Regular graphs with equal matching number and independence number*

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Abstract

Let $r \geq 3$ be an integer and G be a graph. Let $\delta(G), \Delta(G)$, $\alpha(G)$ and $\mu(G)$ denotes minimum degree, maximum degree, independence number and matching number of G, respectively. Recently, Caro, Davila and Pepper proved $\delta(G)\alpha(G) \leq \Delta(G)\mu(G)$. Mohr and Rautenbach characterized the extremal graphs for non-regular graphs and 3-regular graphs. In this note, we characterize the extremal graphs for all r-regular graphs in term of Gallai-Edmonds Structure Theorem, which extends Mohr and Rautenbach's result.

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1 Introduction

In this paper, we consider finite undirected graphs without loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of G is called its *order* and denoted by $|V(G)|$. On the other hand, the number of edges in G is called its *size* and denoted by $e(G)$. For a vertex u of a graph G, the *degree* of u in G is denoted by $d_G(u)$, and the minimum and maximum vertex degrees of G will be denoted $\delta(G)$ and $\Delta(G)$, respectively. The set of vertices adjacent to u in G is denoted by $N_G(u)$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. For two disjoint subsets $S, T \subseteq V(G)$, let $E_G(S, T)$ denote the set of edges of G joining S to T and let $e_G(S, T) = |E_G(S, T)|$. A component is *trivial* if it has no edges; otherwise it is *nontrivial*.

A *matching* of a graph is a set of edges such that no two edges share a vertex in common. For a matching M, a vertex u of G is called *saturated* by M if u is incident to an edge of M. A matching M is a *maximum matching* of G if there does not exist a matching M' of G such that $|M'| > |M|$. A *perfect matching* of a graph is a matching saturating all vertices. The cardinality of a maximum matching is called the *matching number* of G and is denoted by $\mu(G)$. An *independent set* is a set of vertices in a graph, no two of which are adjacent. A *maximum independent set* is an independent set of largest possible size for a given graph G. The cardinality of a maximum independent set is called the *independence number* of G and is denoted by $\alpha(G)$.

There are many relationships between the graph parameters $\alpha(G)$ and $\mu(G)$. It is known that $\lfloor \frac{n}{2} \rfloor + 1 \leq \alpha(G) + \mu(G) \leq |V(G)| \leq \alpha(G) + 2\mu(G)$ holds for every graph G. If $\alpha(G) + \mu(G) = |V(G)|$, then G is called *König-Egerváry graph* [5,10]. It is easy to see that if G is a Kőnig-Egerváry graph, then $\alpha(G) \geq \mu(G)$. The König-Egerváry graph have been extensively studied in [1,3,6,8].

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Recently, Levit et al. [7] showed that $\alpha(G) \leq \mu(G)$ under the condition that G contains an unique odd cycle. Caro, Davial and Pepper [4] obtained the following results.

Theorem 1 (Caro, Davial and Pepper, [4]). *If G is a graph, then*

$$\delta(G)\alpha(G) \leq \Delta(G)\mu(G),$$

and this bound is sharp.

Theorem 2 (Caro, Davial and Pepper, [4]). *If G is a r-regular graph with $r > 0$, then*

$$\alpha(G) \leq \mu(G).$$

They also proposed the following two open problems.

Problem 3 (Caro, Davial and Pepper, [4]). *Characterize $\alpha(G) = \mu(G)$ whenever G is 3-regular.*

Problem 4 (Caro, Davial and Pepper, [4]). *Characterize all graphs G for which $\delta(G)\alpha(G) = \Delta(G)\mu(G)$.*

Mohr and Rautenbach [9] characterized the non-regular extremal graphs as well as 3-regular graphs, which solved Problems 3 and 4. In the note, we characterize r-regular graphs G with $\alpha(G) = \mu(G)$ in term of Gallai-Edmonds Structure Theorem.

Now we firstly introduce Gallai-Edmonds Structure Theorem [11]. For a graph G, denote by D_G the set of all vertices in G which are not saturated by at least one maximum matching of G. Let A_G be the neighbor set of D_G, i.e., the set of vertices in $V(G) - D_G$ adjacent to at least one vertex in D_G. Finally let $C_G = V(G) - D_G - A_G$. Clearly, this partition is well-defined for every graph and dose not rely on the choices of maximum matchings. A graph G is said to be *factor-critical* if $G - v$ has a perfect matching for any vertex $v \in V(G)$. A matching is said to be a *near-perfect matching* if it covers all vertices but one. For a bipartite graph $H = (A, B)$, the set A with *positive surplus* if $|N_H(X)| > |X|$ for every non-empty subset X of A. The subgraph of G induced by a vertex subset S is denoted by $G[S]$.

Theorem 5 (Gallai-Edmonds Structure Theorem, see [11]). *Let G be a graph and let D_G, C_G and A_G be the vertex-partition defined above. Then*

(i) *the component of the subgraph induced by D_G are factor-critical;*

(ii) *the subgraph induced by C_G has a perfect matching;*

(iii) *if M is any maximum matching of G, it contains a near-perfect matching of each component of D_G, a perfect matching of each component of C_G and matches all vertices of A_G with vertices in distinct component of D_G;*

(iv) *the bipartite graph obtained from G by deleting the vertices of C_G and the edges spanned by A_G and by contracting each component of D_G to a single vertex has positive surplus (as viewed from A_G);*

(v) $E_G(C_G, D_G) = \emptyset$.

The partition (D_G, A_G, C_G) is called a *canonical decomposition*. When there are no confusions, we also denote $G[D_G]$, $G[A_G]$ and $G[C_G]$ by D_G, A_G and C_G, respectively. For a maximum matching M and a component of D_i of D_G, we say that D_i is *M-full* if some vertex of D_i is matched with a vertex in A_G, otherwise, D_i is *M-near full*.

Let G be an r-regular graph without perfect matching. A connected component D_i of D_G is called "*good*" if D_i is a non-trivial connected component and satisfies the following two properties:

(i) $\alpha(D_i) = (|V(D_i)| - 1)/2$;

(ii) D_i contains a maximum independent set $I(D_i)$ such that $E_G(I(D_i), A_G) = \emptyset$.

In this note, we character the extremal graphs for all r-regular graphs and obtain the following results.

Theorem 6. *Let G be a connected r-regular graph. Then $\alpha(G) = \mu(G)$ if and only if G is bipartite or (D_G, A_G, C_G) satisfies that*

(i) $C_G = \emptyset$,

(ii) *$A_G \subseteq I(G)$ for any maximum independent set of G,*

(iii) *every nontrivial component of D_G is good.*

2 Proof of Theorem 6

Before proving the Theorem 6, we firstly show the following lemma.

Lemma 7. *Let G be a connected r-regular graph without perfect matching. If $\alpha(G) = \mu(G)$, then*

(i) *$A_G \subseteq I(G)$ for any maximum independent set of G;*

(ii) $C_G = \emptyset$.

Proof. Firstly, we show (i). Let $I(G)$ be an arbitrary maximum independent set of G, let $A'_G = I(G) \cap A_G$ and $B'_G = I(G) \cap B_G$, where $B_G \subseteq D_G$ denotes the set of isolated vertices of D_G. Let q denote the number of connected components of D_G. Let D_i denote the connected component of D_G for $1 \le i \le q$. By Theorem 5 (iii), we have

$$\begin{aligned}\mu(G) &= \mu(C_G) + |A_G| + \mu(D_G)\\ &= \frac{1}{2}|C_G| + |A_G| + \frac{1}{2}\sum_{i=1}^{q}(|D_i| - 1)\end{aligned}$$

i.e.,

$$\mu(G) = \frac{1}{2}|C_G| + |A_G| + \frac{1}{2}\sum_{i=1}^{q}(|D_i| - 1) \tag{1}$$

Since D_i is factor-critical, we have $\alpha(D_i) \le (|D_i| - 1)/2$. Thus we have

$$|I(G) \cap D_G| \le \frac{1}{2}\sum_{i=1}^{q}(|D_i| - 1).$$

By Theorem 5 (ii), C_G has a perfect matching. Thus we infer that

$$\alpha(C_G) \le \frac{1}{2}|C_G|.$$

Hence,

$$\begin{aligned}\alpha(G) = |I(G)| &= |I(G) \cap C_G| + |I(G) \cap A_G| + |I(G) \cap D_G|\\ &\le \alpha(C_G) + |A'_G| + |B'_G| + \alpha(D_G - B_G)\\ &\le \frac{1}{2}|C_G| + |A'_G| + |B'_G| + \frac{1}{2}\sum_{i=1}^{q}(|D_i| - 1),\end{aligned}$$

i.e.,

$$\alpha(G) \leq \frac{1}{2}|C_G| + |A'_G| + |B'_G| + \frac{1}{2}\sum_{i=1}^{q}(|D_i| - 1), \tag{2}$$

Claim 1. $B'(G) = \emptyset$.

By contradiction. Suppose that $B'_G \neq \emptyset$. Note that $\alpha(G) = \mu(G)$. Combining (1) and (2), we have

$$|A_G| = |A'_G| + |B'_G|. \tag{3}$$

Since G is an regular graph and B'_G is an indepednet set, we have $|N_G(B'_G)| \geq |B'_G|$ with equality if and only if $G[N_G(B'_G) \cup B'_G]$ is a connected component of G and $N_G(B'_G)$ is also an independent set. Note that G is connected. So if $|N_G(B'_G)| = |B'_G|$, then $V(G) = B'_G \cup N_G(B'_G)$ and G is an r-regular bipartite graph, which implies that G has a perfect matching by Hall's Theorem, a contradiction. Thus we may assume that $|N_G(B'_G)| > |B'_G|$. Since $A'_G \cup B'_G$ is an independent set, we have $A'_G \subseteq A_G - N_G(B'_G)$. Thus

$$|A'_G| + |B'_G| \leq |A_G - N_G(B'_G)| + |B'_G| < |A_G|,$$

contradicting to (3). This completes the proof of claim 1.

By Claim 1, $|A_G| = |A'_G|$, then we have $A_G = A'_G \subseteq I(G)$. This completes the proof of (i).

Next we show (ii). Suppose that the result does not hold. Since $\alpha(G) = \mu(G)$, by (1) and (2), we have

$$|I(G) \cap C_G| = \alpha(C_G) = \mu(C_G) = \frac{1}{2}|C_G|.$$

Recall that $A_G \subseteq I(G)$. One can see that $E_G(A_G, I(G) \cap C_G) = \emptyset$. Since G is r-regular, we have

$$\frac{1}{2}r|C_G| = r|I(G) \cap C_G| \leq e_G(I(G) \cap C_G, C_G - (I(G) \cap C_G)) \leq r|C_G - (I(G) \cap C_G)| = \frac{1}{2}r|C_G|,$$

which implies $E_G(A_G, C_G - (I(G) \cap C_G)) = \emptyset$. Thus we have $E_G(A_G, C_G) = \emptyset$. Note that $E_G(D_G, C_G) = \emptyset$ by Theorem 5 (v). On the other hand, since G contains no perfect matchings, one can see that $D_G \neq \emptyset$ by definition of D_G. Since G is connected, we may infer that $C_G = \emptyset$. This completes the proof of Lemma 7. □

Proof of the Theorem 6. Firstly, we consider sufficiency. Let G be an r-regular bipartite graph with bipartition (A, B). One can see that $|A| = |B|$ and $\alpha(G) = \frac{|V(G)|}{2}$. By Hall's Theorem, G has a perfect matching, i.e., $\mu(G) = \frac{|V(G)|}{2}$. Therefore, $\alpha(G) = \mu(G)$.

Now we may assume that G is a regular graph and satisfies the following three conditions

(i) $C_G = \emptyset$,

(ii) $A_G \subseteq I(G)$ for any maximum independent set of G,

(iii) every nontrivial component of D_G is good.

Let q denote the number of connected components of D_G and let D_i denote the connected component of D_G for $1 \leq i \leq q$. By Theorem 5 (iii),we have

$$\mu(G) = |A_G| + \frac{1}{2}\sum_{i=1}^{q}(|D_i| - 1). \tag{4}$$

Since D_i is good for $1 \leq i \leq q$, then D_i contains an independent set $I(D_i)$ such that

$$|I(D_i)| = (|V(D_i)| - 1)/2 \text{ and } E_G(I(D_i), A_G) = \emptyset.$$

When D_i is an isolated vertex, $I(D_i) = \emptyset$. So $I(G) = A_G \bigcup \cup_{i=1}^{q} I(D_i)$ is an independent set of G. Note that

$$|I(G)| = |A_G| + \frac{1}{2}\sum_{i=1}^{q}(|D_i| - 1). \tag{5}$$

Since $I(G)$ is a maximum independent set, combining (4) and (5), one can see that

$$\mu(G) = |I(G)| = \alpha(G).$$

Next, we prove the necessity. Let G be an r-regular graph with $\alpha(G) = \mu(G)$. Let $I(G)$ be a maximum independent set of G. We discuss two cases.

Case 1. G has a perfect matching.

Note that

$$\mu(G) = \frac{1}{2}|V(G)| = \alpha(G)$$

and

$$|I(G)| = \frac{|V(G)|}{2} = |V(G) - I(G)|. \tag{6}$$

One can see that

$$e_G(I(G), V(G) - I(G)) = \alpha(G)r = \frac{|V(G)|}{2}r.$$

It follows that $V(G) - I(G)$ is an independent set and G is an r-regular bipartite graph.

Case 2. G has no perfect matching.

By Lemma 7, $C_G = \emptyset$ and $A_G \subseteq I(G)$. Let B_G denote the set of isolated vertices of D_G. Since G is connected, then for every $x \in B_G$, $E_G(\{x\}, A_G) \neq \emptyset$. So we have $B_G \cap I(G) = \emptyset$. So it is sufficient for us to show that every nontrivial component D_i of D_G is good. Since D_i is factor-critical, we have $\alpha(D_i) \leq \frac{1}{2}(|D_i| - 1)$. Recall that $A_G \subseteq I(G)$. Then we have

$$\alpha(G) = |I(G)| = |A_G| + \sum_{i=1}^{p} |V(D_i) \cap I(G)| \leq |A_G| + \frac{1}{2}\sum_{i=1}^{p}(|D_i| - 1),$$

where p denotes the number of connected components of D_G with order at least three. Note that

$$\alpha(G) = \mu(G) = |A_G| + \frac{1}{2}\sum_{i=1}^{p}(|D_i| - 1).$$

Hence we have $|I(G) \cap V(D_i)| = \frac{1}{2}(|D_i| - 1)$ and so $I(G) \cap V(D_i)$ is a maximum independent set of D_i. Moreover, one can see that $E_G(I(G) \cap V(D_i), A_G) = \emptyset$ since $A_G \subseteq I(G)$. This completes the proof. $\square$

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