A NEW APPROACH ON LOCALLY CHECKABLE PROBLEMS

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ABSTRACT. By providing a new framework, we extend previous results on locally checkable problems in bounded treewidth graphs. As a consequence, we show how to solve, in polynomial time for bounded treewidth graphs, double Roman domination and Grundy domination, among other problems for which no such algorithm was previously known. Moreover, by proving that fixed powers of bounded degree and bounded treewidth graphs are also bounded degree and bounded treewidth graphs, we can enlarge the family of problems that can be solved in polynomial time for these graph classes, including distance coloring problems and distance domination problems (for bounded distances).

1. Introduction

Many combinatorial optimization problems in graphs can be classified as vertex partitioning problems. The partition classes have to verify inner-properties and/or inter-properties, and there is an objective function to minimize or maximize. Some of these properties are *locally checkable*, that is, the property that each vertex has to satisfy with respect to the partition involves only the vertex and its neighbors. This is the case of stable set, dominating set and k-coloring, among others.

In the spirit of generalizing this kind of problems, in [6,7] Bodlaender defined the local condition composition (LCC) and edge condition composition (ECC) problems, and showed polynomial-time algorithms to solve LCC problems on bounded treewidth and bounded degree graphs, and ECC problems on bounded treewidth graphs. In [61] Telle defined the locally checkable vertex partitioning (LCVP) problems and in [15], Bui-Xuan, Telle and Vatshelle presented dynamic programming algorithms for LCVP problems that run in polynomial time on many graph classes, including interval graphs, permutation graphs and Dilworth k graphs, and in fixed-parameter single-exponential time parameterized by boolean-width. In [17], Cattanéo and Perdrix defined a different generalization of LCVP problems that allows us to deal with properties of the subset that are not necessarily locally checkable, as for example being connected, and prove hardness results for LCVP problems and such generalizations.

In this paper, we define a new framework for locally checkable problems, which we call r-locally checkable problems. In a r-locally checkable problem, every vertex v has a list of colors L_v that it can receive, along with the cost $W_{v,i}$ of receiving each color $i \in L_v$. There is a function check that, for each vertex v and each coloring c of the closed neighborhood of radius r of v, determines if the colors assigned to v and the vertices at distance at most r from v are permitted for v. We include edge labels ℓ_e , whose values may be involved in the checking functions. For technical reasons, other simple operators are also required, such as one that combines the costs and one that compares them. This approach generalizes LCVP problems, including other problems such as $\{k\}$ -domination (which cannot be expressed as a LCVP problem, at least not in

²⁰¹⁰ Mathematics Subject Classification. 05C15, 05C69, 05C85, 68Q25, 68R10.

Key words and phrases. locally checkable problem, vertex partitioning problem, local condition composition problem, double Roman domination, Grundy domination, treewidth.

an straightforward way). We further consider a set of global properties (which might be, for example, that certain sets of the partition induce a connected or an acyclic subgraph). In this way, ECC problems can be modeled within our framework.

A key notion to our paper is the *treewidth* of a graph, which was introduced by Robertson and Seymour [56] (and previously considered under different names by Bertelè and Brioschi [5] and Halin [41]). Graphs of treewidth at most k are called partial k-trees. Some graph classes with bounded treewidth include: forests (treewidth 1); pseudoforests, cacti, outerplanar graphs, and series-parallel graphs (treewidth at most 2); Halin graphs and Apollonian networks (treewidth at most 3) [8, 11, 50, 56]. In addition, control flow graphs arising in the compilation of structured programs also have bounded treewidth (at most 6) [62].

To solve 1-locally checkable problems in bounded treewidth graphs, we give an algorithm based on a rather simple computation of a recursive function traversing a special tree decomposition, using dynamic programming as it is usual with this kind of problems, but with an abstraction of the "extra" parameters involved in ad-hoc solutions of locally checkable problems. In order to formally describe this abstraction, we introduce the concept of partial neighborhoods. A partial neighborhood system gives us tools to accumulate information from the neighbors of a vertex. According to the sizes of the sets and the time complexity of the functions involved, we distinguish polynomial and constant partial neighborhood systems. Then our algorithm is polynomial when some mild conditions are satisfied. The main result of this paper is the following.

Theorem 5.5.1. Let \mathcal{F} be a family of graphs of bounded treewidth. Consider a family of instances of a 1-locally checkable problem with a polynomial partial neighborhood system, where

- $G \in \mathcal{F}$,
- $C = \max\{|L_v| : v \in V(G)\}$ is polynomial in the input size, and
- the functions \oplus and min can be computed in polynomial time.

Then there exists an algorithm that solves these instances in polynomial time. Furthermore, if we have a constant partial neighborhood system, C is bounded by a constant, and the functions \oplus and min can be computed in constant time, then the time complexity of such algorithm is O(|V(G)|).

Furthermore, by proving that fixed powers of bounded degree and bounded treewidth graphs are also bounded degree and bounded treewidth graphs, we can enlarge the family of problems that can be solved in polynomial time for these graph classes, including distance coloring problems (packing chromatic number [14,30,36,60], L(p,1)-coloring [18,19,38,39]), distance independence [29], distance domination [43], and distance LCVP problems [48], for bounded distances. These results are unified in the following corollary of Section 6.

Corollary 6.0.3. Let \mathcal{F} be a family of graphs of bounded treewidth and bounded degree. Then, for any r-locally checkable problem with $G \in \mathcal{F}$, \mathcal{C} polynomial in the input size, and all functions check, min and \oplus computable in polynomial time, there exists a polynomial-time algorithm that solves it.

We also prove that NP-complete problems can be reduced to some 1-locally checkable problems in complete graphs, even when restricting the sets of colors and edge labels to $\{0,1\}$. Thus, a generalization of the polynomiality to bounded clique-width graphs is not possible unless P=NP.

We show how to model double Roman domination, minimum chromatic violation and Grundy domination as 1-locally checkable problems with polynomial partial neighborhood systems. As a result, we obtain polynomial-time algorithms to solve these problems for bounded treewidth graphs. Until the date and to the best of our knowledge, no such algorithms were previously known.

Courcelle's celebrated theorem (see [24]) states that every graph problem definable in Monadic Second-Order (MSO) logic can be solved in linear time for bounded treewidth graphs. However, its main drawback is that the multiplicative constants in the running time of the algorithm generated with an MSO-formula can be extremely large [33]. In contrast, the statement of our problem is closer to natural language, and the time complexity of the algorithm for bounded treewidth graphs is fully detailed and involves relatively small constants.

A similar approach for a family of problems of different nature was presented in [21]. Namely, the authors define the framework of algebraic path problems, enclosing weighted shortest path, dataflow problems, regular expressions, and other problems arising in program analysis in the area of software engineering, and they present an algorithm to solve this framework of problems in concurrent systems such that each of the components is a bounded treewidth graph.

The present paper is organized as follows. Section 2 contains the necessary preliminaries and basic definitions. The central notion of the paper, r-locally checkable problems, is formally introduced in Section 3. In Section 4, we study 1-locally checkable problems in complete graphs, under different hypothesis. In Section 5 we give an algorithm to solve 1-locally checkable problems parameterized by treewidth. In Section 6 we analyze the time complexity of r-locally checkable problems in bounded treewidth and bounded degree graphs, and prove that fixed powers of such graphs are also bounded treewidth and bounded degree graphs. In Section 7 we extend the algorithm from Section 5 with some global properties. In particular, this recovers the results in [6,7] for LCC problems. In Section 8 we show how to model different problems as 1-locally checkable problems with polynomial partial neighborhood systems, obtaining polynomial-time algorithms to solve these problems for bounded treewidth graphs. We include in Appendix A the definition of the locally checkable problems mentioned throughout the article.

2. Basic definitions and preliminary results

2.1. Algebraic definitions. Let S be a set. A closed binary operation on S is a function $\star\colon S\times S\to S$. It is usual to write $\star(s_1,s_2)$ as $s_1\star s_2$. Such an operation is commutative if $s_1\star s_2=s_2\star s_1$ for all $s_1,s_2\in S$, and it is associative if $(s_1\star s_2)\star s_3=s_1\star (s_2\star s_3)$ for all $s_1,s_2,s_3\in S$. An element $e\in S$ is neutral (also called identity) if $e\star s=s\star e=s$ for all $s\in S$. An element $a\in S$ is absorbing if $a\star s=s\star a=a$ for all $s\in S$. It is easy to prove that if $s\in S$ is a neutral (resp. absorbing) element, then this element is unique. A commutative and associative operation \star can be naturally extended to any nonempty finite subset of S, writing $\bigstar_{x\in X}P(x)$ when $\{P(x):x\in X\}\subseteq S$ and X is finite and nonempty, moreover, if the operation also has a neutral element e then we define $\bigstar_{x\in\emptyset}P(x)=e$.

Let S be a set and \star be a closed binary operation on S. Then (S, \star) is a monoid if \star is associative and has a neutral element. If \star is also commutative then (S, \star) is a commutative monoid.

A binary relation \mathcal{R} on a set S is a subset of the Cartesian product $S \times S$. It is usual to write $(s_1, s_2) \in \mathcal{R}$ as $s_1 \mathcal{R} s_2$. We say that \mathcal{R} is reflexive if $s \mathcal{R} s$ for all

 $s \in S$, antisymmetric if $s_1 \mathcal{R} s_2 \wedge s_2 \mathcal{R} s_1 \Rightarrow s_1 = s_2$ for all $s_1, s_2 \in S$, and transitive if $s_1 \mathcal{R} s_2 \wedge s_2 \mathcal{R} s_3 \Rightarrow s_1 \mathcal{R} s_3$ for all $s_1, s_2, s_3 \in S$. If \mathcal{R} is reflexive, antisymmetric and transitive, then (S, \mathcal{R}) is called a partial order (or partially ordered set). If in addition $s_1 \mathcal{R} s_2 \vee s_2 \mathcal{R} s_1$ for every $s_1, s_2 \in S$, then (S, \mathcal{R}) is a total order (or totally ordered set).

Let (S, \preceq) be a totally ordered set. A maximum element is an element $m \in S$ such that $s \preceq m$ for all $s \in S$. Note that not every totally ordered set has a maximum element, and it is easy to prove that if it does have a maximum element then this element is unique. The minimum operation, min, is the closed binary operation on S such that $\min(s_1, s_2) = s_1$ if $s_1 \preceq s_2$ and $\min(s_1, s_2) = s_2$ if $s_2 \preceq s_1$. It is easy to prove that min is commutative and associative.

A set of natural numbers is *co-finite* if its complement with respect to the set of natural numbers is finite.

We denote by [a, b], with $a, b \in \mathbb{Z}$ and a < b, the set of all integer numbers greater than or equal to a and less than or equal to b, that is $\{a, a + 1, \dots, b\}$.

Given a set S and a function $f: S \to \mathbb{R}$, the weight of the function f (finite or infinite) is defined as $f(S) = \sum_{s \in S} f(s)$.

Throughout this paper we will work with the set Bool = $\{TRUE, FALSE\}$ of boolean values and all the usual logical operators, such as \neg , \land , \lor and \Rightarrow .

Let $f: A \to B$ be a function and let $S \subseteq A$. We denote by $f|_S$ the function f restricted to the domain S, that is, the function $f|_S: S \to B$ is defined as $f|_S(s) = f(s)$ for all $s \in S$.

- 2.2. Automata. A deterministic finite-state automaton is a five-tuple $(Q, \Sigma, \delta, q_0, F)$ that consists of
 - Q: a finite set of states,
 - Σ : a finite set of *input symbols* (often called the *alphabet*),
 - $\delta: Q \times \Sigma \to Q$: a transition function,
 - $q_0 \in Q$: an initial or start state, and
 - $F \subseteq Q$: a set of final or accepting states.

We say that an automaton $M = (Q, \Sigma, \delta, q_0, F)$ accepts a string $s_1 ... s_n$, with $n \ge 1$, if and only if $s_i \in \Sigma$ for all $1 \le i \le n$ and $\delta(... \delta(\delta(q_0, s_1), s_2) ..., s_n) \in F$.

For example, the automaton $M = (\{q_0, q_1\}, \{1\}, \delta, q_0, \{q_1\})$ where $\delta(q_0, 1) = q_1$ and $\delta(q_1, 1) = q_0$, is an automaton that accepts sequences of an odd number of 1s.

For more about automata theory we refer the reader to [47].

2.3. Computability. A function is polynomial time computable if there exists an algorithm A and a polynomial p(n) such that, for all inputs of length n, A computes the function and runs in time less than or equal to p(n).

All sets considered in this article contain elements that can be encoded and passed as parameters to a computable function.

We will not delve further into this topic. For more information we refer the reader to the vast literature on computability theory.

2.4. Basic definitions on graphs. Let G be a finite, simple and undirected graph. We denote by V(G) and E(G) the vertex set and edge set, respectively, of G. For any $W \subseteq V(G)$, we denote by G[W] the subgraph of G induced by G[V]. Let G[V] (closed neighborhood of G[V]) be the set of neighborhood of G[V] and let G[V] = G[V] = G[V] = G[V] (closed neighborhood of G[V]). The closed neighborhood of G[V] is G[V]. The degree of a vertex G[V] is denoted by G[V]. The maximum degree of a vertex in G[V] is denoted by G[V].

A graph class is a collection of graphs that is closed under isomorphism. Given a graph class \mathcal{G} , we say that \mathcal{G} is of bounded degree if $\sup\{\Delta(G) \mid G \in \mathcal{G}\} < \infty$.

A graph G is connected if for every pair of vertices u, v in V(G) there exists a path in G from u to v. A connected component of a graph is an inclusion-wise maximal connected subgraph of it. For two vertices x, y in a connected graph G, we denote by $\operatorname{dist}_G(x,y)$ the distance between x and y, that is, the length (number of edges) of a shortest x,y-path in G. Let $N_G^k[v]$ be the set of vertices at distance at most k from v in G, and $N_G^k(v) = N_G^k[v] - \{v\}$. Let $M_G^k(v)$ be the set of edges whose endpoints at distance at most k from k in k-th power of k is the graph denoted by k-such that for all distinct vertices k-th power of k-such that for all distinct vertices k-th power of k-such that for all distinct vertices k-th power of k-such that for all distinct vertices k-th power of k-such that for all distinct vertices k-th power of k-such that for all distinct vertices k-th power of k-such that for all distinct vertices k-th power of k-such that for all distinct vertices k-th power of k-th power of k-such that for all distinct vertices k-th power of k-th power

A complete graph is a graph whose vertices are pairwise adjacent. We denote by K_r the complete graph on r vertices. A clique (resp. stable set or independent set) in a graph is a set of pairwise adjacent (resp. nonadjacent) vertices. The maximum size of a clique (resp. independent set) in the graph G is denoted by $\omega(G)$ (resp. $\alpha(G)$).

A graph G is bipartite if V(G) can be partitioned into two stable sets V_1 and V_2 , and G is complete bipartite if every vertex of V_1 is adjacent to every vertex of V_2 . We denote by $K_{r,s}$ the complete bipartite graph with $|V_1| = r$ and $|V_2| = s$. The star S_n is the complete bipartite graph $K_{1,n-1}$.

A proper k-coloring of a graph is a partition of its vertices into at most k stable sets, each of them called color class. Equivalently, a proper k-coloring is an assignment of colors to vertices such that adjacent vertices receive different colors, and the number of colors used is at most k. The chromatic number $\chi(G)$ of a graph G is the minimum k that allows a proper k-coloring of G. In the more general LIST-COLORING problem, each vertex v has a list L(v) of available colors for it.

A pair of vertices or a pair of edges dominate each other when they are either equal or adjacent, while a vertex and an edge dominate each other when the vertex belongs to the edge. We will denote by $\gamma_{U,W}(G)$, for U,W sets of elements of G, the minimum cardinality or weight of a subset S of U which dominates W. The parameter $\gamma_{V,V}$ is also denoted simply by γ , and the associated problem is known as MINIMUM DOMINATING SET. Parameters $\gamma_{V,E}$ and $\gamma_{E,V}$ are associated with the MINIMUM VERTEX COVER and MINIMUM EDGE COVER, respectively. In the MINIMUM $\{k\}$ -DOMINATION problem, given a graph G we want to find the minimum weight of a function $f: V(G) \to \{0, 1, \ldots, k\}$ such that $\sum_{u \in N_G[v]} f(u) \geq k$ for all $v \in V(G)$.

For a graph G and $uv \in E(G)$, the graph obtained by subdividing uv in G arises from G by adding a new vertex w, making w adjacent to u and v, and then deleting the edge uv. The subdivision graph of G, obtained by subdividing each of the edges of G, is S(G) = (V', E') where $V' = V(G) \cup E(G)$ and $E' = \{ve : v \in V(G), e \in E(G), \text{ and } v \text{ is an endpoint of } e\}$.

The jagged graph of G is J(G) = (V', E') where $V' = V(G) \cup E(G)$ and $E' = E(G) \cup \{ve : v \in V(G), e \in E(G), \text{ and } v \text{ is an endpoint of } e\}.$

The *line graph* of a graph G is denoted by L(G) and has as vertex set E(G), where two vertices are adjacent in L(G) if and only if the corresponding edges have a common endpoint (i.e., are adjacent) in G. The *total graph* of G, denoted by T(G), is defined similarly: its vertex set is $V(G) \cup E(G)$, V(G) induces G, E(G) induces L(G), and $v \in V(G)$, $uw \in E(G)$ are adjacent in T(G) if and only if either v = u or v = w.

A graph, or a subgraph of a graph, is *acyclic* if it does not contain a cycle of length at least three. An acyclic graph is called a *forest*. A *tree* is a connected acyclic graph. In a tree T, we usually call the elements in V(T) nodes. A node with degree at most 1

is called a *leaf* and a node of degree at least 2 is called an *internal node*. A tree is called a *rooted tree* if one vertex has been designated the *root*, in which case the edges have a natural orientation, towards or away from the root. For a rooted tree T and $u \in V(T)$, the neighbor of u on the path to the root is called the *parent* of u and a vertex v is a *child* of u if u is the parent of v. A *binary tree* is a rooted tree where every internal node has at most two children.

- 2.5. **Definitions and preliminary results on treewidth.** A tree-decomposition of a graph G is a family $\{X_i : i \in I\}$ of subsets of V(G) (called bags), together with a tree T with V(T) = I, satisfying the following properties:
 - (W1) $\bigcup_{i \in I} X_i = V(G)$.
 - (W2) Every edge of G has both its ends in X_i for some $i \in I$.
 - (W3) For all $v \in V(G)$, the set of nodes $\{i \in I : v \in X_i\}$ induces a subtree of T.

The width of the tree-decomposition is $\max\{|X_i|-1:i\in I\}$. The treewidth of G, denoted tw(G), is the minimum $w\geq 0$ such that G has a tree-decomposition of width less or equal w.

Given a graph class \mathcal{G} , the treewidth of \mathcal{G} is $tw(\mathcal{G}) = \sup\{tw(G) \mid G \in \mathcal{G}\}$. We say that \mathcal{G} is of bounded treewidth if $tw(\mathcal{G}) < \infty$.

We will often make use of the following basic properties of the treewidth, some of which can be easily deduced.

Proposition 2.5.1. Let \mathcal{G} be a family of graphs of bounded treewidth. If $G \in \mathcal{G}$ then |E(G)| is O(|V(G)|).

Proposition 2.5.2. If H is a subgraph of a graph G then $tw(H) \leq tw(G)$.

Theorem 2.5.3 ([57, pages 1 and 2]). For every graph G, $tw(G) \ge \omega(G) - 1$. Moreover, $tw(G) \ge \chi(G) - 1$.

Theorem 2.5.4 ([50, page 76]). The treewidth of S(G) is equal to the treewidth of G.

Theorem 2.5.5. The treewidth of J(G) is less than or equal to tw(G) + 1.

A tree-decomposition $(T, \{X_t\}_{t \in V(T)})$ is nice [50, Definition 13.1.4] if

- T is a rooted binary tree;
- if a node i has two children j and k then $X_i = X_j = X_k$; (join node)
- if a node i has one child j, then either
 - $|X_i| = |X_j| 1$ and $X_i \subset X_j$, or (forget node) • $|X_i| = |X_j| + 1$ and $X_i \supset X_j$. (introduce node)

Let T_i be the subtree of T rooted at node i. We will denote by G_i the subgraph of G induced by $\bigcup_{i \in V(T_i)} X_i$.

Theorem 2.5.6 ([9, Theorem 1]). There exists an algorithm, that given an n-vertex graph G and an integer k, in time $O(c^k n)$ for some $c \in \mathbb{N}$, either outputs that the treewidth of G is larger than k, or constructs a tree-decomposition of G of width at most 5k + 4.

Theorem 2.5.7 ([50, Lemma 13.1.3]). For constant k, given a tree-decomposition of a graph G of width k and O(n) nodes, where n is the number of vertices of G, one can find a nice tree-decomposition of G of width k and with at most 4n nodes in O(n) time.

However, we will work with a slight modification of nice tree decompositions, where the bags of the root and leaves have only one vertex each. **Definition 2.5.8.** A tree-decomposition $(T, \{X_t\}_{t \in V(T)})$ is called *easy* if

- T is a binary tree rooted at r such that $|X_r|=1$;
- if a node i has two children j and k then $X_i = X_j = X_k$; (join node)
- if a node i has one child j, then either
 - $|X_i| = |X_j| 1$ and $X_i \subset X_j$, or (forget node)
 - $|X_i| = |X_j| + 1$ and $X_i \supset X_j$; (introduce node)
- if a node i has no children, then $|X_i|=1$.

(leaf node)

It is straightforward to prove that, given a nice tree-decomposition $(T, \{X_t\}_{t \in V(T)})$ of width k and O(n) nodes, one can construct in O(kn) time an easy tree-decomposition of width k and O(kn) nodes.

- 2.6. Definitions and preliminary results on frameworks for locally checkable problems. Throughout this article we will make special emphasis on two previous frameworks for locally checkable problems. In this section we review their definitions and results related to our work.
- 2.6.1. LCVP problems. Let σ and ρ be finite or co-finite subsets of non-negative integer numbers. A subset S of vertices of a graph G is a sigma-rho set, or simply (σ, ρ) -set, of G if for every v in S, $|N(v) \cap S| \in \sigma$, and for every v in $V(G) \setminus S$, $|N(v) \cap S| \in \rho$. The locally checkable vertex subset problems [61] consist of finding a minimum or maximum (σ, ρ) -set in an input graph G, possibly on vertex weighted graphs.

A generalization of these problems asks for a partition of V(G) into q classes, with each class satisfying a certain (σ, ρ) -property, as follows. A degree constraint matrix D_q is a $q \times q$ matrix with entries being finite or co-finite subsets of non-negative integer numbers. A D_q -partition of a graph G is a partition $\{V_1, V_2, \dots, V_q\}$ of V(G) such that for $1 \leq i, j \leq q$ it holds that for every $v \in V_i$, $|N(v) \cap V_j| \in D_q[i,j]$. A locally checkable vertex partitioning (LCVP) problem [61] consists of deciding if G has a D_q partition. Optimization versions can be defined, possibly on vertex weighted graphs.

The distance-r locally checkable vertex partitioning problems [48] further generalize LCVP problems by considering, for each vertex v, $N_G^r(v)$ instead of $N_G(v)$.

- In [15], Bui-Xuan, Telle and Vatshelle presented dynamic programming algorithms for LCVP problems that run in polynomial time on many graph classes, including interval graphs, permutation graphs and Dilworth k graphs, and in fixed-parameter singleexponential time parameterized by boolean-width. In [48] Jaffke, Kwon, Strømme and Telle presented dynamic programming algorithms for distance-r locally checkable vertex partitioning problems in graphs of bounded mim-width.
- 2.6.2. LCC and ECC problems. In [6,7] Bodlaender defined the local condition composition (LCC) and edge condition composition (ECC) problems, and showed polynomialtime algorithms to solve LCC problems on bounded treewidth and bounded degree graphs, and ECC problems on bounded treewidth graphs.

Definition 2.6.1 ([6, Definition 2.9]). Let Π be a graph decision problem, and let D_{Π} be the set of instances of Π , Y_{Π} the set of instances for which the answer is "yes", and $s:D_{\Pi}\to\mathbb{N}$ be a function that assigns sizes to instances of Π . We say Π is a basic local condition problem, if and only if there exist

- non-negative integers $m, c \in \mathbb{N}$,
- m commutative monoids $(M^1, \oplus^1), \ldots, (M^m, \oplus^m)$, and a tuple $(M^{m+1}, \oplus^{m+1}, \preceq)$ such that (M^{m+1}, \oplus^{m+1}) is a commutative monoid, (M^{m+1}, \preceq) is a total order and $a \preceq b \Rightarrow a \oplus^{m+1} c \preceq b \oplus^{m+1} c$ for all $a, b, c \in M^{m+1}$

such that

- each $D \in D_{\Pi}$ is of the form $(G, (X, Y, R_1, \dots, R_m, K, I))$, where
 - G is an undirected graph
 - X is a finite set with $s(D) \ge |X|$
 - Y is a finite set with s(D) > |Y|
 - for all $i, 1 \leq i \leq m, R_i$ denotes a subset of M^i
 - $K \in M^{m+1}$
- for all $i, 1 \leq i \leq m+1$, there exists a function val_i , that maps all 4-tuples, consisting of an instance $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}$, a vertex $v \in V(G)$, and functions $f: N_G^c[v] \to X$, $g: M_G^c(v) \to Y$, to elements of M_i , such that for all (constants) $d \in \mathbb{N}^+$:
 - (1) there exists an algorithm that calculates $val_i(D, v, f, g)$, for all D = (G, (X, G)) $Y, R_1, \ldots, R_m, K, I) \in D_{\Pi}, v \in V(G), f: N_G^c[v] \to X, g: M_G^c(v) \to Y$ with degree(G) $\leq d$, in time, polynomial in s(D).
 - $R_1, \ldots, R_m, K, I) \in D_{\Pi}$, with degree $(G) \leq d$ and subsets $W \subseteq V(G)$: $|\{\bigoplus_{w\in W}^{i} val_{i}(D, w, f|_{N_{G}^{c}[w]}, g|_{M_{G}^{c}(w)}) \mid f : N_{G}^{c}[W] \to X, g : M_{G}^{c}(W) \to Y\}| \le |\{\{\bigoplus_{w\in W}^{i} val_{i}(D, w, f|_{N_{G}^{c}[w]}, g|_{M_{G}^{c}(w)}) \mid f : N_{G}^{c}[W] \to X, g : M_{G}^{c}(W) \to Y\}| \le |\{\{\{\bigoplus_{w\in W}^{i} val_{i}(D, w, f|_{N_{G}^{c}[w]}, g|_{M_{G}^{c}(w)}) \mid f : N_{G}^{c}[W] \to X, g : M_{G}^{c}(W) \to Y\}| \le |\{\{\{\{\{\{\}\}\}\}\}\}\}|_{M_{G}^{c}(w)} = |\{\{\{\}\}\}\}\}|_{M_{G}^{c}(w)} = |\{\{\{\}\}\}\}|_{M_{G}^{c}(w)} = |\{\{\}\}\}|_{M_{G}^{c}(w)} = |\{\{\}\}|_{M_{G}^{c}(w)} = |\{\{\}\}|_{M_{G}^{c}(w)} = |\{\{\}\}|_{M_{G}^{c}(w)} = |\{\{\}\}|_{M_{G}^{c}(w)} = |\{\}\}|_{M_{G}^{c}(w)} = |\{\{\}\}|_{M_{G}^{c}(w)} = |\{\}\}|_{M_{G}^{c}(w)} = |\{\}|_{M_{G}^{c}(w)} = |\{\}|_{M_{G}^{c}(w$ $p_i(s(D)).$
 - (3) there exists an algorithm that calculates $a \oplus^i b$ for given a, b, such that there are $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}$, with degree $(G) \leq$ $d, \ W_1 \ \subseteq \ V(G), \ W_2 \ \subseteq \ V(G), \ W_1 \cap W_2 \ = \ \emptyset, \ f \colon N_G^c[W_1 \cup W_2] \ \to \ X,$ $g: M_G^c(W_1 \cup W_2) \to Y, \ a = \bigoplus_{w \in W_1}^i val_i(D, w, f|_{N_G^c[w]}, g|_{M_G^c(w)}) \text{ and } b = 0$ $\bigoplus_{w\in W_2}^{i} val_i(D, w, f|_{N_G^c[w]}, g|_{M_G^c(w)})$, in time, polynomial in s(D).
 - (4) if i = m + 1, then there exists an algorithm, that calculates whether $a \leq b$ for given a, b, such that there are $D = (G, (X, Y, R_1, \dots, R_m, K, I)) \in D_{\Pi}$, with degree(G) $\leq d$, $W \subseteq V(G)$, $f_1 \colon N_G^c[W] \to X$, $f_2 \colon N_G^c[W] \to X$, $g_1 \colon M_G^c(W) \to Y$, $g_2 \colon M_G^c(W) \to Y$, $a = \bigoplus_{w \in W}^{m+1} val_{m+1}(D, w, f_1|_{N_G^c[w]})$ $g_1|_{M_G^c(w)}$) and $b = \bigoplus_{w \in W}^{m+1} val_{m+1}(D, w, f_2|_{N_G^c[w]}, g_2|_{M_G^c(w)})$ or b = K, in time polynomial in s(D).
 - (5) if $1 \leq i \leq m$, there exists an algorithm that calculates for all D = (G, G) $(X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}$ with degree $(G) \leq d, f : V(G) \rightarrow X$, $g: E(G) \to Y$ and given $a = \bigoplus_{w \in V(G)}^{i} val_i(D, w, f|_{N_G^c[w]}, g|_{M_G^c(w)})$ whether $a \in R_i$, in time polynomial in s(D).
- For all $D = (G, (X, Y, R_1, \dots, R_m, K, I)) \in D_{\Pi} : D \in Y_{\Pi}$, if and only if there exists functions $f: V(G) \to X, g: E(G) \to Y$, with
 - (1) $\forall i, 1 \leq i \leq m : \bigoplus_{v \in V(G)}^{i} val_{i}(D, v, f|_{N_{G}^{c}[v]}, g|_{M_{G}^{c}(v)}) \in R_{i}$ (2) $\bigoplus_{v \in V(G)}^{m+1} val_{m+1}(D, v, f|_{N_{G}^{c}[v]}, g|_{M_{G}^{c}(v)}) \leq K.$

Definition 2.6.2 ([6, Definition 2.10]). Let Π be a graph decision problem. We say II is a local condition composition problem, if and only if there exists a basic local condition composition problem Π' and a graph-invariant polynomial transformation from Π to Π' . The class of local condition composition problems is denoted by LCC.

Definition 2.6.3 ([6, Definition 2.11]). Let Π be a graph decision problem, and let D_{Π} be the set of instances of Π , Y_{Π} the set of instances for which the answer is "yes", and $s:D_{\Pi}\to\mathbb{N}$ be a function that assigns sizes to instances of Π . We say Π is a basic edge condition problem, if and only if there exist

- a non-negative integer $m \in \mathbb{N}$,
- m commutative monoids $(M^1, \oplus^1), \ldots, (M^m, \oplus^m)$, and

- a tuple $(M^{m+1}, \oplus^{m+1}, \preceq)$ such that (M^{m+1}, \oplus^{m+1}) is a commutative monoid, (M^{m+1}, \preceq) is a total order and $a \preceq b \Rightarrow a \oplus^{m+1} c \preceq b \oplus^{m+1} c$ for all $a, b, c \in M^{m+1}$ such that
 - each $D \in D_{\Pi}$ is of the form $(G, (X, Y, R_1, \dots, R_m, K, I))$, where
 - G is an undirected graph
 - X is a finite set with $s(D) \ge |X|$
 - Y is a finite set with s(D) > |Y|
 - for all $i, 1 \leq i \leq m, R_i$ denotes a subset of M^i
 - $K \in M^{m+1}$
 - for all $i, 1 \leq i \leq m+1$, there exists a function val_i , that maps all 4-tuples, consisting of an instance $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}$, an edge $uv \in E(G)$, and functions $f: \{u, v\} \to X$, $g: \{uv\} \to Y$, to elements of M_i , such that:
 - (1) there exists an algorithm that calculates $val_i(D, e, f, g)$, for all $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}$, $uv \in E(G)$, $f : \{u, v\} \to X$, $g : \{uv\} \to Y$ in time, polynomial in s(D).
 - (2) if $1 \leq i \leq m$, there is a polynomial p_i , such that for all $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}$, and subsets $E' \subseteq E(G) : |\{\bigoplus_{uv \in E'}^i val_i(D, uv, f(u), f(v), g(uv)) \mid f : N_G(E) \to X, g : E \to Y\}| \leq p_i(s(D))$.
 - (3) there exists an algorithm that calculates $a \oplus^i b$ for given a, b, such that there are $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}$, $E_1 \subseteq E(G)$, $E_2 \subseteq E(G)$, $E_1 \cap E_2 = \emptyset$, $f: N_G(E_1 \cup E_2) \to X$, $g: E_1 \cup E_2 \to Y$, $a = \bigoplus_{uv \in E_1}^i val_i(D, uv, f(u), f(v), g(uv))$ and $b = \bigoplus_{uv \in E_2}^i val_i(D, uv, f(u), f(v), g(uv))$, in time, polynomial in s(D).
 - (4) if i = m + 1, then there exists an algorithm, that calculates whether $a \leq b$ for given a, b, such that there are $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}, E' \subseteq E(G), f_1 \colon N_G(E') \to X, f_2 \colon N_G(E') \to X, g_1 \colon E' \to Y, g_2 \colon E' \to Y, a = \bigoplus_{uv \in E'}^{m+1} val_{m+1}(D, uv, f_1(u), f_1(v), g_1(uv))$ and $b = \bigoplus_{uv \in E'}^{m+1} val_{m+1}(D, uv, f_2(u), f_2(v), g_2(uv))$ or b = K, in time polynomial in s(D).
 - (5) if $1 \leq i \leq m$, there exists an algorithm that calculates for all $D = (G, (X, Y, R_1, \ldots, R_m, K, I)) \in D_{\Pi}, f: V(G) \to X, g: E(G) \to Y$ and given $a = \bigoplus_{uv \in E(G)}^{i} val_i(D, uv, f(u), f(v), g(uv))$ whether $a \in R_i$, in time polynomial in s(D).
 - For all $D = (G, (X, Y, R_1, \dots, R_m, K, I)) \in D_{\Pi} : D \in Y_{\Pi}$, if and only if there exists functions $f: V(G) \to X$, $g: E(G) \to Y$, with
 - (1) $\forall i, 1 \leq i \leq m : \bigoplus_{uv \in E(G)}^{i} val_i(D, uv, f(u), f(v), g(uv)) \in R_i$
 - (2) $\bigoplus_{uv \in E(G)}^{m+1} val_{m+1}(D, uv, f(u), f(v), g(uv)) \leq K.$

Definition 2.6.4 ([6, Definition 2.12]). Let Π be a graph decision problem. We say Π is an edge condition composition problem, if and only if there exists a basic edge condition composition problem Π' and a graph-invariant polynomial transformation from Π to Π' . The class of edge condition composition problems is denoted by ECC.

Theorem 2.6.5 ([6, Theorem 2.5]). $ECC \subseteq LCC$.

Theorem 2.6.6 ([6, Theorem 3.7]). (i) Let $\Pi \in LCC$, and let $k, d \in \mathbb{N}^+$. Let Θ be a class of graphs with $G \in \Theta \Rightarrow degree(G) \leq d \wedge treewidth(G) \leq k$. Then Π restricted to Θ can be solved in polynomial time.

(ii) Let $\Pi \in ECC$, and let $k \in \mathbb{N}^+$. Let Θ be a class of graphs with $G \in \Theta \Rightarrow treewidth(G) < k$. Then Π restricted to Θ can be solved in polynomial time.

3. r-locally checkable problems

Let $r \in \mathbb{N}$. Let G be a simple undirected graph. Then suppose we have the following:

- a set LABELS and, for each edge $e \in E(G)$, a label $\ell_e \in \text{LABELS}$;
- a set Colors and, for every vertex $v \in V(G)$, a nonempty set (also called list) $L_v \subseteq \text{Colors}$ of possible colors for v;
- a totally ordered set (WEIGHTS, \leq) with a maximum element (called ERROR), together with the minimum operation of the order \leq (called min) and a closed binary operation on WEIGHTS (called \oplus) that is commutative and associative, has a neutral element (called e_{\oplus}) and an absorbing element that is equal to ERROR, and is such that $s_1 \leq s_2 \Rightarrow s_1 \oplus s_3 \leq s_2 \oplus s_3$ for all $s_1, s_2, s_3 \in$ WEIGHTS;
- for every vertex $v \in V(G)$ and for every color $i \in L_v$, a weight (or cost) $W_{v,i} \in WEIGHTS \{ERROR\}$ of assigning color i to vertex v; and
- a function check that, given a vertex $v \in V(G)$ and given a color assignment $c \colon N_G^r[v] \to \bigcup_{u \in N_G^r[v]} L_u$ such that $c(u) \in L_u$ for all $u \in N_G^r[v]$, returns True if the vertex v together with its neighborhood of radius r (considering the labels of the edges uv with $u \in N_G^r(v)$) satisfies a certain condition, and FALSE otherwise.

We say that an assignment of colors to vertices c is valid in V if V is the domain of c and $c(v) \in L_v$ for all $v \in V$, and it is proper if it is valid in V(G) and $check\left(v, c|_{N_G^r[v]}\right)$ is true for every $v \in V(G)$. The weight of a color assignment c valid in V is $W(c) = \bigoplus_{v \in V} W_{v,c(v)}$.

Given all the previously defined G, ℓ_e , L_v , (WEIGHTS, \leq , \oplus), $W_{v,i}$ and check, an r-locally checkable problem consists of finding the minimum weight (according to the order \leq) of a proper assignment of colors to vertices. If no such coloring exists, the answer should be Error.

If we further consider a set $\Pi(c)$ of global properties (such as "the subgraph induced by the set $\{v:c(v)=i\}$ is connected and $|\{v:c(v)=j\}| \leq 1$ "), then a generalized r-locally checkable problem consists of finding the minimum weight (according to the order \prec) of a proper color assignment c that satisfies the properties in $\Pi(c)$.

3.1. **Examples.** Many different optimization and decision problems can be modeled as r-locally checkable problems (for a decision problem we can say, for example, that the answer is "no" if and only if the minimum weight of a proper coloring is Error.

For the examples shown throughout this paper, we will assume that, otherwise stated, the definitions of L_v are for all $v \in V(G)$, of $W_{v,i}$ for all $v \in V(G)$ and all $i \in L_v$, and of check(v,c) for all $v \in V(G)$ and all color assignments c valid in $N_G[v]$. Also, if the labels ℓ_e are not specified, we can assume they are all equal to 1.

In Table 1 we show some examples of coloring, domination, independence and packing problems as 1-locally checkable problems. Their definitions can be found in Appendix A.

Observe that for list-coloring and H-coloring we are only interested in determining whether such coloring exists or not, so we do not use the weights for optimizing the solution, instead we use them precisely for determining if a solution is correct. For k-coloring we can make use of weights to determine the smallest $j \leq k$ for which there exists a j-coloring in G (in particular, we could set k as a known upper bound for the chromatic number to obtain it).

Problem	$\text{Weights}, \preceq, \oplus$	L_v	$\mathbf{W}_{v,i}$	check(v,c)	
k-coloring	$\mathbb{N} \cup \{+\infty\}, \leq, \max$	$[\![1,k]\!]$	i	$\forall u \in N_G(v). c(u) \neq c(v)$	
k-chromatic sum	$\mathbb{N} \cup \{+\infty\}, \leq, +$	$[\![1,k]\!]$	i	$\forall u \in N_G(v). c(u) \neq c(v)$	
List-coloring	$\mathbb{N} \cup \{+\infty\}, \leq, +$	input	0	$\forall u \in N_G(v). c(u) \neq c(v)$	
H-coloring	$\mathbb{N} \cup \{+\infty\}, \leq, +$	V(H)	0	$\forall u \in N_G(v). c(u) \in N_H(c(v))$	
k-tuple domination	$\mathbb{N} \cup \{+\infty\}, \leq, +$	$\{0, 1\}$	i	$\sum_{u \in N_G[v]} c(u) \ge k$	
Total k -tuple domination	$\mathbb{N} \cup \{+\infty\}, \leq, +$	$\{0, 1\}$	i	$\sum_{u \in N_G(v)} c(u) \ge k$	
k-domination	$\mathbb{N} \cup \{+\infty\}, \leq, +$	$\{0, 1\}$	i	$c(v) = 0 \Rightarrow \sum_{u \in N_G(v)} c(u) \ge k$	
$\{k\}$ -domination	$\mathbb{N} \cup \{+\infty\}, \leq, +$	$\llbracket 0,k rbracket$	i	$\sum_{u \in N_G[v]} c(u) \ge k$	
k-rainbow domination	$\mathbb{N} \cup \{+\infty\}, \leq, +$	$2^{[\![1,k]\!]}$	i	$\left \bigcup_{u \in N_G[v]} c(u) \right = k$	
Roman domination	$\mathbb{N} \cup \{+\infty\}, \leq, +$	$\{0, 1, 2\}$	i	$c(v) = 0 \Rightarrow \exists u \in N_G(v). c(u) = 2$	
Independent set	$\mathbb{N} \cup \{-\infty\}, \geq, +$	$\{0, 1\}$	i	$c(v) = 1 \Rightarrow \sum_{u \in N_G(v)} c(u) = 0$	
$\{k\}$ -packing function	$\mathbb{N} \cup \{-\infty\}, \geq, +$	$\llbracket 0,k rbracket$	i	$\sum_{u \in N_G[v]} c(u) \le k$	
$\{k\}$ -limited packing	$\mathbb{N} \cup \{-\infty\}, \geq, +$	$\{0, 1\}$	i	$\sum_{u \in N_G[v]} c(u) \le k$	

Table 1. Examples of 1-locally checkable problems.

For the total version of most of these domination problems we only need to replace N[v] by N(v).

For weighted versions of these problems, weights are part of the input.

For problems where the input graph is directed, we can consider edge labels that also carry the direction of the edge, and the *check* function can use it to distinguish in-neighbors and out-neighbors.

- 3.2. **LCVP problems.** The problem of deciding if a graph G has a D_q partition can be modeled as a 1-locally checkable problem in the following way:
 - $L_v = [1, q];$
 - (Weights, \preceq , \oplus) = ($\mathbb{N} \cup \{+\infty\}$, \leq , +);
 - $W_{v,i} = 0$; and
 - $check(v, c) = \forall j \in [1, q]. | \{u : u \in N(v) \land c(u) = j\} | \in D_q[c(v), j].$

Therefore, our framework is a generalization of LCVP problems. Furthermore, it allows us to model more problems, like $\{k\}$ -domination.

3.3. **LCC and ECC problems.** Consider an ECC problem Π . By definition, there exists a basic ECC problem Π' such that Π can be reduced to Π' . We will show that Π' can be reduced to a generalized 1-locally checkable problem in the jagged graph of the input graph.

Construct J(G) from the input graph G. Let $(M^*, \oplus^*, \preceq^*)$ be obtained by adding a maximum element to $(M^{m+1}, \oplus^{m+1}, \preceq)$. Then

- $L_v = X$ for all $v \in V(G)$, and $L_{uv} = L_u \times L_v \times Y$ for all $uv \in E(G)$;
- (Weights, \leq , \oplus) = (M^*, \leq^*, \oplus^*) ;
- $W_{v,x} = e_{\oplus}$ for all $v \in V(G)$, and $W_{uv,(x_u,x_v,y)} = val_{m+1}(D,uv,x_u,x_v,y)$ for all $uv \in E(G)$;

- check(v,c) = TRUE for all $v \in V(G)$ and all color assignments c valid in $check(uv,c) = (c(uv)_1 = c(u) \land c(uv)_2 = c(v))$ for all $uv \in E(G)$ and all color assignments c valid in $N_{J(G)}[e]$; and
- Global properties $\Pi(c)$: $\bigoplus_{uv \in E(G)}^{i} val_i(D, uv, c(uv)_1, c(uv)_2, c(uv)_3) \in R_i$, for all $i \in [1, m]$.

With a similar approach, we can reduce a LCC problem in bounded degree graphs to a generalized r-locally checkable problem in the jagged graph of the input graph (for an appropriate r). Construct J(G) from the input graph G. Let $(M^*, \oplus^*, \preceq^*)$ be obtained by adding a maximum element to $(M^{m+1}, \oplus^{m+1}, \preceq)$. For all $v \in V(G)$, let \mathbb{F}_v and \mathbb{G}_v be, respectively, the sets of all functions $f: N_G^r[v] \to X$ and $g: M_G^r(v) \to Y$ (notice that \mathbb{F}_v and \mathbb{G}_v are finite sets that can be computed in polynomial time). Then

- $L_v = \mathbb{F}_v \times \mathbb{G}_v$ for all $v \in V(G)$, and $L_{uv} = Y$ for all $uv \in E(G)$;
- (Weights, \leq , \oplus) = (M^*, \leq^*, \oplus^*) ;
- $W_{v,(f,g)} = val_{m+1}(D, v, f, g)$ for all $v \in V(G)$, and $W_{uv,y} = e_{\oplus} \text{ for all } uv \in E(G);$
- for all $v \in V(J(G))$ and all color assignments c valid in $N^r_{J(G)}[v]$, check(v,c)checks that $f_u(x) = f_w(x)$ and $g_u(y) = g_w(y)$ (where $(f_u, g_u) = c(u)$ and $(f_w, g_w) = c(w)$) for all $u, w \in N^r_{J(G)}[v] \cap V(G), x \in N^r_{J(G)}[v] \cap N^r_G[u] \cap N^r_G[w]$ and $y \in N_{J(G)}^r[v] \cap M_G^r(u) \cap M_G^r(w)$; and
- Global properties $\Pi(c)$: $\bigoplus_{v \in V(G)}^{i} val_i(D, v, c(v)_1, c(v)_2) \in R_i$, for all $i \in [1, m]$.

4. 1-LOCALLY CHECKABLE PROBLEMS IN COMPLETE GRAPHS

It is easy to see that we can polynomially reduce NP-complete problems in graphs to particular 1-locally checkable problems in complete graphs, even when restricting the sets of colors and edge labels to $\{0,1\}$. Indeed, we can transform the classical domination problem in a graph G to the following 1-locally checkable problem in complete graphs. We construct a complete graph G' such that V(G') = V(G) and set

- $\ell_{uv} = 1$ if $uv \in E(G)$ and $\ell_{uv} = 0$ otherwise;
- $L_v = \{0, 1\};$
- (Weights, \leq , \oplus) = ($\mathbb{R} \cup \{+\infty\}, \leq$, +);
- $W_{v,i} = i$; and
- $check(v,c) = \left(c(v) + \sum_{u \in N_{G'}(v)} (c(u) \cdot \ell_{vu}) \ge 1\right).$

It is clear that the minimum weight of a proper coloring in this instance equals $\gamma(G)$, and this transformation can be performed in polynomial time.

If we restrict LABELS to {1}, we can still find a polynomial-time reduction from the domination problem in a graph G to a 1-locally checkable problem in a complete graph G' such that V(G') = V(G), by setting:

- $L_v = \{\emptyset, N_G[v]\};$
- (WEIGHTS, \preceq , \oplus) = ($\mathbb{R} \cup \{+\infty\}$, \leq , +); $W_{v,\emptyset} = 0$ and $W_{v,N_G[v]} = 1$; and
- $check(v, c) = \left(v \in \bigcup_{u \in N_{G'}[v]} c(u)\right).$

Of course, 1-locally checkable problems are polynomial-time solvable under appropriate conditions.

Theorem 4.0.1. Consider a 1-locally checkable problem and a family of instances where

- G is a complete graph;
- $\ell_e = 1$ for all $e \in E(G)$;
- the number of all possible colors (that is, the size of the set $\bigcup_{v \in V(G)} L_v$) is bounded by a constant; and
- check(v,c) can be computed in polynomial time and only depends on v, c(v) and the number of neighbors of each color that v has.

Then this problem can be solved in polynomial time in |V(G)| for these instances.

Proof. Assume COLORS = $\{c_1, \ldots, c_{\mathcal{C}}\}$. Notice that since G is a complete graph then $N_G[v] = V(G)$ for all $v \in V(G)$. Therefore, by the restrictions imposed to check, we can assume there exists a function check' such that $check'(v, c(v), (k_1, \ldots, k_{\mathcal{C}})) = check(v, c)$, where $k_i = |\{u \in V(G) : c(u) = c_i\}|$ for all $i \in [1, \mathcal{C}]$.

For every distribution of colors (k_1, \ldots, k_C) , with $k_i \in \mathbb{N}_0$ for all $i \in [1, C]$ and such that $\sum_{i=1}^{C} k_i = |V(G)|$, we need to verify if it can actually be achieved (that is, there exists a proper color assignment c such that $k_i = |\{u \in V(G) : c(u) = c_i\}|$ for all $i \in [1, C]$), and if so, find one such proper assignment of colors to vertices of minimum weight. To this end, we construct a directed capacitated network F with vertices i for all $i \in \text{Colors}$, (v, i) for all $v \in V(G)$ and all $i \in L_v$, v for all $v \in V(G)$, and s and t. There is a directed edge of capacity $k_{\sigma(i)}$ and cost 0 from s to i for all $i \in \text{Colors}$. There is a directed edge of capacity 1 and cost 0 from i to (v, i) if $k_i \geq 1$ and $check'(v, i, (k_1, \ldots, k_C))$ is true. There is a directed edge of capacity 1 and cost 0 from v to v for all $v \in V(G)$, v and v are no more edges than these ones. Note that the distribution v to v for all v and v are no more edges than these ones. Note that the distribution v and v and v are no more edges than these ones. Note that the distribution v is achievable if and only if the maximum flow in v is achievable to the maximum flow in v of minimum cost.

Finally, the answer to the problem is obtained by finding the minimum proper color assignment among the ones found for all achievable distributions of colors.

Since \mathcal{C} is bounded by a constant, the number of distributions of colors is polynomial in |V(G)| (because it is $\binom{|V(G)|+\mathcal{C}-1}{\mathcal{C}-1} < (|V(G)|+1)^{\mathcal{C}}$), $check'(v,i,(k_1,\ldots,k_{\mathcal{C}}))$ can be computed in polynomial time, constructing F takes polynomial time, and the problem of finding the maximum flow of minimum cost is polynomial-time solvable, then the statement holds.

5. 1-LOCALLY CHECKABLE PROBLEMS IN BOUNDED TREEWIDTH GRAPHS

In this section we give a polynomial-time algorithm to solve 1-locally checkable problems in bounded treewidth graphs, under mild conditions. In Section 6 we show that these results can be extended to r-locally checkable problems (for any fixed $r \geq 1$) in bounded treewidth and bounded degree graphs. In Section 7 we will explain how to modify the algorithm in order to add some global properties.

We will solve the problem in a dynamic programming fashion, traversing an easy tree decomposition (see Definition 2.5.8) of the input graph and describing how to proceed with each type of node of the tree decomposition, as it is usual with this kind of problems, but with an abstraction of the "extra" parameters involved in adhoc solutions of locally checkable problems. In order to describe this abstraction, and hence the algorithm, we first need to introduce the concept of partial neighborhoods.

5.1. **Partial neighborhoods.** We define a system that, roughly speaking, gives us tools to accumulate information from the neighbors of a vertex.

Definition 5.1.1. A partial neighborhood system for an instance of a 1-locally checkable problem consists of:

- A set $\mathcal{N}_{v,i}$, for every $v \in V(G)$ and $i \in L_v$, together with a closed binary operation $\boxplus^{v,i}$ on $\mathcal{N}_{v,i}$ that is commutative and associative and has a neutral element $e_{v,i}$.
- A function $new N_{v,i}$, for every $v \in V(G)$ and $i \in L_v$, that given $u \in N_G(v)$ and $j \in L_u$ returns an element of $\mathcal{N}_{v,i}$ (possibly making use of the label of the edge vu).
- A function $check_{v,i} : \mathcal{N}_{v,i} \to \text{Bool}$, for every $v \in V(G)$ and $i \in L_v$. This function must satisfy $check_{v,c(v)} \left(\coprod_{u \in N_G(v)}^{v,c(v)} new N_{v,c(v)}(u,c(u)) \right) = check(v,c)$ for every vertex $v \in V(G)$ and every color assignment c valid in $N_G[v]$.

In words, with $new N_{v,i}(u,j)$ we create new information, that says how u having color j affects v when having color i. The operation $\boxplus^{v,i}$ combines two pieces of information. For a color assignment c valid in $N_G[v]$, $check_{v,c(v)}\left(\coprod_{u \in N_G(v)}^{v,c(v)} new N_{v,c(v)}(u,c(u)) \right)$ simply verifies a condition over all the information collected from the neighbors of v. Finally, we require the equality $check_{v,c(v)}\left(\coprod_{u \in N_G(v)}^{v,c(v)} new N_{v,c(v)}(u,c(u)) \right) = check(v,c)$ to make these tools analogous to the use of check(v,c). We refer to the elements of $\mathcal{N}_{v,i}$ as $partial\ neighborhoods$ of vertex v with color i.

Remark 5.1.2. For every instance of a 1-locally checkable problem there exists a partial neighborhood system. We will show how to construct one. The idea behind the following partial neighborhood system is to store all the colors assigned to the neighbors of v, where \bot represents that a neighbor has not yet been assigned a color, and \times can be thought as an error sign. Let $v \in V(G)$, $i \in L_v$ and assume $N_G(v) = \{u_1, \ldots, u_{d_G(v)}\}$. Let $\mathcal{N}_{v,i}$ be the set of all $d_G(v)$ -tuples x such that $x_h \in L_{u_h} \cup \{\bot, \times\}$ for all $h \in [1, d_G(v)]$. Let $\boxplus^{v,i}$ be such that

$$(n \boxplus^{v,i} n')_h = \begin{cases} n_h & \text{if } n_h = n'_h \text{ or } n'_h = \bot \\ n'_h & \text{if } n_h = \bot \text{ and } n'_h \neq \bot \\ \times & \text{otherwise} \end{cases}$$

for all $h \in [1, d_G(v)]$. Let $new N_{v,i}(u_h, j)$ be the $d_G(v)$ -tuple that has j in its position h and \bot in all its other positions. Let $x \in \mathcal{N}_{v,i}$. If $x = (j_1, \ldots, j_{d_G(v)})$ with $j_h \in L_{u_h}$ for all $h \in [1, d_G(v)]$, let c be the color assignment in $N_G[v]$ such that c(v) = i and $c(u_h) = j_h$ for all $h \in [1, d_G(v)]$, and then define $check_{v,i}(x) = check(v, c)$. Otherwise, let $check_{v,i}(x) = FALSE$.

Finding partial neighborhood systems that have smaller sets $\mathcal{N}_{v,i}$ is of extreme importance because it reduces the time complexity of the algorithm given in Section 5.3. We say a partial neighborhood system is polynomial (resp. constant) if it is such that $\max\{|\mathcal{N}_{v,i}|:v\in V(G),i\in L_v\}$ is polynomial (resp. constant) in the size of the input of the problem, and all the functions $check_{v,i}$, eq_n , $\boxminus^{v,i}$ and $newN_{v,i}$ can be computed in time polynomial (resp. constant) in the input size. In Table 2 we show constant partial neighborhood systems of the locally checkable problems listed in Table 1.

Another key concept is the following. Let $X \subseteq V(G)$ and c be a color assignment valid in X. Given a partial neighborhood system, a valid partial neighborhood mapping for c is a function η of domain X such that $\eta(v) \in \mathcal{N}_{v,c(v)}$ for all $v \in X$.

Problem	$\mathcal{N}_{v,i}$	$n \boxplus^{v,i} n'$	$newN_{v,i}(u,j)$	$check_{v,i}(n)$
k-coloring	Воог	$n \wedge n'$	$j \neq i$	n
k-chromatic sum	Воог	$n \wedge n'$	$j \neq i$	n
List-coloring	Воог	$n \wedge n'$	$j \neq i$	n
H-coloring	Bool	$n \wedge n'$	$j \in N_H(i)$	n
k-tuple domination	$\llbracket 0,k rbracket$	$\min(n+n',k)$	j	$n+i \ge k$
Total k-tuple domination	$\llbracket 0,k rbracket$	$\min(n+n',k)$	j	$n \ge k$
k-domination	$\llbracket 0,k rbracket$	$\min(n+n',k)$	j	$i = 0 \Rightarrow n \ge k$
$\{k\}$ -domination	$\llbracket 0,k rbracket$	$\min(n+n',k)$	j	$n+i \ge k$
k-rainbow domination	$2^{[\![1,k]\!]}$	$n \cup n'$	j	$ n \cup i = k$
Roman domination	Bool	$n \vee n'$	j=2	$i = 0 \Rightarrow n$
Independent set	$\{0, 1\}$	$\min(n+n',1)$	j	$i = 1 \Rightarrow n = 0$
$\{k\}$ -packing function	[0, k+1]	$\min(n+n',k+1)$	j	$n \le k$
$\{k\}$ -limited packing	[0, k+1]	$\min(n+n',k+1)$	j	$n \le k$

Table 2. Examples of partial neighborhood systems for 1-locally checkable problems.

Finally, given $v \in V(G)$ and $i \in L_v$, any function $f: \mathcal{N}_{v,i} \to \text{Bool}$ is called a checking function for (v, i).

- 5.2. Notation and definitions. The following definitions and notation will be useful throughout the rest of the article. To make the notation less cumbersome, we write \check{c}_v instead of $\check{c}(v)$.
 - Function extended with one element in its domain. Let $f: X \to Y$ and $x \notin X$. Then the function $f^{x \to y}: X \cup \{x\} \to Y \cup \{y\}$ is such that $f^{x \to y}(x) = y$ and $f^{x \to y}(z) = f(z)$ for all $z \in X$.
 - Function with one element less in its domain. Let $f: X \to Y$ and $x \in X$. Then the function $f^{-x}: X \{x\} \to Y$ is such that $f^{-x}(z) = f(z)$ for all $z \in X \{x\}$.
 - Graph where some edges are removed. Let H be a graph. Then $H^{\neg S}$ is the graph such that $V(H^{\neg S}) = V(H)$ and $E(H^{\neg S}) = E(H) \{uv : u, v \in S\}$.
 - Neutral weight mapping. Let $X \subseteq V(G)$. Then the function $\omega_X^e \colon X \to W$ EIGHTS is such that $\omega_X^e(v) = e_{\oplus}$ for all $v \in X$.
 - Equality checking function. For every $v \in V(G)$, every $i \in L_v$ and every $n \in \mathcal{N}_{v,i}$, let $eq_n : \mathcal{N}_{v,i} \to \text{Bool}$ be the function such that $eq_n(n') = (n = n')$ for all $n' \in \mathcal{N}_{v,i}$.
 - Function that returns equality checking functions. Let $X \subseteq V(G)$, c be a color assignment valid in X and η be a valid partial neighborhood mapping for c. Then let \check{c}^{η} be the function of domain X such that \check{c}^{η}_v is $eq_{\eta(v)}$ for all $v \in X$.
 - Reduction of a partial neighborhood mapping. Let $X \subseteq V(G)$, c be a color assignment valid in X, η be a valid partial neighborhood mapping for c and $v \in X$. Then the function $\eta^{\sim v}$ of domain $X \{v\}$ is such that

 $\eta^{\sim v}(u) = \eta(u) \coprod^{u,c(u)} new N(v,c(v))$ if $u \in X \cap N_G(v)$ and $\eta^{\sim v}(u) = \eta(u)$ otherwise.

- Neutral partial neighborhood mapping. Let $X \subseteq V(G)$ and c be a color assignment valid in X. Then the function η_c^e of domain X is such that $\eta_c^e(v) = e_{v,c(v)}$ for all $v \in X$. Observe that η_c^e is a valid partial neighborhood mapping for c.
- Partial neighborhood in a subgraph. Let H be a subgraph of G and c be a color assignment valid in V(H). Then we define NS such that $NS(v, c, H) = \coprod_{u \in N_H(v)}^{v,c(v)} new N_{v,c(v)}(u,c(u))$ for all $v \in V(H)$. Roughly speaking, NS(v,c,H) is the information we can obtain from the neighbors of v in H and the color assignment c.
- 5.3. **Algorithm.** Consider an instance of a 1-locally checkable problem with a partial neighborhood system. Let G be the input graph and let $(T, \{X_t\}_{t \in V(T)})$ be an easy tree decomposition of G.

For every $X \subseteq V(G)$, every G' subgraph of G such that $X \subseteq V(G')$ and $N_G[V(G') - X] \subseteq V(G')$, every color assignment c valid in X, every η that is a valid partial neighborhood mapping for c, and every \check{c} such that $\check{c}_v \in \{check_{v,c(v)}\} \cup \{eq_n : n \in \mathcal{N}_{v,c(v)}\}$ for all $v \in X$, we say that a function f is a (X, c, η, \check{c}) -coloring in G' if

- f is a color assignment valid in V(G'),
- f(v) = c(v) for all $v \in X$,
- $\check{c}_v\left(\eta(v) \boxplus^{v,f(v)} \operatorname{NS}(v,f,G')\right) = \operatorname{TRUE} \text{ for all } v \in X, \text{ and}$
- $check(u, f|_{N_G[u]}) = \text{True for all } u \in V(G') X.$

For a (X, c, η, \check{c}) -coloring f in G' and a function $\omega \colon X \to WEIGHTS$, we define the weight under ω of f as $W_{\omega}(f) = \left(\bigoplus_{v \in X} \omega(v)\right) \oplus \left(\bigoplus_{v \in V(G') - X} W_{v, f(v)}\right)$.

For every node t, let \mathcal{D}_t be the set of all tuples $(S, c, \omega, \eta, \check{c})$ such that

- $S \subseteq X_t$,
- c is a color assignment valid in X_t ,
- $\omega: X_t \to \text{Weights}$ is such that $\omega(v) \in \{e_{\oplus}, W_{v,c(v)}\}$ for all $v \in X_t$,
- η is a valid partial neighborhood mapping for c, and
- \check{c} is a function that, given a vertex $v \in X_t$, returns a checking function for (v, c(v)) such that $\check{c}_v \in \{check_{v,c(c)}\} \cup \{eq_n : n \in \mathcal{N}_{v,c(v)}\};$

and let $\lambda_t \colon \mathcal{D}_t \to \text{Weights}$ be defined by

$$\lambda_t(S, c, \omega, \eta, \check{c}) = \min\{W_{\omega}(f) : f \text{ is a } (X_t, c, \eta, \check{c}) \text{--coloring in } G_t^{\neg S}\}$$

for every $(S, c, \omega, \eta, \check{c}) \in \mathcal{D}_t$.

Remark 5.3.1. Notice that if there are no $(X_t, c, \eta, \check{c})$ -colorings in $G_t^{\neg S}$ then we have $\lambda_t(S, c, \omega, \eta, \check{c}) = \text{Error}.$

The following result is immediate from the previous definitions.

Corollary 5.3.2. If r is the root of T then the minimum weight of a proper coloring in G is

$$\min\{\lambda_r(\emptyset, c, \omega, \eta_c^e, \check{c}) : (\emptyset, c, \omega, \eta_c^e, \check{c}) \in \mathcal{D}_r, and$$

$$\omega(v) = W_{v,c(v)} \text{ and } \check{c}_v = check_{v,c(v)} \text{ for all } v \in X_r\}.$$

We will show how to compute λ in a recursive way.

Lemma 5.3.3 (Leaf node). Let t be a leaf node and $X_t = \{v\}$. Then

$$\lambda_t(S, c, \omega, \eta, \check{c}) = \begin{cases} \omega(v) & \text{if } \check{c}_v(\eta(v)) \\ \text{ERROR} & \text{otherwise} \end{cases}$$

Proof. Notice that in this case $V(G_t^{\neg S}) = \{v\} = X_t$.

If $\check{c}_v(\eta(v)) = \text{FALSE}$ then, by definition, there are no $(X_t, c, \eta, \check{c})$ -colorings in G_t^{-S} . Therefore, $\lambda_t(S, c, \omega, \eta, \check{c}) = \text{Error}$, leading to the desired equality.

If $\check{c}_v(\eta(v)) = \text{True}$ then, by definition, c is the only possible $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$. Therefore, $\lambda_t(S, c, \omega, \eta, \check{c}) = \min\{W_\omega(f) : f \text{ is a } (X_t, c, \eta, \check{c}) - \text{coloring in } G_t^{\neg S}\} =$ $W_{\omega}(c) = \omega(v).$

Lemma 5.3.4 (Forget node). Let t be a forget node, s be the child of t and $X_s - X_t =$ $\{v\}$. Then

$$\lambda_t(S, c, \omega, \eta, \check{c}) = \min_{i \in L_v} \{\lambda_s(S, c^{v \to i}, \omega^{v \to W_{v,i}}, \eta^{v \to e_{v,i}}, \check{c}^{v \to check_{v,i}})\}.$$

Proof. Notice that $v \notin S$ and also $G_t^{\neg S} = G_s^{\neg S}$. By definition of λ_s we have

$$\min_{i \in L_v} \{ \lambda_s(S, c^{v \to i}, \omega^{v \to W_{v,i}}, \eta^{v \to e_{v,i}}, \check{c}^{v \to check_{v,i}}) \}$$

- $= \min_{i \in L_v} \{ \min \{ \mathbf{W}_{\omega^{v \to \mathbf{W}_{v,i}}}(f) : f \text{ is a } (X_s, c^{v \to i}, \eta^{v \to e_{v,i}}, \check{c}^{v \to check_{v,i}}) \text{coloring in } G_s^{\neg S} \} \}$
- $= \min\{\mathbf{W}_{\omega^{v \to \mathbf{W}_{v,i}}}(f) : i \in L_v \land f \text{ is a } (X_s, c^{v \to i}, \eta^{v \to e_{v,i}}, \check{c}^{v \to check_{v,i}}) \text{coloring in } G_s \cap S\}.$

Let $i \in L_v$. We claim that every f that is a $(X_s, c^{v \to i}, \eta^{v \to e_{v,i}}, \check{c}^{v \to check_{v,i}})$ -coloring in $G_s^{\neg S}$ is also a $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$. Conversely, every f that is a $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$ is also a $(X_s, c^{v \to f(v)}, \eta^{v \to e_{v, f(v)}}, \check{c}^{v \to check_{v, f(v)}})$ -coloring in $G_s^{\neg S}$. We prove the first claim (the second one is similar) by showing each of the items of the definition of $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$ holds:

- f(w) = c(w) for all $w \in X_t$ (because it is true for all $w \in X_s$),
- f is a color assignment valid in $V(G_s^{\neg S}) = V(G_t^{\neg S})$, $check(u, f) = \text{True for all } u \in V(G_s^{\neg S}) X_s = V(G_t^{\neg S}) X_t \{v\}$ and $check(v, f) = check_{v,i} \left(NS(v, f, G_s^{\neg S}) \right)$ $= check_{v,i} \left(e_{v,i} \boxtimes^{v,i} \operatorname{NS}(v, f, G_s^{\neg S}) \right)$ $=\check{c}_{v}^{v \to check_{v,i}} \left(\eta(v) \boxplus^{v,i} \operatorname{NS}(v, f, G_s^{\neg S}) \right)$ = True

(because f(v) = i), and

• $\check{c}_w\left(\eta(w) \stackrel{\sim}{\mathbb{H}^{w,f(w)}} \operatorname{NS}(w,f,G_t^{-S})\right) = \text{True for all } w \in X_t \text{ (because it is true for } M$ all $w \in X_s$ and $E(G_t^{\neg S}) = E(G_s^{\neg S})$.

Clearly, $W_{\omega^{v\to w_{v,f(v)}}}(f) = W_{\omega}(f)$ for every f that is a $(X_t, c, \eta, \check{c})$ -coloring in G_t^{-S} . Therefore

$$\min\{\mathbf{W}_{\omega^{v \to \mathbf{W}_{v,i}}}(f) : i \in L_v \land f \text{ is a } (X_s, c^{v \to i}, \eta^{v \to e_{v,i}}, \check{c}^{v \to check_{v,i}}) - \text{coloring in } G_s^{\neg S}\}$$

- = $\min\{W_{\omega}(f): f \text{ is a } (X_t, c, \eta, \check{c})\text{-coloring in } G_t^{\neg S}\}$
- $=\lambda_t(S,c,\omega,\eta,\check{c}).$

Lemma 5.3.5 (Introduce node). Let t be an introduce node, s be the child of t and $X_t - X_s = \{v\}$. Let $n_v = \eta(v) \boxplus^{v,c(v)} \operatorname{NS}(v,c,G_t^{\neg S}[X_t])$. Then

$$\lambda_t(S, c, \omega, \eta, \check{c}) = \begin{cases} \omega(v) \oplus \lambda_s(S - \{v\}, c^{-v}, \omega^{-v}, \eta^{\sim v}, \check{c}^{-v}) & \text{if } \check{c}_v(n_v) \\ \text{ERROR} & \text{otherwise} \end{cases}$$

Proof. Observe that $v \notin V(G_s^{\neg(S-\{v\})})$ and $N_{G_t^{\neg S}}[v] \subseteq X_t$ (because $(T, \{X_i\}_{i \in V(T)})$ is a tree-decomposition), and that this implies that $NS(v, c, G_t^{\neg S}) = NS(v, c, G_t^{\neg S}[X_t])$.

If $\check{c}_v(n_v) = \text{FALSE}$ then, by definition, there are no $(X_t, c, \eta, \check{c})$ -colorings in $G_t^{\neg S}$ and we also have $\lambda_t(S, c, \omega, \eta, \check{c}) = \text{ERROR}$.

Now assume that $\check{c}_v(n_v) = \text{True}$. It is straightforward to prove that if a function f is a $(X_s, c^{-v}, \eta^{\sim v}, \check{c}^{-v})$ -coloring in $G_s^{\neg (S - \{v\})}$ then the function $f^{v \to c(v)}$ is a $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$ and $W_{\omega}(f^{v \to c(v)}) = \omega(v) \oplus W_{\omega^{-v}}(f)$. Furthermore, it is also straightforward to prove that for every $(X_t, c, \eta, \check{c})$ -coloring g in $G_t^{\neg S}$, the function g^{-v} is a $(X_s, c^{-v}, \eta^{\sim v}, \check{c}^{-v})$ -coloring in $G_s^{\neg (S - \{v\})}$ and $\omega(v) \oplus W_{\omega^{-v}}(g^{-v}) = W_{\omega}(g)$. Therefore

$$\lambda_{t}(S, c, \omega, \eta, \check{c})$$

$$= \min\{W_{\omega}(g) : g \text{ is a } (X_{t}, c, \eta, \check{c}) \text{-coloring in } G_{t}^{\neg S}\}$$

$$= \min\{\omega(v) \oplus W_{\omega^{-v}}(f) : f \text{ is a } (X_{s}, c^{-v}, \eta^{\sim v}, \check{c}^{-v}) \text{-coloring in } G_{s}^{\neg (S - \{v\})}\}$$

$$= \omega(v) \oplus \min\{W_{\omega^{-v}}(f) : f \text{ is a } (X_{s}, c^{-v}, \eta^{\sim v}, \check{c}^{-v}) \text{-coloring in } G_{s}^{\neg (S - \{v\})}\}$$

$$= \omega(v) \oplus \lambda_{s}(S - \{v\}, c^{-v}, \omega^{-v}, \eta^{\sim v}, \check{c}^{-v}).$$

Lemma 5.3.6 (Join node). Let t be a join node and r and s be the children of t. We say that a pair (η_r, η_s) of valid partial neighborhood mappings for c is good if $\check{c}_v(\eta(v) \boxplus^{v,c(v)} \operatorname{NS}(v, c, G_t^{\neg S}[X_t]) \boxplus^{v,c(v)} \eta_r(v) \boxplus^{v,c(v)} \eta_s(v))$ is true for all $v \in X_t$. Let $W = \bigoplus_{v \in X_t} \omega(v)$. Then

$$\lambda_t(S, c, \omega, \eta, \check{c}) = \min_{(\eta_r, \eta_s) \text{ is good}} \{ W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\eta_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\eta_s}) \}.$$

Proof. Recall that $X_t = X_r = X_s$.

For every color assignment f valid in $V(G_t^{\neg S})$, denote by $f|_r$ (resp. $f|_s$) the restriction of f to $V(G_r^{\neg X_r})$ (resp. $V(G_s^{\neg X_s})$). Let $W = \bigoplus_{v \in X_t} \omega(v)$. Notice that $W_{\omega}(f) = W \oplus W_{\omega_{X_r}^e}(f|_r) \oplus W_{\omega_{X_s}^e}(f|_s)$ for every color assignment f valid in $V(G_t^{\neg S})$.

Suppose there exists a $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$ and let f be one of them. Let η_r and η_s be functions of domain X_t such that $\eta_r(v) = \text{NS}(v, f, G_r^{\neg X_r})$ and $\eta_s(v) = \text{NS}(v, f, G_s^{\neg X_s})$ for all $v \in X_t$.

Since $V(G_r^{\neg X_r}) \cap V(G_s^{\neg X_s}) = X_t$ then $V(G_r^{\neg X_r}) \cap V(G_s^{\neg X_s}) \cap N_{G^{\neg X_t}}(v) = \emptyset$. Moreover, since f is a $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$ then we know that, for all $v \in X_t$, $\operatorname{NS}(v, c, G_t^{\neg S}[X_t]) \boxplus^{v,c(v)} \eta_r(v) \boxplus^{v,c(v)} \eta_s(v) = \operatorname{NS}(v, f, G_t^{\neg S})$ and thus $\check{c}_v(\eta(v) \boxplus^{v,c(v)} \operatorname{NS}(v, c, G_t^{\neg S}[X_t]) \boxplus^{v,c(v)} \eta_r(v) \boxplus^{v,c(v)} \eta_s(v)) = \operatorname{TRUE}$. Therefore, the functions η_r and η_s form a good pair of valid partial neighborhood mappings for c.

It is straightforward to prove that $f|_r$ is a $(X_r, c, \eta_c^e, \check{c}^{\eta_r})$ -coloring in $G_r^{-X_r}$, and that $f|_s$ is a $(X_s, c, \eta_c^e, \check{c}^{\eta_s})$ -coloring in $G_s^{-X_s}$. Hence,

$$\begin{split} \mathbf{W}_{\omega}(f) &= W \oplus \mathbf{W}_{\omega_{X_r}^e}(f|_r) \oplus \mathbf{W}_{\omega_{X_s}^e}(f|_s) \\ &\geq W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\eta_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\eta_s}) \\ &\geq \min_{(\eta_r, \eta_s) \text{ is good}} \{W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\eta_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\eta_s})\}. \end{split}$$

Since $\min\{W_{\omega}(f): f \text{ is a } (X_t, c, \eta, \check{c})\text{-coloring in } G_t^{\neg S}\} = \text{Error if there are no}$ $(X_t, c, \eta, \check{c})$ -colorings in $G_t^{\neg S}$, we obtain

$$\lambda_t(S, c, \omega, \eta, \check{c})$$

$$= \min\{W_{\omega}(f) : f \text{ is a } (X_t, c, \eta, \check{c}) \text{--coloring in } G_t^{\neg S}\}$$

$$\geq \min_{(\eta_r, \eta_s) \text{ is good}} \{W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\eta_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\eta_s})\}.$$

To conclude, we will show that the other inequality holds.

If $\min_{(\eta_r,\eta_s) \text{ is good}} \{ W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\eta_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\eta_s}) \} = \text{Error then}$ the statement trivially holds. Otherwise, the minimum is realized by a good pair $(\widehat{\eta}_r, \widehat{\eta}_s)$, and $\lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\widehat{\eta}_r}) \neq \text{ERROR}$ and $\lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\widehat{\eta}_s}) \neq \text{ERROR}$. Therefore there exists a $(X_r, c, \eta_c^e, \check{c}^{\widehat{\eta_r}})$ -coloring $\widehat{f_r}$ in $G_r^{\neg X_r}$ and a $(X_s, c, \eta_c^e, \check{c}^{\widehat{\eta_s}})$ -coloring $\widehat{f_s}$ in $G_s^{\neg X_s}$ such that $W_{\omega_{X_r}^e}(\widehat{f}_r) = \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\widehat{\eta}_r})$ and $W_{\omega_{X_s}^e}(\widehat{f}_s) = \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\widehat{\eta}_s})$.

Let \widehat{f} be a function of domain $V(G_t^{\neg S})$ such that

- $\widehat{f}(v) = c(v)$ for all $v \in X_t$,
- $\widehat{f}(v) = \widehat{f}_r(v)$ for all $v \in V(G_r^{\neg X_r}) X_r$, and $\widehat{f}(v) = \widehat{f}_s(v)$ for all $v \in V(G_s^{\neg X_s}) X_s$.

It is straightforward to prove that \widehat{f} is a $(X_t, c, \eta, \check{c})$ -coloring in $G_t^{\neg S}$, and also that \widehat{f}_r (resp. \widehat{f}_s) is the restriction of \widehat{f} to $V(G_r^{\neg X_r})$ (resp. $V(G_s^{\neg X_s})$). Therefore,

$$\min_{(\eta_r,\eta_s) \text{ is good}} \{ W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\eta_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\eta_s}) \}$$

$$= W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\widehat{\eta}_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\widehat{\eta}_s})$$

$$= W \oplus W_{\omega_{X_r}^e}(\widehat{f}_r) \oplus W_{\omega_{X_s}^e}(\widehat{f}_s)$$

$$= W_{\omega}(\widehat{f})$$

$$\geq \min\{W_{\omega}(f) : f \text{ is a } (X_t, c, \eta, \check{c}) \text{--coloring in } G_t^{\neg S}\}$$

$$= \lambda_t(S, c, \omega, \eta, \check{c}).$$

Consequently,

$$\lambda_t(S, c, \omega, \eta, \check{c}) = \min_{(\eta_r, \eta_s) \text{ is good}} \{ W \oplus \lambda_r(X_r, c, \omega_{X_r}^e, \eta_c^e, \check{c}^{\eta_r}) \oplus \lambda_s(X_s, c, \omega_{X_s}^e, \eta_c^e, \check{c}^{\eta_s}) \}.$$

Then there is a simple algorithm to compute λ . Indeed, based on the recurrence given in lemmas 5.3.3, 5.3.4, 5.3.5 and 5.3.6, the algorithm is executed in a bottomup fashion (that is, first for all the leaf nodes, then for their parents, and so on) by computing $\lambda_t(S, c, \omega, \eta, \check{c}) \in \text{Weights}$ for every node t and every $(S, c, \omega, \eta, \check{c}) \in \mathcal{D}_t$. Finally, the result is obtained by finding the minimum among all $\lambda_r(\emptyset, c, \omega, \eta_c^e, \check{c})$ such that r is the root of T, c is a color assignment valid in X_r , $\omega: X_r \to \text{Weights}$ is such that $\omega(v) = W_{v,c(v)}$ for all $v \in X_r$, and $\check{c}_v = check_{v,c(v)}$ for all $v \in X_r$.

5.4. Time complexity. Let $k = \max\{|X_t| : t \in V(T)\}, \mathcal{N} = \max\{|\mathcal{N}_{v,i}| : v \in V(T)\}$ $V(G), i \in L_v$ and $C = \max\{|L_v| : v \in V(G)\}$. Let $t_{\check{c}}, t_{\boxplus}, t_{newN}, t_{\oplus}, \text{ and } t_{\min} \text{ be upper }$ bounds for the executing time of all the functions $check_{v,i}$ and eq_n , $\boxminus^{v,i}$, $newN_{v,i}$, \oplus , and min, respectively. Other operations are assumed to run in constant time. In particular, we are assuming that we access $W_{v,i}$ in constant time.

Traversing the tree T requires O(|V(T)|) time. In $O(k^2|V(T)|+k|E(G)|)$ time we can construct the adjacency matrices of all the graphs $G[X_t]$ with $t \in V(T)$ (by traversing

T top-down and computing $N_{G_t}(v) \cap X_t$ only for nodes t that are the child of a forget nodes s with $X_t - X_s = \{v\}$). Also, in O(k) time we can construct each of the necessary function extensions and restrictions, and in $O((t_{\mathbb{H}} + t_{newN})k)$ we can construct each of the necessary $\eta^{\sim v}$.

We analyze four separate cases, and the proof in each one of them is straightforward.

- Leaf node: $O(t_{\check{c}})$
- Forget node: $O(k^2 + (k + t_{\min})C)$
- Introduce node: $O((t_{\boxplus} + t_{newN})k + t_{\check{c}} + t_{\oplus} + k^2)$
- Join node: $O((t_{\boxplus} + t_{newN})k^2 + t_{\oplus}k + ((t_{\boxplus} + t_{\check{c}})k + t_{\oplus} + t_{\min})\mathcal{N}^{2k})$

Each one of them is computed for every possible tuple $(S, c, \omega, \eta, \check{c})$. We know that there are no more than $2^k \cdot C^k \cdot 2^k \cdot N^k \cdot (1 + N)^k$ of such tuples, and that constructing each of them requires O(k) operations.

In summary, the time complexity of this algorithm is $O((t_{\oplus}k + (k + t_{\min})\mathcal{C} + (t_{\boxplus} + t_{newN})k^2 + ((t_{\boxplus} + t_{\check{c}})k + t_{\oplus} + t_{\min})\mathcal{N}^{2k})4^k\mathcal{C}^k\mathcal{N}^k(\mathcal{N} + 1)^kk|V(T)| + k|E(G)|).$

5.5. **Special cases.** The next result easily follows from the previous time complexity analysis, Proposition 2.5.1, Theorem 2.5.6 and Theorem 2.5.7.

Theorem 5.5.1. Let \mathcal{F} be a family of graphs of bounded treewidth. Consider a family of instances of a 1-locally checkable problem with a polynomial partial neighborhood system, where

- $G \in \mathcal{F}$,
- $\mathcal{C} = \max\{|L_v| : v \in V(G)\}$ is polynomial in the input size, and
- the functions \oplus and min can be computed in polynomial time.

Then there exists an algorithm that solves these instances in polynomial time. Furthermore, if we have a constant partial neighborhood system, C is bounded by a constant, and the functions \oplus and min can be computed in constant time, then the time complexity of such algorithm is O(|V(G)|).

In particular, this result, together with Table 2, implies that all the problems listed in Table 1 can be solved in O(|V(G)|) time. For other problems, under certain hypothesis we can give a more generic polynomial partial neighborhood system, hence the next result.

Corollary 5.5.2. Let \mathcal{F} be a family of graphs of bounded treewidth. Consider a family of instances of a 1-locally checkable problem where

- $G \in \mathcal{F}$,
- $|\bigcup_{v \in V(G)} L_v|$ is bounded by a constant,
- \oplus and min can be computed in polynomial time, and
- check(v, c) can be computed in polynomial time and only depends on v, c(v) and the number of neighbors of each color that v has.

Then, for such instances, the problem can be solved in polynomial time.

Proof. For each instance, let Colors = $\bigcup_{v \in V(G)} L_v$ and define the following partial neighborhood system:

- $\mathcal{N}_{v,i} = [0, d_G(v)]^{|\text{Colors}|};$
- $(n \boxplus^{v,i} n')_j = \min(n_j + n'_j, d_G(v))$ for all $j \in \text{Colors}$;
- $new N_{v,i}(u,j)_j = 1$ and $new N_{v,i}(u,j)_h = 0$ for all $h \in Colors \{j\};$

• $check_{v,i}(n) = check(v,c)$ where c is any color assignment valid in $N_G[v]$ such that c(v) = i and $|\{u : u \in N_G(v) \land c(u) = j\}| = n_j$ for all $j \in Colors$. Note that we can construct c in polynomial time using flow algorithms.

By Theorem 5.5.1, the statement holds.

5.6. **LCVP problems.** In Section 3.2 we have seen how to model the problem of deciding if G has a D_q partition as a 1-locally checkable problem, and now we extend it with a partial neighborhood system. Let m(S) be the maximum of S if S is finite, or the maximum of \overline{S} if S is co-finite. Then

- $\mathcal{N}_{v,i}$ is the Cartesian product of the sets $[1, m(D_q[i,j]) + 1]$ for all $1 \leq j \leq q$;
- $n \boxplus^{v,i} n'$ is such that $(n \boxplus^{v,i} n')_j = \min(n_j + n'_j, m(D_q[i,j]) + 1)$ for all $1 \le j \le q$;
- $newN_{v,i}(u,j)$ is the tuple such that its jth entry is 1 and all its other entries are 0; and
- $check_{v,i}(n) = (\forall j \in [1, q], n_j \in D_q[i, j]).$

Therefore, with the algorithm in Section 5.3, we can solve LCVP problems in bounded treewidth graphs in $O(q^c|V(G)|)$ time, for some constant c (which equals k+2 if k is the width of a given tree-decomposition of G). This recovers the results obtained by Telle in [61].

6. r-locally checkable problems in bounded treewidth and bounded degree graphs

Recall the partial neighborhood system defined in Remark 5.1.2, for which $\mathcal{N} \leq (\mathcal{C}+2)^{\Delta(G)}$. Hence, the next result easily follows from Theorem 5.5.1.

Corollary 6.0.1. Let \mathcal{F} be a family of graphs of bounded treewidth and bounded degree. Then for any 1-locally checkable problem with input graph $G \in \mathcal{F}$, \mathcal{C} polynomial in the input size, and all functions check, min and \oplus computable in polynomial time, there exists a polynomial-time algorithm that solves it.

Furthermore, the next lemma shows that fixed powers of bounded treewidth and bounded degree graphs are also bounded treewidth and bounded degree graphs, therefore extending the results of the previous sections to more problems in these graph classes.

Lemma 6.0.2. Let G be a graph and p > 2. Then

$$\Delta(G) \le \Delta(G^p) \le \Delta(G)^p$$

and

$$\max(tw(G), \Delta(G)) \le tw(G^p) \le (tw(G) + 1)(\Delta(G) + 1)^{\lceil \frac{p}{2} \rceil} - 1.$$

Proof. The inequality $\Delta(G) \leq \Delta(G^p) \leq \Delta(G)^p$ follows easily from the definition of power of a graph. Let v be a vertex of G of maximum degree. In G^p , the graph induced by $N_G[v]$ is a clique of size $\Delta(G) + 1$ and, by Theorem 2.5.3, we get that $tw(G^p) \geq \Delta(G)$. Since G is a subgraph of G^p and by Proposition 2.5.2, we have $tw(G^p) \geq tw(G)$.

Now assume $(T, \{X_t\}_{t \in V(T)})$ is a tree decomposition of G. For every $t \in V(T)$, let Y_t be the set of vertices that are at distance less than or equal to $\lceil \frac{p}{2} \rceil$ from a vertex of X_t . We will prove that $(T, \{Y_t\}_{t \in V(T)})$, is a tree decomposition of G^p .

Clearly, $\bigcup_{t \in V(T)} Y_t = V(G) = V(G^p)$, so property (W1) holds.

Let u, v be two vertices that are neighbors in G^p . If they are also neighbors in G, then there exists a bag X_t that contains both of the vertices and since $X_t \subseteq Y_t$, we

get that property (W2) holds in this case. If not, there exists a vertex w that is at distance at most $\lceil \frac{p}{2} \rceil$ from both u and v in G. Therefore, since there exists a bag X_t that contains w, this implies that $w \in Y_t$ (because $X_t \subseteq Y_t$) and $u, v \in Y_t$ (because u, v are at distance at most $\lceil \frac{p}{2} \rceil$ from $w \in X_t$). Consequently, property (W2) also holds in this remaining case.

Now we will prove that (W3) holds. For all $u \in V(G)$, let $T_u^X = T[\{t \in V(T) : u \in X_t\}]$ and $T_u^Y = T[\{t \in V(T) : u \in Y_t\}]$. Applying property (W3) to $(T, \{X_t\}_{t \in V(T)})$, we obtain that T_u^X is a subtree of T for every $u \in V(G)$. Let $v \in V(G)$. We will prove that T_v^Y is connected. By definition of the bags Y_t , we know that $v \in Y_t$ if and only if $t \in V(T_v^X)$ or $t \in V(T_u^X)$ for some u such that $d(v, u) \leq \lceil \frac{p}{2} \rceil$. Let $t \in V(T_v^X)$. Notice that in order to prove that T_v^Y is connected it is sufficient to prove that there exists a path in T between t and every $s \in V(T_v^Y) - V(T_v^X)$. Let $s \in V(T_v^Y) - V(T_v^X)$ and let $v_s \in Y_s$ be such that $d(v, v_s) \leq \lceil \frac{p}{2} \rceil$. Since T_u^X is a subtree of T for every $u \in V(G)$, and $V(T_u^X) \cap V(T_w^X) \neq \emptyset$ for all $uw \in E(G)$ (because of property (W2) applied to $(T, \{X_t\}_{t \in V(T)})$ and the edge uw), and there exists a path in G between v and v_s , we get that there exists a path in T between t and t. Therefore (W3) holds for $t \in V(T_t)$.

Since every bag Y_t has at most $(tw(G)+1)(\Delta(G)+1)^{\lceil \frac{p}{2} \rceil}$ vertices, we obtain $tw(G^p) \leq (tw(G)+1)(\Delta(G)+1)^{\lceil \frac{p}{2} \rceil}-1$.

Directly from Lemma 6.0.2 and Corollary 6.0.1, by reducing the problem to a 1-locally checkable problem in G^r , the next result easily follows.

Corollary 6.0.3. Let \mathcal{F} be a family of graphs of bounded treewidth and bounded degree. Then, for any r-locally checkable problem with $G \in \mathcal{F}$, \mathcal{C} polynomial in the input size, and all functions check, min and \oplus computable in polynomial time, there exists a polynomial-time algorithm that solves it.

As a result, the algorithm of Section 5 can be instantiated to solve, in polynomial time for bounded treewidth and bounded degree graphs, distance coloring problems [14, 18, 19, 30, 36, 38, 39, 60], distance independence [29], distance domination problems [43], and distance LCVP problems [48], for bounded distances.

A similar result has been obtained by Jaffke, Kwon, Strømme and Telle for the distance versions of the LCVP problems in bounded MIM-width graphs [48].

7. DEALING WITH GLOBAL PROPERTIES IN BOUNDED TREEWIDTH GRAPHS

In this section we explain how to modify the previous algorithm in order to handle some global properties. The general idea in all of these cases is to modify λ_t by extending it with new parameters (that is, at each node t we compute $\lambda_t(S, c, \omega, \eta, \check{c}, \ldots)$). For simplicity, in the following subsections we omit some parts of the original algorithm, writing only the necessary changes.

7.1. The size of a color class is an element of a particular set. Suppose we want the class of color j to have a size that is an element of a set $\sigma \subseteq \mathbb{N}_0$.

Consider a deterministic finite-state automaton $(Q, \{1\}, \delta, q_0, F)$ that accepts a string of n consecutive characters 1 if and only if $n \in \sigma$. Notice that for all finite sets $\sigma \subseteq \mathbb{N}_0$ there exists such an automaton: let m be the maximum element of σ , $Q = \{s_0, \ldots, s_{m+1}\}$, $q_0 = s_0$, $F = \{s_i : i \in \sigma\}$, and $\delta(s_i, 1) = s_{i+1}$ for all $0 \le i \le m$ and $\delta(s_{m+1}, 1) = s_{m+1}$. Although, for time complexity issues, when m is not a constant we might be interested in another automata, with constant number of states (for example, if σ is the set of odd numbers in [0, |V(G)|], we only need two states).

In the algorithm, at each node t, we add a parameter $state_j$ that stores the state of the partial size of the color class j, and also a parameter $accept_j$ that checks if we are in the desired state, and then proceed in the following way.

- Leaf node: Now we also need to check if $accept_i(state_i)$ is true.
- Forget node: For all $i \in L_v$, let $state_j^i = state_j$ if $i \neq j$ and $state_j^i = \delta(state_j, 1)$ otherwise. Then

$$\lambda_t(\ldots, state_j, accept_j) = \min\{\lambda_s(\ldots, state_j^i, accept_j) : i \in L_v\}.$$

- Introduce node: Remains the same (with $state_i$ and $accept_i$ added to λ_s).
- Join node: For all $q \in Q$, let $eq_q: Q \to BOOL$ be such that $eq_q(q') = (q = q')$ for all $q' \in Q$. Then

```
\lambda_t(\ldots, state_j, accept_j) = \min\{W \oplus \lambda_r(\ldots, state_j, eq_q) \oplus \lambda_s(\ldots, q, accept_j) : q \in Q \land \ldots\}.
```

• At the root r where $X_r = \{v\}$, we compute all $\lambda_r(\ldots, s_r, a)$, with a such that $a(s) = (s \in F)$, and $s_r = q_0$ if $c(v) \neq j$ and $s_r = \delta(q_0, 1)$ otherwise.

Note that it is easy to generalize this idea to more classes by simply adding as many $state_j$ and $accept_j$ as needed (each of them with its own automaton), and even to a set J of classes by replacing statements of the form " $i \neq j$ " with " $i \notin J$ ".

The time complexity now depends on the number of states and color classes to restrict. We can assume that checking if a state is an accepting one is a constant-time operation and so is computing $\delta(s,1)$. Let \mathcal{R} be the number of color classes (or sets of color classes) to restrict and let \mathcal{S} be the size of the largest set of states among all considered automata. The only changes in complexity are:

- Leaf node: add \mathcal{R} .
- Forget node: add $2\mathcal{RC}$.
- Introduce node: add $2\mathcal{R}$.
- Join node: multiply by $\mathcal{S}^{\mathcal{R}}$.
- When we multiply by the number of all possible combinations of the parameters of λ_t : add a factor $(\mathcal{S}(\mathcal{S}+1))^{\mathcal{R}}$.

In particular, the complexity of the algorithm remains polynomial in |V(G)| if \mathcal{R} is bounded by a constant, allowing us to, for example, ask for a color class to be non-empty or to have at most one element.

7.2. **LCC-like properties.** Let (M, \oplus) be a commutative monoid. Suppose we want to satisfy an expression of the form $\bigoplus_{v \in V(G)} f(v, c) \in X$ for some $X \subseteq M$ and function f that receives a vertex v and a color assignment valid in $N_G[v]$. For all $V \subseteq V(G)$, let M(V) be the set of different values that $\bigoplus_{v \in V} f(v, c)$ can have. Assume that, for every $V \subseteq V(G)$, |M(V)| is bounded by some polynomial in the size of the input. Also assume that computing $x \oplus y$ and $x \in X$ can be done in polynomial time for all $x, y \in M$ and $X \subseteq M$.

Let (M^*, \oplus^*) be obtained by adding an absorbing element E^* to (M, \oplus) , and let e^* be the neutral element. Consider other items similar to the partial neighborhood system for check, but for f:

- A set $\widetilde{\mathcal{N}_{v,i}}$, for every $v \in V(G)$ and $i \in L_v$, together with a closed binary operation $\widetilde{\boxplus}^{v,i}$ on $\widetilde{\mathcal{N}_{v,i}}$ that is commutative and associative and has a neutral element $\widetilde{e}_{v,i}$.
- A function $\widetilde{newN_{v,i}}$, for every $v \in V(G)$ and $i \in L_v$, that given $u \in N_G(v)$ and $j \in L_u$ returns an element of $\widetilde{N_{v,i}}$.

• A function $f_{v,i} : \widetilde{\mathcal{N}_{v,i}} \to M$, for every $v \in V(G)$ and $i \in L_v$. This function must satisfy $f_{v,c(v)}\left(\widetilde{\coprod}_{u \in N_G(v)}^{v,c(v)} \widetilde{newN}_{v,c(v)}(u,c(u))\right) = f(v,c)$ for every vertex $v \in V(G)$ and every color assignment c valid in $N_G[v]$.

In the algorithm, at each node t, we add to λ_t the following parameters: x, that stores the state of a partial accumulation of values f(v, c); accept, that checks if we are in the desired state (similar to the idea in the previous subsection); $\tilde{\eta}$, that carries the values of the "partial neighborhood for f" of the vertices in X_t ; and \tilde{f} , that behaves like \tilde{c} except that instead of providing functions of codomain BOOL, it provides functions of codomain M^* . Then proceed in the following way.

- Leaf node: Now we also need to check if $accept(x \oplus^* \widetilde{f}_v(\widetilde{\eta}(v)))$ is true.
- Forget node:

$$\lambda_t(\ldots, x, accept, \widetilde{\eta}, \widetilde{f}) = \min\{\lambda_s(\ldots, x, accept, \widetilde{\eta}^{v \to \widetilde{e}_{v,i}}, \widetilde{f}^{v \to f_{v,i}}) : i \in L_v\}.$$

- Introduce node: Accumulate in x the value of $\widetilde{f}_v(\widetilde{\eta}(v))$, remove v from the domain of the new functions, and compute the new partial neighborhoods of the vertices in X_s .
- Join node: For all $m \in M$, let $eq_m : M \to \text{Bool}$ be such that $eq_m(m') = (m = m')$ for all $m' \in M$. For all $\widetilde{\eta}$, let $\widetilde{f}^{\widetilde{\eta}}$ be the function defined as $\widetilde{f}^{\widetilde{\eta}}_v(n) = e^*$ if $n = \widetilde{\eta}(v)$ and $\widetilde{f}^{\widetilde{\eta}}_v(n) = E^*$ otherwise. For all $X \subseteq V(G)$ and color assignment c valid in X, let $\widetilde{\eta}^{\widetilde{e}}_c$ be the function of domain X such that $\widetilde{\eta}^{\widetilde{e}}_c(v) = \widetilde{e}_{v,c(v)}$ for all $v \in X$. For all H subgraph of G and all color assignment c valid in V(H), let $\widetilde{NS}(v,c,H) = \widetilde{\coprod}_{u \in N_H(v)} \widetilde{new} N_{v,c(v)}(u,c(u))$ for all $v \in V(H)$. Then

 $\lambda_t(\ldots, x, accept, \widetilde{\eta}, \widetilde{f}) = \min\{W \oplus \lambda_r(\ldots, e, eq_{m_r}, \widetilde{\eta}_c^{\widetilde{e}}, \widetilde{f}^{\widetilde{\eta}_r}) \oplus \lambda_s(\ldots, e, eq_{m_s}, \widetilde{\eta}_c^{\widetilde{e}}, \widetilde{f}^{\widetilde{\eta}_s}) : m_r \in M(V(G_r) - X_r) \land m_s \in M(V(G_s) - X_s) \land accept((\bigoplus_{v \in X_t}^* \widetilde{f}_v(\widetilde{\eta}(v) \widetilde{\boxplus}^{v,c(v)} \widetilde{\eta}_r(v) \widetilde{\boxplus}^{v,c(v)} \widetilde{\eta}_r(v) \widetilde{\boxplus}^{v,c(v)} \widetilde{\eta}_s(v) \widetilde{\mathbb{Q}}_s(v) \widetilde{$

• At the root r, with $X_r = \{v\}$, we compute all $\lambda_r(\ldots, e, a, \widetilde{\eta}_c^{\widetilde{e}}, \widetilde{f})$, where a is such that $a(x) = (x \in X)$, and $\widetilde{f_v} = f_{v,c(v)}$.

Note that it is easy to generalize this idea to more properties like this, by simply adding as many groups of parameters as needed. If the number of this properties is bounded by a constant, then the complexity of the algorithm remains polynomial.

This, along with the ideas and results in sections 3.3, 5 and 6, recovers the main results from Bodlaender in [6,7].

7.3. One color class is connected. Suppose we want the class of color j to be connected.

At each node t, we add the parameter $comp_j: X_t \to [\![1,k]\!]$ that maps vertices of color j to natural numbers that represent connected components.

In the following items, let X_t^j denote $\{u: u \in X_t \land c(u) = j\}$, and $N_t^j(v)$ denote $N_G(v) \cap X_t^j$.

- Leaf node: Remains the same.
- Forget node: $\lambda_t(\ldots, comp_j) = \min\{\lambda_s(\ldots, comp_j^i) : i \in L_v\}$ where $comp_j^i$ is such that:
 - if $i \neq j$ then $comp_j^i(u) = comp_j(u)$ for all $u \in X_t^j$, and

• $comp_j^j(v) = \begin{cases} \min_{u \in N_t^j(v)} \{comp_j(u)\} & \text{if } N_t^j(v) \neq \emptyset \\ \text{any value in } \llbracket 1, k \rrbracket - \{comp_j(u) : u \in X_t^j\} & \text{otherwise} \end{cases}$ and, for all $u \in X_t^j$, $comp_j^j(u) = comp_j^j(v)$ if there exists $z \in N_t^j(v)$ such that $comp_j(u) = comp_j(z)$, and $comp_j^j(u) = comp_j(u)$ otherwise.

In other words, if v is a neighbor of two or more vertices of different connected components then those connected components can be unified, and if v is not a neighbor of any other vertex of color j then it is in a new connected component.

- Introduce node: If $c(v) \neq j$ then it remains the same (adding $comp_j$ to λ_s). Otherwise, we split the case related to $\check{c}_v(n_v) = \text{True}$ in the following ones:
 - If there exists $u \in X_s^j$ such that $comp_j(v) = comp_j(u)$ then $\lambda_t(\ldots, comp_j) = \ldots \lambda_s(\ldots, comp_j^{-v})$.
 - If there does not exist $u \in X_s^j$ such that $comp_j(v) = comp_j(u)$, and $X_s^j \neq \emptyset$ then $\lambda_t(\ldots, comp_j) = \text{Error}$.
 - If $X_s^j = \emptyset$ then let $M = (\{q_0, q_1\}, \{1\}, \delta, q_0, \{q_0\})$ be an automaton such that $\delta(q_0, 1) = q_1$ and $\delta(q_1, 1) = q_1$. We use M to request that the class of color j is empty in G_s , therefore $\lambda_t(\ldots, comp_j) = \ldots \lambda_s(\ldots, q_0, eq_{q_0})$.

That is, if v belongs to a different connected component than all the vertices of color j in X_s , then there is no way to connect v with them and we get an error. Also, if there are no vertices of color j in X_s then there cannot be any vertices of color j in G_s , because $N_G(V(G) - V(G_s)) \cap V(G_s) \subseteq X_s$.

• **Join node:** For every $S \subseteq \{comp_j(v) : v \in X_t^j\}$ let $comp_j^S$ be a function such that, for all $v \in X_t^j$,

$$comp_j^s(v) = \begin{cases} \min(S) & \text{if } comp_j(v) \in S \\ comp_j(v) & \text{otherwise.} \end{cases}$$

At this node we need to unify the different connected components. To do so, on one branch we unify a set S of them while on the other branch we unify a set R of them, such that $S \cup R = \{comp_j(v) : v \in X_t^j\}$ and $|S \cap R| = 1$. Then $\lambda_t(\ldots, comp_j) = \min\{W \oplus \lambda_r(\ldots, comp_j^S) \oplus \lambda_s(\ldots, comp_j^R) : S \cup R = \{comp_j(v) : v \in X_t^j\} \land |S \cap R| = 1 \land \ldots\}.$

• At the root r where $X_r = \{v\}$, the function $comp_j$ is such that $comp_j(v) = 1$.

As before, notice that it is easy to generalize this idea to more classes or sets of classes by adding as many $comp_i$ functions as needed.

Let \mathcal{J} be the number of color classes (or sets of color classes) to restrict. The only changes in complexity are:

- Leaf node: add \mathcal{J} .
- Forget node: add $(C-1+k^2)\mathcal{J}$.
- Introduce node: add $k\mathcal{J}$.
- Join node: multiply by $(k2^k)^{\mathcal{J}}$.
- When we multiply by the number of all possible combinations of the parameters of λ_t : add a factor $(k^k)^{\mathcal{J}}$.

Note that because in the introduce node we require that the class of color j is empty, we also need to compute $\lambda_t(\ldots, q_0, eq_{q_0})$ for all $t \in T$, but this addition does not change the time complexity here (due to the fact that both \mathcal{S} and \mathcal{R} are bounded by a constant in this case).

7.4. One color class is acyclic. (For undirected graphs.)

It can be done in essentially the same way as for the connected property. The only difference is that in the introduce node we do as in the original algorithm, and in the forget node we check if v is a neighbor of at least two vertices that belong to the same component (in which case there is a cycle and we raise an error).

8. Applications

In this section we show how to model different problems as 1-locally checkable problems with polynomial partial neighborhood systems. As a result, we obtain polynomial-time algorithms to solve these problems for bounded treewidth graphs. For double Roman domination, minimum chromatic violation and Grundy domination (and their variants), no such algorithms were previously known (until the date and to the best of our knowledge). As regards the problems that were already known to be polynomial-time solvable for the before-mentioned classes, it is worth to mention how to restate these problems as 1-locally checkable problems, even when the time complexity of the proposed solution is worse than the best one known, because problems modeled this way can be easily modified or combined, adding global properties or more restrictions, or even dealing with some distance versions, and they can inspire the statement of other problems as 1-locally checkable problems.

Throughout this section, we will assume that, otherwise stated, the definitions of $\mathcal{N}_{v,i}$ are for all $v \in V(G)$ and $i \in L_v$, of $n \boxplus^{v,i} n'$ for all $v \in V(G)$, $i \in L_v$ and $n, n' \in \mathcal{N}_{v,i}$, of $new N_{v,i}(u,j)$ for all $v \in V(G)$, $i \in L_v$, $u \in N_G(v)$ and $j \in L_u$, and of $check_{v,i}(n)$ for all $v \in V(G)$, $i \in L_v$ and $n \in \mathcal{N}_{v,i}$.

8.1. **Double Roman domination.** This problem was first defined in [4] and proved to be NP-complete for bipartite and chordal graphs in [2].

A double Roman dominating function on a graph G is a function $f: V(G) \to \{0,1,2,3\}$ having the property that if f(v) = 0, then vertex v must have at least two neighbors assigned 2 under f or one neighbor w with f(w) = 3, and if f(v) = 1, then vertex v must have at least one neighbor w with $f(w) \ge 2$. The weight of a double Roman dominating function f is the sum $f(V(G)) = \sum_{v \in V(G)} f(v)$, and the minimum weight of a double Roman dominating function on G is the double Roman domination number of G.

We can model the Double Roman domination problem as a 1-locally checkable problem in the following way:

```
• L_v = \{0, 1, 2, 3\};
```

- (Weights, \preceq , \oplus) = ($\mathbb{R} \cup \{+\infty\}$, \leq , +);
- $W_{v,i} = i$;
- $check(v,c) = (c(v) = 0 \Rightarrow (\exists u, w \in N_G(v). u \neq w \land c(u) = c(w) = 2)$ $\lor (\exists u \in N_G(v). c(u) = 3))$ $\land (c(v) = 1 \Rightarrow \exists u \in N_G(v). c(u) \geq 2).$

It is easy to see that some of its variants, such as perfect [27], independent [53], outer independent [1] and total [59], can be modeled by making slight modifications to the previous items.

By Corollary 5.5.2, the Double Roman domination problem can be solved in polynomial time for graphs of bounded treewidth. However, we show a constant partial neighborhood system and therefore obtain a linear-time algorithm:

```
• \mathcal{N}_{v,i} = \{0, 1, 2\} \times \{0, 1\};
• n \boxplus^{v,i} n' = (\min(n_1 + n'_1, 2), \min(n_2 + n'_2, 1));
```

•
$$newN_{v,i}(u,j) = \begin{cases} (0,0) & \text{if } j \leq 1\\ (1,0) & \text{if } j = 2\\ (0,1) & \text{if } j = 3; \end{cases}$$

• $check_{v,i}(n) = (i = 0 \Rightarrow n_1 \geq 2 \lor n_2 \geq 1) \land (i = 1 \Rightarrow n_1 + n_2 \geq 1).$

Notice that this partial neighborhood system simply counts (until it saturates) the number of neighbors assigned with 2 and also the ones assigned with 3. For other versions of the double Roman domination problem, we might require to also count the number of neighbors assigned with 0 or 1.

8.2. **Minimum chromatic violation problem.** This NP-hard problem was first defined in [10] as a generalization of the k-coloring problem.

Given a graph G, a set of weak edges $F \subseteq E(G)$ and a positive integer k, the minimum chromatic violation problem asks for a k-coloring of the graph G' = (V(G), E(G) - F) minimizing the number of weak edges with both endpoints receiving the same color.

We can reduce this problem to the following 1-locally checkable problem in the subdivision graph S(G):

```
L<sub>v</sub> = [1, k] for all v ∈ V(G),

L<sub>uv</sub> = L<sub>u</sub> × L<sub>v</sub> for all uv ∈ E(G);
(WEIGHTS, ≤, ⊕) = (ℝ ∪ {+∞}, ≤, +);
W<sub>v,i</sub> = 0 for all v ∈ V(G), i ∈ L<sub>v</sub>,

W<sub>uv,(i,i)</sub> = 1 for all uv ∈ E(G), i ∈ L<sub>u</sub> ∩ L<sub>v</sub>,

W<sub>uv,(i,j)</sub> = 0 for all uv ∈ E(G), i ∈ L<sub>u</sub>, j ∈ L<sub>v</sub> − {i};
check(v, c) = TRUE for all v ∈ V(G),

check(uv, c) = (c(uv) = (c(u), c(v))) for all uv ∈ F, and

check(uv, c) = (c(uv) = (c(u), c(v)) ∧ c(u) ≠ c(v)) for all uv ∈ E(G) − F.
```

Basically, every edge in G is colored with a pair of colors and checks if these are the colors of its endpoints. Edges in F are allowed to have endpoints of the same color, while edges not in F always produce an error when colored with a pair of equal colors. If an edge is colored with a pair of equal colors then its weight is 1, otherwise is 0.

In this way we have $C = k^2$. We give a constant partial neighborhood system for this model:

```
• \mathcal{N}_{v,i} = \text{Bool for all } v \in V(S(G)), i \in L_v;

• n \boxplus^{v,i} n' = (n \land n') \text{ for all } v \in V(S(G)), i \in L_v \text{ and } n, n' \in \mathcal{N}_{v,i};

• new N_{v,i}(e,j) = \text{True for all } v \in V(G), i \in L_v, e \in N_{S(G)}(v), j \in L_e,

new N_{uv,(c_u,c_v)}(u,j) = (j=c_u) \text{ for all } uv \in E(G), c_u \in L_u, c_v \in L_v, j \in L_u,

new N_{uv,(c_u,c_v)}(v,j) = (j=c_v) \text{ for all } uv \in E(G), c_u \in L_u, c_v \in L_v, j \in L_v;

• check_{v,i}(n) = \text{True for all } v \in V(G), i \in L_v, n \in \mathcal{N}_{v,i},
```

• $check_{v,i}(n) = 1$ RUE for all $v \in V(G), i \in L_v, n \in \mathcal{N}_{v,i}$, $check_{uv,(i,j)}(n) = n$ for all $uv \in F, i \in L_u, j \in L_v, n \in \mathcal{N}_{uv,(i,j)}$, $check_{uv,(i,j)}(n) = (n \land i \neq j)$ for all $uv \in E(G) - F, i \in L_u, j \in L_v, n \in \mathcal{N}_{uv,(i,j)}$.

Therefore, when k is bounded by a constant the minimum chromatic violation problem can be solved in linear time for graphs of bounded treewidth.

8.3. **Grundy domination number.** This problem was introduced in [12] and proved to be NP-complete even for chordal graphs.

A sequence $S = (v_1, \ldots, v_k)$ of distinct vertices of a graph G is a dominating sequence if $\{v_1, \ldots, v_k\}$ is a dominating set of G, and S is called a legal (dominating) sequence if (in addition) $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$ for each i. We say that v_i footprints the vertices in $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j]$, and that v_i is the footprinter of every $u \in N[v_i] - \bigcup_{j=1}^{i-1} N[v_j]$

(notice that every vertex in V(G) has a unique footprinter). We are interested in the maximum length of a legal dominating sequence in G.

Given a legal sequence S, every vertex $v \in V(G)$ can be associated with a pair (p_v, f_v) , where p_v is the position of v in S (or \perp if v is not in S) and f_v is the position in S of the footprinter of v. Directly from the definition of legal sequences we can deduce the following statement. A set $\{(p_v, f_v) : v \in V(G)\}$ determines a legal sequence if and only if the following conditions are satisfied:

- (1) for all $v \in V(G)$, there exists a unique $u \in N_G[v]$ such that $f_v = p_u$, and $f_v \leq p_w$ for all $w \in N_G[v]$ (i.e., v is properly footprinted),
- (2) for all $v \in V(G)$, if $p_v \neq \perp$ then there exists $u \in N_G[v]$ such that $f_u = p_v$ (i.e., if v appears in the sequence then it footprints at least one vertex), and
- (3) $p_v \neq p_u$ for all $u, v \in V(G)$ such that $u \neq v$ and $p_u \neq \bot$ (i.e., two vertices that appear in the sequence cannot have the same position in it).

Notice that conditions 1 and 2 are locally checkable, but the last one is not. However, we claim that the Grundy domination problem can be reduced to the following 1-locally checkable problem. Let n = |V(G)| and $\perp = n + 1$.

- $L_v = (\{1, ..., n\} \cup \{\bot\}) \times \{1, ..., n\};$ (Weights, \preceq , \oplus) = ($\mathbb{R} \cup \{-\infty\}, \ge, +$);
- for all $f \in \{1, ..., n\}$, $W_{v,(\perp,f)} = 0$ and $W_{v,(p,f)} = 1$ for all $p \neq \perp$;
- $check(v, c) = (\exists ! u \in N_G[v]. c(v)_2 = c(u)_1)$ $\land \quad (\forall u \in N_G[v]. \ c(v)_2 \le c(u)_1)$ $\land \quad (c(v)_1 \ne \bot \Rightarrow \exists u \in N_G[v]. \ c(v)_1 = c(u)_2)$ $\land (c(v)_1 \neq \bot \Rightarrow \forall u \in N_G(v). c(v)_1 \neq c(u)_1).$

Let G be a graph. Let c be a proper coloring of G in the previous 1-locally checkable problem, and let $(p_v, f_v) = c(v)$ for all $v \in V(G)$. Conditions 1 and 2 are trivially satisfied. Suppose that the last condition is not satisfied. Then there are two different vertices u, v such that $p_u = p_v \neq \bot$. By definition of check, we know that

- $u \notin N_G|v|$,
- if there exists a vertex w such that $f_w > p_v = p_u$ then $v \notin N_G[w]$ and $u \notin N_G[w]$ (thus, $w \notin N_G[v]$ and $w \notin N_G[u]$), and
- if there exists a vertex $w \in N_G[u]$ such that $f_w = p_v = p_u$ then $w \notin N_G[v]$.

Therefore, $N_G[u] \cap N_G[v] = F$ such that $f_w < p_u = p_v$ for all $w \in F$. Now we can assign colors (p'_w, f'_w) for every $w \in V(G)$, such that

$$p'_{w} = \begin{cases} p_{w} & \text{if } p_{w} \leq p_{u} \text{ and } w \neq v \\ p_{w} + 1 & \text{otherwise} \end{cases}$$

and

$$f'_w = \begin{cases} f_w & \text{if } f_w < p_u, \text{ or } f_w = p_u \text{ and } w \notin N_G[v] \\ f_w + 1 & \text{otherwise.} \end{cases}$$

That is, we move one position forward all the vertices that appear after u in S and increase the necessary f_w . It is easy to see that this new assignment preserves the "legality" of u (i.e., if $N_G[u] - \{z : z \in N_G[w] \land p_w < p_u\} \neq \emptyset$ then $N_G[u] - \{z : z \in P_G[w] \land p_w < p_u\}$ $N_G[w] \wedge p'_w < p'_v \neq \emptyset$) and also of all the other vertices.

In order to give a polynomial-time algorithm for the Grundy domination problem in bounded treewidth graphs, we define a constant partial neighborhood system as follows:

•
$$\mathcal{N}_{v,(p,f)} = \{0,1,2\} \times \text{Bool} \times \text{Bool} \times \text{Bool};$$

```
• n \boxplus^{v,(p,f)} n' = (\min(n_1 + n'_1, 2), n_2 \wedge n'_2, n_3 \vee n'_3, n_4 \wedge n'_4);

• newN_{v,(p_v,f_v)}(u,(p_u,f_u)) = n where:

• n_1 = \begin{cases} 1 & \text{if } f_v = p_u \\ 0 & \text{otherwise,} \end{cases}

• n_2 = (f_v \leq p_u),

• n_3 = (p_v = f_u),

• n_4 = (p_v \neq p_u);

• check_{v,(p,f)}(n) = (n_1 < 2 \wedge (n_1 = 1 \Rightarrow f \neq p) \wedge (n_1 = 0 \Rightarrow f = p))

\wedge (n_2 \wedge f \leq p)

\wedge (p \neq \bot \Rightarrow (n_3 \vee p = f))

\wedge (p \neq \bot \Rightarrow n_4).
```

We can also model the Grundy total domination problem (defined in [13]) in a very similar way, by simply removing the cases where a vertex can footprint itself.

For both problems, since C is $O(|V(G)|^2)$, and we defined a constant partial neighborhood system, the time complexity of the algorithm is polynomial in |V(G)| for a graph G in a family of bounded treewidth graphs.

8.4. Additive coloring. Let η be an upper bound of the additive chromatic number. It was shown in [3] that the additive chromatic number in a graph G is at most $\Delta(G)^2 - \Delta(G) + 1$.

To model the additive coloring problem as a 1-locally checkable problem with a partial neighborhood system, we define the colors to be pairs of integers (n, s), where n represents the number assigned to the vertex and s the sum of the numbers assigned to its neighbors. Formally:

```
• L_v = [\![1, \eta]\!] \times [\![1, \Delta(G)\eta]\!];

• (\text{Weights}, \preceq, \oplus) = (\mathbb{R} \cup \{+\infty\}, \leq, \text{max});

• W_{v,i} = i_1;

• check(v, c) = (\forall u \in N_G(v). c(u)_2 \neq c(v)_2) \wedge \left(c(v)_2 = \sum_{u \in N_G(v)} c(u)_1\right).
```

It is straightforward to derive a polynomial partial neighborhood system for this model. One possibility is the following:

```
• \mathcal{N}_{v,i} = \text{Bool} \times [1, \Delta(G)\eta + 1];

• n \boxplus^{v,i} n' = (n_1 \wedge n'_1, \min(n_2 + n'_2, \Delta(G)\eta + 1));

• new N_{v,i}(u,j) = (i_2 \neq j_2, j_1)

• check_{v,i}(n) = n_1 \wedge (i_2 = n_2).
```

Then C is $O(\Delta(G)\eta^2)$ and N is $O(\Delta(G)\eta)$, implying that there exists a polynomial-time algorithm to compute $\eta(G)$ for graphs G in a class of graphs of bounded treewidth. Another polynomial time algorithm was obtained by R. Grappe, L. N. Grippo, and M. Valencia-Pabon (personal communication).

8.5. Distance domination problems. These problems can be naturally expressed as r-locally checkable problems, for some r depending on the characteristics of the problem. Moreover, some of them are in fact 1-locally checkable problems.

We will start by showing how to model the distance k-domination problem as a 1-locally checkable problem. To do this, we restate the problem in the following way: vertices receive integers from 0 to k (that represent their distance to a vertex of the distance k-dominating set), and vertices with a number greater than 0 must satisfy the condition of having a neighbor with the preceding number assigned. Then

•
$$L_v = [0, k];$$

```
• (WEIGHTS, \leq, \oplus) = (\mathbb{R} \cup \{+\infty\}, \leq, +);
• W<sub>v,0</sub> = 1 and
W<sub>v,i</sub> = 0 for all i \in [\![1,k]\!];
• check(v,c) = (c(v) > 0 \Rightarrow \exists u \in N_G(v). c(u) = c(v) - 1).
```

It only remains to show a constant partial neighborhood system in order to get a linear-time algorithm for this problem in bounded treewidth graphs:

```
• \mathcal{N}_{v,i} = \text{Bool};

• n \boxplus^{v,i} n' = n \lor n';

• newN_{v,i}(u,j) = (j=i-1);

• check_{v,i}(n) = (i > 0 \Rightarrow n).
```

Notice that with a similar argument we can model similar problems involving more than two sets in the partition. The idea is to make the colors indicate how far the vertices are from every other color. For example, if we have to color the graph with $\{r, g, b\}$ in such a way that every vertex is at distance at most 3 from a vertex of color r and at distance at most 2 from a vertex of color g, following this idea to model the problem as a 1-locally checkable problem, our set of colors is $\{r_{0,1}, r_{0,2}, g_{1,0}, g_{2,0}, g_{3,0}, b_{1,1}, b_{2,1}, b_{3,1}, b_{1,2}, b_{2,2}, b_{3,2}\}$.

However, when there are restrictions over the distance between vertices of the same color, the previous approach would not work. We will now explain how to model these problems when the required distance is 2.

Let us work with the semitotal domination problem. We first restate the problem in order to differentiate two possible situations for a vertex in D (colors D_1 and D_2) and two possible situations for a vertex not in D (colors \overline{D} and \overline{D}_*). Vertices of color D_1 represent those vertices in D that have a neighbor in D, vertices of color D_2 represent those vertices in D that are at distance 2 of another vertex in D, vertices of color \overline{D}_* represent those vertices not in D that are the nexus between two vertices in D, and vertices of color \overline{D} represent all the remaining ones. Then we can set

```
• L_v = \{D_1, D_2, \overline{D}, \overline{D}_*\};

• (WEIGHTS, \preceq, \oplus) = (\mathbb{R} \cup \{+\infty\}, \leq, +);

• W_{v,D_1} = W_{v,D_2} = 1, and

W_{v,\overline{D}} = W_{v,\overline{D}_*} = 0;

• check(v,c) = (c(v) \in \{\overline{D}, D_1\} \Rightarrow \exists u \in N_G(v). c(u) \in \{D_1, D_2\})

\land (c(v) = D_2 \Rightarrow \exists u \in N_G(v). c(u) = \overline{D}_*)

\land (c(v) = \overline{D}_* \Rightarrow \exists u, w \in N_G(v). u \neq w \land c(u), c(w) \in \{D_1, D_2\}).
```

And we can naturally give a constant partial neighborhood system:

```
• \mathcal{N}_{v,i} = \{0, 1, 2\};

• n \boxplus^{v,i} n' = \min(n + n', 2);

• If i \in \{D_1, \overline{D}, \overline{D}_*\}:

• newN_{v,i}(u,j) = \begin{cases} 1 & \text{if } j \in \{D_1, D_2\} \\ 0 & \text{otherwise,} \end{cases}

if i = D_2:

• newN_{v,i}(u,j) = \begin{cases} 1 & \text{if } j = \overline{D}_* \\ 0 & \text{otherwise;} \end{cases}

• check_{v,i}(n) = (n \ge 1) \text{ for } i \in \{\overline{D}, D_1, D_2\}, \text{ and } check_{v,\overline{D}_*}(n) = (n \ge 2).
```

Notice that in the restatement of semitotal domination we changed the "duties" of the vertices: the ones in charge of checking that a vertex of color D_2 is at distance 2 of another vertex in D are now the vertices of color \overline{D}_* .

Combining the ideas of the two previous problems we can model other related problems (such as total distance 2-domination) as 1-locally checkable problems.

8.6. **Problems involving edges.** Consider locally checkable problems where, for every edge, a certain condition comprising their endpoints and their consecutive edges must be satisfied. We will show how to reduce these problems to 1-locally checkable problems in the jagged graph J(G) of the input graph G. We consider two kind of problems: when edges do not have requirements over other edges, and when they do.

For the first class of problems, the reduction is straightforward. We might need to duplicate the colors in order to distinguish the colors assigned to edges from the colors assigned to vertices. As an example, we show how to model vertex cover:

```
L<sub>v</sub> = {0, 1} for all v ∈ V(G) and L<sub>uv</sub> = {0} for all uv ∈ E(G);
(WEIGHTS, ≤) = (ℝ ∪ {+∞}, ≤) and ⊕ = +;
W<sub>v',i</sub> = i for all v' ∈ V(J(G)), i ∈ L<sub>v'</sub>;
check(v, c) = TRUE for all v ∈ V(G), and check(uv, c) = (c(u) + c(v) > 1) for all uv ∈ E(G).
```

And we give a constant partial neighborhood system:

```
• \mathcal{N}_{v',i} = \{0, 1\};

• n \boxplus^{v',i} n' = \min(n + n', 1);

• new N_{v',i}(u', j) = j; and

• check_{v,i}(n) = \text{True for all } v \in V(G), i \in L_v, \text{ and } check_{uv,0}(n) = (n > 1) \text{ for all } uv \in E(G).
```

Edge cover is basically the same as vertex cover but interchanging vertices and edges. As regards the second class of these problems, since two edges in E(G) that share an endpoint are at distance two in J(G), we can use the ideas from semitotal domination: the neighbors in J(G) of the edges in E(G) (which are vertices in V(G)) are the ones in charge of checking the requirements of the edges in E(G). We illustrate the maximum matching problem, for which we can set:

```
• L_v = \{0\} for all v \in V(G) and L_{uv} = \{0,1\} for all uv \in E(G);

• (WEIGHTS, \preceq) = (\mathbb{R} \cup \{-\infty\}, \geq) and \oplus = +;

• W_{v',i} = i for all v' \in V(J(G)), i \in L_{v'};

• check(v,c) = \left(\sum_{u \in N_G(v)} c(uv) \leq 1\right) for all v \in V(G), and check(uv,c) = \text{True} for all uv \in E(G).
```

And we give a constant partial neighborhood system:

```
• \mathcal{N}_{v',i} = \{0,1\};

• n \boxplus^{v',i} n' = \min(n+n',1);

• newN_{v',i}(u',j) = j; and

• check_{v,0}(n) = (n \le 1) for all v \in V(G), and check_{uv,i}(n) = \text{True} for all uv \in E(G), i \in L_{uv}.
```

ACKNOWLEDGEMENTS

This work was partially supported by ANPCyT PICT-2015-2218, UBACyT Grants 20020190100126BA, 20020170100495BA and 20020160100095BA (Argentina), and Programa Regional MATHAMSUD MATH190013. Carolina L. Gonzalez is partially supported by a CONICET doctoral fellowship.

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APPENDIX A. PROBLEMS DEFINITIONS

We define here the decision versions of the problems mentioned along the paper. Other similar problems can also be modeled as r-locally checkable problems.

A.1. Domination problems.

Dominating set [55]

Instance: A (weighted) graph G and a positive integer k.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most k and such that $|N[v] \cap S| \ge 1$ for every $v \in V(G)$?

Total domination [22]

Instance: A (weighted) graph G and a positive integer k.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most k and such that $|N(v) \cap S| \ge 1$ for every $v \in V(G)$?

k-Tuple domination [42]

Instance: A (weighted) graph G and a positive integer ℓ .

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most ℓ and such that $|N[v] \cap S| \ge k$ for every $v \in V(G)$?

Total k-tuple domination [46]

Instance: A (weighted) graph G and a positive integer ℓ .

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most ℓ and such that $|N(v) \cap S| \ge k$ for every $v \in V(G)$?

k-domination [31]

Instance: A (weighted) graph G and a positive integer ℓ .

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most ℓ and such that $|N(v) \cap S| \ge k$ for every $v \in V(G) \setminus S$?

 $\{k\}$ -DOMINATION [44]

Instance: A graph G and a positive integer ℓ .

Question: Does there exist a function $f: V(G) \to \{0, 1, ..., k\}$ of weight at most ℓ and such that $f(N[v]) \ge k$ for every $v \in V(G)$?

k-rainbow domination [20]

Instance: A graph G and a positive integer ℓ .

Question: Does there exist a function $f: V(G) \to 2^{\{1,\dots,k\}}$ of weight $(\sum_{v \in V(G)} |f(v)|)$ at most ℓ and such that for every vertex $v \in V(G)$ for which $f(v) = \emptyset$ we have $\bigcup_{u \in N_G[v]} f(u) = \{1,\dots,k\}$?

Semitotal dominating set [37]

Instance: A (weighted) graph G with no isolated vertex and a positive integer k.

Question: Does there exist a dominating set $D \subseteq V(G)$ of size (weight) at most k and such that every vertex in D is within distance two of another vertex in D?

DISTANCE k-DOMINATION (also called k-Covering) [45, 54]

Instance: A (weighted) graph G and a positive integer ℓ .

Question: Does there exist a set $S \subseteq V(G)$ of size (weight) at most ℓ and such that every vertex in G is within distance k from a vertex in S?

Connected dominating set [58]

Instance: A (weighted) graph G and a positive integer k.

Question: Does there exist a dominating set of G of size (weight) at most k that induces a connected subgraph of G?

ROMAN DOMINATION [23]

Instance: A graph G and a positive integer k.

Question: Does there exist a function $f: V(G) \to \{0, 1, 2\}$ of weight at most k and such that every vertex $u \in V(G)$ for which f(u) = 0 is adjacent to at least one vertex $v \in V(G)$ for which f(v) = 2?

Grundy total domination [13]

Instance: A graph G and a positive integer k.

Question: Does there exist a sequence (v_1, \ldots, v_ℓ) of distinct vertices of G such that $\ell \geq k$, $\{v_1, \ldots, v_\ell\}$ is a dominating set of G and $N(v_i) - \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ for each i?

A.2. Coloring problems.

k-coloring [35]

Instance: A graph G.

Question: Does there exist a function $c: V(G) \to \{1, ..., k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$?

List-coloring [28, 63]

Instance: A graph G and a set L(v) of colors for each vertex $v \in V(G)$.

Question: Does there exist a proper coloring c such that $c(v) \in L_v$ for all $v \in V(G)$?

k-CHROMATIC SUM [51]

Instance: A graph G.

Question: Is there a proper coloring c of the graph G such that $\sum_{v \in V} c(v) \leq k$?

Additive coloring (also called lucky labeling) [25]

Instance: A graph G and a positive integer k.

Question: Does there exist a function $f: V(G) \to \{1, ..., k\}$ such that for every two adjacent vertices u, v the sums of numbers assigned to their neighbors are different (that is, $\sum_{w \in N(u)} f(w) \neq \sum_{z \in N(v)} f(z)$)?

ACYCLIC COLORING [40]

Instance: A graph G and a positive integer k.

Question: Does there exist a k-coloring of G such that of every subgraph of G spanned by vertices of two of the colors is acyclic (in other words, is a forest)?

L(h, k)-labeling [16]

Instance: A graph G and a positive integer s.

Question: Does there exist a labeling of its vertices by integers between 0 and s such that adjacent vertices of G are labeled using colors at least h apart, and vertices having a common neighbor are labeled using colors at least k apart?

A.3. Independence problems.

Independent set [35]

Instance: A (weighted) graph G and a positive integer k.

Question: Does there exist an independent set of G of size (weight) at least k?

k-independent set (also called distance d-independent set) [29, 32]

Instance: A graph G and a positive integer s.

Question: Does there exist $X \subseteq V(G)$ of size at least s such that the distance between every two vertices of X is at least k + 1?

A.4. Packing problems.

 $\{k\}$ -PACKING FUNCTION [52]

Instance: A graph G and a positive integer ℓ .

Question: Does there exist a function $f: V(G) \to \{0, 1, ..., k\}$ of weight at least ℓ and such that $f(N[v]) \le k$ for every $v \in V(G)$?

k-LIMITED PACKING [34]

Instance: A graph G and a positive integer ℓ .

Question: Does there exist a function $f: V(G) \to \{0,1\}$ of weight at least ℓ and such that $f(N[v]) \le k$ for every $v \in V(G)$?

PACKING CHROMATIC NUMBER [14]

Instance: A graph G and a positive integer k.

Question: Can G be partitioned into disjoint classes X_1, \ldots, X_k where vertices in X_i have pairwise distance greater than i?

A.5. Problems involving edges.

Matching [26]

Instance: A(n) (edge weighted) graph G and a positive integer k.

Question: Does there exist a set $M \subseteq E(G)$ of pairwise non-adjacent edges of size (weight) at least k?

EDGE DOMINATION [64]

Instance: A(n) (edge weighted) graph G and a positive integer k.

Question: Does there exist $F \subseteq E(G)$ of size (weight) at most k and such that every edge in E(G) shares an endpoint with at least one edge in F?

Vertex cover [49]

Instance: A (weighted) graph G and a positive integer k.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most k and such that every edge in E(G) has at least one endpoint in S?

Edge cover [35]

Instance: A(n) (edge weighted) graph G and a positive integer k.

Question: Does there exist $F \subseteq E(G)$ of size (weight) at most k and such that every vertex in V(G) belongs to at least one edge in F?

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