Maximum 0-1 Timed Matching on Temporal Graphs^{*}

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Abstract

Temporal graphs are graphs where the topology and/or other properties of the graph change with time. They have been used to model applications with temporal information in various domains. Problems on static graphs become more challenging to solve in temporal graphs because of dynamically changing topology, and many recent works have explored graph problems on temporal graphs. In this paper, we define a type of matching called 0-1 timed matching for temporal graphs, and investigate the problem of finding a maximum 0-1 timed matching for different classes of temporal graphs. We assume that only the edge set of the temporal graph changes with time. Thus, a temporal graph can be represented by associating each edge with one or more non-overlapping discrete time intervals for which that edge exists. We first prove that the problem is NP-complete for rooted temporal trees when each edge is associated with two or more time intervals. We then propose an $O(n \log n)$ time algorithm for the problem on a rooted temporal tree with n vertices when each edge is associated with exactly one time interval. The problem is then shown to be NP-complete also for bipartite temporal graphs even when each edge is associated with a single time interval and degree of each vertex is bounded by a constant $k \geq 3$. We next investigate approximation algorithms for the problem for temporal graphs where each edge is associated with more than one time intervals. It is first shown that there is no $\frac{1}{n^{1-\epsilon}}$ -factor approximation algorithm for the problem for any $\epsilon > 0$ even on a rooted temporal tree with n vertices unless NP = ZPP. We then present a $\frac{5}{2N^*+3}$ -factor approximation algorithm for the problem for general temporal graphs where \mathcal{N}^* is the average number of edges overlapping in time with each edge in the temporal graph. The same algorithm is also a constant-factor approximation algorithm for temporal graphs with degree of each vertex bounded by a constant.

Keywords: 0-1 timed matching, temporal matching, time dependent matching, temporal graph, time varying graph

1 Introduction

Graphs are an important tool for modelling systems with a set of objects and pairwise relationships between those objects. In many applications, the objects and the relations between them have temporal properties, resulting in dynamically changing graphs where the topology and/or other properties of the graph change with time. Static graphs are not suitable for modelling such applications as they cannot represent the temporal information. Some examples of domains where such dynamic graphs with temporal properties arise are networks with intermittent inter-vertex connectivity such as delay tolerant networks (DTN) [53], vehicular ad-hoc networks (VANET) [19], mobile ad-hoc networks (MANET) [21], social networks [31, 2, 8], biological networks [37, 26, 50], and transportation networks [39, 25]. While modelling these systems, the temporal information in the system must be represented in the graph model used. Temporal graphs [35] have been proposed as a tool for modelling such systems with temporal properties.

Solving many well known graph problems on temporal graphs introduces new challenges. For many graph problems, the usual definitions and algorithms for the problem on static graphs do not apply directly to temporal graphs. As an example, consider the notion of paths in a graph, which is used extensively in applications using graphs. Unlike static graphs, in a temporal graph, an edge can only be

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used for traversal if it exists at the time of the traversal, and the definition of a path in a static graph no longer holds in a temporal graph. In a temporal graph, a path from a vertex u to vertex v must contain a sequence of edges from u to v that exist in strictly increasing order of time, unlike a static graph where just a sequence of edges from u to v is sufficient. Various other notions of paths have been defined on temporal graphs [11, 9, 52]. This makes many problems that use notions of paths more complex to solve in temporal graphs. Similar examples exist for many other graph related problems. Several works have addressed graph problems such as finding paths and trees [9, 29, 42], computing dominating sets [40], travelling salesman problem [45], finding vertex separator [55], computing sparse spanners [12], etc. on temporal graphs.

In this paper, we investigate the problem of finding matchings in a temporal graph. A matching in a static graph is defined as a subset of edges of the graph such that no two edges share a common vertex. A maximum matching is a matching with the maximum cardinality among all matchings. Finding a maximum matching for a static graph [28, 44] is a well-studied problem in the area of static graphs due to its wide application. Assignment problems [6] form an important class of applications that use the notion of maximum matching. Some example applications of assignment problems that occur in dynamic graph topologies are task assignment problem in ad allocation [7, 54], crowdsourcing market [27], dynamic assignment problem [51] etc. Some of these applications may require that assignments be available for all applicable time periods. As an example, consider a problem of matching the maximum number of consumers with service providers, where each consumer requests for service in a number of time slots, and may want to accept a matched service provider only if it can get the service at all its requested time slots. The traditional definition of matching for static graphs is not enough to address such assignment problems due to its temporal nature. However, this can be modelled by a temporal graph with edges between consumers and service providers, with each edge marked with the time intervals for which the service is requested. An allocation is valid if it ensures that no two consumers allocated to a service provider has any common requested time slot. In this paper, we define a type of matching called 0-1 timed matching for a temporal graph to model such situations, and investigate the complexity and algorithms for finding a maximum 0-1 timed matching for different types of temporal graphs. We have assumed that only the edge set of the temporal graph changes with time. Thus, a temporal graph can be represented by labelling each edge with non-overlapping discrete time intervals for which that edge exists. The underlying graph of a temporal graph is a static graph for which the vertex set includes all the vertices of the temporal graph and the edge set includes each edge which is present in the temporal graph for at least one timestep. The specific contributions of this paper are as follows.

- 1. We prove that the problem of finding a maximum 0-1 timed matching for a rooted temporal tree is NP-complete when each edge of the tree is associated with 2 or more time intervals.
- 2. We show that the problem is solvable in polynomial time if each edge of the rooted temporal tree is associated with a single time interval. In particular, we propose a dynamic programming based $O(n \log n)$ time algorithm to solve the problem on such a rooted temporal tree with n vertices.
- 3. Next, we study the computational complexity of the problem when each edge of the temporal graph is associated with a single time interval. We prove that the problem is NP-complete in this case even for bounded degree bipartite temporal graphs when degree of each vertex is bounded by 3 or more. This automatically proves that the problem is NP-complete for a bipartite temporal graph with a single time interval per edge, and hence, for a general temporal graph with a single time interval per edge.
- 4. We investigate the hardness of approximation of the problem when each edge of the temporal graph is associated with multiple time intervals. We prove that there is no approximation algorithm with approximation ratio $\frac{1}{n^{1-\epsilon}}$, for any $\epsilon > 0$, for finding a maximum 0-1 timed matching even on a rooted temporal tree with n vertices when each edge is associated with multiple time intervals unless NP = ZPP.
- 5. We propose an approximation algorithm to address the problem for a temporal graph when each edge is associated with multiple time intervals. The approximation ratio of the proposed algorithm is $\frac{5}{2\mathcal{N}^*+3}$ where \mathcal{N}^* is the average number of edges overlapping with each edge in the temporal graph. Two edges are overlapping with each other if both are incident on the same vertex and there exists at least one timestep when both the edges exist¹. We also show that the same algorithm

¹Formal definition of overlapping edges is given in Section 4.

is a constant factor approximation algorithm for a temporal graph when degree of each vertex is bounded by a constant.

The rest of this paper is organised as follows. Section 2 describes some related work in the area. Section 3 describes the system model used. Section 4 formally defines the problem. Section 5 presents the results related to the problem of finding a maximum 0-1 timed matching for a rooted temporal tree. Section 6 presents the results related to the problem of finding a maximum 0-1 timed matching for a general temporal graph. Finally Section 7 concludes the paper.

2 Related Work

The problem of finding a maximum matching is a well studied problem for static graphs. In [16], Edmonds proposed a $O(n^4)$ time algorithm, where *n* is the number of vertices in the input graph. Since then, many algorithms have been proposed to address the problem on both general graphs and other restricted classes of graphs [28, 44, 18, 17, 32, 13, 47, 48, 49]. One generalisation of the maximum matching problem is the maximum packing problem where the goal is to find the maximum number of vertex disjoint subgraphs of a given graph such that each subgraph is isomorphic to an element of a given class of graphs. This is a well studied problem for static graphs, with many different variants of this problem addressed for both general graphs and other restricted classes of graphs [33, 38, 46, 22].

Matchings in temporal graphs have recently attracted the attention of researchers. One variant of the matching problem on temporal graphs is *multistage matching* [14]. In this problem, the temporal graph is viewed as a sequence of static graphs, one for each timestep of the total duration, with the static graph for timestep t containing all the vertices and edges that exist at t. A sequence of maximum matchings is then found, one for each static graph in this representation of the temporal graph. The work in [14] attempts to optimize different parameters such as *intersection profit* and *union cost* while finding this sequence of matchings. In the case of intersection profit, the goal is to minimize the number of changes incurred to construct a maximum matching for a static graph at a certain timestep from a maximum matching for the static graph at the previous timestep. In the case of union cost, the goal is to minimize the total number of edges in the union of the maximum matchings for the graphs at each timestep. Several other works [23, 3] have studied different variants of multistage matchings.

Other than multistage matching, a few other definitions of matchings for temporal graphs are available in literature. For a given temporal graph, Michail et al. [45] consider the decision problem of finding if there exists a maximum matching M in the underlying graph such that a single label can be assigned to each edge of M with the constraint that the assigned label to an edge is chosen from the timesteps when that edge exists in the temporal graph and no two edges of M are assigned the same label. It is then proved that the problem is NP-hard. Baste et al. [4] define another type of temporal matching called γ -matching. γ -edges are defined as edges which exist for at least γ consecutive timesteps. The maximum γ -matching is defined as a maximum cardinality subset of γ -edges such that no two γ -edges in the subset share any vertex at any timestep. They have shown that the problem of finding a maximum γ -matching is NP-hard when $\gamma > 1$, and proposed a 2-approximation algorithm for the problem. Mertzios et al. [43] define another type of temporal matching called Δ -matching where two edge instances at timesteps t, t' can be included in a matching if either those two edge instances do not share any vertex or |t - t'|is greater than or equal to a positive integer Δ . They prove that this problem is APX-hard for any $\Delta \geq 2$ when the lifetime of the temporal graph is at least 3, and the problem remains NP-hard even when the underlying graph of the temporal graph is a path. An approximation algorithm is proposed to find a $\frac{\Delta}{2\Delta-1}$ -approximate maximum Δ -matching for a given temporal graph with n vertices, m edges and lifetime \mathcal{T} in $O(\mathcal{T}m(\sqrt{n}+\Delta))$ time. In [1], Akrida et al. address the maximum matching problem on stochastically evolving graphs represented using stochastic arrival departure model. In this model, each vertex in the temporal graph arrives and departs at certain times, and these arrival and departure times of each vertex are determined using independent probability distributions. The vertex exists in the time interval between the arrival time and the departure time. An edge between two vertices can exist if there is an intersection between the time intervals for which those two vertices exist. A matching on a stochastically evolving graph is defined as the subset of edges such that no two edges are incident on the same vertex. A fully randomized approximation scheme (FPRAS) has been derived to approximate the expected size of maximum cardinality matching on a stochastically evolving graph. A probabilistic optimal algorithm is proposed when the model is defined over two timesteps. They have also defined price of stochasticity and proved that the upper bound on the price of stochasticity is $\frac{2}{2}$.

In this paper, we propose another type of matching for temporal graphs called 0-1 timed matching. We



Figure 1: A temporal tree rooted at vertex r

then investigate the complexity and algorithms for finding a maximum 0-1 timed matching in temporal graphs.

3 System Model

We represent a temporal graph by the evolving graphs [20] model. In this model, a temporal graph is represented as a finite sequence of static graphs, each static graph being an undirected graph representing the graph at a discrete timestep. The total number of timesteps is called the *lifetime* of the temporal graph. In this paper, we assume that the vertex set of the temporal graph remains unchanged throughout the lifetime of the temporal graph; only the edge set changes with time. For simplicity, it is assumed that all the changes in the edge set are known a-priori (this assumption has also been used in other existing works [9, 43]). Also, there are no self-loops and at most one edge exists between any two vertices at any timestep. Thus, a temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with set of vertices \mathcal{V} and set of edges \mathcal{E} is represented as a sequence of graphs $(G_0, G_1, \dots, G_{T-1})$, where $G_i = (\mathcal{V}, \mathcal{E}_i)$ is the static graph at timestep *i* with set of edges \mathcal{E}_i that exist at timestep *i*, and \mathcal{T} is the lifetime of \mathcal{G} . As only the edge set changes with time, each edge in \mathcal{E} of a temporal graph \mathcal{G} can be represented by specifying the time intervals for which the edge exists. Thus an edge $e \in \mathcal{E}$ between vertices u and v can be represented as $e(u, v, (s_1, f_1), d_1)$ $(s_2, f_2), \dots, (s_k, f_k))$, where $u, v \in \mathcal{V}, u \neq v, f_k \leq \mathcal{T}$ and a pair (s_i, f_i) indicates that the edge exists for the time interval $[s_i, f_i)$, where $0 \leq s_i < f_i \leq \mathcal{T}$. In other words, each interval (s_i, f_i) associated with edge e between vertices u and v denotes that the edge e exists in all the static graphs $(G_{s_i}, G_{s_i+1}, \cdots, G_{f_i-1})$. Also, if an edge e has two such pairs (s_i, f_i) and (s_j, f_j) , $s_i \neq s_j$ and if $s_i < s_j$ then $f_i < s_j$. Thus, the maximum number of time intervals for an edge can be $\lfloor \frac{T}{2} \rfloor$. An edge at a single timestep is called an instance of that edge. For simplicity, we also denote the edge e between vertices $u, v \in \mathcal{V}$ by e_{uv} when the exact time intervals for which e exists are not important. The corresponding instance of e_{uv} at time t is denoted by e_{uv}^t .

4 **Problem Definition**

In this section, we define a 0-1 timed matching for temporal graphs. We first define some terminologies related to temporal graphs that we will need.

For a given temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with lifetime \mathcal{T} , the *underlying graph* of \mathcal{G} is defined as the static graph $\mathcal{G}_U = (\mathcal{V}, \mathcal{E}_U)$, where $\mathcal{E}_U = \{(u, v) \mid \exists t \text{ such that}, e_{uv}^t \text{ is an instance of } e_{uv} \in \mathcal{E}\}$. Next, we define different types of temporal graphs.

Definition 4.1. Temporal Tree: A temporal graph \mathcal{G} is a temporal tree if the underlying graph of \mathcal{G} is a tree.

Definition 4.2. Rooted Temporal Tree: A rooted temporal tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ rooted at vertex r is a temporal tree with one vertex $r \in \mathcal{V}$ chosen as root of the tree.

Note that, the underlying graph of a rooted temporal tree is also a rooted tree. For any vertex v, let p(v) denote the parent vertex of v and child(v) denote the set of children vertices of v in the underlying graph of the rooted temporal tree. For the root vertex r, p(r) = NULL. Depth of a vertex v in a rooted temporal tree \mathcal{G} rooted at r is the path length from r to v in \mathcal{G}_U . Height of a rooted temporal tree \mathcal{G} is the maximum depth of any vertex in \mathcal{G} .



Figure 2: A bipartite temporal graph

Figure 1, shows a temporal tree rooted at vertex r. In this temporal tree p(a), p(b), p(c) is r and $child(r) = \{a, b, c\}$. The depth of vertex r is 0. Depth of a, b, c is 1, depth of d is 2 and depth of f is 3. The height of this rooted temporal tree is 3.

Definition 4.3. Bipartite Temporal Graph: A temporal graph \mathcal{G} is bipartite if the underlying graph of \mathcal{G} is a bipartite graph.

Definition 4.4. Bounded Degree Temporal Graph: A temporal graph \mathcal{G} is a bounded degree temporal graph where degree of each vertex is bounded by some positive integer k, if the underlying graph of \mathcal{G} is a bounded degree graph where degree of each vertex is bounded by k.

Definition 4.5. Bounded Degree Bipartite Temporal Graph: A temporal graph \mathcal{G} is a bounded degree bipartite temporal graph if it is both a bipartite temporal graph and a bounded degree temporal graph.

Definition 4.6. Overlapping Edge: Given a temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, any edge $e_{vw} \in \mathcal{E}$ is said to be overlapping with another edge e_{uv} if there exists a timestep t such that both e_{vw}^t and e_{uv}^t exist.

Note that if e_{uv} is overlapping with edge e_{vw} , then e_{vw} is also overlapping with e_{uv} . We refer to such pair of edges as *edges overlapping with each other*.

In Figure 2, e_{dg} is an overlapping edge with e_{fg} because both edges are incident on g and both e_{dg}^1 and e_{fg}^1 exist. On the other hand, edges e_{ab} and e_{ad} are incident on the same vertex a, but there is no timestep t when both e_{ab}^t and e_{ad}^t exist. Thus e_{ab} and e_{ad} are non-overlapping with each other. For any two sets of edges E_1 , E_2 , if $E_1 \subseteq E_2$ and no two edges in E_1 are overlapping with each other, then E_1 is called a non-overlapping subset of E_2 .

Definition 4.7. Overlapping Number of Edge e_{uv} : Given a temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the overlapping number of an edge $e_{uv} \in \mathcal{E}$, denoted by $\mathcal{N}(e_{uv})$, is the number of edges overlapping with e_{uv} in \mathcal{E} .

Definition 4.8. 0-1 Timed Matching: A 0-1 timed matching M for a given temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a non-overlapping subset of \mathcal{E} .

Definition 4.9. Maximum 0-1 Timed Matching: A maximum 0-1 timed matching for a given temporal graph is a 0-1 timed matching with the maximum cardinality.

Definition 4.10. *Maximal 0-1 Timed Matching:* A 0-1 timed matching M for a given temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is maximal if for any edge $e_{uv} \in \mathcal{E} \setminus M$, $M \cup \{e_{uv}\}$ is not a 0-1 timed matching for \mathcal{G} .

For the bipartite temporal graph \mathcal{G} shown in Figure 2, a maximum 0-1 timed matching M is $\{e_{ab}, e_{ad}, e_{cd}, e_{fg}\}$. M is also a maximal 0-1 timed matching for \mathcal{G} . Consider another 0-1 timed matching $M' = \{e_{ab}, e_{ad}, e_{dg}\}$ for \mathcal{G} . M' is a maximal 0-1 timed matching for \mathcal{G} but not a maximum 0-1 timed matching. Note that, the edges in a 0-1 timed matching for a given temporal graph may not be a matching for its underlying graph.

In the next section, we explore the problem of finding a maximum 0-1 timed matching for a given rooted temporal tree.

5 Finding a Maximum 0-1 Timed Matching for Rooted Temporal Tree

In this section, we first analyse the hardness of computing a maximum 0-1 timed matching for a given rooted temporal tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We show that this problem is NP-complete for \mathcal{G} when each edge in \mathcal{E} is associated with 2 or more time intervals. We then explore the problem when each edge of \mathcal{E} is associated with a single time interval. We find that this problem is solvable in polynomial time and propose a dynamic programming based algorithm for it.

5.1 Complexity of Finding a Maximum 0-1 Timed Matching for Rooted Temporal Tree

We first show that the problem of finding a maximum 0-1 timed matching is NP-complete for a rooted temporal tree even when the number of intervals associated with each edge is at most 2. We refer to the problem of finding a maximum 0-1 timed matching for a rooted temporal tree when each edge is associated with at most 2 time intervals as MAX-0-1-TMT-2. We first define the decision version of MAX-0-1-TMT-2, referred to as the D-MAX-0-1-TMT-2 problem.

Definition 5.1. *D-MAX-0-1-TMT-2:* Given a rooted temporal tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with lifetime \mathcal{T} , where each edge in \mathcal{E} is associated with at most 2 time intervals, and a positive integer g, does there exist a 0-1 timed matching M for \mathcal{G} such that |M| = g?

We show that there is a polynomial time reduction from the decision version of the problem of finding a maximum rainbow matching for a properly edge coloured path, referred to as D-MAX-RBM-P [36], to the D-MAX-0-1-TMT-2 problem. Before describing the details of the reduction, we define a properly edge coloured path and the D-MAX-RBM-P problem.

Definition 5.2. Properly Edge Coloured Path: A path P is a properly edge coloured path if each edge of P is coloured in such a way that no two edges incident on the same vertex are coloured with the same colour.

Definition 5.3. *D-MAX-RBM-P:* Given a properly edge coloured path P = (V, E) and a positive integer h, does there exist a set $R \subset E$ of size h, such that R is a matching for P and no two edges in R are coloured with the same colour?

The D-MAX-RBM-P problem is known to be NP-complete [36].

Theorem 5.1. D-MAX-0-1-TMT-2 is NP-complete.

Proof. We first show that the problem is in NP. Consider a certificate $\langle \langle \mathcal{G} = (\mathcal{V}, \mathcal{E}), g \rangle, M \rangle$, where \mathcal{G} is a rooted temporal tree with lifetime \mathcal{T} , each edge in \mathcal{E} is associated with at most 2 time intervals, g is a given integer and M is a subset of \mathcal{E} . We consider one edge $e_{uv} \in M$ at a time and compare associated time intervals of e_{uv} with associated time intervals of all the other edges in M to find any edge overlapping with e_{uv} in M. We perform this check for all the edges in M to find any such overlapping edges with each other. This checking can be done in polynomial time. Whether |M| = g can also be easily checked in polynomial time. Hence, the D-MAX-0-1-TMT-2 problem is in NP.

Next, we prove that there is a polynomial time reduction from the D-MAX-RBM-P problem to the D-MAX-0-1-TMT-2 problem. Consider an instance $\langle P = (V, E), h \rangle$ of the D-MAX-RBM-P problem where P is a properly edge coloured path, $V = \{v_0, v_1, \dots, v_{n-1}\}, |V| = n, E = \{(v_0, v_1), (v_1, v_2), \dots, (v_{n-2}, v_{n-1})\}, |E| = n-1$, and h is a positive integer. For our reduction, we assign a sequence of positive integers $A = \{a_i \mid a_i \in \mathbb{N}, a_i < n\}$, to vertices in V in the following way. We assign $a_i = i$ to the vertex v_i . The path P is a properly edge coloured path. Let c be the number of different colours used to colour P. As the number of edges in P is n-1, then $c \leq n-1$. We represent these colours with different integers from n to n + c - 1.

From this given instance of the D-MAX-RBM-P problem, we construct an instance of the D-MAX-0-1-TMT-2 problem as follows.

- The temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is constructed as follows.
 - We add a vertex $\nu(v_i v_{i+1})$ for each edge $(v_i, v_{i+1}) \in E$. Additionally we add a vertex r. Thus, $\mathcal{V} := \{\nu(v_i v_{i+1}) | \forall (v_i, v_{i+1}) \in E\} \cup \{r\}.$



Figure 3: (a) A properly edge coloured path where vertices are assigned some integers (dashed and continuous lines are representing two different colours; integer along an edge is representing the colour of it), (b) Corresponding temporal tree rooted at vertex r.

- We add an edge between each added vertex $\nu(v_i v_{i+1}) \in \mathcal{V}$ and r. This edge exists for time intervals $(a_i, a_i + 2)$ and $(c_j, c_j + 1)$, where a_i is the integer assigned to v_i , c_j is the integer representing the colour by which (v_i, v_{i+1}) is coloured. Thus,
- $\mathcal{E} := \{ e(\nu(v_i v_{i+1}), r, (a_i, a_i + 2), (c_j, c_j + 1)) \, | \, \forall (v_i, v_{i+1}) \in E \},\$

 $-\mathcal{T}:=n+c$

• g := h

As each edge in \mathcal{E} connects a vertex in $\mathcal{V} \setminus \{r\}$ to r, \mathcal{G} is a temporal tree rooted at r (we choose r as the root vertex). According to the construction, the number of intervals associated with each edge in \mathcal{E} is 2. Any edge $e(u, v, (s_1, f_1)(s_2, f_2)) \in \mathcal{E}$ is also denoted as e_{uv} when the time intervals for which this edge exists are not important. Figure 3 shows the construction of a rooted temporal tree from a properly edge coloured path.

We first show that, if there is a solution for the instance of the D-MAX-0-1-TMT-2 problem, then there is a solution for the instance of the D-MAX-RBM-P problem. For a 0-1 timed matching M, |M| = g, for \mathcal{G} , we construct a rainbow matching R, |R| = h, for P as follows. Consider the set of edges $R = \{(v_i, v_{i+1}) | e_{r\nu(v_i v_{i+1})} \in M\}$. As |M| = g and g = h, |R| = h. We prove that R is a rainbow matching for P. We prove this by contradiction. Assume that, R is not a rainbow matching for P. This is possible in two cases.

- I. There are at least two edges $(v_{i-1}, v_i), (v_i, v_{i+1})$ incident on the same vertex v_i included in R. As P is a path, according to the construction, v_{i-1}, v_i, v_{i+1} are assigned three consecutive integers a_{i-1}, a_i, a_{i+1} respectively. This implies that, time intervals $(a_{i-1}, a_{i-1} + 2)$ and $(a_i, a_i + 2)$ are associated with edges $e_{r\nu(v_{i-1}v_i)}$ and $e_{r\nu(v_iv_{i+1})}$ respectively, and both are included in M. As $a_i = a_{i-1} + 1$, edges $e_{r\nu(v_{i-1}v_i)}$ and $e_{r\nu(v_iv_{i+1})}$ exist at the timestep a_i . This contradicts the fact that M is a 0-1 timed matching for \mathcal{G} .
- II. There are two edges $(v_i, v_{i+1}), (v_j, v_{j+1})$ which are coloured with the same colour, c_k . This implies that edges $e_{r\nu(v_iv_{i+1})}$ and $e_{r\nu(v_jv_{j+1})}$ are included in M when both exist at timestep c_k . This also contradicts the fact that M is a 0-1 timed matching for \mathcal{G} .

Next, we show that if there is no solution for the instance of the D-MAX-0-1-TMT-2 problem, then there is no solution for the instance of the D-MAX-RBM-P problem. To show this, we prove that if we have a solution for the instance of the D-MAX-RBM-P problem, then we have a solution for the instance of the D-MAX-0-1-TMT-2 problem. For a rainbow matching R, |R| = h, for P, we construct a 0-1 timed matching M, for \mathcal{G} as follows. Consider the set $M = \{e_{r\nu(v_iv_{i+1})} \mid (v_i, v_{i+1}) \in R\}$. As |R| = h and h = g, |M| = g. Next we show that, M is a 0-1 timed matching for \mathcal{G} . We prove this by contradiction. Assume that, M is not a 0-1 timed matching for \mathcal{G} . This implies that, there are at least two edges $e_{r\nu(v_iv_{i+1})}$ and $e_{r\nu(v_jv_{j+1})}$ in M such that there exists at least one timestep t when both the edges exist. According to our construction of \mathcal{G} , t < n + c. There can be two possible cases.

I. $0 \le t \le n-1$: As $0 \le t \le n-1$, t is assigned to some vertex in V. According to our assumption, edges $e_{r\nu(v_iv_{i+1})}$ and $e_{r\nu(v_jv_{j+1})}$ in M and both exist at timestep t. This implies that, timestep t is present in both intervals $(a_i, a_i + 2)$ and $(a_j, a_j + 2)$ which are associated with $e_{r\nu(v_iv_{i+1})}$ and $e_{r\nu(v_jv_{j+1})}$ respectively (according to construction of \mathcal{G}). Without loss of generality, we can assume that $a_i < a_j$. In this scenario, $a_i + 1 = a_j = t$. Then, according to the assignment of integers to the vertices in V, $v_{i+1} = v_j$. This implies that, the edges $(v_i, v_{i+1}), (v_j, v_{j+1}) \in R$ are incident on the same vertex v_{i+1} . This contradicts the fact that R is a rainbow matching for P.

II. $n \leq t \leq n + c - 1$: As $n \leq t \leq n + c - 1$, t represents one colour which is used to colour the edges in E. According to the assignment of integers to the colours of each edge, $n \leq t \leq n + c - 1$ implies that edges $(v_i, v_{i+1}), (v_j, v_{j+1}) \in R$ are coloured with the same colour which is represented by t. This also contradicts the fact that R is a rainbow matching for P.

This completes the proof of this theorem.

5.2 Finding a Maximum 0-1 Timed Matching for a Rooted Temporal Tree with Single Time Interval per Edge

We present a dynamic programming based algorithm for finding a maximum 0-1 timed matching for a rooted temporal tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with root $r \in \mathcal{V}$ where each edge in \mathcal{E} is associated with a single time interval.

Consider a vertex $v \in \mathcal{V}$ with parent p(v). Let $\mathbb{T}_v = (\mathbb{V}_v, \mathbb{E}_v)$ be the temporal subtree rooted at v. Consider the temporal tree $T_v = (\mathcal{V}_v, \mathcal{E}_v)$, where $\mathcal{V}_v = \mathbb{V}_v \cup \{p(v)\}, \mathcal{E}_v = \mathbb{E}_v \cup \{e_{vp(v)}\}$. For the purpose of computing the depths of vertices in T_v , we consider p(v) as the root of T_v . Let M_v denote a maximum 0-1 timed matching for T_v . For any non-root vertex v, two cases are possible for M_v .

- 1. Edge $e_{vp(v)}$ is not included in M_v .
- 2. Edge $e_{vp(v)}$ is included in M_v .

Let TM1[v] and TM2[v] denote the maximum 0-1 timed matchings for T_v which does not include edge $e_{vp(v)}$ (Case 1), and which includes edge $e_{vp(v)}$ (Case 2) respectively. In the rest of this paper, $T_{u_i} = (\mathcal{V}_{u_i}, \mathcal{E}_{u_i})$ denotes the temporal tree with vertex set $\mathcal{V}_{u_i} = \mathbb{V}_{u_i} \cup \{p(u_i)\}$, and edge set $\mathcal{E}_{u_i} = \mathbb{E}_{u_i} \cup \{e_{u_i p(u_i)}\}$ where $\mathbb{T}_{u_i} = (\mathbb{V}_{u_i}, \mathbb{E}_{u_i})$ is the temporal subtree rooted at u_i . The algorithm first orders the vertices in \mathcal{V} in non-increasing order of depths. It then computes TM1[v] and TM2[v] for each T_v , $v \in \mathcal{V}$ in this order.

We first describe the method for computing TM1[v] for T_v when for all $T_{u_i}, u_i \in child(v), TM1[u_i]$ and $TM2[u_i]$ are already computed. Note that for any $T_{u_i}, u_i \in child(v)$, if $|TM2[u_i]| < |TM1[u_i]| + 1$, then there is a maximum 0-1 timed matching for T_v which does not include $e_{vu_i}^2$. Thus, for any $u_i \in child(v)$, the edge e_{vu_i} can be considered for inclusion in M_v only if $|TM2[u_i]| = |TM1[u_i]| + 1$. Based on this, we define the set of *feasible edges* as follows.

Definition 5.4. Feasible Edges for T_v : The feasible edges for T_v , denoted by F_v , is the set of edges e_{vu_i} such that $u_i \in child(v)$ and $|TM2[u_i]| = |TM1[u_i]| + 1$.

To compute TM1[v] for T_v , the algorithm first computes the feasible edges F_v for T_v . The algorithm then finds the maximum cardinality subset $F'_v \subseteq F_v$ such that no two edges in F'_v are overlapping with each other. TM1[v] is then computed using the following equation.

$$TM1[v] := \left(\bigcup_{\forall u_i \in child(v) \land e_{vu_i} \in F'_v} TM2[u_i]\right) \cup \left(\bigcup_{\forall u_i \in child(v) \land e_{vu_i} \notin F'_v} TM1[u_i]\right) \tag{1}$$

Note that if v is a leaf vertex, $child(v) = \emptyset$. Thus by Equation 1, $TM1[v] = \emptyset$ for any T_v where v is a leaf vertex.

We next describe the method for computing TM2[v] for any non-root vertex v, when for all T_{u_i} , $u_i \in child(v)$, $TM1[u_i]$ and $TM2[u_i]$ are already computed. We first define the following.

Definition 5.5. Compatible Edges for T_v : Compatible edges for T_v , denoted by C_v , is the set of edges e_{vu_i} such that $e_{vu_i} \in F_v$ and e_{vu_i} is non-overlapping with $e_{vp(v)}$.

 $^{^{2}}$ This statement is formally proved in Lemma 5.2 later in the paper.



Figure 4: A temporal tree rooted at r

To compute TM2[v] for T_v , the algorithm first computes the set of compatible edges C_v for T_v . The algorithm then computes the maximum cardinality subset $C'_v \subseteq C_v$ such that no two edges in C'_v are overlapping with each other. TM2[v] is then computed using the following equation.

$$TM2[v] := e_{vp(v)} \cup \left(\bigcup_{\forall u_i \in child(v) \land e_{vu_i} \in C'_v} TM2[u_i]\right) \cup \left(\bigcup_{\forall u_i \in child(v) \land e_{vu_i} \notin C'_v} TM1[u_i]\right)$$
(2)

Note that, if v is a leaf vertex, $child(v) = \emptyset$. Thus by Equation 2, $TM2[v] = \{e_{vp(v)}\}$ for any T_v where v is a leaf vertex.

For the root vertex r of \mathcal{G} , p(r) = NULL, and hence T_r is not defined as the edge $e_{rp(r)}$ does not exist. However, for simplicity, we still compute TM1[r] for r using Equation 1 as the parent of v is not required for computation of TM1[v] for any vertex v. The algorithm returns TM1[r] as a maximum 0-1 timed matching for \mathcal{G} .

For example, in Figure 4, while computing TM1[b] for T_b , $TM1[g] = \emptyset$, $TM2[g] = \{e_{bg}\}$ for T_g , and $TM1[h] = \emptyset$, $TM2[h] = \{e_{bh}\}$ for T_h where g and h are two children of b. Thus by Equation 1, $TM1[b] = \{e_{bg}\}$ or $\{e_{bh}\}$ and by Equation 2, $TM2[b] = \{e_{bg}, e_{rb}\}$. Similarly for T_a , $TM1[a] = \{e_{ci}, e_{cj}, e_{dl}, e_{dq}\}$ and $TM2[a] = \{e_{ar}, e_{ci}, e_{cj}, e_{dl}, e_{dq}\}$ and for T_b , $TM1[b] = \{e_{bh}\}$ and $TM2[b] = \{e_{br}, e_{bg}\}$. Thus, $TM1[r] = \{e_{ar}, e_{ci}, e_{cj}, e_{dl}, e_{dq}, e_{br}, e_{bg}\}$ which is a maximum 0-1 timed matching for the rooted temporal tree shown in Figure 4.

Algorithm 1 describes the pseudocode of the proposed algorithm. The algorithm invokes a function createLevelList to put all the vertices in \mathcal{G} in different lists, with two vertices put in the same list if and only if their depths in \mathcal{G} are the same. Each T_v is processed in non-increasing order of the depths of vertex v in \mathcal{G} , starting from the vertex with the maximum depth. For each T_v , the algorithm computes TM1[v] and TM2[v] following Equations 1 and 2 respectively. It returns TM1[r] as a maximum 0-1 timed matching for the given rooted temporal tree where r is the root vertex. While computing TM1[v] and TM2[v] for any T_v , Algorithm 1 internally invokes two other functions. Function maxNonOverlap(S) returns the maximum cardinality non-overlapping subset of edges from a set of edges S. Each edge of the rooted temporal tree is associated with a single time interval. Thus, the problem of finding a maximum cardinality non-overlapping subset of edges of a given set of edges reduces to the interval scheduling problem [34, 15]. Hence, maxNonOverlap(S) finds a maximum cardinality non-overlapping subset of edges S maximum cardinality non-overlapping subset of edges of a given set of edges reduces to the interval scheduling problem [34, 15].

Algorithm 1 dp0-1TimedMatching

Input: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, root vertex r **Output:** $M \subseteq \mathcal{E}$, a maximum 0-1 timed matching for \mathcal{G} 1: if r = NULL or $child(r) = \emptyset$ then $M := \emptyset; \quad return(M)$ 2: 3: nList := createLevelList(r)4: for $level = max_depth \rightarrow 0$ do $\triangleright max_depth = maximum depth of a vertex$ while $(v := nList[level].extractVertex())! = \emptyset$ do 5: $TM1[v] := \emptyset; TM2[v] := \emptyset$ 6: 7: $F_v := \emptyset; \quad C_v := \emptyset$ 8: for all $u_i \in child(v)$ do 9: if $|TM2[u_i]| = |TM1[u_i]| + 1$ then 10: $F_v := F_v \cup \{e_{vu_i}\}$ 11: $F'_v := maxNonOverlap(F_v)$ 12:for all $u_i \in child(v)$ do ${\bf if} \ e_{vu_i} \in F'_v \ {\bf then} \\$ 13: $TM1[v] := TM1[v] \cup TM2[u_i]$ 14:15:else $TM1[v] := TM1[v] \cup TM1[u_i]$ 16:17:if $p(v) \neq NULL$ then for all $e_{vu_i} \in F_v$ and $interSect(e_{vu_i}, e_{vp(v)}) = 0$ do 18: 19: $C_v := C_v \cup \{e_{vu_i}\}$ 20: $C'_v := maxNonOverlap(C_v)$ $TM2[v] := TM2[v] \cup \{e_{vp(v)}\}$ 21:for all $u_i \in child(v)$ do 22: 23:if $e_{vu_i} \in C'_v$ then 24: $TM2[v] := TM2[v] \cup TM2[u_i]$ 25:else $TM2[v] := TM2[v] \cup TM1[u_i]$ 26:27: M := TM1[r]28: return(M)

The function $interSect(e_{uv}, e_{vw})$ returns 1 if e_{uv}, e_{vw} are overlapping with each other, returns 0 otherwise.

5.3 **Proof of Correctness**

Lemma 5.2. Given a rooted temporal tree \mathcal{G} , for any T_v in \mathcal{G} such that $p(v) \neq NULL$, if |TM2[v]| < |TM1[v]| + 1, then there exists a maximum 0-1 timed matching for \mathcal{G} which does not include the edge $e_{vp(v)}$.

Proof. Let M be a maximum 0-1 timed matching for \mathcal{G} . Since \mathcal{G} is a tree, M must include a 0-1 timed matching for T_v . Let M_v^* be the 0-1 timed matching of T_v included in M. Then two cases are possible:

- M_v^* does not include $e_{vp(v)}$: In this case, M is a maximum 0-1 timed matching of \mathcal{G} that does not include $e_{vp(v)}$. So the lemma holds.
- M_v^* includes $e_{vp(v)}$: In this case, by definition of TM2[v], $|M_v^*| \leq |TM2[v]|$. However, since M is a maximum 0-1 timed matching for \mathcal{G} , $|M_v^*| = |TM2[v]|$, as otherwise we can replace M_v^* by TM2[v] in M to get a larger 0-1 timed matching, which is a contradiction. Therefore, $|M_v^*| \leq |TM1[v]|$, as |TM2[v]| < |TM1[v]| + 1. Consider $M' = (M \setminus M_v^*) \cup TM1[v]$. Then M' is a 0-1 timed matching of \mathcal{G} and $|M'| \geq |M|$. However, |M'| cannot be greater than |M| as M is a maximum 0-1 timed matching of \mathcal{G} . Hence, |M'| = |M|. Hence, M' is a maximum 0-1 timed matching of \mathcal{G} . Also, M' does not include the edge $e_{vp(v)}$. Hence, there exists a maximum 0-1 timed matching of \mathcal{G} that does not include the edge $e_{vp(v)}$.

The following lemma follows from Lemma 5.2

Lemma 5.3. Given a rooted temporal tree \mathcal{G} , for any T_v in \mathcal{G} such that $p(v) \neq NULL$, if |TM2[v]| < |TM1[v]| + 1, then TM1[v] is a maximum 0-1 timed matching for T_v .

Lemma 5.4. Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a rooted temporal tree such that $v \in \mathcal{V}$, for each T_{u_i} , $u_i \in child(v)$, $TM1[u_i]$ and $TM2[u_i]$ are already computed. Then, Equation 1 correctly computes TM1[v] for T_v .

Proof. We partition the set child(v) into the following two sets A and B:

- $A = \{u_i \mid u_i \in child(v) \land |TM2[u_i]| < |TM1[u_i]| + 1\}$
- $B = \{u_i \mid u_i \in child(v) \land |TM2[u_i]| = |TM1[u_i]| + 1\}$

Note that $A \cup B = child(v)^3$ and $A \cap B = \emptyset$. Also, by Definition 5.4, $F_v = \{e_{vu_i} \mid u_i \in B\}$. Hence, for all nodes $u_i \in A$, $e_{vu_i} \notin F_v$.

By definition, TM1[v] is a maximum 0-1 timed matching for T_v when edge $e_{vp(v)}$ is not included. Since T_v is a temporal tree, for each $u_i \in child(v)$, TM1[v] should include the maximum possible sized 0-1 timed matching for T_{u_i} , without violating the properties of a 0-1 timed matching. Two cases are possible for including the maximum 0-1 timed matching for T_{u_i} for a node $u_i \in child(v)$:

- Case 1: $u_i \in A$: In this case, From Lemma 5.3, $TM1[u_i]$ is a maximum 0-1 timed matching for T_{u_i} . Also, since $TM1[u_i]$ does not include the edge e_{vu_i} , inclusion of $TM1[u_i]$ does not violate the 0-1 timed matching property of TM1[v] irrespective of any choice made for including either $TM1[u_k]$ or $TM2[u_k]$ for any other T_{u_k} , $u_k \in child(v)$. Hence, in this case, $TM1[u_i]$ should be included in TM1[v].
- Case 2: $u_i \in B$: In this case, $|TM2[u_i]| > |TM1[u_i]|$ and hence, $TM2[u_i]$ should be included in TM1[v] instead of $TM1[u_i]$ if possible. However, including $TM2[u_k]$ for all T_{u_k} , $u_k \in B$ may violate the 0-1 timed matching property as e_{vu_x} , e_{vu_y} can be overlapping for two distinct trees T_{u_x} , T_{u_y} , such that $u_x, u_y \in B$. Since $|TM2[u_i]| = |TM1[u_i]| + 1$ in this case, the maximum number of edges can be included in TM1[v] from the 0-1 timed matchings of trees T_{u_i} , $u_i \in B$ in the following way: (i) include $TM2[u_r]$ in TM1[v] for the maximum possible number of trees T_{u_r} , $u_r \in B$ such that there are no overlapping edges, and (ii) include $TM1[v_s]$ in TM1[v] for the remaining trees T_{u_s} , $u_s \in B$.

Equation 1 includes $TM1[u_i]$ for all $u_i \in A$ in TM1[v] (as required in Case 1 above for nodes in A) as it includes $TM1[u_i]$ for all trees T_{u_i} , $u_i \in child(v)$ such that $e_{vu_i} \notin F'_v$ (since for all $u_i \in A$, $e_{vu_i} \notin F_v$ and hence $e_{vu_i} \notin F'_v$). Also, F'_v is the maximum cardinality subset of F_v with no overlapping edges, and Equation 1 also includes $TM2[u_i]$ for all trees T_{u_i} , such that $e_{vu_i} \in F'_v$. Hence, since $F_v = \{e_{vu_i} | u_i \in B\}$, Equation 1 includes (as required in Case 2 above for nodes in B) $TM2[u_r]$ for maximum possible number of trees T_{u_r} , $u_r \in B$. Also, for all other trees T_{u_s} , $u_s \in B$ it includes $TM1[u_s]$ as $u_s \notin F'_v$ (as required in Case 2 above). Hence Equation 1 correctly chooses the 0-1 timed matchings for all $u_i \in child(v)$ for maximizing the size of TM1[v]. Hence Equation 1 computes TM1[v] correctly.

The following lemma can be proved using similar arguments used to prove Lemma 5.4.

Lemma 5.5. Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a rooted temporal tree such that $v \in \mathcal{V}$, for each T_{u_i} , $u_i \in child(v)$, $TM1[u_i]$ and $TM2[u_i]$ are already computed. Then, Equation 2 correctly computes TM2[v] for T_v .

Theorem 5.6. Algorithm 1 correctly computes a maximum 0-1 timed matching for a given rooted temporal tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Proof. We first prove that the algorithm correctly computes TM1[v] and TM2[v] for each T_v where $v \in \mathcal{V}$. We prove this by induction on the height of the temporal tree T_v .

Base Case: The height of T_v is 1: Algorithm 1, in Line 6, initializes the values of TM1[v] and TM2[v] to \emptyset for any T_v . As height of T_v is 1, v is a leaf vertex. Thus, the computed $TM1[v] = \emptyset$ and $TM2[v] = \{e_{vp(v)}\}$ (added at Line 21) are correct.

Inductive Step: Assume that Algorithm 1 correctly computes TM1[x] and TM2[x] for any temporal tree T_x with height up to l. We need to prove that Algorithm 1 correctly computes TM1[v] and TM2[v] for temporal tree T_v with height l+1. Since the height of T_v is l+1, the depth of v is at most max_depth-l, where max_depth is the maximum depth of any vertex in \mathcal{G} . Then, for any $w \in child(v)$, the depth of w is at most max_depth -l+1, and hence, the height of w is at most l. Since the algorithm processes vertices in non-increasing order of their depths (Lines 3 and 4 in Algorithm 1), when TM1[v] and TM2[v] are computed, TM1[w] and TM2[w] are already computed correctly for all T_w , $w \in child(v)$ by the induction hypothesis. Hence, by Lemma 5.4 and 5.5, TM1[v] and TM2[v] are correctly computed using Equation 1 (Lines 8 to 16 in Algorithm 1) and Equation 2 (Lines 8 to 10, and Lines 17 to 26 in Algorithm 1).

Thus, Algorithm 1 correctly computes TM1[r] where r is the root vertex of \mathcal{G} and returns it as a maximum 0-1 timed matching for \mathcal{G} . Hence the theorem holds.

³For any T_{u_i} , if $|TM2[u_i]| > |TM1[u_i]| + 1$, we can get a 0-1 timed matching $M'' = TM2[u_i] \setminus \{e_{p(u_i)u_i}\}$ for T_{u_i} such that M'' does not include edge $e_{p(u_i)u_i}$ and $|TM1[u_i]| < |M''|$. According to definition of $TM1[u_i]$, this is impossible.

Theorem 5.7. The time complexity of Algorithm 1 is $O(n \log n)$.

Proof. Algorithm 1 stores the vertices in the rooted temporal tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in different lists according to their depth in \mathcal{G} such that two vertices with the same depth are in the same list. This can be done in O(n) time by traversing the rooted temporal tree in level order where $|\mathcal{V}| = n$.

While computing $TM1[v_i]$ and $TM2[v_i]$ for T_{v_i} where $v_i \in \mathcal{V}$, $TM1[v_j]$ and $TM2[v_j]$ of T_{v_j} are available for all $v_j \in child(v_i)$. In this scenario, as intermediate steps, Algorithm 1 computes F_{v_i} and C_{v_i} at Lines 8 to 10 and Lines 18 to 19 respectively. The number of edges incident on any vertex v_i is $d(v_i)$ where $d(v_i)$ is the degree of vertex v_i in \mathcal{G}_U , and each edge is associated with a single time interval. Thus, computing F_{v_i} and C_{v_i} take $O(d(v_i))$ time. Since the maximum size of F_{v_i} and C_{v_i} are $O(d(v_i))$, finding the maximum cardinality non-overlapping subsets F'_{v_i} and C'_{v_i} take $O(d(v_i) \log d(v_i))$ time using the function maxNonOverlap. Thus, the computation of $TM1[v_i]$ and $TM2[v_i]$ for any T_{v_i} takes $O(d(v_i) \log d(v_i))$ time. Hence, total time taken for computing $TM1[v_i]$ and $TM2[v_i]$ for all T_{v_i} , $v_i \in \mathcal{V}$, in the temporal tree is $\sum_{i=1}^n O(d(v_i) \log d(v_i)) = O(\sum_{i=1}^n d(v_i) \log d(v_i))$. Since for any $v_i \in \mathcal{V}$, $d(v_i) < n, \sum_{i=1}^n d(v_i) \log d(v_i) \le \sum_{i=1}^n d(v_i) \log n$. Hence, $\sum_{i=1}^n d(v_i) \log d(v_i) \le \log n \sum_{i=1}^n d(v_i)$. Since \mathcal{G}_U is a tree, $\sum_{i=1}^n d(v_i) = 2(n-1)$. Hence, $\sum_{i=1}^n d(v_i) \log d(v_i) \le 2(n-1) \log n$. Hence, the time complexity of Algorithm 1 is $O(n \log n)$.

6 Finding a Maximum 0-1 Timed Matching for Temporal Graphs

In this section, we address the problem of finding a maximum 0-1 timed matching for a given temporal graph which is not a tree. We first analyse the computational complexity of the problem. After that, we analyse the approximation hardness of the problem of finding a maximum 0-1 timed matching for a given temporal graph when multiple time intervals are associated with each edge. Finally, we propose an approximation algorithm for the problem.

6.1 Complexity of Finding Maximum 0-1 Timed Matching in Temporal Graphs

We first prove that the decision version of the problem of finding a maximum 0-1 timed matching for temporal graphs when each edge is associated with multiple time intervals, referred to as D-MAX-0-1-TM-MULT, is in NP. The D-MAX-0-1-TM-MULT problem is defined as follows.

Definition 6.1. *D-MAX-0-1-TM-MULT:* Given a temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with lifetime \mathcal{T} , where each edge in \mathcal{E} is associated with multiple time intervals, and a positive integer f, does there exist a 0-1 timed matching M for \mathcal{G} such that |M| = f?

Lemma 6.1. The D-MAX-0-1-TM-MULT problem is in NP.

Proof. Consider a certificate $\langle \langle \mathcal{G} = (\mathcal{V}, \mathcal{E}), f \rangle, M \rangle$, where \mathcal{G} is a temporal graph with lifetime \mathcal{T} such that each edge is associated with multiple time intervals, f is a given positive integer, and M is a given set of edges. We consider one edge $e_{uv} \in M$ at a time and compare the associated time intervals of e_{uv} with associated time intervals of all the other edges in M. We perform this check for all the edges in M to find if there exists any pair of edges overlapping with each other. As discussed in Section 3, the maximum number of intervals associated with an edge is $\lfloor \frac{\mathcal{T}}{2} \rfloor$. Hence, this checking can be done in polynomial time. Whether |M| = f and $M \subseteq \mathcal{E}$ can also be easily checked in polynomial time. Hence, the problem of finding a maximum 0-1 timed matching for temporal graphs when each edge is associated with multiple time intervals is in NP.

We have already proved in Theorem 5.1 that the problem of finding a maximum 0-1 timed matching for a given rooted temporal tree when each edge is associated with multiple time intervals is NP-complete. Thus, from Theorem 5.1 and Lemma 6.1, we get the following result:

Corollary 6.2. The problem of finding a maximum 0-1 timed matching for temporal graphs when each edge is associated with multiple time intervals is NP-complete.

Next, we investigate the computational complexity of the problem of finding a maximum 0-1 timed matching for temporal graphs when each edge is associated with a single time interval. We prove that this problem is also NP-complete. In order to show that, we prove that this problem is NP-complete even when the given temporal graph is a degree-bounded bipartite temporal graph such that degree of each vertex is bounded by 3 and each edge is associated with a single time interval. We refer to this problem as D-MAX-0-1-TMBD3B-1 problem. We first define the D-MAX-0-1-TMBD3B-1 problem. In the rest of

this paper, we refer to a bounded degree bipartite temporal graph when degree of each vertex is bounded by 3 as BDG3B.

Definition 6.2. *D-MAX-0-1-TMBD3B-1:* Given a BDG3B $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with lifetime \mathcal{T} , where each edge in \mathcal{E} is associated with a single time interval, and a positive integer z, does there exist a 0-1 timed matching M for \mathcal{G} such that |M| = z?

Next, we prove NP-completeness of the D-MAX-0-1-TMBD3B-1 problem by showing that there is a polynomial time reduction from the 2P2N-3SAT problem [10] to the D-MAX-0-1-TMBD3B-1 problem. The 2P2N-3SAT problem is known to be NP-complete [10]. The 2P2N-3SAT problem is defined as follows.

Definition 6.3. 2P2N-3SAT: Let $U = \{v_1, v_2, \dots, v_m\}$ be a set of m boolean variables and let Ψ be a 3-CNF formula with d clauses C_1, C_2, \dots, C_d such that each variable occurs exactly twice positively and twice negatively in Ψ . Does there exist a truth assignment which satisfies Ψ ?

Theorem 6.3. The D-MAX-0-1-TMBD3B-1 problem is NP-complete.

Proof. We first show that the D-MAX-0-1-TMBD3B-1 problem is in NP. Consider a certificate $\langle \langle \mathcal{G} = (\mathcal{V}, \mathcal{E}), z \rangle, M \rangle$, where \mathcal{G} is a BDG3B with lifetime \mathcal{T} such that each edge in \mathcal{E} is associated with a single time interval, z is a given integer, and M is a given set of edges. We consider one edge $e_{uv} \in M$ at a time and compare the associated time interval of e_{uv} with associated time intervals of all the other edges in M. We perform this check for all the edges in M to find if there exist any edges overlapping with each other. This checking can be done in polynomial time. Whether |M| = z and $M \subseteq \mathcal{E}$ can also be easily checked in polynomial time. Hence, the D-MAX-0-1-TMBD3B-1 problem is in NP.

Next, we prove that, there is a polynomial time reduction from the 2P2N-3SAT problem to the D-MAX-0-1-TMBD3B-1 problem. Let $\langle U, \Psi \rangle$ be an instance of the 2P2N-3SAT problem where $U = \{v_1, v_2, \dots, v_m\}$ be a set of *m* variables and Ψ be a 3-CNF formula with $d = \frac{4m}{3}$ clauses C_1, C_2, \dots, C_d such that each clause C_i consists of exactly 3 literals and each variable occurs exactly twice positively and twice negatively in Ψ . A literal is a variable or negation of a variable in *U*. Without loss of generality, we assume that each clause in Ψ consists of distinct literals. From this instance of the 2P2N-3SAT problem, we construct an instance of the D-MAX-0-1-TMBD3B-1 problem as follows.

- We construct a temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows:
 - 1. We add a vertex c_i to a set A_1 for each clause C_i in Ψ . Thus, $A_1 := \{c_i \mid \forall C_i \in \Psi\}$ We add a vertex a_i to a set A_2 for each variable v_i in U. Thus, $A_2 := \{a_i \mid \forall v_i \in U\}$ We add two vertices b_i, \bar{b}_i to a set B for each variable v_i in U. Thus, $B := \{b_i, \bar{b}_i \mid \forall v_i \in U\}$ Then the set of vertices \mathcal{V} in the temporal graph \mathcal{G} is $\mathcal{V} := A \cup B$ where $A := A_1 \cup A_2$
 - 2. We add an edge between vertex c_i and vertex b_j which exists for time interval (i 1, i), if v_j is a literal in clause C_i . Thus,

 $\mathcal{E}_{cb} := \{ e(c_i, b_j, (i-1, i)) \mid \forall v_j \in U, C_i \in \Psi, v_j \text{ is a literal in } C_i \}$

We add an edge between vertex c_i and vertex \bar{b}_j which exists for time interval (i-1,i), if \bar{v}_j is a literal in clause C_i . Thus,

 $\mathcal{E}_{c\bar{b}} := \{ e(c_i, \bar{b}_j, (i-1, i)) \, | \, \forall v_j \in U, C_i \in \Psi, \, \bar{v}_j \text{ is a literal in } C_i \}$

We add an edge between each vertex a_i and vertex b_i which exists for time interval (0, d). Thus,

 $\mathcal{E}_{ab} := \{ e(a_i, b_i, (0, d)) \, | \, \forall a_i, b_i \in \mathcal{V} \}$

We add an edge between each vertex a_i and vertex \bar{b}_i which exists for time interval (0, d). Thus,

 $\mathcal{E}_{a\bar{b}} := \{ e(a_i, \bar{b}_i, (0, d)) \, | \, \forall a_i, \bar{b}_i \in \mathcal{V} \}$

Then the set of edges ${\mathcal E}$ in the temporal graph ${\mathcal G}$ is

$$\mathcal{E} := \mathcal{E}_{cb} \cup \mathcal{E}_{c\bar{b}} \cup \mathcal{E}_{ab} \cup \mathcal{E}_{a\bar{b}}$$

3. $\mathcal{T} := d$



Figure 5: Construction of a BDG3B from the input of an instance of the 2P2N-3SAT problem.

• z := d + m

As there are exactly 3 literals in each clause C_j and for each variable v_i , literals v_i and \bar{v}_i are present in exactly two clauses each, \mathcal{G} is a BDG3B. Any edge $e(u, v, (s, f)) \in \mathcal{E}$ is also denoted as e_{uv} when the time interval for which this edge exists is not important. Figure 5 shows the construction of a BDG3B from an instance of the 2P2N-3SAT problem.

We first prove that if there is a solution for the D-MAX-0-1-TMBD3B-1 problem, then there is a truth assignment which satisfies Ψ . For a 0-1 timed matching M, |M| = z, for \mathcal{G} , we construct a truth assignment \mathcal{S} for Ψ as follows.

- 1. For each $e_{a_ib_i} \in M$, assign $v_i :=$ false.
- 2. For each $e_{a_i\bar{b}_i} \in M$, assign $v_i := \mathbf{true}$.

We first show that, for each vertex $v \in A$, there is exactly one edge incident on v is in M. According to our construction, \mathcal{G} is bipartite and |A| = m + d. Again, any two edges incident on the same vertex in A are overlapping with each other. Thus, no two edges incident on the same vertex in A can be in M. As |M| = m + d and |A| = m + d, for each vertex $v \in A$, there is exactly one edge incident on v in M. This shows that, for each vertex $a_i \in A_2$ there is exactly one edge incident on a_i in M. This implies that, each variable $v_i \in U$ is assigned to a truth value and no variable in U is assigned to both **true** and **false**.

Next we show that S satisfies all the clauses in Ψ . For any clause C_k , we show that at least one literal $v_i \in C_k$ or $\bar{v}_i \in C_k$ is satisfied. We have shown that, for each vertex $c_k \in A_1$, there is one edge incident on c_k in M. There are two possible cases:

- I. Some edge $e_{c_k b_i}$ incident on c_k is in M: This implies that if variable v_i is set to **true**, the clause C_k is satisfied. According to the truth assignment in S, if $e_{a_i \bar{b}_i}$ is in M, then v_i is set to **true**. Again we have already proved that, for each $a_i \in A_2$, there is an edge incident on a_i in M. If $e_{a_i b_i}$ is in M, then $e_{c_k b_i}$ cannot be in M because these two edges are overlapping with each other. Hence, $e_{a_i \bar{b}_i} \in M$ and C_k is satisfied.
- II. Some edge $e_{c_k \bar{b}_i}$ incident on c_k is in M: This implies that if variable v_i is set to **false**, the clause C_k is satisfied. According to the truth assignment in S, if $e_{a_i b_i}$ is in M, then v_i is set to **false**. Again we have already proved that, for each $a_i \in A_2$, there is an edge incident on a_i in M. If $e_{a_i \bar{b}_i}$ is in M, then $e_{c_k \bar{b}_i}$ cannot be in M because these two edges are overlapping with each other. Hence, $e_{a_i b_i} \in M$ and C_k is satisfied.

Thus, each clause C_k in Ψ is satisfied by S and each variable in U is assigned to either **true** or **false**. Hence, S is a truth assignment which satisfies Ψ .

Next, we prove that if there is no solution for the instance of the D-MAX-0-1-TMBD3B-1 problem, then there is no truth assignment which satisfies Ψ . We prove this by showing that if there is a truth assignment which satisfies Ψ , then there is a solution for the instance of the D-MAX-0-1-TMBD3B-1 problem. For a satisfying truth assignment S of Ψ , we construct a 0-1 timed matching M of size m + dfor \mathcal{G} as follows.

- 1. For any variable $v_i \in U$, if $v_i =$ **false** is in S, we include $e_{a_i b_i}$ in M.
- 2. For any variable $v_i \in U$, if $v_i = \mathbf{true}$ is in \mathcal{S} , we include $e_{a_i\bar{b}_i}$ in M.
- 3. For any clause C_l in Ψ , we select any one literal v_i in C_l such that v_i evaluates to **true** following assignment S, then we include $e_{c_l b_i}$ in M. If the selected literal is \bar{v}_i in C_l such that \bar{v}_i evaluates to **true** following assignment S, then we include $e_{c_l \bar{b}_i}$ in M. Note that, for each clause C_l , we include exactly one such edge incident on c_l in M.

We first prove that M is a 0-1 timed matching for \mathcal{G} . We prove this by contradiction. Assume that, M is not a 0-1 timed matching for \mathcal{G} . Then there are at least two edges in M, which are overlapping with each other. According to the construction of \mathcal{G} , any two edges incident on two different vertices in A_1 are non-overlapping with each other. M includes only one edge incident on each vertex in A_1 . As each variable v_i is assigned to either **true** or **false**, only one edge incident on a vertex in A_2 is included in M. Thus, two edges can overlap in following two cases:

- I. There is some vertex $b_i \in B$ such that $e_{a_ib_i}$ and some edge $e_{c_jb_i}$ are both included in M. According to the construction of M, $e_{a_ib_i} \in M$ implies that v_i is assigned to **false**. Again $e_{c_jb_i} \in M$ implies that, $v_i =$ **true** and $v_i \in C_j$. Thus, $e_{a_ib_i}$ and $e_{c_jb_i}$ both can be included in M only when v_i is assigned to both **true** and **false**. This is a contradiction.
- II. There is some vertex $\bar{b}_i \in B$ such that $e_{a_i\bar{b}_i}$ and some edge $e_{c_j\bar{b}_i}$ are both included in M. According to the construction of M, $e_{a_i\bar{b}_i} \in M$ implies that v_i is assigned to **true**. Again $e_{c_j\bar{b}_i} \in M$ implies that, $v_i =$ **false** and $\bar{v}_i \in C_j$. Thus, $e_{a_i\bar{b}_i}$ and $e_{c_j\bar{b}_i}$ both can be included in M only when v_i is assigned to both **true** and **false**. This is also a contradiction.

Next, we prove that |M| = m + d = z. We prove this by contradiction. There can be following two possible cases:

- I. |M| > m + d: According to the construction of M, for each vertex $v \in A$, only one edge incident on v is included in M. As \mathcal{G} is bipartite and |A| = m + d, |M| > m + d is impossible.
- II. |M| < m + d: S satisfies Ψ . There are d clauses in Ψ . Hence, according to the construction of M, one edge incident on each vertex in A_1 is included in M. Thus |M| < m + d implies that, there is at least one vertex $a_i \in A_2$ such that no edge incident on a_i is included in M. This implies that, there is a variable $v_i \in U$ which is not assigned to any truth value in S. This is a contradiction.

Hence, |M| = m + d = z. This completes the proof.

The following result directly follows from Theorem 6.3.

Corollary 6.4. The problem of finding a maximum 0-1 timed matching for temporal graphs when each edge is associated with a single time interval is NP-complete.

6.2 Approximation Hardness of Finding Maximum 0-1 Timed Matching for Temporal Graphs

Next, we study the approximation hardness of the problem of finding a maximum 0-1 timed matching for a temporal graph when multiple time intervals are associated with each edge. In order to show inapproximability of the problem of finding maximum 0-1 timed matching for temporal graphs, we prove that there is no $\frac{1}{n^{1-\epsilon}}$, for any $\epsilon > 0$, factor approximation algorithm for finding a maximum 0-1 timed matching for a rooted temporal tree when each edge is associated with multiple time intervals unless NP = ZPP. We refer to the problem of finding a maximum 0-1 timed matching for a rooted temporal tree when multiple time intervals are associated with each edge as the MAX-0-1-TMT-MULT problem.



Figure 6: (a) A static graph where edges are labelled by integers, g and h are two 0 degree vertices which are also labelled with integers (b) Corresponding temporal tree

We prove this by showing that there exists an approximation preserving reduction from the maximum independent set (MAX-IS) problem to the MAX-0-1-TMT-MULT problem. It is already known that there is no $\frac{1}{n^{1-\epsilon}}$, for any $\epsilon > 0$, factor approximation algorithm for the MAX-IS problem unless NP = ZPP [30, 5]. We define these two problems first.

Definition 6.4. *MAX-0-1-TMT-MULT:* Given a rooted temporal tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with lifetime \mathcal{T} , where each edge in \mathcal{E} is associated with multiple time intervals, find a maximum sized 0-1 timed matching M for \mathcal{G} .

Definition 6.5. *MAX-IS:* Given a static graph G = (V, E), find a maximum sized set $I \subseteq V$ such that no two vertices in I are connected by an edge in E.

Theorem 6.5. There exists an approximation preserving reduction from the MAX-IS problem to the MAX-0-1-TMT-MULT problem.

Proof. Consider an instance $\langle G = (V, E) \rangle$ of the MAX-IS problem where G is a static graph with |V| = n, and |E| = m. For our reduction, we assume that each edge $(u, v) \in E$ is labelled with a distinct integer a_{uv} between 0 to m - 1. We assume that the set V_0 of 0 degree vertices, $V_0 \subseteq V$, is given and $|V_0| = n_0$. If V_0 is not given, it can be easily computed in polynomial time. We also assume that, each vertex $v \in V_0$ is labelled with a distinct integer a_v , between m to $m + n_0 - 1$. For each vertex $v \in V$, the *neighbouring vertex set of* v, N_v , includes each vertex w_i such that edge $(v, w_i) \in E$. Note that, for any vertex $u \in V_0$, $N_u = \emptyset$.

From the given instance of the MAX-IS problem, we construct an instance of the MAX-0-1-TMT-MULT problem as follows.

- We construct the temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows
 - We add a vertex $\nu(v)$ for each vertex $v \in V$. Additionally add a vertex r. Thus, $\mathcal{V} := \{\nu(v) \mid \forall v \in V\} \cup \{r\}.$
 - We add an edge between each $\nu(v)$ added to \mathcal{V} for each vertex $v \in V_0$ to r. This edge exists for the time interval $(a_v, a_v + 1)$. Thus,
 - $\mathcal{E}_{deg0} := \{ e(\nu(v), r, (a_v, a_v + 1)) \mid \forall v \in V_0 \}$
 - We add an edge between each $\nu(v)$ added to \mathcal{V} for each vertex $v \in V \setminus V_0$ and r. This edge exists for the time intervals $(a_{vw_1}, a_{vw_1} + 1), (a_{vw_2}, a_{vw_2} + 1), \cdots, (a_{vw_k}, a_{vw_k} + 1)$ where each $w_i \in N_v$, k is the degree of v (deg(v)), and each a_{vw_i} is the integer labelling edge (v, w_i) incident on v. Thus,

$$\mathcal{E}_{degk} := \{ e(\nu(v), r, (a_{vw_1}, a_{vw_2} + 1), \cdots, (a_{vw_k}, a_{vw_k} + 1)) \mid \forall v \in (V \setminus V_0), \forall w_i \in N_v, k = deg(v) \}$$

- Then the set of edges \mathcal{E} in the temporal graph \mathcal{G} is
 - $\mathcal{E} := \mathcal{E}_{deg0} \cup \mathcal{E}_{degk}$
- Lifetime $\mathcal{T} = m + n_0$

There is an edge between each vertex in $\mathcal{V} \setminus \{r\}$ and r, and there is no edge between any other vertices. Thus, \mathcal{G} is a temporal tree rooted at r (we choose r as the root of \mathcal{G}). Any edge $e(u, v, (s_1, f_1), (s_2, f_2), \dots, (s_i, f_i)) \in \mathcal{E}$ is also denoted as e_{uv} when the time intervals for which this edge exists are not important. Figure 6 shows the construction of a rooted temporal tree from a given static graph where each edge in the temporal tree is associated with multiple time intervals. For a 0-1 timed matching M for \mathcal{G} , we construct a solution I for the MAX-IS problem on G such that |I| = |M| as follows. Consider the set of vertices $I = \{v \mid e_{r\nu(v)} \in M\}$. We show that I is an independent set for G and |I| = |M|. As for each edge in M a vertex in I is selected, |I| = |M|. We prove that I is an independent set for G by contradiction. Assume that I is not an independent set for G. This implies that there are at least two vertices $u, v \in I$ such that $(u, v) \in E$. As $u, v \in I$, both $e_{\nu(u)r}$ and $e_{\nu(v)r}$ are in M. As $(u, v) \in E$, (u, v) is labelled with integer a_{uv} , according to our construction, the time interval $(a_{uv}, a_{uv} + 1)$ is associated with both $e_{\nu(u)r}$ and $e_{\nu(v)r}$, and both are incident on r. This implies that M is not a 0-1 timed matching for \mathcal{G} . This is a contradiction. This completes the proof of the theorem. \Box

Therefore, from Theorem 6.5 and using the approximation hardness result of the MAX-IS [30, 5] problem, we conclude that, there is no $\frac{1}{n^{1-\epsilon}}$, for any $\epsilon > 0$, factor approximation algorithm for the MAX-0-1-TMT-MULT problem unless NP = ZPP. From this result for a rooted temporal tree, we get the following result:

Corollary 6.6. There is no $\frac{1}{n^{1-\epsilon}}$, for any $\epsilon > 0$, factor approximation algorithm for the problem of finding maximum 0-1 timed matching problem in temporal graphs when each edge is associated with multiple time intervals unless NP=ZPP.

6.3 Approximation Algorithm for Finding Maximum 0-1 Timed Matching for Temporal Graphs

In this section, we present an approximation algorithm for finding a maximum 0-1 timed matching for temporal graphs. We first show that there exists an approximation preserving reduction from the maximum 0-1 timed matching (MAX-0-1-TM-MULT) problem to the maximum independent set (MAX-IS) problem. We first define the MAX-0-1-TM-MULT problem.

Definition 6.6. *MAX-0-1-TM-MULT:* Given a temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with lifetime \mathcal{T} , where each edge in \mathcal{E} is associated with multiple time intervals, find a maximum sized 0-1 timed matching M for \mathcal{G} .

Theorem 6.7. There exists an approximation preserving reduction from the MAX-0-1-TM-MULT problem to the MAX-IS problem.

Proof. Consider an instance $\langle \mathcal{G} = (\mathcal{V}, \mathcal{E}) \rangle$ of MAX-0-1-TM-MULT problem where \mathcal{G} is a temporal graph with $|\mathcal{V}| = n$, $|\mathcal{E}| = m$ and lifetime \mathcal{T} . We construct an instance of the MAX-IS problem as follows. We construct the static graph G = (V, E) from $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in the following way:

- For each edge $e_{uv} \in \mathcal{E}$, add a vertex $\nu_{uv} \in V$. Thus,
 - $V := \{ \nu_{uv} \, | \, \forall e_{uv} \in \mathcal{E} \}$
- Add an edge (ν_{uv}, ν_{vw}) between ν_{uv} and ν_{vw} if e_{uv} and e_{vw} are overlapping with each other. Thus, $E := \{(\nu_{uv}, \nu_{vw}) | e_{uv}, e_{vw} \in \mathcal{E} \text{ are overlapping with each other}\}$

Given an independent set I for G, we can construct a 0-1 timed matching M for \mathcal{G} as $M = \{e_{uv} | \nu_{uv} \in I\}$. We show that M is a 0-1 timed matching for \mathcal{G} and |I| = |M|. As M includes an edge for each vertex in I, |I| = |M|. We prove that M is a 0-1 timed matching for \mathcal{G} by contradiction. Let there be two edges $e_{uv}, e_{vw} \in M$ such that e_{uv} and e_{vw} are overlapping with each other. This implies that, $\nu_{uv}, \nu_{vw} \in I$ and $(\nu_{uv}, \nu_{vw}) \in E$. Thus, I is not an independent set for G. This is a contradiction. Hence, M is a 0-1 timed matching for \mathcal{G} .

In Theorem 6.7, it can be noted that the number of vertices in the constructed static graph G = (V, E)is $|\mathcal{E}| = m$ and degree of each vertex in V is the overlapping number of the corresponding edge in \mathcal{E} . Thus, the average degree of a vertex in G is the average overlapping number of an edge in \mathcal{G} .

In [24], an approximation algorithm is proposed to find a maximum independent set for a given graph G with approximation ratio $\frac{5}{2d^*+3}$ where d^* is the average degree of a vertex in G. Thus, applying that algorithm on the static graph constructed from a given temporal graph using the steps described in Theorem 6.7, we can get the following result.

Theorem 6.8. There is an approximation algorithm to find a maximum 0-1 timed matching for a given temporal graph \mathcal{G} with approximation ratio $\frac{5}{2N^*+3}$, where \mathcal{N}^* is the average overlapping number of an edge in \mathcal{G} .

It can be observed that the overlapping number of any edge $e_{uv} \in \mathcal{E}$ is dependent on the degree of u and v in the underlying graph \mathcal{G}_U . Thus, the average overlapping number \mathcal{N}^* of edges in a temporal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is also dependent on the degree of vertices in \mathcal{G}_U . This indicates that this algorithm produces a 0-1 timed matching with good approximation ratio for sparse graphs. The algorithm produces a 0-1 timed matching with good approximation ratio even when there exists a small number of edges for which the overlapping number is high if the average overlapping number of the edge in \mathcal{E} is small.

It is proved earlier in Theorem 6.3 that the problem of finding a maximum 0-1 timed matching for a bounded degree bipartite temporal graph when degree of each vertex is bounded by 3 and each edge is associated with a single time interval is NP-complete. From Theorem 6.8, we obtain the following result for the problem of finding a maximum 0-1 timed matching for a given bounded degree temporal graph.

Corollary 6.9. There is an approximation algorithm to find a maximum 0-1 timed matching with approximation ratio $\frac{5}{2B+3}$ for a given bounded degree temporal graph where degree of each vertex is bounded by B and each edge is associated with multiple time intervals. Thus, for bounded degree temporal graphs where degree of each vertex is bounded by a constant, this is a constant factor approximation algorithm.

7 Conclusion

In this paper, we have defined 0-1 timed matching on temporal graphs, and investigated the problem of finding a maximum 0-1 timed matching for different types of temporal graphs. We have proved that this problem is NP-complete for a rooted temporal tree when each edge is associated with 2 or more time intervals, and proposed a $O(n \log n)$ time dynamic programming based algorithm for a rooted temporal tree with n vertices when each edge is associated with a single time interval. It is also proved that this problem is NP-complete for a bounded degree bipartite temporal graph where degree of each vertex is bounded by 3 or more such that each edge is associated with a single time interval. We have also proved that there is no $\frac{1}{n^{1-\epsilon}}$, for any $\epsilon > 0$, factor approximation algorithm for the problem of finding a maximum 0-1 timed matching even for a rooted temporal tree when each edge is associated with multiple time intervals unless NP = ZPP. Then, we have proposed an approximation algorithm to address the problem for a temporal graph when each edge is associated with multiple time intervals. The work can be extended to consider the problem on other classes of temporal graphs.

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