any vertex v of G, there are two Hamilton v-paths P and Q such that the two edges incident to v on P and Q are distinct.

We will need the following two lemmas.

Lemma 1. *Let v be a vertex of degree 3 in a doubly homogeneously traceable graph G of order n and circumference c. Suppose G' is the graph obtained from G by blowing up v into K_3. Then G' is also doubly homogeneously traceable. If v lies in a longest cycle of G, then G' has circumference $c + 2$.*

Proof. Let $N(v) = \{x_1, x_2, x_3\}$ and suppose v is blown up into K_3 whose vertices are v_1, v_2, v_3 such that v_i is adjacent to x_i for $i = 1, 2, 3$. Let $u \in V(G')$. If $u \notin \{v_1, v_2, v_3\}$, there exist two Hamilton u-paths $P : u, \ldots, x_i, v, x_j, \ldots$ and $Q : u, \ldots, x_s, v, x_t, \ldots$ of G where the two edges incident to u on P and Q are distinct. Then G' has two Hamilton u-paths $u, \ldots, x_i, v_i, v_f, v_j, x_j, \ldots$ and $u, \ldots, x_s, v_s, v_g, v_t, x_t, \ldots$ where the two edges incident to u are distinct.

Next suppose $u \in \{v_1, v_2, v_3\}$. Without loss of generality suppose G has two Hamilton v-paths $v, x_1, \ldots$ and $v, x_2, \ldots$. Then G' has two Hamilton v_3-paths: $v_3, v_2, v_1, x_1, \ldots$ and $v_3, v_1, v_2, x_2, \ldots$. Since G is doubly homogeneously traceable, G has a Hamilton x_1-path $x_1, y, \ldots, x_i, v, x_j, \ldots$ with $y \neq v$. It follows that G' has two Hamilton v_1-paths: $v_1, x_1, y, \ldots, x_i, v_i, v_j, x_j, \ldots$ and $v_1, v_3, v_2, x_2, \ldots$, where the two edges $v_1 x_1$ and $v_1 v_3$ are distinct. Similarly we can show that G' has two Hamilton v_2-paths where the two edges incident to v_2 are distinct. This completes the proof that G' is doubly homogeneously traceable.

Now suppose v lies in a longest cycle of G. Let $\ldots, x_i, v, x_j, \ldots$ be a cycle of G with length c. Then G' contains the cycle $\ldots, x_i, v_i, v_f, v_j, x_j, \ldots$ which has length $c + 2$. Thus $\mathrm{circum}(G') \geq c + 2$. On the other hand, let C be a longest cycle of G', which has length at

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least $c+2$. If C contains no vertex from the set $S = \{v_1, v_2, v_3\}$, it is also a cycle in G and hence has length at most c, a contradiction. Observe that every vertex in S has exactly one neighbor outside S. If C contains a vertex in S, then C contains at least two vertices in S. Note that the vertices in $V(C) \cap S$ appear consecutively on C. Since S is a clique and C is a longest cycle, we deduce that $|V(C) \cap S| = 3$. Thus $v_r v_s v_t$ is a path on C with $\{r, s, t\} = \{1, 2, 3\}$. Replacing this path by the vertex v we obtain a cycle in G which has length at most c. Hence C has length at most $c + 2$, implying that $\mathrm{circum}(G') \le c + 2$. Combining this inequality with $\mathrm{circum}(G') \ge c + 2$ we obtain $\mathrm{circum}(G') = c + 2$. $\quad\square$

Lemma 2. *Let v be a vertex of degree 4 in a doubly homogeneously traceable graph G of order n and circumference c. Suppose G' is the graph obtained from G by blowing up v into K_4. Then G' is also doubly homogeneously traceable. If v lies in a longest cycle of G and v lies in a clique of cardinality 4, then G' has circumference $c + 3$ and G' contains a vertex v' that lies in a longest cycle of G' and also lies in a clique of cardinality 4.*

Proof. The proof that G' is doubly homogeneously traceable is similar to that in the above proof of Lemma 1 (but easier).

Next suppose v lies in a longest cycle of G and v lies in a clique of cardinality 4. Let $N(v) = \{x_1, x_2, x_3, x_4\}$ where v, x_1, x_2, x_3 form a clique and suppose v is blown up into K_4 whose vertices are v_1, v_2, v_3, v_4 such that v_i is adjacent to x_i for $i = 1, 2, 3, 4$. See Figure 2 for the change of local structures.
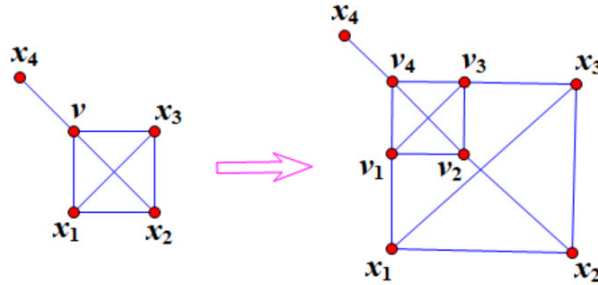

Fig. 2. Local changes

Let $\ldots, x_i, v, x_j, \ldots$ be a longest cycle of G with length c. Then G' contains the cycle $\ldots, x_i, v_i, v_s, v_t, v_j, x_j, \ldots$ which has length $c + 3$. Thus $\mathrm{circum}(G') \ge c + 3$. We then prove the reverse inequality. Let C be a longest cycle of G', which has length at least $c + 3$. Denote $S = \{v_1, v_2, v_3, v_4\}$. If C contains no vertex from the set S, it is also a cycle in G and hence has length at most c, a contradiction. Note that every vertex in S has exactly one neighbor outside S. Thus, if a cycle contains a vertex in S, it contains at least two. We have $|V(C) \cap S| \ge 2$. If $w \in V(C) \cap S$, then at least one neighbor of w on C belongs to S.

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Since S is a clique and C is a longest cycle, we deduce that $|V(C) \cap S| = 4$. On the cycle C, using the vertex v instead of v_4 and replacing a path of length 5 by a path of length 2, or replacing two paths of length 3 and 2 respectively by two edges we obtain a cycle of G, where we have used the fact that v, x_1, x_2, x_3 form a clique in G. Since $\text{circum}(G) = c$, the cycle C has length at most $c + 3$. This proves $\text{circum}(G') = c + 3$.

Finally we may choose v_4 as the vertex v'. □

Now we are ready to state and prove the main results.

Theorem 3. *For every even integer $n \geq 10$, there exists a cubic homogeneously traceable nonhamiltonian graph of order n; for every integer $p \geq 18$, there exists a 4-regular homogeneously traceable graph of order p and circumference $p - 4$.*

Proof. The Petersen graph P depicted in Figure 3 is a cubic doubly homogeneously traceable graph of order 10 and circumference 9.
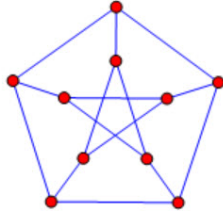

Fig. 3. The Petersen graph

Note that every vertex of P lies in a longest cycle. Thus, choosing any vertex v of P and blowing up v into K_3 we obtain a cubic graph P_{12} of order 12. By Lemma 1, P_{12} is doubly homogeneously traceable and has circumference 11. Let u be a vertex of P_{12} that lies in a longest cycle. In P_{12}, blowing up u into K_3 we obtain a cubic graph P_{14} of order 14. By Lemma 1, P_{14} is doubly homogeneously traceable and has circumference 13. Continuing this process we can construct a cubic homogeneously traceable graph of order n and circumference $n - 1$ for any even integer $n \geq 10$.

It is easy to verify that the three graphs in Figures 4, 5 and 6 are 4-regular doubly homogeneously traceable graphs of order p and circumference $p - 4$ for $p = 18, 19, 20$ respectively, where the vertices x, y, z lie in a longest cycle and in a clique of cardinality 4.
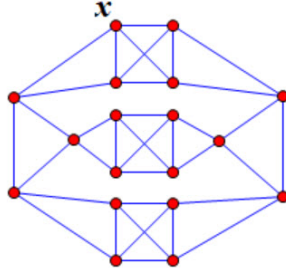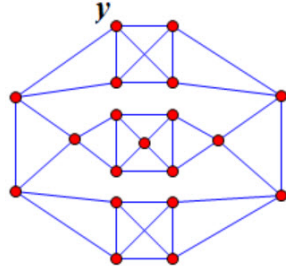
Fig. 4. The 4-regular base graph of order 18

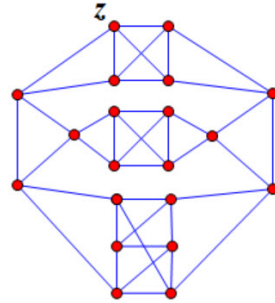Fig. 5. The 4-regular base graph of order 19

Fig. 6. The 4-regular base graph of order 20

Next we apply Lemma 2 repeatedly. Starting from the graph in Figure 4 and the vertex x, successively blowing up a vertex that lies in a longest cycle and in a clique of cardinality 4, we can construct a 4-regular homogeneously traceable graph of order p and circumference $p - 4$ for every integer $p \geq 18$ with $p \equiv 0 \mod 3$. Starting from the graph in Figure 5 and the vertex y, successively blowing up a vertex that lies in a longest cycle and in a clique of cardinality 4, we can construct a 4-regular homogeneously traceable graph of order p and circumference $p - 4$ for every integer $p \geq 19$ with $p \equiv 1 \mod 3$. Starting from the graph in Figure 6 and the vertex z, successively blowing up a vertex that lies in a longest cycle and in a clique of cardinality 4, we can construct a 4-regular homogeneously traceable graph of order p and circumference $p - 4$ for every integer $p \geq 20$ with $p \equiv 2 \mod 3$. This completes the proof. $\qquad\square$

Remark. The above proof of Theorem 3 shows that in the statement of Theorem 3, we may replace "homogeneously traceable" by "doubly homogeneously traceable". But we prefer the current version, since the term "doubly homogeneously traceable" is technical in some sense.

3 Unsolved problems

It is known ([2, Theorem 4] and [4, pp. 9-11]) that the minimum size of a homogeneously traceable nonhamiltonian graph of order n is $\lceil 5n/4 \rceil$.

The extremal problem concerning the independence number is easy.

Theorem 4. *The maximum independence number of a homogeneously traceable graph of order n is $\lfloor n/2 \rfloor$.*

Proof. Let G be a homogeneously traceable graph of order n, and let $v_1, v_2, \ldots, v_n$ be a Hamilton path. Suppose S is an independent set of G. If n is even, S contains at most one vertex in each of the edges $v_1 v_2$, $v_3 v_4$, $\ldots$, $v_{n-1} v_n$ and hence $|S| \leq n/2$. Now suppose n is odd. Similarly, we have $|S| \leq (n+1)/2$. We will show that $|S|$ cannot equal $(n+1)/2$. To the contrary, assume $|S| = (n+1)/2$. Then $S = \{v_1, v_3, v_5, \ldots, v_n\}$. Since G is homogeneously traceable, there is a Hamilton path $v_2, v_{i_2}, \ldots, v_{i_n}$. Since n is odd, there exists an integer k with $2 \leq k \leq n-1$ such that both i_k and i_{k+1} are odd. But v_{i_k} and $v_{i_{k+1}}$ are adjacent and they both belong to S, contradicting the condition that S is an independent set. It follows that $|S| \leq (n-1)/2$. We have proved that $|S| \leq \lfloor n/2 \rfloor$.

This upper bound $\lfloor n/2 \rfloor$ can be attained by the cycle C_n which is homogeneously traceable, and hence it is indeed the maximum value. $\qquad\square$

Finally we pose two unsolved problems.

Conjecture 1. *The minimum circumference of a homogeneously traceable graph of order n is $\lceil 2n/3 \rceil + 2$.*

The circumference $\lceil 2n/3 \rceil + 2$ in Conjecture 1 is attained by the graph in Figure 7 where $p = \lfloor (n-6)/3 \rfloor$ and when $p \geq 2$ the vertices u and v are distinct, x and y are distinct and w and z are distinct.
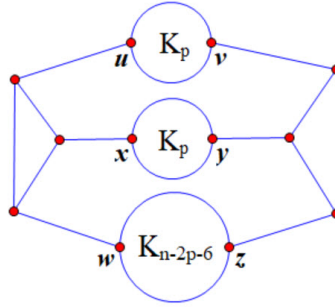
Fig. 7. A homogeneously traceable graph of a small circumference

Problem 2. *Given an integer $k \geq 4$, determine the integers n such that there exists a k-regular homogeneously traceable nonhamiltonian graph of order n.*

Theorem 3 solves the case $k = 3$ of Problem 2. A computer search shows that there exists no 4-regular homogeneously traceable nonhamiltonian graph of order ≤ 15. Thus, according to Theorem 3, only the two orders 16 and 17 are uncertain for $k = 4$.

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