# Antimagic orientation of subdivided caterpillars

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June 22, 2021

### Abstract

Let  $m \geq 1$  be an integer and G be a graph with m edges. We say that G has an antimagic orientation if G has an orientation D and a bijection  $\tau : A(D) \to \{1, 2, \ldots, m\}$ such that no two vertices in D have the same vertex-sum under  $\tau$ , where the vertex-sum of a vertex v in D under  $\tau$  is the sum of labels of all arcs entering v minus the sum of labels of all arcs leaving v. Hefetz, Mütze and Schwartz [J. Graph Theory, 64: 219-232, 2010] conjectured that every connected graph admits an antimagic orientation. The conjecture was confirmed for certain classes of graphs such as regular graphs, graphs with minimum degree at least 33, bipartite graphs with no vertex of degree zero or two, and trees including caterpillars and complete k-ary trees. We prove that every subdivided caterpillar admits an antimagic orientation, where a subdivided caterpillar is a subdivision of a caterpillar T such that the edges of T that are not on the central path of T are subdivided the same number of times.

**Keywords**: caterpillar, subdivided caterpillar, antimagic labeling, antimagic orientation

## 1 Introduction

For two integers p and q, let  $[p,q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$ . Let G be a graph with  $m \geq 1$  edges. An *antimagic labeling* of G is a bijection  $\tau : E(G) \to [1,m]$  such that for any two distinct vertices u and v, the *vertex-sum*, the sum of the labels of the edges incident with a given vertex, of u is distinct from the vertex-sum of v. A graph is said to be *antimagic* if it admits an antimagic labeling.

The idea of antimagic labeling was introduced in 1990 by Hartsfield and Ringel [6] and they conjectured that every connected graph and every tree other than  $K_2$  is antimagic. One of the groups of researchers to continue this investigation is Kaplan, Lev and Roditty [8], who proved that any tree having more than two vertices and at most one vertex of degree two is antimagic (also see [10]). Other findings include proof of antimagic labelings for all regular graphs [2, 3], graphs with average degree greater than or equal to  $d_0$  for some constant  $d_0$  with no isolated edge and at most one isolated vertex [4], and all complete multipartite graphs other than  $K_2$  [1].

In 2010 Hefetz, Mütze and Schwartz introduced a variation of antimagic labeling, specifically the labeling of digraphs [7]. Let D be a digraph. We denote the vertex set and the arc set of D by V(D) and A(D), respectively. Let |A(D)| = m. An antimagic labeling of D is a bijection  $\tau : A(D) \to [1, m]$  such that no two vertices receive the same oriented vertex-sum, where the oriented vertex-sum of a vertex  $u \in V(D)$  is the sum of labels of all arcs entering u minus the sum of labels of all arcs leaving u. We use  $s_{[D,\tau]}(u)$  to denote the oriented vertex-sum of the vertex  $u \in V(D)$  under the bijection  $\tau$ . For simplicity we refer to the oriented vertex-sum as the vertex-sum in the remainder of this paper as we are exclusively working with antimagic orientations. We say a graph G admits an antimagic orientation if it has an orientation D such that D has an antimagic labeling. Hefetz, Mütze and Schwartz [7] proposed the following conjecture.

### **Conjecture 1.** Every connected graph admits an antimagic orientation.

Note that all antimagic bipartite graphs admit an antimagic orientation where all edges are oriented in the same direction between the partite sets. In [7], Hefetz, Mütze and Schwartz proved Conjecture 1 for some classes of graphs, such as stars, wheels, and graphs of order n with minimum degree at least  $c \log n$  for an absolute constant c. In the process the authors proved a stronger result that every orientation of these graphs is antimagic as well. Additional cases for this conjecture that have been proved already include regular graphs [7, 9, 16, 15], biregular bipartite graphs with minimum degree at least two [13], Halin graphs [18], graphs with large maximum degree [17], graphs with minimum degree at least 33 and bipartite graphs with no vertex of degree 0 or 2 [12]. Researchers have taken particular interest in investigating trees, as we do. For antimagic orientation, it is proved that Conjectur 1 is true for caterpillars [11] (it was actually proved that caterpillars are antimagic), complete k-ary trees [14], and lobsters [5]. A caterpillar is a tree of order at least 3 such that the removal of it's leaves produces a path. We will call a longest path in a caterpillar the *spine* and denote it by P (it is easy to see that all the vertices of the caterpillar not contained in P are leaves of the caterpillar), and call all the edges incident with an internal vertex of P a leq of the caterpillar. A subdivided caterpillar  $T^*$  is a subdivision of a caterpillar T such that all the legs of the caterpillar are subdivided the same number of times. We again call the corresponding subdivision of the spine of T the spine of  $T^*$ , and call the corresponding subdivisions of the legs of T the legs of  $T^*$ .

It was proved in [12] that every bipartite graph without vertices of degree 0 or 2 admits an antimagic orientation. It suggests that constructing antimagic orientations for graphs with

many vertices of degree 2 is very difficult in general. In this paper, we confirm Conjecture 1 for subdivided caterpillars whose most vertices are of degree 2.

#### **Theorem 2.** Every subdivided caterpillar admits an antimagic orientation.

The remainder of the paper is organized as follows: in next section, we prepare some notation and preliminaries, and in Section 3, we prove Theorem 2.

# 2 Notation and preliminary lemmas

Let T be a subdivided caterpillar with m edges for some integer  $m \ge 2$ . Throughout the remainder of this paper, we will denote by P the spine of T. Let

p = |E(P)|, s = the total number of legs of T, and k = the length of each leg.

Thus we have

$$m = p + ks. \tag{1}$$

As it was already proved that every caterpillar admits an antimagic orientation [11], we will assume from now on that

$$s \ge 1$$
 and  $k \ge 2$ . (2)

Furthermore, we let  $P = v_0 v_1 \dots v_p$  and  $U = \{v_{h_1}, v_{h_2}, \dots, v_{h_t}\} \subseteq V(P)$  be the set of vertices of degree at least 3 in T, where  $h_1 < h_2 < \dots < h_t$ . We call each vertex in U a *joint* of T, and a joint with degree 3 in T a *small joint* and a *big joint* otherwise. A leg with an endvertex as a joint is *attaching* at the joint. A leg attaching at a big joint is a *big leg*, and a leg attaching at a small joint is a *small leg*. Note that  $t \leq s$  and when t = s, then all the t joints are small joints. Let  $L_1, \dots, L_s$  be the s legs of T and we let

$$L_i = x_{i0} x_{i1} \dots x_{ik},$$

for each  $i \in [1, s]$ , where  $x_{i0} \in U$ . Furthermore, we may assume that these legs are ordered in consistent with the joints of T along P from  $v_0$  to  $v_p$ : for two joints  $v_{h_i}$  and  $v_{h_j}$  with i < j, the indices of legs attaching at  $v_{h_i}$  are smaller than the indices of legs attaching at  $v_{h_j}$ .

Let  $L_i$  be any leg of T for some  $i \in [1, s]$ . In all the proofs later,  $L_i$  is oriented using the following pattern:

For each odd j with  $j \in [1, k], x_{i(j-1)} \leftarrow x_{ij}$  and  $x_{ij} \rightarrow x_{i(j+1)},$  (3)

where the second arrow is defined only if  $j \leq k - 1$ . See Figure 1 below for an illustration of T, notation defined on T, and the orientation of the legs of T.

We will need the following result to orient and label the spine P of T.



Figure 1: Notation for T when p = 6, s = 3, and k = 3.

**Lemma 3** ([5]). Let  $P = v_0 v_1 \cdots v_p$  be a path and  $U = \{v_{h_1}, v_{h_2}, \cdots, v_{h_t}\} \subseteq V(P)$  with  $|U| \ge 1$ , where  $p \ge 2$  and  $h_1 < h_2 < \cdots < h_t$ . Then P has an orientation  $\overrightarrow{P}$  and a labeling  $\tau : A(\overrightarrow{P}) \to [1, p]$  such that

- (i)  $s_{[\overrightarrow{P},\tau]}(v) \ge 1$  for any vertex  $v \in U$ , and  $s_{[\overrightarrow{P},\tau]}(v) \le p-1$  for every  $v \in U$  with  $v \ne v_{h_t}$ and  $s_{[\overrightarrow{P},\tau]}(v_{h_t}) \ge 3$ ;
- (ii)  $1 \leq |s_{[\overrightarrow{P},\tau]}(v)| \leq p$  for every vertex  $v \in V(P) \setminus U$ ; and
- $(iii) \ s_{[\overrightarrow{P},\tau]}(u) \neq s_{[\overrightarrow{P},\tau]}(v) \ for \ any \ two \ distinct \ vertices \ u,v \in V(P) \setminus U.$

The second part of Lemma 3 (i) was not specified in [5]. However, the conclusion is a direct consequence of the orientation of P: for every  $v \in U \setminus \{v_{h_t}\}$  and the two edges incident with v, one of the edge is entering v and the other is leaving v in  $\overrightarrow{P}$ ; and for the vertex  $v_{h_t}$ , both the edges incident with  $v_{h_t}$  are entering  $v_{h_t}$  in  $\overrightarrow{P}$ .

We first construct an antimalized orientation for T when s = 1.

Lemma 4. Each subdivided caterpillar with a single leg admits an antimagic orientation.

*Proof.* Let T be a subdivided caterpillar. We adopt the notation introduced earlier. Thus  $s = 1, k \ge 2$  by (2), and m = p + k by (1). We define an antimagic orientation of T in two steps.

Step 1 Orient the edges in E(P) and label them.

By Lemma 3, P has an orientation  $\overrightarrow{P}$  and a labeling  $\tau_1$  using numbers  $1, 2, \ldots, p$  satisfying the three properties described in Lemma 3.

Step 2 Orient and label the leg of T.

Recall  $L_1 = x_{10} \dots x_{1k}$  is the leg of T and  $x_{10} = v_{h_1}$ . We orient  $L_1$  using the pattern described in (3) and define  $\tau_2 : E(L_1) \to [p+1,m]$  according to two different cases below:

$$\begin{cases} \tau_2(x_{1j}x_{1(j+1)}) = m - j, \quad j \in [0, k-1], & \text{if } s_{[\overrightarrow{P}, \tau_1]}(v_{h_1}) \ge m - 2, \\ \tau_2(x_{10}x_{11}) = m_1, \quad \tau_2(x_{11}x_{12}) = m_2, \quad \text{and} \\ \tau_2(x_{1j}x_{1(j+1)}) = m - j, \quad j \in [2, k-1], & \text{if } s_{[\overrightarrow{P}, \tau_1]}(v_{h_1}) < m - 2, \end{cases}$$

where  $m_1 \in \{m, m-1\}$  is the number such that  $m_1 + s_{[\vec{P}, \tau_1]}(v_{h_1})$  is even, and  $\{m_2\} = \{m, m-1\} \setminus \{m_1\}.$ 

Let  $T^*$  be the orientation of T obtained through the two steps above, and let

$$\tau: A(T^*) \to [1, m] \text{ such that } \tau(e) = \begin{cases} \tau_1(e) & \text{if } e \in A(\overrightarrow{P}), \\ \tau_2(e) & \text{if } e \in A(T^*) \setminus A(\overrightarrow{P}), \end{cases}$$

where note that we treat an arc e of  $T^*$  and the corresponding un-directed e as the same element in defining  $\tau$ . (This convention will be used throughout the paper when we defining an antimagic labeling of an orientation of a graph.)

We show next that  $\tau$  is an antimagic labeling of  $T^*$ . Let  $u, v \in V(T^*)$  be any two distinct vertices. By the labeling, it is clear that  $s_{[T^*,\tau]}(u) \neq s_{[T^*,\tau]}(v)$  if  $u, v \in V(P)$  or  $u, v \in V(L_1) \setminus \{v_{h_1}\}$ . If  $u \in V(P) \setminus \{v_{h_1}\}$  and  $v \in V(L_1) \setminus \{v_{h_1}\}$ , then  $|s_{[T^*,\tau]}(u)| \in [1,p]$  and  $|s_{[T^*,\tau]}(v)| \in [p+1, 2m-1]$ , and so they are different. Thus we assume that  $u = v_{h_1}$  and  $v \in V(L_1) \setminus \{v_{h_1}\}$ . If  $s_{[\overrightarrow{P},\tau_1]}(v_{h_1}) \geq m-2$ , then  $s_{[T^*,\tau]}(u) \geq m-2+\tau_2(x_{10}x_{11}) = m-2+m = 2m-2$ . Note that  $s_{[T^*,\tau]}(x_{1j}) = (-1)^j(2m-2j+1)$  for each  $j \in [1, k-1]$  and  $|s_{[T^*,\tau]}(x_{1k})| = p+1$ . Thus for even  $j \in [1, k]$ ,  $s_{[T^*,\tau]}(x_{1j}) \leq 2m-3$ . Hence  $s_{[T^*,\tau]}(v_{h_1}) \neq s_{[T^*,\tau]}(v)$ . Therefore we assume  $s_{[\overrightarrow{P},\tau_1]}(v_{h_1}) < m-2$ . Then by Lemma 3 (i) and the definition of  $\tau_2$ , we know that  $m+2=3+m-1 \leq s_{[T^*,\tau]}(v_{h_1}) \leq m-3+m=2m-3$  and  $s_{[T^*,\tau]}(v_{h_1})$  is even. Since  $s_{[T^*,\tau]}(x_{12}) \in \{2m-2, 2m-3\}$ ,  $s_{[T^*,\tau]}(x_{1j}) = (-1)^j(2m-2j+1)$  is odd for  $j \in [3, k-1]$ , and  $s_{[T^*,\tau]}(x_{1k}) \leq m$ , we again have  $s_{[T^*,\tau]}(v_{h_1}) \neq s_{[T^*,\tau]}(v)$ . This finishes the proof.

# 3 Proof of Theorem 2

*Proof.* Let T be a subdivided caterpillar. We adopt the notation introduced in Section 2. By Lemma 4 and (2), we assume  $k \ge 2$  and  $s \ge 2$ . We construct an antimagic orientation of T below.

### Step 1: Orient the edges in E(P) and label them.

By Lemma 3, P has an orientation  $\overrightarrow{P}$  and a labeling  $\tau_1$  using numbers  $1, 2, \ldots, p$  satisfying the three properties described in Lemma 3.

### Step 2: Orient and label the legs of T.

We orient each leg of T using the pattern described in (3). Together with the orientation of P in Step 1, we have obtained an orientation  $\overrightarrow{T}$  of T.

Next, we will assign labels in [p+1,m] to the edges contained in legs of T. Define

$$E_1 = \{x_{i0}x_{i1} : i \in [1, s]\}$$

to be the set of the edges from the legs that are incident with the joints of T.

### Step 2.1: Assign labels in [m - s + 1, m] to edges in $E_1$ .

We let  $M \subseteq E_1$  be a matching of T with size |U| = t and saturating U, recall  $U = \{v_{h_1}, \ldots, v_{h_t}\}$  is the set of vertices of T of degree at least 3 in T. We arbitrarily assign labels in [m - (s - t) + 1, m] to edges in  $E_1 \setminus M$  such that distinct edge receive distinct label. Denote by  $\tau_2^*$  the current labeling of  $T_1^*$ , where  $T_1^*$  consists of  $\overrightarrow{P}$  and the oriented edges from  $E_1 \setminus M$ . Now for each vertex  $v \in U$ , we compute  $s_{[T_1^*, \tau_2^*]}(v)$ , and assume that

$$s_{[T_1^*,\tau_2^*]}(x_1) \ge \ldots \ge s_{[T_1^*,\tau_2^*]}(x_t),$$

where  $\{x_1, \ldots, x_t\}$  is a permutation of the vertices of U. Let  $T_1$  be the union of  $T_1^*$  and those edges from M together with their orientation. Now define

$$\tau_2: A(T_1) \to [m-s+1, m] \text{ with } \tau_2(e) = \begin{cases} \tau_2^*(e), & \text{if } e \in A(T_1^*), \\ m-(s-t)+1-i, & \text{if } e \in M \text{ and is incident with } x_i \end{cases}$$

By this definition of  $\tau_2$ , we have

$$s_{[T_1,\tau_2]}(x_1) > \ldots > s_{[T_1,\tau_2]}(x_t).$$
 (4)

Step 2.2: Assign labels in [p+1,m] to edges in  $E(T) \setminus E(P)$ .

Let  $i \in [1, s]$ . For the leg  $L_i$ , assume  $s_{[T_1, \tau_2]}(x_{i0}x_{i1}) = a_i$ . Then the labels will be used for edges of  $L_i$  will be the set

$$A_i = \{a_i, a_i - s, a_i - 2s, \dots, a_i - (k-1)s\}.$$

It is clear that  $\bigcup_{i=1}^{s} A_i = [p+1,m]$ . Let  $f : E(T) \setminus E(P) \to [p+1,m]$  be a bijection. For  $e \in E(T) \setminus E(P)$ , assume  $e \in E(L_i)$  for some  $i \in [1,s]$ . We define f according to three different cases (case 2 and case 3 can be combined, but we separate them for clarity), see Figure 2 for an illustration.

(I) For even  $j \in [0, k-2]$ , let

$$f(x_{ij}x_{i(j+1)}) = a_i - \frac{j}{2}s, \quad f(x_{ij+1}x_{i(j+2)}) = a_i - \left(\left\lceil \frac{k}{2} \right\rceil + \frac{j}{2}\right)s$$

When k is odd, let  $f(x_{i(k-1)}x_{ik}) = a_i - \frac{k-1}{2}s$ .



Figure 2: Three different labeling patterns of leg  $L_i$ .

(II) k is even, for even  $j \in [0, k-2]$ , let

$$f(x_{ij}x_{i(j+1)}) = a_i - (k-1)s + \left(\frac{k}{2} + \frac{j}{2}\right)s, \quad f(x_{ij+1}x_{i(j+2)}) = a_i - (k-1)s + \frac{j}{2}s.$$

(III) k is odd, let  $f(x_{i(k-1)}x_{ik}) = a_i - \frac{k-1}{2}s$  and for even  $j \in [0, k-3]$ , let

$$f(x_{ij}x_{i(j+1)}) = a_i - (k-1)s + \frac{j}{2}s, \quad f(x_{ij+1}x_{i(j+2)}) = a_i - (k-1)s + \left(\frac{k+1}{2} + \frac{j}{2}\right)s.$$

We define a bijection  $\tau_3 : E(T) \setminus E(P) \to [p+1,m]$  as follows: for  $e \in E(T) \setminus E(P)$ , assume  $e \in E(L_i)$  for some  $i \in [1, s]$ . Then

 $\tau_{3}(e) = \begin{cases} f(e) & \text{in (I)}, & \text{if } L_{i} \text{ is a big leg}, \\ f(e) & \text{in (II)}, & \text{if } k \text{ is even and } L_{i} \text{ is a small leg such that } x_{i0} \neq v_{h_{t}}, \\ f(e) & \text{in (III)}, & \text{if } k \text{ is odd and } L_{i} \text{ is a small leg such that } x_{i0} \neq v_{h_{t}}. \end{cases}$ 

Assume now that  $L_i$  is a small leg and  $x_{i0} = v_{h_t}$ . Let

$$\tau_{3}(e) = \begin{cases} f(e) & \text{in (I)}, & \text{if } s_{[\overrightarrow{P},\tau_{1}]}(v_{h_{t}}) \geq m-s, \\ f(e) & \text{in (II)}, & \text{if } k \text{ is even and } s_{[\overrightarrow{P},\tau_{1}]}(v_{h_{t}}) \leq p, \\ f(e) & \text{in (III)}, & \text{if } k \text{ is odd and } s_{[\overrightarrow{P},\tau_{1}]}(v_{h_{t}}) \leq p. \end{cases}$$

Lastly, assume  $L_i$  is a small leg,  $x_{i0} = v_{h_t}$ , and  $p+1 \leq s_{[\vec{P},\tau_1]}(v_{h_t}) \leq m-s-1$ . Assume, without loss of generality that  $x_{i0} = v_{h_t} = x_\ell$  for some  $\ell \in [1,s]$ , where recall  $x_\ell$  is defined

in (4). Since  $s_{[\vec{P},\tau_1]}(v_{h_t}) \ge p+1$  and  $s_{[\vec{P},\tau_1]}(v_{h_t}) \le p$  for  $v_{h_t} \in U \setminus \{v_{h_t}\}$ , and by the definition of  $\tau_2$ , it follows that if  $\ell \ge 2$ , then T has big legs, and if T has no big leg, then  $\ell = 1$ . We define

$$\tau_3(e) = \begin{cases} f(e) & \text{in (I)}, & \text{if } \ell \ge 2, \\ f(e) & \text{in (II)}, & \text{if } k \text{ is even and } \ell = 1, \\ f(e) & \text{in (III)}, & \text{if } k \text{ is odd and } \ell = 1. \end{cases}$$

#### Step 2.3: Modify $\tau_3$ defined in Step 2.2 to avoid same vertex-sums.

For the bijection  $\tau_3$  defined in Step 2.2, it might happen that one vertex-sum of some degree 2 vertices from the legs of T is the same as the vertex-sum of the vertex  $v_{h_t}$ . For this reason, we will modify  $\tau_3$  slightly so that under the modification, the vertex-sum of  $v_{h_t}$  is not the same as those of degree 2 vertices  $x_{ij}$  for all  $i \in [1, s]$  and all  $j \in [1, k - 1]$ .

Recall that  $v_{h_t} = x_{\ell}$ . When  $v_{h_t}$  is a small joint, we assume  $L_q$ , for some  $q \in [1, s]$ , is the leg of T with  $x_{q0} = v_{h_t}$ . If  $\ell \ge 2$  and so  $q \ge 2$ , assume, by relabeling the legs if necessary, that  $L_{q-1}$  is the leg of T with  $x_{(q-1)0} = x_{\ell-1}$  and  $x_{(q-1)0}x_{(q-1)1} \in M$ . Thus by the definition of  $\tau_2$  in Step 2.1, we have

$$\tau_2(x_{(q-1)0}x_{(q-1)1}) = m - (s-t) + 1 - (\ell - 1)$$
 and  $\tau_2(x_{q0}x_{q1}) = m - (s-t) + 1 - \ell.$ 

If  $\ell = 1$ , then s = t and q = 1 by the way of naming the legs of T. Thus  $L_{q+1} = L_2$  is the leg of T with  $x_{(q+1)0} = x_{\ell+1}$  and  $x_{(q+1)0}x_{(q+1)1} \in M$ . Thus by the definition of  $\tau_2$  in Step 2.1, we have

$$\tau_2(x_{q0}x_{q1}) = m$$
 and  $\tau_2(x_{(q+1)0}x_{(q+1)1}) = m - 1.$ 

We define  $\tau_4 : E(T) \setminus E(P) \to [p+1,m]$  by modifying  $\tau_3$  as below. If  $v_{h_t}$  is a big joint, we let  $\tau_4 = \tau_3$ . Thus we assume  $v_{h_t}$  is a small joint. If  $s_{[\vec{P},\tau_1]}(v_{h_t}) \leq p$  or  $s_{[\vec{P},\tau_1]}(v_{h_t}) \geq m-s$ , we let  $\tau_4 = \tau_3$ . Thus we assume  $v_{h_t}$  is a small joint and  $p+1 \leq s_{[\vec{P},\tau_1]}(v_{h_t}) \leq m-s-1$ . Under this assumption, we modify  $\tau_3$  in two different subcases.

- (a) If (a): s is even and  $s_{[\vec{P},\tau_1]}(v_{h_t}) + m (s-t) + 1 \ell$  is odd, (b): s is odd, k is even and  $s_{[\vec{P},\tau_1]}(v_{h_t}) + m (s-t) + 1 \ell$  has a different parity than k/2 1, or (c): s is odd, k is odd and  $s_{[\vec{P},\tau_1]}(v_{h_t}) + m (s-t) + 1 \ell$  has a different parity than (k-1)/2, then we let  $\tau_4 = \tau_3$ .
- (b) In all other cases we let  $\tau_4$  be obtained from  $\tau_3$  by switching the labels on the leg  $L_{q-1}$  with that of  $L_q$  if  $\ell \geq 2$ , and by switching the labels on the leg  $L_q$  with that of  $L_{q+1}$  if  $\ell = 1$ .

From now on, we will call the leg  $L_q$  big if it was labeled using the pattern f in (I), and small otherwise.

We now let  $\tau : A(\overrightarrow{T}) \to [1, m]$  be the bijection obtained from  $\tau_1$  and  $\tau_4$  by letting  $\tau(e) = \tau_1(e)$  if  $e \in A(\overrightarrow{P})$  and  $\tau(e) = \tau_4(e)$  if  $e \in A(\overrightarrow{T}) \setminus A(\overrightarrow{P})$ . For notation simplicity, we write  $s_{[\overrightarrow{T},\tau]}$  as  $s_{\tau}$  in the rest of the proof. Let  $u, v \in V(\overrightarrow{T})$  be any two distinct vertices. We show  $s_{\tau}(u) \neq s_{\tau}(v)$  in the following 6 cases, which implies that  $\tau$  is an antimagic labeling of  $\overrightarrow{T}$ .

Case 1: both u and v are leaves of T.

Case 2: u is a leaf of T and  $v \in U$ .

Case 3:  $u, v \in U$ .

Case 4: u is a leaf of T and v is a degree 2 vertex of T.

Case 5: both u and v are degree 2 vertices of T.

Case 6: u is a degree 2 vertex of T and  $v \in U$ .

**Case 1**: both u and v are leaves of T.

As  $\tau$  is a bijection and so distinct edges receive distinct labels under  $\tau$ , it is clear that  $s_{\tau}(u) \neq s_{\tau}(v)$ .

**Case 2**: u is a leaf of T and  $v \in U$ .

Note that  $s_{\tau}(v) \geq p+1$ . Thus  $s_{\tau}(u) \neq s_{\tau}(v)$  when  $u \in V(P)$  or k is odd, since  $s_{\tau}(u) \leq p$  or  $s_{\tau}(u) < 0$ . So assume  $u \in V(T) \setminus V(P)$  and k is even. When k is even, then by (I) and (II) of the definition of f, we have  $s_{\tau}(u) \leq m - ks/2s$ . However,  $s_{\tau}(v) \geq s_{[\vec{P},\tau_1]}(v) + (m-s+1) - (k-1)s + ks/2 > m+1 - ks/2 > s_{\tau}(u)$ . Thus again,  $s_{\tau}(u) \neq s_{\tau}(v)$ .

Case 3:  $u, v \in U$ .

Recall  $T_1$  consists of  $\overrightarrow{P}$  and the oriented edges from  $E_1$ . We let  $\pi : A(T_1) \to [1, p] \cup \{\tau_3(x_{i0}x_{i1}) : i \in [1, s]\}$  be the bijection obtained from  $\tau_1$  and  $\tau_3$  by letting  $\pi(e) = \tau_1(e)$  if  $e \in A(\overrightarrow{P})$  and  $\pi(e) = \tau_3(e)$  if  $e \in A(T_1) \setminus A(\overrightarrow{P})$ .

By the definition of  $\tau_3$ , we have  $\tau_3(e) \leq \tau_2(e)$  for each  $e \in E_1$ . Thus by (4), we have

$$s_{[T_1,\pi]}(x_1) > \ldots > s_{[T_1,\pi]}(x_t).$$
 (5)

Recall  $\tau_4$  was obtained by modifying  $\tau_3$  either in terms of  $\tau_4 = \tau_3$  or by exchanging the labels on  $L_q$  with that on  $L_{q-1}$  if  $\ell \geq 2$  and with that on  $L_{q+1}$  if  $\ell = 1$ , where recall  $x_\ell = v_{h_\ell}$ . If  $\tau_4 = \tau_3$ , then for each  $i \in [1, t]$ , we have  $s_\tau(x_i) = s_{[T_1, \pi]}(x_i)$  and thus  $s_\tau(u) \neq s_\tau(v)$  by (5).

Thus we assume  $\tau_4 \neq \tau_3$ , which implies that  $v_{h_t}$  is a small joint and  $p+1 \leq s_{[\vec{P},\tau_1]}(v_{h_t}) \leq m-s-1$  by the definition of  $\tau_4$ . When  $\ell \geq 2$ , we have  $s_{\tau}(x_i) = s_{[T_1,\pi]}(x_i)$  for all  $i \in$ 

 $[1,t] \setminus \{\ell-1,\ell\}, s_{\tau}(x_{\ell-1}) = s_{[T_1,\pi]}(x_{\ell-1}) - 1 \text{ and } s_{\tau}(x_{\ell}) = s_{[T_1,\pi]}(x_{\ell}) + 1.$  Note that  $x_{\ell-1}$  is a big joint and  $x_{\ell} = v_{h_{\ell}}$  is a small joint. Thus

$$s_{[T_1,\pi]}(x_{\ell-1}) \ge s_{[\overrightarrow{P},\tau_1]}(x_{\ell-1}) + m - s + t + 1 + m - (s-t) + 1 - (\ell-1) > 2m + 3 - (s-t) - \ell - s$$
  
and

$$s_{[T_1,\pi]}(x_\ell) = s_{[\overrightarrow{P},\tau_1]}(x_\ell) + m - (s-t) + 1 - \ell \le m - s - 1 + m - (s-t) + 1 - \ell = 2m - (s-t) - \ell - s.$$

Therefore,  $s_{\tau}(x_{\ell-1}) = s_{[T_1,\pi]}(x_{\ell-1}) - 1 > s_{[T_1,\pi]}(x_{\ell}) + 1 = s_{\tau}(x_{\ell})$  and so the strict inequalities in (5) still hold with respect to  $\tau$ . As a consequence  $s_{\tau}(u) \neq s_{\tau}(v)$ .

We then assume  $\ell = 1$ . We have  $s_{\tau}(x_i) = s_{[T_1,\pi]}(x_i)$  for all  $i \in [1,t] \setminus \{1,2\}$ ,  $s_{\tau}(x_1) = s_{[T_1,\pi]}(x_1) - 1$  and  $s_{\tau}(x_2) = s_{[T_1,\pi]}(x_2) + 1$ . In this cases, both  $x_1 = v_{h_t}$  and  $x_2$  are small joints. Thus

$$s_{[T_1,\pi]}(x_1) = s_{[\overrightarrow{P},\tau_1]}(x_1) + m \ge p + 1 + m$$

and

$$s_{[T_1,\pi]}(x_2) = s_{[\overrightarrow{P},\tau_1]}(x_2) + m - 1 \le p - 1 + m - 1$$

by Lemma 3 (i). Therefore,  $s_{\tau}(x_1) = s_{[T_1,\pi]}(x_1) - 1 > s_{[T_1,\pi]}(x_2) + 1 = s_{\tau}(x_2)$  and so the street inequalities in (5) still hold with respect to  $\tau$ . This again gives  $s_{\tau}(u) \neq s_{\tau}(v)$ .

Case 4: u is a leaf of T and v is a degree 2 vertex of T.

If  $u, v \in V(P)$ , then we have  $s_{\tau}(u) \neq s_{\tau}(v)$  by Lemma 3. Thus we assume that  $|V(P) \cap \{u, v\}| \leq 1$ . Note that although  $\tau_4$  is a modification of  $\tau_3$ , the set of vertex-sums induced on the vertices of  $V(T) \setminus V(P)$  are the same under both of them, and the bijection f. We here derive formulas for those vertex-sums. Let  $i \in [1, s]$  and  $L_i$  be a leg. By the definition of f, when k is even, we have

$$\begin{cases} s_{\tau}(x_{ij}) = (-1)^{j} \left( 2a_{i} - \left(\frac{k}{2} + j - 1\right)s \right), \ j \in [1, k - 1], & \text{if } L_{i} \text{ is a big leg,} \\ s_{\tau}(x_{ij}) = (-1)^{j} \left( 2a_{i} - (2(k - 1) - \frac{k}{2} - (j - 1))s \right), \ j \in [1, k - 1], & \text{if } L_{i} \text{ is a small leg.} \\ = (-1)^{j} \left( 2a_{i} - \left(\frac{3k}{2} - 1 - j\right)s \right). \end{cases}$$

$$(6)$$

By the definition of f, when k is odd, we have

$$\begin{cases} s_{\tau}(x_{ij}) = (-1)^{j} \left( 2a_{i} - \left(\frac{k-1}{2} + j\right)s \right), \ j \in [1, k-1], & \text{if } L_{i} \text{ is a big leg,} \\ s_{\tau}(x_{ij}) = (-1)^{j} \left( 2a_{i} - \left(2(k-1) - \frac{k+1}{2} - (j-1)\right)s \right), \ j \in [1, k-1], & \text{if } L_{i} \text{ is a small leg.} \\ = (-1)^{j} \left( 2a_{i} - \left(\frac{3(k-1)}{2} - j\right)s \right). \end{cases}$$

$$(7)$$

Similarly, when k is even, we have

$$\begin{cases} s_{\tau}(x_{ik}) = a_i - (k-1)s, & \text{if } L_i \text{ is a big leg,} \\ s_{\tau}(x_{ik}) = a_i - \frac{k}{2}s, & \text{if } L_i \text{ is a small leg.} \end{cases}$$
(8)

When k is odd, we have

$$\begin{cases} s_{\tau}(x_{ik}) = -(a_i - \frac{k-1}{2}s), & \text{if } L_i \text{ is a big leg,} \\ s_{\tau}(x_{ik}) = -(a_i - \frac{k-1}{2}s), & \text{if } L_i \text{ is a small leg.} \end{cases}$$
(9)

By the formulas above we see that  $|s_{\tau}(v)| \geq 2(m-s+1)-3s(k-1)/2 \geq p+1$  if  $d_T(v) = 2$ and  $v \notin V(P)$ . By the definition of  $\tau_1$  and Lemma 3, we have  $|s_{\tau}(v)| \leq p$  if  $d_T(v) = 2$  and  $v \in V(P)$ . For any leaves u of T, we have  $p+1 \leq |s_{\tau}(u)| \leq m-s(k-1)/2$  if  $u \notin V(P)$ and  $|s_{\tau}(u)| \leq p$  if  $u \in V(P)$ . Thus we have  $s_{\tau}(u) \neq s_{\tau}(v)$  if  $|V(P) \cap \{u, v\}| \leq 1$ . Hence we assume  $u, v \notin V(P)$ . Then

$$|s_{\tau}(v)| - |s_{\tau}(u)| \ge 2(m - s + 1) - 3s(k - 1)/2 - (m - s(k - 1)/2) = m + 2 - s - ks > 0,$$
  
as  $m = p + ks > s + ks$ . Thus  $s_{\tau}(u) \ne s_{\tau}(v)$ .

**Case 5**: both u and v are degree 2 vertices of T.

If  $u, v \in V(P)$ , then we have  $s_{\tau}(u) \neq s_{\tau}(v)$  by Lemma 3. If  $u \in V(P)$  and  $v \notin V(P)$ , then  $|s_{\tau}(u)| \leq p < |s_{\tau}(v)|$ . Thus we assume  $u, v \in V(T) \setminus V(P)$  and  $u = x_{ij} \in V(L_i)$  and  $v = x_{hr} \in V(L_h)$  for some  $i, h \in [1, s]$  and  $j, r \in [1, k - 1]$ . It is clear that  $s_{\tau}(u) \neq s_{\tau}(v)$  if jand r have different parities. Thus we assume  $j \equiv r \pmod{2}$ .

We have three subcases to analysis: both  $L_i$  and  $L_h$  are big legs,  $L_i$  is a big leg and  $L_h$  is a small leg, and both  $L_i$  and  $L_h$  are small legs (recall that the leg  $L_q$  with  $x_{q0} = v_{h_t}$  is called small only if it is labeled using the definition of f in (II) or (III)). When k is even, by (6),

$$|s_{\tau}(x_{ij}) - s_{\tau}(x_{hr})| = \begin{cases} |2a_i - 2a_h + (r - j)s|, & \text{if both } L_i \text{ and } L_h \text{ are big legs,} \\ |2a_i - 2a_h + (k - r - j)s|, & \text{if } L_i \text{ is big and } L_h \text{ is small,} \\ |2a_i - 2a_h + (j - r)s|, & \text{if both } L_i \text{ and } L_h \text{ are small legs.} \end{cases}$$

Since k is even and  $j \equiv r \pmod{2}$ , we know that both |r - j| and |k - r - j| are even. Since u and v are distinct vertices, we know that if i = h, then  $j \neq r$  and if j = r then  $i \neq h$ . If  $L_i$  is big and  $L_h$  is small, then  $a_i \neq a_h$ . These facts together with the fact that  $a_i, a_h \in [m - s + 1, m]$  and so  $|2a_i - 2a_h| \leq 2(s - 1)$ , imply  $|s_\tau(x_{ij}) - s_\tau(x_{hr})| \neq 0$ . Thus  $s_\tau(u) \neq s_\tau(v)$ .

We then assume that k is odd. Then by (7),

$$|s_{\tau}(x_{ij}) - s_{\tau}(x_{hr})| = \begin{cases} |2a_i - 2a_h + (r - j)s|, & \text{if both } L_i \text{ and } L_h \text{ are big legs,} \\ |2a_i - 2a_h + (k - 1 - r - j)s|, & \text{if } L_i \text{ is big and } L_h \text{ is small,} \\ |2a_i - 2a_h + (j - r)s|, & \text{if both } L_i \text{ and } L_h \text{ are small legs.} \end{cases}$$

Since k - 1 is even and  $j \equiv r \pmod{2}$ , we know that both |r - j| and |k - 1 - r - j| are even. Since u and v are distinct vertices, we know that if i = h, then  $j \neq r$  and if j = r then

 $i \neq h$ . If  $L_i$  is big and  $L_h$  is small, then  $a_i \neq a_h$ . These facts together with the fact that  $a_i, a_h \in [m - s + 1, m]$  and so  $|2a_i - 2a_h| \leq 2(s - 1)$ , imply  $|s_\tau(x_{ij}) - s_\tau(x_{hr})| \neq 0$ . Thus  $s_\tau(u) \neq s_\tau(v)$ .

**Case 6**: u is a degree 2 vertex of T and  $v \in U$ .

Since  $s_{\tau}(v) \ge p+1$ , it follows that if  $u \in V(P)$  or  $s_{\tau}(u) < 0$ , then  $s_{\tau}(u) \ne s_{\tau}(v)$ . Thus we assume  $u \in V(T) \setminus V(P)$  and  $s_{\tau}(u) > 0$ , where  $s_{\tau}(u) > 0$  in particular, implies  $k \ge 3$ .

By (6) and (7), when  $s_{\tau}(u) > 0$  and u is contained in a big leg  $L_i$  for some  $i \in [1, s]$ , we have

$$2a_i - \left\lceil \frac{k+2}{2} \right\rceil s \ge s_\tau(u) \ge \begin{cases} 2a_i - \frac{3(k-1)}{2}s, & \text{if } k \text{ is odd,} \\ 2a_i - \frac{3(k-2)}{2}s, & \text{if } k \text{ is even.} \end{cases}$$

When  $s_{\tau}(u) > 0$  and u is contained in a small leg  $L_i$  for some  $i \in [1, s]$ , we have

$$2a_i - \frac{3(k-2)}{2}s \le s_\tau(u) \le 2a_i - \frac{k-1}{2}s.$$

Let  $e_v \in M$  be the edge incident with v, where recall M is the matching defined in Step 2.1. Then by the definition of  $\tau_2$ , we have

$$s_{\tau}(v) > m - s + t + 1 + \tau_2(e_v).$$

As  $\tau_2(e_v) \ge m - s + 1$  and  $k \ge 3$ , we have  $s_\tau(v) > 2m - 2s + 3 > 2m - \lceil (k+2)/2 \rceil s \ge s_\tau(u)$ when u is contained in a big leg. When u is contained in a small leg  $L_i$ , by the definition of  $\tau_2$ , we have  $a_i < \tau_2(e_v) < m - s + t + 1$ . Thus  $s_\tau(v) > m - s + t + 1 + a_i > 2a_i - (k-1)s/2 \ge s_\tau(u)$ . Again, this gives  $s_\tau(u) \ne s_\tau(v)$ . When  $v = v_{h_t}$  is a small joint but  $s_{[\overrightarrow{P},\tau_1]}(v) \ge m - s$ , we have  $s_\tau(v) \ge m - s + \tau_2(e_v)$ . As  $\tau_2(e_v) \ge m - s + 1$  and  $k \ge 3$ , we have  $s_\tau(v) \ge 2m - 2s + 1 > 2m - 2s \ge s_\tau(u)$  when u is contained in a big leg. When u is contained in a small leg  $L_i$ , by the definition of  $\tau_2$ , we have  $a_i < \tau_2(e_v)$  and  $a_i - s < m - s$  (note that  $a_i \ne \tau_2(e_v)$  as  $L_q$  is labeled using the definition of f in (I) when  $s_{[\overrightarrow{P},\tau_1]}(v) \ge m - s$  and so is treated as a big leg). Thus  $s_\tau(u) \le a_i + a_i - s < \tau_2(e_v) + m - s$  and so  $s_\tau(u) \ne s_\tau(v)$ .

When v is a small joint with  $s_{[\vec{P},\tau_1]}(v) \leq p$ , by the definition of f in (II) and (III), we have  $s_{\tau}(v) \leq p + m - (k-2)s/2$  if k is even and  $s_{\tau}(v) \leq p + m - (k-1)s$  when k is odd. On the other hand, as m = p + ks, we get  $s_{\tau}(u) \geq 2(m - s + 1) - (3k - 6)s/2 =$  $m + p + 2 - (k - 2)s/2 > s_{\tau}(v)$  if  $s_{\tau}(u) > 0$  and u is contained in a small leg or k is even. Thus  $s_{\tau}(u) \neq s_{\tau}(v)$  if u is contained in a small leg or k is even. So we assume  $s_{\tau}(u) > 0$ , u is contained in a big leg and k is odd. Then as  $(k + 3)s/2 \leq ks$  (recall  $k \geq 3$ ), we have  $s_{\tau}(u) \geq 2(m - s + 1) - (3k - 3)s/2 \geq m + p + 2 - (k - 1)s > s_{\tau}(v)$ . Thus  $s_{\tau}(u) \neq s_{\tau}(v)$ .

Lastly, we assume v is a small joint with  $s_{[\vec{P},\tau_1]}(v) \ge p+1$ . By Lemma 3, we conclude that  $v = v_{h_t}$ . Also by the argument when  $v = v_{h_t}$  is a small joint with  $s_{[\vec{P},\tau_1]}(v) \ge m-s$ , we

assume here that  $s_{[\vec{P},\tau_1]}(v) \leq m-s-1$ . By the definition of  $\tau_4$  in Step 2.3 and (6) and (7), we know that  $s_{\tau}(u)$  and  $s_{\tau}(v)$  have different parities and so  $s_{\tau}(u) \neq s_{\tau}(v)$ .

The proof is now finished.

# 4 Open problem

In this paper, it is shown that every subdivided caterpillar admits an antimagic orientation. Since every bipartite antimagic graph G admits an antimagic orientation, it is natural to ask that whether subdivided caterpillars are antimagic. We propose the following conjecture.

**Conjecture 5.** Every subdivided caterpillar is antimagic.

# References

- N. Alon, G. Kaplan, A. Lev, Y. Roditty, and R. Yuster. Dense graphs are antimagic. J. Graph Theory, 47(4):297–309, 2004.
- [2] K. Bérczi, A. Bernáth, and M. Vizer. Regular graphs are antimagic. *Electron. J. Combin.*, 22(3):Paper 3.34, 6, 2015.
- [3] F. Chang, Y. Liang, Z. Pan, and X. Zhu. Antimagic labeling of regular graphs. J. Graph Theory, 82(4):339–349, 2016.
- [4] T. Eccles. Graphs of large linear size are antimagic. J. Graph Theory, 81(3):236-261, 2016.
- [5] Y. Gao and S. Shan. Antimagic orientation of lobsters. Discrete Appl. Math., 287:21–26, 2020.
- [6] N. Hartsfield and G. Ringel. *Pearls in graph theory*. Academic Press, Inc., Boston, MA, 1990. A comprehensive introduction.
- [7] D. Hefetz, T. Mütze, and J. Schwartz. On antimagic directed graphs. J. Graph Theory, 64(3):219–232, 2010.
- [8] G. Kaplan, A. Lev, and Y. Roditty. On zero-sum partitions and anti-magic trees. Discrete Math., 309(8):2010–2014, 2009.
- [9] T. Li, Z. Song, G. Wang, D. Yang, and C. Zhang. Antimagic orientations of even regular graphs. J. Graph Theory, 90(1):46–53, 2019.

- [10] Y. Liang, T. Wong, and X. Zhu. Anti-magic labeling of trees. Discrete Math., 331:9–14, 2014.
- [11] A. Lozano, M. Mora, C. Seara, and J. Tey. Caterpillars are antimagic. Mediterr. J. Math., 18(2):Paper No. 39, 2021.
- [12] S. Shan. Antimagic orientation of graphs with minimum degree at least 33. arXiv:2005.01829, 2020.
- [13] S. Shan and X. Yu. Antimagic orientation of biregular bipartite graphs. *Electron. J. Combin.*, 24(4):#P4.31, 2017.
- [14] C. Song and R. Hao. Antimagic orientations for the complete k-ary trees. J. Comb. Optim., 38(4):1077–1085, 2019.
- [15] C. Song and R. Hao. Antimagic orientations of disconnected even regular graphs. Discrete Math., 342(8):2350–2355, 2019.
- [16] D. Yang. A note on antimagic orientations of even regular graphs. Discrete Appl. Math., 267:224–228, 2019.
- [17] D. Yang, J. Carlson, A. Owens, K. E. Perry, I. Singgih, Z. Song, F. Zhang, and X. Zhang. Antimagic orientations of graphs with large maximum degree. *Discrete Math.*, 343(12), 2020.
- [18] X. Yu, Y. Chang, and S. Zhou. Antimagic orientation of Halin graphs. Discrete Math., 342(11):3160–3165, 2019.