

Rainbow triangles in arc-colored digraphs ^{*}

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October 16, 2018

Abstract

Let D be an arc-colored digraph. The arc number $a(D)$ of D is defined as the number of arcs of D . The color number $c(D)$ of D is defined as the number of colors assigned to the arcs of D . A rainbow triangle in D is a directed triangle in which every pair of arcs have distinct colors. Let $f(D)$ be the smallest integer such that if $c(D) \geq f(D)$, then D contains a rainbow triangle. In this paper we obtain $f(\overleftrightarrow{K}_n)$ and $f(T_n)$, where \overleftrightarrow{K}_n is a complete digraph of order n and T_n is a strongly connected tournament of order n . Moreover we characterize the arc-colored complete digraph \overleftrightarrow{K}_n with $c(\overleftrightarrow{K}_n) = f(\overleftrightarrow{K}_n) - 1$ and containing no rainbow triangles. We also prove that an arc-colored digraph D on n vertices contains a rainbow triangle when $a(D) + c(D) \geq a(\overleftrightarrow{K}_n) + f(\overleftrightarrow{K}_n)$, which is a directed extension of the undirected case.

Keywords: arc-colored digraph, rainbow triangle, color number, complete digraph, strongly connected tournament

1 Introduction

In this paper we only consider finite digraphs without loops or multiple arcs. For terminology and notations not defined here, we refer the readers to [2] and [3].

^{*}The first author is supported by GXNSF (Nos. 2016GXNSFFA38001 and 2018GXNSFAA138152) and Program on the High Level Innovation Team and Outstanding Scholars in Universities of Guangxi Province; the second author is supported by NSFC (Nos. 11571135 and 11671320) and the third author is supported by the Fundamental Research Funds for the Central Universities (No. 31020180QD124).

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Let $D = (V, A)$ be a digraph. We use $a(D)$ to denote the number of arcs of D . If $uv \in A(D)$, then we say that u *dominates* v (or v is *dominated* by u) and uv is an *in-arc* of v (or uv is an *out-arc* of u). For a vertex v of D , the *in-neighborhood* $N_D^-(v)$ of v is the set of vertices dominating v , and the *out-neighborhood* $N_D^+(v)$ of v is the set of vertices dominated by v . The *in-degree* $d_D^-(v)$ and *out-degree* $d_D^+(v)$ of v are defined as the cardinality of $N_D^-(v)$ and $N_D^+(v)$, respectively. The *degree* $d_D(v)$ of v is defined as the sum of $d_D^-(v)$ and $d_D^+(v)$. A *complete digraph* is a digraph obtained from a complete graph K_n by replacing each edge xy of K_n with a pair of arcs xy and yx , denoted by \overleftrightarrow{K}_n . A *complete bipartite digraph* is a digraph obtained from a complete bipartite graph $K_{m,n}$ by replacing each edge xy of $K_{m,n}$ with a pair of arcs xy and yx , denoted by $\overleftrightarrow{K}_{m,n}$. A *tournament* is a digraph obtained from a complete graph K_n by replacing each edge xy of K_n with exactly one of the arcs xy and yx . A digraph D is *strongly connected* if, for each pair of distinct vertices x and y in D , there exists an (x, y) -path. The subdigraph of D induced by $S \subseteq V(D)$ is denoted by $D[S]$. An *arc-coloring* of D is a mapping $C : A(D) \rightarrow \mathbb{N}$, where \mathbb{N} is the set of natural numbers. We call D an *arc-colored digraph* if it is assigned such an arc-coloring C . We use $C(D)$ and $c(D)$ (called the *color number* of D) to denote the set and the number of colors assigned to the arcs of D , respectively. If $c(D) = k$, then we call D a *k-arc-colored digraph*. Let D be an arc-colored digraph and i a color in $C(D)$. We use D^i to denote the arc-colored subdigraph of D induced by all the arcs of color i . For a vertex $v \in D$, we use $CN_D^-(v)$ and $CN_D^+(v)$ to denote the set of colors assigned to the in-arcs and the out-arcs of v , respectively. The *color neighbor* $CN_D(v)$ of v is defined as $CN_D(v) = CN_D^-(v) \cup CN_D^+(v)$. The *in-color degree* $d_D^{-c}(v)$ and the *out-color degree* $d_D^{+c}(v)$ of v are the cardinality of $CN_D^-(v)$ and $CN_D^+(v)$, respectively. If there is no ambiguity, we often omit the subscript D in the above notations. A *rainbow digraph* is a digraph in which every pair of arcs have distinct colors. A *rainbow triangle* is a directed triangle which is rainbow.

The existence of rainbow subgraphs has been widely studied, see the survey papers [7, 11]. In particular, the existence of rainbow triangles attracts much attention during the past decades. For an edge-colored complete graph K_n , Gallai [8] characterized the coloring structure of K_n containing no rainbow triangles. Gyarfas and Simonyi [9] showed that each edge-colored K_n with $\Delta^{mon}(K_n) < \frac{2n}{5}$ contains a rainbow triangle and this bound is tight. Fujita et al. [6] proved that each edge-colored K_n with $\delta^c(K_n) > \log_2 n$ contains a rainbow triangle and this bound is tight. For a general edge-colored graph G of order n , Li and Wang [14] proved that if $\delta^c(G) \geq \frac{\sqrt{7}+1}{6}n$, then G contains a rainbow triangle.

Li [13] and Li et al. [12] improved the condition to $\delta^c(G) > \frac{n}{2}$ independently, and showed that this bound is tight. Li et al. [15] further proved that if G is an edge-colored graph of order n satisfying $d^c(u) + d^c(v) \geq n + 1$ for every edge $uv \in E(G)$, then it contains a rainbow triangle. In [16], Li et al. gave some maximum monochromatic degree conditions for an arc-colored strongly connected tournament T_n to contain rainbow triangles, and to contain rainbow triangles passing through a given vertex. For more results on rainbow cycles, see [1, 4, 5, 10].

In this paper, we mainly study the existence of rainbow triangles in arc-colored digraphs. Let D be an arc-colored digraph on n vertices. Sridharan [18] proved that the maximum number of arcs among all digraphs of order n with no directed triangles is $\lfloor \frac{n^2}{2} \rfloor$. Thus D contains a rainbow triangle if $c(D) \geq \lfloor \frac{n^2}{2} \rfloor + 1$. This lower bound is sharp by considering the complete bipartite digraph $\overleftrightarrow{K}_{\lfloor \frac{n}{2}, \lceil \frac{n}{2} \rceil}$ with arcs assigned pairwise distinct colors.

For an edge-colored graph G , we use $e(G)$ and $c(G)$ to denote the number of edges of G and the number of colors assigned to the edges of G , respectively. Let $f(G)$ be the smallest integer such that if $c(G) \geq f(G)$, then G contains a rainbow triangle. In [9], the authors proved that $f(K_n) = n$. Li et al. [12] proved that if $e(G) + c(G) \geq \frac{n(n+1)}{2}$, then G contains a rainbow triangle. Note that $\frac{n(n+1)}{2} = \frac{n(n-1)}{2} + n = e(K_n) + f(K_n)$. Motivated by this result, we wonder whether an arc-colored digraph D on n vertices contains a rainbow triangle when

$$a(D) + c(D) \geq a(\overleftrightarrow{K}_n) + f(\overleftrightarrow{K}_n).$$

First we calculate $f(\overleftrightarrow{K}_n)$ for $n \geq 3$.

Theorem 1. *Let \overleftrightarrow{K}_n be an arc-colored complete digraph of order $n \geq 3$ and $f(\overleftrightarrow{K}_n)$ be the smallest integer such that \overleftrightarrow{K}_n with $c(\overleftrightarrow{K}_n) \geq f(\overleftrightarrow{K}_n)$ contains a rainbow triangle. Then*

$$f(\overleftrightarrow{K}_n) = \begin{cases} \lfloor \frac{n^2}{4} \rfloor + 3, & n = 3, 4; \\ \lfloor \frac{n^2}{4} \rfloor + 2, & n \geq 5. \end{cases}$$

We also investigate the structure of the arc-colored complete digraphs \overleftrightarrow{K}_n with $c(\overleftrightarrow{K}_n) = f(\overleftrightarrow{K}_n) - 1$ and containing no rainbow triangles.

Theorem 2. *Let \mathcal{G}_n be the class of arc-colored complete digraphs of order n such that for each $D \in \mathcal{G}_n$, $c(D) = f(D) - 1$ and D contains no rainbow triangles. Then each D in \mathcal{G}_3 can be decomposed into two arc-disjoint 2-arc-colored triangles Δ_1 and Δ_2 such that $C(\Delta_1) \cap C(\Delta_2) = \emptyset$. For each D in \mathcal{G}_4 , there exists a permutation of the vertex set of D ,*

say $v_1v_2v_3v_4$, such that

$$\begin{cases} C(v_1v_2) = C(v_2v_3) = C(v_3v_4) = C(v_4v_1) = a, \\ C(v_1v_4) = C(v_4v_3) = C(v_3v_2) = C(v_2v_1) = b, \\ C(v_1v_3) = c, \quad C(v_3v_1) = d, \\ C(v_2v_4) = e, \quad C(v_4v_2) = f, \end{cases}$$

where a, b, c, d, e, f are pairwise distinct colors.

Each D in \mathcal{G}_5 belongs to one of the following three types of digraphs:

- *Type I:* There is a vertex $v \in V(D)$ such that all arcs incident to v are colored by a same color c , $D - v \in \mathcal{G}_4$ and $c \notin C(D - v)$;
- *Type II:* The vertex set of D can be partitioned into two subsets $\{a_1, a_2\}$ and $\{b_1, b_2, b_3\}$ such that the spanning subdigraph H of D with $A(H) = \{a_i b_j | i = 1, 2; j = 1, 2, 3\}$ (or $A(H) = \{b_j a_i | i = 1, 2; j = 1, 2, 3\}$) is rainbow and all arcs in $A(D) \setminus A(H)$ are colored by a same new color;
- *Type III:* The vertex set of D can be partitioned into two subsets $\{a_1, a_2\}$ and $\{b_1, b_2, b_3\}$ such that $C(D[\{a_1, a_2\}]) = \{a, b\}$, $D[\{b_1, b_2, b_3\}] \in \mathcal{G}_3$, $C(D[\{b_1, b_2, b_3\}]) = \{c, d, e, f\}$ and all arcs between $\{a_1, a_2\}$ and $\{b_1, b_2, b_3\}$ are colored by g , where a, b, c, d, e, f, g are pairwise distinct colors.

For each $D \in \mathcal{G}_n$, $n \geq 6$, the vertex set of D can be partitioned into two subsets $\{a_1, a_2, \dots, a_{\lfloor \frac{n}{2} \rfloor}\}$ and $\{b_1, b_2, \dots, b_{\lceil \frac{n}{2} \rceil}\}$ such that the spanning subdigraph H of D with

$$A(H) = \{a_i b_j | i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n}{2} \rceil\}$$

or

$$A(H) = \{b_j a_i | i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n}{2} \rceil\}$$

is rainbow and all arcs in $A(D) \setminus A(H)$ are colored by a same new color.

Furthermore, we study the " $a(D) + c(D)$ " condition for the existence of rainbow triangles in arc-colored digraphs (not necessarily complete).

Theorem 3. *Let D be an arc-colored digraph on n vertices. If*

$$a(D) + c(D) \geq \begin{cases} n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 3, & n = 3, 4; \\ n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 2, & n \geq 5, \end{cases}$$

then D contains a rainbow triangle.

Remark 1. By the definition of $f(\overleftrightarrow{K}_n)$ and Theorem 1, we can see that the bound of $a(D) + c(D)$ in Theorem 3 is sharp.

Finally, we give a color number condition for the existence of rainbow triangles in strongly connected tournaments.

Theorem 4. *Let D be an arc-colored strongly connected tournament on n vertices. If $c(D) \geq \frac{n(n-1)}{2} - n + 3$, then D contains a rainbow triangle.*

Remark 2. The bound of $c(D)$ in Theorem 4 is sharp. Let D be a digraph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and arc set $A = (\{v_i v_j | 1 \leq i < j \leq n\} \setminus \{v_1 v_n\}) \cup \{v_n v_1\}$. Then D is a strongly connected tournament. Color all the arcs incident to v_1 by a same color and color the remaining arcs by pairwise distinct new colors. Then $c(D) = \frac{n(n-1)}{2} - (n-1) + 1 = \frac{n(n-1)}{2} - n + 2$. But there is no rainbow triangle in D .

2 Proofs of the theorems

Let v be a vertex in D , and c a color in $C(D)$. If all the arcs with color c are incident to v , then we call c a color *saturated* by v . We use $C^s(v)$ to denote the set of colors saturated by v and define $d^s(v) = |C^s(v)|$. If a color in $C(D)$ is not saturated by v , then it is also a color in $C(D - v)$. This implies that $c(D - v) = c(D) - d^s(v)$.

Observation 1. *Let D be an arc-colored complete digraph. For a vertex $v \in D$, if there are two vertices $u \neq w$ such that $C(uv) \neq C(vw)$ and $C(uv), C(vw) \in C^s(v)$, then uvw is a rainbow triangle.*

Proof. Since the arc wu is not incident to v , we have $C(wu) \notin C^s(v)$. Namely, $C(uv)$, $C(vw)$ and $C(wu)$ are pairwise distinct colors. Thus, uvw is a rainbow triangle. \square

Before presenting the proof of Theorem 1, we first prove the following lemmas.

Lemma 1. *Let D be an arc-colored digraph of order $n \geq 4$ without rainbow triangles. For a vertex $v \in D$, if $D - v \cong \overleftrightarrow{K}_{n-1}$ and $d^s(v) \geq 3$, then $CN^-(v) \cap C^s(v) = \emptyset$ or $CN^+(v) \cap C^s(v) = \emptyset$. Moreover, if $D \cong \overleftrightarrow{K}_4$, then $c(D) \leq 5$.*

Proof. Let $C^s(v) = \{1, 2, \dots, k\}$, $k \geq 3$. If $CN^+(v) \cap C^s(v) \neq \emptyset$, without loss of generality, assume that $C(vw) = 1$. We will show that $CN^-(v) \cap C^s(v) = \emptyset$. By contradiction, assume that there is a vertex $u \neq w$ such that $C(uv) \in C^s(v) \setminus \{1\}$, then uvw is a rainbow triangle, a contradiction. Suppose that $C(uv) \in C^s(v) \setminus \{1\}$, such as $C(uv) = 2$. Since

$d^s(v) \geq 3$, there is an arc colored by 3 incident to v . By the above argument, this arc must be an out-arc of v , so we can assume that $C(vu) = 3$. But now $wvuw$ is a rainbow triangle, a contradiction. Thus $CN^-(v) \cap (C^s(v) \setminus \{1\}) = \emptyset$. Namely, $2 \in CN^+(v)$, by similar analysis we have $CN^-(v) \cap (C^s(v) \setminus \{2\}) = \emptyset$. Hence, we have $CN^-(v) \cap C^s(v) = \emptyset$.

If $n = 4$, then assume that $V(D) = \{v, x, y, z\}$, $\{1, 2, 3\} \subseteq C^s(v)$, $C(vx) = 1$, $C(vy) = 2$, $C(vz) = 3$, $D[\{x, y, z\}]$ is a \overleftrightarrow{K}_3 , $C(xv) = a$, $C(yv) = b$ and $C(zv) = c$. Since D contains no rainbow triangles, we have $C(yx) = C(zx) = a$, $C(xy) = C(zy) = b$ and $C(xz) = C(yz) = c$. So $C(D) = \{1, 2, 3\} \cup \{a\} \cup \{b\} \cup \{c\}$. If a, b, c are pairwise distinct, then $xyzx$ is a rainbow triangle, a contradiction. So two of a, b and c must be a same color. Then $c(D) \leq 5$. \square

Lemma 2. *Let D be an arc-colored complete digraph of order $n \geq 4$. If D contains no rainbow triangles, then there must be a vertex $v \in V(D)$ such that $d^s(v) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Suppose for every vertex $v \in V(D)$, we have $d^s(v) \geq \lfloor \frac{n}{2} \rfloor + 1$. Let v be a vertex of D . Since $n \geq 4$, we have $d^s(v) \geq 3$. By Lemma 1, either $CN^-(v) \cap C^s(v) = \emptyset$ or $CN^+(v) \cap C^s(v) = \emptyset$. Without loss of generality, suppose $C^s(v) = \{1, 2, \dots, k\}$, $k \geq \lfloor \frac{n}{2} \rfloor + 1$ and $C(vw_i) = i$, for $i = 1, \dots, k$. For $j \neq 1$, since D contains no rainbow triangles, $C(w_1w_j) \neq 1$ and $C(w_jv) \neq 1$, we have $C(w_1w_j) = C(w_jv)$. Thus, $C(w_1w_j) \notin C^s(w_1)$ for $j = 2, \dots, k$. Similarly, for $j \neq 1$, since D contains no rainbow triangles, $C(w_jw_1) \neq j$ and $C(w_1v) \neq j$, we have $C(w_jw_1) = C(w_1v)$. Since $C(w_jw_1) \in CN^-(w_1) \cap CN^+(w_1)$, we can see that $C(w_jw_1) \notin C^s(w_1)$ for $j = 2, \dots, k$. So, all colors assigned to the arcs between w_1 and $\{w_2, \dots, w_k\}$ do not belong to $C^s(w_1)$. Note that for a pair of arcs uw_1 and w_1u , at most one of them has a color in $C^s(w_1)$. So,

$$|V(D - \{w_1, \dots, w_k\})| \geq d^s(w_1) \geq \lfloor \frac{n}{2} \rfloor + 1.$$

But now

$$|V(D)| = n \geq \lfloor \frac{n}{2} \rfloor + 1 + \lfloor \frac{n}{2} \rfloor + 1 \geq n + 1,$$

a contradiction. \square

Now we can give the proof of Theorem 1.

Proof of Theorem 1. We divide the proof into four cases.

Case 1. $n = 3$.

If $n = 3$, then we have $a(D) = 6$. If $c(D) \geq \lfloor \frac{n^2}{4} \rfloor + 3 = 5$, then at most two arcs have a same color, other arcs all have pairwise distinct new colors. Since there are

two arc-disjoint triangles in D , at least one of them is rainbow. Let $V(D) = \{u, v, w\}$, $C(uv) = C(vw) = 1$, $C(wu) = 2$, $C(vu) = C(uw) = 3$ and $C(wv) = 4$. Then $c(D) = 4$ and neither of two triangles are rainbow. So we have $f(\overleftrightarrow{K}_3) = \lfloor \frac{n^2}{4} \rfloor + 3 = 5$.

Case 2. $n = 4$.

For $n = 4$, if $c(D) \geq \lfloor \frac{n^2}{4} \rfloor + 3 = 7$ but D contains no rainbow triangles, then for every vertex $v \in V(D)$, the complete digraph $D - v$ contains no rainbow triangles either. Since $f(\overleftrightarrow{K}_3) = 5$, we have $c(D - v) = c(D) - d^s(v) \leq 4$. So for every vertex $v \in V(D)$, we have $d^s(v) \geq 3$. By Lemma 1, we have $c(D) \leq 5$, a contradiction.

Let $V(D) = \{v, x, y, z\}$, $C(vx) = C(xy) = C(yz) = C(zv) = 1$, $C(vz) = C(zy) = C(yx) = C(xv) = 2$, $C(vy) = 3$, $C(yv) = 4$, $C(xz) = 5$ and $C(zx) = 6$. Then $c(D) = 6$ and D contains no rainbow triangles. So we have $f(\overleftrightarrow{K}_4) = \lfloor \frac{n^2}{4} \rfloor + 3 = 7$.

Claim 1. Let D be an arc-colored \overleftrightarrow{K}_4 without rainbow triangles. If $c(D) = 6$, then there must be a permutation of the vertex set of D , say $v_1v_2v_3v_4$, such that

$$\begin{cases} C(v_1v_2) = C(v_2v_3) = C(v_3v_4) = C(v_4v_1) = a, \\ C(v_1v_4) = C(v_4v_3) = C(v_3v_2) = C(v_2v_1) = b, \\ C(v_1v_3) = c, \quad C(v_3v_1) = d, \\ C(v_2v_4) = e, \quad C(v_4v_2) = f, \end{cases}$$

where a, b, c, d, e, f are pairwise distinct colors.

Proof. Since $f(\overleftrightarrow{K}_3) = 5$, $c(D) = 6$ and D contains no rainbow triangles, we have $d^s(v) \geq 2$ for each vertex $v \in V(D)$. If there is a vertex $v \in V(D)$ such that $d^s(v) \geq 3$, then by Lemma 1, we have $c(D) \leq 5$, a contradiction. So we have $d^s(v) = 2$ for every vertex $v \in V(D)$. Thus for each $v \in V(D)$, $D - v$ belongs to \mathcal{G}_3 .

By the structure of \mathcal{G}_3 , we know that for each color i the arc-colored digraph D^i must be connected (otherwise, we recolor a component of D^i by a new color, then the obtained arc-colored complete digraph has $f(\overleftrightarrow{K}_4)$ colors but contains no rainbow triangles, a contradiction) and belong to one of the following four types.

Type 1: an arc;

Type 2: a directed path of length 2;

Type 3: a directed path of length 3;

Type 4: a directed cycle of length 4.

Let $X_j = \{i \in C(D) : D^i \text{ belongs to Type } j\}$ and $x_j = |X_j|$ for $j = 1, 2, 3, 4$. Then

we have

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = c(D) \\ x_1 + 2x_2 + 3x_3 + 4x_4 = a(D) \\ x_2 + 2x_3 + 4x_4 = 2\binom{4}{3} \quad (\text{the number of directed triangles in } D) \\ x_j \in \mathbb{N} \text{ for } j = 1, 2, 3, 4. \end{cases}$$

By these equations, we get $x_1 = 4, x_2 = x_3 = 0$ and $x_4 = 2$. Without loss of generality, let $X_1 = \{1, 2, 3, 4\}$, $X_4 = \{5, 6\}$ and let $uxyzu$ be the directed cycle of length 4 colored by 5. If $C(xu) \in X_1$, then $C(yx), C(uz) \notin X_1$ (otherwise, $yxuy$ or $xuzx$ is a rainbow triangle). This forces $C(yx) = C(uz) = 6$. Note that D^6 is a directed cycle of length 4. We have $C(xu) = 6 \in X_4$. This contradicts to the assumption that $C(xu) \in X_1$. Thus $C(xu) \notin X_1$. This forces $C(xu) = 6$. By the symmetry of the cycle $uxyzu$, we get $C(xu) = C(uz) = C(zy) = C(yx) = 6$. For each color $i = 1, 2, 3, 4$, D^i is an arc. \square

Let D be an arc-colored complete digraph of order $n \geq 5$ with vertex set $\{v_1, v_2, \dots, v_n\}$.

Let

$$R = \{v_{2i-1}v_{2j} \mid i = 1, 2, \dots, \lceil \frac{n}{2} \rceil, j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}.$$

Color the arcs in R with pairwise distinct colors and color the remaining arcs with a same new color. Then $c(D) = \lfloor \frac{n^2}{4} \rfloor + 1$ and D contains no rainbow triangles. So $f(\overleftrightarrow{K}_n) \geq \lfloor \frac{n^2}{4} \rfloor + 2$ for $n \geq 5$.

Case 3. $n = 5$.

For $n = 5$, if $c(D) \geq \lfloor \frac{n^2}{4} \rfloor + 2 = 8$ but D contains no rainbow triangles, then for every vertex $v \in V(D)$, the complete digraph $D - v$ contains no rainbow triangles either. Since $f(\overleftrightarrow{K}_4) = 7$, we have $c(D - v) = c(D) - d^s(v) \leq 6$. So for every vertex $v \in V(D)$, we have $d^s(v) \geq 2$. On the other hand, by Lemma 2, there must be a vertex $v \in V(D)$ such that $d^s(v) \leq \lfloor \frac{n}{2} \rfloor = 2$. So there exists a vertex $v \in V(D)$ such that $d^s(v) = 2$. Let $D' = D - v$, then D' is an arc-colored \overleftrightarrow{K}_4 without rainbow triangles and $c(D') = 6$. By Claim 1, we can assume that $V(D') = \{u, x, y, z\}$ and

$$\begin{cases} C(ux) = C(xy) = C(yz) = C(zu) = 5, \\ C(uz) = C(zy) = C(yx) = C(xu) = 6, \\ C(uy) = 1, \quad C(yu) = 2, \\ C(xz) = 3, \quad C(zx) = 4. \end{cases}$$

Let $C^s(v) = \{7, 8\}$. Without loss of generality, we can assume that $C(vu) = 7$. Considering the triangle $vuxv$, we have $C(xv) \neq 8$. If $C(vx) = 8$, then considering the triangles $vuzv$ and $vxzv$, we have $C(zv) \in \{6, 7\} \cap \{3, 8\}$, a contradiction. So $C(vx) \neq 8$. Similarly, we have

$$8 \notin \{C(vy)\} \cup \{C(yv)\} \cup \{C(vz)\} \cup \{C(zv)\}.$$

So $C(uv) = 8$. By similar analysis, we have

$$7 \notin \{C(vx)\} \cup \{C(xv)\} \cup \{C(vy)\} \cup \{C(yv)\} \cup \{C(vz)\} \cup \{C(zv)\}.$$

Considering the triangles $vuxv$ and $vyuv$, we have $C(xv) = 5$ and $C(vy) = 2$. But now $xvyx$ is a rainbow triangle, a contradiction. Thus, we have $f(\overleftrightarrow{K}_5) = \lfloor \frac{n^2}{4} \rfloor + 2 = 8$.

Case 4. $n \geq 6$.

Suppose Theorem 1 is true for $\overleftrightarrow{K}_{n-1}$, now we consider \overleftrightarrow{K}_n , $n \geq 6$. Let D be an arc-colored complete digraph of order $n \geq 6$. If $c(D) \geq \lfloor \frac{n^2}{4} \rfloor + 2$ but D contains no rainbow triangles, then for every vertex $v \in V(D)$, the digraph $D - v$ contains no rainbow triangles either. Thus, we have $c(D - v) = c(D) - d^s(v) \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 1$. So for every vertex $v \in V(D)$, we have

$$\begin{aligned} d^s(v) &\geq \lfloor \frac{n^2}{4} \rfloor + 2 - \left(\lfloor \frac{(n-1)^2}{4} \rfloor + 1 \right) \\ &= \begin{cases} \frac{n}{2} + 1, & n \text{ is even;} \\ \frac{n+1}{2}, & n \text{ is odd.} \end{cases} \end{aligned}$$

On the other hand, by Lemma 2, there must be a vertex $v \in V(D)$ such that $d^s(v) \leq \lfloor \frac{n}{2} \rfloor$, a contradiction. So we have $f(\overleftrightarrow{K}_n) = \lfloor \frac{n^2}{4} \rfloor + 2$ for $n \geq 5$.

The proof is complete. \square

Proof of Theorem 2. Let $D \in \mathcal{G}_3$. Since the two arc-disjoint directed triangles Δ_1 and Δ_2 are not rainbow, we have $c(\Delta_1) \leq 2$ and $c(\Delta_2) \leq 2$. Thus $4 = c(D) \leq c(\Delta_1) + c(\Delta_2) \leq 4$, the equality holds if and only if $c(\Delta_1) = c(\Delta_2) = 2$ and $C(\Delta_1) \cap C(\Delta_2) = \emptyset$.

We have already characterized \mathcal{G}_4 by Claim 1 in Theorem 1.

Let $D \in \mathcal{G}_5$. Then $c(D) = f(\overleftrightarrow{K}_5) - 1 = 8 - 1 = 7$. If $d^s(v) = 1$ for a vertex $v \in V(D)$, then we have $c(D - v) = 6$. By Claim 1 in Theorem 1, we can assume that

$V(D - v) = \{u, x, y, z\}$ and

$$\begin{cases} C(ux) = C(xy) = C(yz) = C(zu) = 5, \\ C(uz) = C(zy) = C(yx) = C(xu) = 6, \\ C(uy) = 1, \quad C(yu) = 2, \\ C(xz) = 3, \quad C(zx) = 4. \end{cases}$$

Without loss of generality, we can assume that $C(vu) = 7 \in C^s(v)$. Considering triangles $vuxv$, $vuyv$ and $vuzv$, we have $C(xv) = 5$ or 7 , $C(yv) = 1$ or 7 and $C(zv) = 6$ or 7 . Considering triangles $vz xv$ and $vzyv$, we have $C(vz) \in \{4, 5, 7\} \cap \{1, 6, 7\}$, and hence $C(vz) = 7 = C(xv) = C(yv)$. Considering triangles $vxzv$ and $vxyv$, we have $C(vx) \in \{5, 7\} \cap \{3, 6, 7\}$, and hence $C(vx) = 7 = C(zv)$. Considering triangles $vyzv$ and $vyxv$, we have $C(vy) \in \{5, 7\} \cap \{6, 7\}$, and hence $C(vy) = 7$. Finally, considering triangles $uvxu$ and $uvyu$, we have $C(uv) \in \{6, 7\} \cap \{2, 7\}$, and hence $C(uv) = 7$. Thus, all arcs incident to v are colored by 7 and D belongs to Type I.

Now let us consider the case that $d^s(v) \geq 2$ for each vertex $v \in V(D)$. Let

$$X = \{i \in C(D) : C(uv) = i \text{ and } i \in C^s(u) \cap C^s(v)\},$$

$$Y = \{i \in C(D) : i \in C^s(v) \text{ and } i \notin C^s(u) \text{ if } u \neq v\},$$

$$Z = \{i \in C(D) : i \notin C^s(v) \text{ for any } v \in V(D)\}.$$

Let x, y and z be the cardinality of X, Y and Z , respectively. Then we have

$$\begin{cases} x + y + z = c(D) \\ 2x + y = \sum_{v \in V(D)} d^s(v). \end{cases}$$

Recall that $c(D) = 7$ and $d^s(v) \geq 2$ for each vertex $v \in V(D)$. We have

$$\begin{cases} x + y + z = 7 \\ 2x + y \geq 10. \end{cases}$$

Thus $x \geq z + 3 \geq 3$.

Let H be an arc-colored spanning subdigraph of D with the arcs that are assigned colors in X . Then $a(H) \geq x \geq 3$. Since each directed path uvw in H implies a rainbow triangle $uvwu$, there is no directed path of length 2 in H . Let \hat{H} be the underlying graph of H .

Case 1. $Z = \emptyset$.

If $uv, ab \in A(H)$ for four distinct vertices u, v, a, b , then without loss of generality, we can assume that $C(va) = 1$. Since $vauv$ and $abva$ are not rainbow triangles, it is easy to see that $C(au) = C(bv) = 1$. Thus $1 \in Z$. This contradicts that $Z = \emptyset$. Thus there exists a vertex u such that each arc in H is incident to u and hence $d^s(u) \geq x$.

Let uv be an arc such that $C(uv) \in X$. Let $\{a, b, c\} = V(D) \setminus \{u, v\}$. Since D contains no rainbow triangles and $Z = \emptyset$, we can assume that $C(va) = C(au) = 1$, $C(vb) = C(bu) = 2$ and $C(vc) = C(cu) = 3$. This implies that $\{1, 2, 3\} \subseteq Y$ and $y \geq 3$. Now we have $x, y \geq 3$ and $x + y = 7$. Thus either $x = 3, y = 4$ or $x = 4, y = 3$.

If $x = 3, y = 4$, then $10 = 2x + y = \sum_{v \in V(D) \setminus \{u\}} d^s(v) + d^s(u) \geq 11$, a contradiction.

If $x = 4, y = 3$, then $11 = 2x + y = \sum_{v \in V(D) \setminus \{u\}} d^s(v) + d^s(u) \geq 12$, a contradiction.

Case 2. $x \geq 5$.

If H contains a cycle uvu , namely, $C(uv), C(vu) \in X$, where $v \neq u$, then it is easy to see that none of the arcs in H appears between $\{u, v\}$ and $V(D) \setminus \{u, v\}$. Let $\{a, b, c\} = V(D) \setminus \{u, v\}$. Then either the triangle $abca$ or the triangle $cbac$ contains two arcs of H . In both cases, we get a rainbow triangle. So H contains no two oppositely oriented arcs. Moreover, there is no odd cycle in \hat{H} (otherwise, there must be a directed path of length 2 in H , a contradiction.)

Note that $a(H) \geq x \geq 5$. The graph \hat{H} must contain a cycle, which has to be of length 4, say $a_1b_1a_2b_2a_1$. Let $\{u\} = V(D) \setminus \{a_1, a_2, b_1, b_2\}$. Since there is no directed path of length 2 in H , we can assume that $a_1b_1, a_1b_2, a_2b_1, a_2b_2 \in A(H)$ and all the other arcs in $D[a_1, a_2, b_1, b_2]$ are not contained in H . Assume that $C(a_1a_2) = 1$. Then $1 \in Y \cup Z$. Consider triangles $a_1a_2b_1a_1$ and $a_1a_2b_2a_1$. We get $C(b_1a_1) = C(b_2a_1) = 1$. Consider $b_2a_1b_1b_2$ and $b_1a_1b_2b_1$. We get $C(b_1b_2) = C(b_2b_1) = 1$. By similar analyzing process, we finally see that all the arcs in $A(D - u) \setminus A(H)$ are of color 1. Recall that $a(H) \geq 5$. By the symmetry, we can assume that u is either incident to a_1 or b_1 in H .

If u is incident to b_1 in H , then the situation has to be $ub_1 \in H$ (since H contains no path of length 2). Now consider triangles ub_1a_1u , ub_1b_2u and ub_1a_2u . We get $C(a_1u) = C(b_2u) = C(a_2u) = 1$. Consider triangles ua_1b_2u and ua_2b_2u . We get $C(ua_1) = C(ua_2) = 1$. Again, consider the triangle ua_1b_1u . We get $C(b_1u) = 1$. Now $c(D - ub_2) = 6$. Since $c(D) = 7$, there holds $C(ub_2) \notin C(D - ub_2)$. Thus $ub_2 \in A(H)$. If u is incident to a_1 in H , then by a similar analyzing process, we can obtain that $a_1u, a_2u \in A(H)$ and all the other arcs incident to u are of color 1.

In summary, H is an orientation of $K_{2,3}$ with partite sets A and B such that $|A| = 2, |B| = 3$ and all the arcs are from A to B or from B to A . The remaining arcs in D are

all colored by a same new color. So D belongs to Type II.

Case 3. $z \geq 1$ and $x \leq 4$.

Recall that $x \geq z + 3 \geq 4$ and $x + y + z = 7$. We have $x = 4$, $y = 2$ and $z = 1$. Note that

$$10 = 2x + y = \sum_{v \in V(D)} d^s(v) \geq 2 * 5 = 10.$$

So $d^s(v) = 2$ for each vertex $v \in V(D)$. Now we assert that $d^s(v) \geq d_{\hat{H}}(v)$. If each color in $C^s(v) \cap X$ is only assigned to one arc in D , then there is nothing to prove. If there is a color in $C^s(v) \cap X$ assigned to more than two arcs, then by the definition of X , we know that these arcs must be two oppositely oriented arcs, say vw and wv . Recolor vw by a new color. Then the obtained arc-colored complete digraph D' satisfies that $c(D') = f(\overrightarrow{K}_5)$ but D' contains no rainbow triangles, a contradiction. So we have $d^s(v) \geq d_{\hat{H}}(v)$ for each vertex $v \in V(D)$. Thus the maximum degree of \hat{H} is at most 2.

If \hat{H} contains a path of length 3, then without loss of generality, we can assume that $uvws$ is a path with $uv, vw, ws \in A(H)$. Let $C(vu) = 1$, $C(vw) = c$ and let p be the vertex in D different from u, v, w and s . Since D contains no rainbow triangles, we obtain that $C(uw) = C(su) = C(vs) = C(sw) = 1$ and $C(wu) = C(sv) = c$. It is easy to observe that $1, c \in Z$. Since $z = 1$, we have $c = 1$ and $Z = \{1\}$. Consider triangles $uvpu$, $wvpw$ and $wspw$. We get $C(vp) = C(pu) = C(pw) = C(sp) = a$. If $a \in C^s(p)$, then considering triangles $upwu$, $pvsp$, $wpuw$ and $vpsv$, we can get

$$\{C(up)\} \cup \{C(pv)\} \cup \{C(wp)\} \cup \{C(ps)\} \subseteq \{1, a\}.$$

Thus, we have $d^s(p) = 1$, a contradiction. So $a \notin C^s(p)$ and hence $a = 1$. If $C(up) = 2 \in C^s(p)$, then considering triangles $vupv$ and $sups$, we can get $C(pv), C(ps) \in \{1, 2\}$. Since $d^s(p) = 2$, we have $C(wp) \neq 2$ and $C(wp) \in C^s(p)$. Let $C(wp) = 3$. Consider triangles $wpvw$ and $wpsw$. We can get $C(pv), C(ps) \in \{1, 3\}$. So $C(pv) = C(ps) = 1$. But now $\{2, 3, C(uv), C(wv), C(ws)\} \subseteq X$. This contradicts that $x = 4$. Thus $C(up) \notin C^s(p)$. By similar analyzing process, we can see that $C(pv), C(wp), C(ps) \notin C^s(p)$. This implies that $C^s(p) = \emptyset$, a contradiction.

If the longest path in \hat{H} is of length 1, then the arcs of H form two vertex-disjoint cycles of length 2, say $A(H) = \{uv, vu, pq, qp\}$. Since $z = 1$, it is easy to check that all the arcs between $\{u, v\}$ and $\{p, q\}$ has a same color, namely, the unique color in Z . Let $Y = \{1, 2\}$ and $V(D) \setminus \{u, v, p, q\} = \{w\}$. Then there holds $C^s(w) = \{1, 2\}$. Since D contains no rainbow triangles, we have $C(uw) = C(wv), C(vw) = C(wu), C(pw) = C(wq), C(qw) =$

$C(wp)$. By the symmetry, we can assume that $C(uw) = C(wv) = 1$. Consider triangles $uwpu, uwqu, pwvp$ and $qwvq$. We can see that the color 2 does not appear between w and $\{p, q\}$. This forces $C(vw) = C(wu) = 2$ and all the arcs between w and $\{p, q\}$ are colored by the unique color in Z . So D belongs to Type III.

The remaining case is that \hat{H} is composed of a path of length 2 and a cycle of length 2. Let $V(D) = \{u, v, w, p, q\}$ and $A(H) = \{uv, vw, pq, qp\}$. Assume that $C(vp) = a$ and $C(pv) = b$. Then it is easy to check that each arcs between $\{u, v, w\}$ and $\{p, q\}$ are of color a or b , and $a, b \in Z$. This forces $a = b$ (since $z = 1$). Now the arcs vu, uw, wv are the only possible arcs that are assigned the colors in Y . Thus $c(D[v, u, w]) = 4$ and each color in $D[v, u, w]$ does not appears on $A(D) \setminus A(D[v, u, w])$. So $D[v, u, w] \in \mathcal{G}_3$ and D belongs to Type III.

Let $D \in \mathcal{G}_6$. Since D contains no rainbow triangles, we have $c(D - v) \leq 7$, so $d^s(v) \geq 3$ for every $v \in V(D)$. On the other hand, by Lemma 2, there is a vertex $v \in V(D)$ such that $d^s(v) \leq \lfloor \frac{n}{2} \rfloor = 3$. So there is a vertex $v \in V(D)$ such that $d^s(v) = 3$ and $c(D - v) = 7$. Since $D - v$ contains no rainbow triangles, by the above arguments, $D - v \in \mathcal{G}_5$ and thus belongs to one of the three types of digraphs.

Case 1. $D - v$ belongs to Type I.

Let $V(D) = \{u, v, w, x, y, z\}$, We can assume that

$$\begin{cases} C(uy) = 1, & C(yu) = 2, & C(xz) = 3, & C(zx) = 4, \\ C(ux) = C(xy) = C(yz) = C(zu) = 5, \\ C(uz) = C(zy) = C(yx) = C(xu) = 6, \\ C(wx) = C(wy) = C(wz) = C(wu) = C(uw) = C(xw) = C(yw) = C(zw) = 7. \end{cases}$$

Since $d^s(v) = 3$, by Lemma 1, we can assume that $CN^-(v) \cap C^s(v) = \emptyset$. Let $C^s(v) = \{8, 9, 10\}$ and $C(vz) = 8$. Considering the triangle $vzww$, we have $C(wv) = 7$. But now $d^s(w) \leq 2$, a contradiction.

Case 2. $D - v$ belongs to Type III.

Let $V(D) = \{v, a_1, a_2, b_1, b_2, b_3\}$. We can assume that $C(a_1a_2) = 1$, $C(a_2a_1) = 2$ and $C(\{a_1, a_2\}, \{b_1, b_2, b_3\}) = \{3\}$. Since $d^s(v) = 3$, by Lemma 1, we can assume that $CN^-(v) \cap C^s(v) = \emptyset$. Then there must be a vertex b_j such that $C(vb_j) \in C^s(v)$. Without loss of generality, we can assume that $C(vb_1) = 8 \in C^s(v)$. Considering triangles vb_1a_1v and vb_1a_2v , we have $C(a_1v) = C(a_2v) = 3$. Considering the triangle $a_1va_2a_1$, we have

$C(va_2) \in \{2, 3\}$. Since $C(a_1b_1) = 3$, we can see that $3 \notin C^s(a_2)$ and $d^s(a_2) \leq 2$, a contradiction.

Case 3. $D - v$ belongs to Type II.

Let $V(D) = \{v, a_1, a_2, b_1, b_2, b_3\}$. We can assume that

$$\begin{cases} C(a_1b_1) = 1, & C(a_1b_2) = 2, & C(a_1b_3) = 3, \\ C(a_2b_1) = 4, & C(a_2b_2) = 5, & C(a_2b_3) = 6, \end{cases}$$

and the remaining arcs of $D - v$ are all colored by 7.

Case 3.1. $CN^+(v) \cap C^s(v) = \emptyset$.

Since $d^s(v) = 3$, there must be a vertex b_j such that $C(b_jv) \in C^s(v)$. Without loss of generality, we can assume that $C(b_1v) = 8 \in C^s(v)$. Considering triangles va_1b_1v and va_2b_1v , we have $C(va_1) = 1$ and $C(va_2) = 4$. Considering triangles $a_1b_2va_1$ and $a_2b_2va_2$, we have $C(b_2v) \in \{1, 2\} \cap \{4, 5\}$, a contradiction.

Case 3.2. $CN^-(v) \cap C^s(v) = \emptyset$.

Let $C^s(v) = \{8, 9, 10\}$. If $C(va_1) = 8$, then considering triangles va_1b_1v and va_1b_2v , we have $C(b_1v) = 1$ and $C(b_2v) = 2$. Considering triangles $a_2b_1va_2$ and $a_2b_2va_2$, we have $C(va_2) \in \{1, 4\} \cap \{2, 5\}$, a contradiction. So $C(va_1) \neq 8$. Similarly we can prove that

$$(\{C(va_1)\} \cup \{C(va_2)\}) \cap C^s(v) = \emptyset.$$

Thus $C^s(v) \subseteq \{C(vb_1), C(vb_2), C(vb_3)\}$. Without loss of generality, we can assume that $C(vb_1) = 8$, $C(vb_2) = 9$ and $C(vb_3) = 10$. Considering the triangle set

$$\{vb_1uv | u \in \{a_1, a_2, b_2, b_3\}\} \cup \{vb_2b_1v\},$$

we have

$$C(\{uv | u \in \{a_1, a_2, b_1, b_2, b_3\}\}) = \{7\}.$$

Considering triangles va_1b_1v , va_1b_2v , va_2b_1v and va_2b_2v , we have

$$C(va_1) \in \{1, 7\} \cap \{2, 7\} = \{7\} \text{ and } C(va_2) \in \{4, 7\} \cap \{5, 7\} = \{7\}.$$

Let $v = a_3$. Then we can see that the spanning subdigraph H of D with $A(H) = \{a_i b_j | i = 1, 2, 3; j = 1, 2, 3\}$ is rainbow and all the remaining arcs are colored by a same new color 7. So the theorem is true for $3 \leq n \leq 6$.

Let $D \in \mathcal{G}_n$, $n \geq 7$. Suppose the theorem is true for $\overleftrightarrow{K}_{n-1}$. Now we consider \overleftrightarrow{K}_n , $n \geq 7$.

If $D = \overleftrightarrow{K}_n$ contains no rainbow triangles and $c(D) = \lfloor \frac{n^2}{4} \rfloor + 1$, then $c(D - v) \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 1$ and $d^s(v) \geq \lfloor \frac{n}{2} \rfloor$ for every $v \in V(D)$. On the other hand, by Lemma 2, there is a vertex $v \in V(D)$ such that $d^s(v) \leq \lfloor \frac{n}{2} \rfloor$. So there is a vertex $v \in V(D)$ such that $d^s(v) = \lfloor \frac{n}{2} \rfloor$ and $c(D - v) = \lfloor \frac{(n-1)^2}{4} \rfloor + 1$. By induction hypothesis, the vertex set of $D - v$ can be partitioned into two subsets $\{a_1, a_2, \dots, a_{\lfloor \frac{n-1}{2} \rfloor}\}$ and $\{b_1, b_2, \dots, b_{\lceil \frac{n-1}{2} \rceil}\}$ such that the spanning subdigraph H of D with $A(H) = \{a_i b_j | i = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil\}$ (or $A(H) = \{b_j a_i | i = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil\}$) is rainbow and all arcs in $A(D) \setminus A(H)$ are colored by a same new color c . By symmetry, we only discuss the case $A(H) = \{a_i b_j | i = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil\}$. If n is odd, then we divide the rest of the proof into two cases.

Case 1. $CN^+(v) \cap C^s(v) = \emptyset$.

If there is a vertex b_j such that $C(b_j v) \in C^s(v)$. Without loss of generality, we can assume that $C(b_1 v) \in C^s(v)$. Considering triangles $va_1 b_1 v$ and $va_2 b_1 v$, we have $C(va_1) = C(a_1 b_1)$ and $C(va_2) = C(a_2 b_1)$. Considering triangles $a_1 b_2 v a_1$ and $a_2 b_2 v a_2$, we have

$$C(b_2 v) \in \{C(a_1 b_1), C(a_1 b_2)\} \cap \{C(a_2 b_1), C(a_2 b_2)\}.$$

But $A(H) = \{a_i b_j | i = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil\}$ is rainbow, a contradiction. So $C(b_j v) \notin C^s(v)$, for $j = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil$. Thus $C^s(v) \subseteq \{C(a_1 v), \dots, C(a_{\lfloor \frac{n-1}{2} \rfloor} v)\}$. Since $d^s(v) = \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$, we can see that $C^s(v) = \{C(a_1 v), \dots, C(a_{\lfloor \frac{n-1}{2} \rfloor} v)\}$. Considering the triangle set

$$\{vua_1 v | u \in V(D) \setminus \{v, a_1\}\} \cup \{va_1 a_2 v\},$$

we have

$$C(\{vu | u \in V(D) \setminus \{v\}\}) = \{c\}.$$

Considering triangles $va_1 b_j v$ and $va_2 b_j v$, for $j = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil$, we have

$$C(b_j v) \in \{C(a_1 b_j), c\} \cap \{C(a_2 b_j), c\} = \{c\}.$$

Let $v = b_{\lceil \frac{n}{2} \rceil}$. Then we can see that the spanning subdigraph H of D with $A(H) = \{a_i b_j | i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n}{2} \rceil\}$ is rainbow and all the remaining arcs are colored by a same new color c .

Case 2. $CN^-(v) \cap C^s(v) = \emptyset$.

By similar analysis, we can see that the spanning subdigraph H of D with $A(H) = \{a_i b_j | i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n}{2} \rceil\}$ is rainbow and all the remaining arcs are colored by a same new color c , where $v = a_{\lceil \frac{n}{2} \rceil}$.

If n is even, then by similar analysis we can see that the spanning subdigraph H of D with $A(H) = \{a_i b_j | i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor; j = 1, 2, \dots, \lceil \frac{n}{2} \rceil\}$ is rainbow and all the remaining arcs are colored by a same new color c , where $v = a_{\lfloor \frac{n}{2} \rfloor}$.

The proof is complete. \square

Proof of Theorem 3. Suppose the contrary. Let D be a counterexample with the smallest number of vertices, and then with the smallest number of arcs.

Claim 1. D contains two arcs uv and xy with a same color, where $xy \neq vu$.

Proof. Recall that the maximum number of arcs among all digraphs of order n without directed triangles is $\lfloor \frac{n^2}{2} \rfloor$ (see [18]). If $c(D) \geq \lfloor \frac{n^2}{2} \rfloor + 1$, then D contains a rainbow triangle, a contradiction. So $c(D) \leq \lfloor \frac{n^2}{2} \rfloor$. Thus, we have

$$a(D) - \lfloor \frac{n^2}{2} \rfloor \geq n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 2 - 2\lfloor \frac{n^2}{2} \rfloor = \begin{cases} \frac{n(n-4)}{4} + 2 > 0, & n \text{ is even;} \\ \frac{(n-1)(n-3)}{4} + 2 > 0, & n \text{ is odd.} \end{cases}$$

So $a(D) > \lfloor \frac{n^2}{2} \rfloor$. Namely, D contains a directed triangle Δ and at least two arcs of Δ are colored by a same color. Note that two arcs of a triangle can only have one common end. So D contains two arcs uv and xy with a same color, where $xy \neq vu$. \square

Claim 2.

$$a(D) + c(D) = \begin{cases} n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 3, & n = 3, 4; \\ n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 2, & n \geq 5. \end{cases}$$

Proof. By Claim 1, let a_1 and a_2 be two arcs with a same color. Then $a(D - a_1) = a(D) - 1$ and $c(D - a_1) = c(D)$. If

$$a(D) + c(D) \geq \begin{cases} n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 4, & n = 3, 4; \\ n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 3, & n \geq 5, \end{cases}$$

then

$$a(D - a_1) + c(D - a_1) \geq \begin{cases} n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 3, & n = 3, 4; \\ n(n-1) + \lfloor \frac{n^2}{4} \rfloor + 2, & n \geq 5. \end{cases}$$

Note that $D - a_1$ contains no rainbow triangles either. Thus $D - a_1$ is a counterexample with fewer arcs, a contradiction. \square

Claim 3. For every $v \in V(D)$, we have

$$d(v) + d^s(v) \geq \begin{cases} 2(n-1) + \frac{n}{2} + 1, & n \text{ is even;} \\ 2(n-1) + \frac{n-1}{2} + 1, & n \text{ is odd and } n \neq 5; \\ 10, & n = 5. \end{cases}$$

Proof. Note that $a(D - v) = a(D) - d(v)$ and $c(D - v) = c(D) - d^s(v)$. If

$$d(v) + d^s(v) \leq \begin{cases} 2(n-1) + \frac{n}{2}, & n \text{ is even;} \\ 2(n-1) + \frac{n-1}{2}, & n \text{ is odd and } n \neq 5; \\ 9, & n = 5, \end{cases}$$

then

$$\begin{aligned} a(D - v) + c(D - v) &= a(D) + c(D) - (d(v) + d^s(v)) \\ &\geq \begin{cases} (n-1)(n-2) + \lfloor \frac{(n-1)^2}{4} \rfloor + 3, & n = 4, 5; \\ (n-1)(n-2) + \lfloor \frac{(n-1)^2}{4} \rfloor + 2, & n \geq 6. \end{cases} \end{aligned}$$

Note that $D - v$ does not contain a rainbow triangle. Thus $D - v$ is a counterexample with fewer vertices, a contradiction. \square

Claim 4. $\sum_{v \in V(D)} d^s(v) \leq 2c(D) - 1$.

Proof. Let c be an arbitrary color in $C(D)$. Note that each color c can only be saturated by at most two vertices. So $\sum_{v \in V(D)} d^s(v) \leq 2c(D)$. Moreover, c is saturated by exactly two vertices if and only if c appears on only one arc or on a pair of arcs between two vertices. By Claim 1, D contains two arcs uv and xy with a same color, where $xy \neq vu$. Thus, at least one color cannot be saturated by exactly two vertices. So $\sum_{v \in V(D)} d^s(v) \leq 2c(D) - 1$. \square

By Claims 2-4, we can get that if $n \geq 6$ is even, then

$$2n(n-1) + \frac{n^2}{2} + n \leq \sum_{v \in V(D)} (d(v) + d^s(v)) \leq 2a(D) + 2c(D) - 1 = 2n(n-1) + \frac{n^2}{2} + 3. \quad (1)$$

This implies that $n \leq 3$, a contradiction.

If $n \geq 7$ is odd, then

$$\begin{aligned} 2n(n-1) + \frac{n(n-1)}{2} + n &\leq \sum_{v \in V(D)} (d(v) + d^s(v)) \leq 2a(D) + 2c(D) - 1 \\ &= 2n(n-1) + \frac{(n-1)(n+1)}{2} + 3. \end{aligned} \quad (2)$$

This implies that $n \leq 5$, a contradiction. So it suffices to consider the cases $n = 3, 4, 5$.

For $n = 3$, since $a(D) + c(D) = 11$ and $a(D) \leq 6$, we have $c(D) \geq 5 = \lfloor \frac{n^2}{2} \rfloor + 1$. So D contains a rainbow triangle, a contradiction.

For $n = 4$, we have $a(D) + c(D) \geq 19$. If $a(D) = 12$, then $D \cong \overleftrightarrow{K}_4$ and $c(D) \geq 7$. By Theorem 1, D contains a rainbow triangle, a contradiction. If $a(D) \leq 10$, then $c(D) \geq 9 = \lfloor \frac{4^2}{2} \rfloor + 1$. We know that D contains a rainbow triangle, a contradiction. The

only case left is that $a(D) = 11 = a(\overleftrightarrow{K}_4) - 1$ and $c(D) = 8$. Let u be a vertex in D such that $D - u \cong \overleftrightarrow{K}_3$. Since $f(\overleftrightarrow{K}_3) = 5$, we have $d^s(u) \geq 4$. Let $V(D - u) = \{x, y, z\}$. Then there must exist two vertices in $V(D - u)$ (say x and y) such that $c(ux)$ and $c(yu)$ are two distinct colors in $C^s(u)$. This implies that $uxyu$ is a rainbow triangle, a contradiction.

Lemma 3. *Let D be an arc-colored digraph of order 3. If $a(D) + c(D) = 10$ and D contains no rainbow triangle, then $D \cong \overleftrightarrow{K}_3$.*

Proof. Since D contains no rainbow triangle, we have $c(D) \leq \lfloor \frac{n^2}{2} \rfloor = 4$ and $a(D) \geq 6$. So $c(D) = 4$, $a(D) = 6$ and $D \cong \overleftrightarrow{K}_3$. \square

Lemma 4. *Let D be an arc-colored digraph of order 4. If $a(D) + c(D) = 18$ and D contains no rainbow triangle, then $D \cong \overleftrightarrow{K}_4$.*

Proof. For every $v \in V(D)$, since $D - v$ contains no rainbow triangles, we have $a(D - v) + c(D - v) \leq 10$ and hence $d(v) + d^s(v) \geq 8$. If $d(v) + d^s(v) \geq 9$ for every $v \in V(D)$, then

$$36 \leq \sum_{v \in V(D)} (d(v) + d^s(v)) \leq 2a(D) + 2c(D) - 1 = 35, \quad (3)$$

a contradiction. So there is a vertex $v \in V(D)$ such that $d(v) + d^s(v) = 8$. Let $V(D) = \{v, x, y, z\}$ and $d(v) + d^s(v) = 8$. Then $a(D - v) + c(D - v) = 10$. By Lemma 3, $D - v \cong \overleftrightarrow{K}_3$, and thus $D - v \in \mathcal{G}_3$. Furthermore, by Theorem 2, we know that the color sets of the two directed triangles in $D - v$ is disjoint. Let $C(D - v) = \{1, 2, 3, 4\}$. If $D \not\cong \overleftrightarrow{K}_4$, then $d(v) \leq 5$ and $d^s(v) \geq 3$. Let $\{5, 6, 7\} \subseteq C^s(v)$. If there exist two vertices in $V(D - v)$ (say x and y) such that $c(vx)$ and $c(yv)$ are two distinct colors in $C^s(v)$, then we have $vxyv$ is a rainbow triangle, a contradiction. So we can assume that $C(vx) = 5, C(vy) = 6$ and $C(vz) = 7$. If $yv \in A(D)$, then consider triangles $vxyv$ and $vzyv$. We get $C(xy) = C(yv)$ and $C(zv) = C(yv)$. Thus $C(xy) = C(zv)$. This contradicts the structure of $D - v \in \mathcal{G}_3$. So we have $yv \notin A(D)$. Similarly, we can get $xv, zv \notin A(D)$. Thus $d(v) = d^s(v) = 3$. This contradicts that $d(v) + d^s(v) = 8$. \square

For $n = 5$, we have $a(D) + c(D) \geq 28$. For each integer p , let $X_p = \{u \in V(D) : a(D - u) + c(D - u) = p\}$ and let $x_p = |X_p|$. Since D contains no rainbow triangle, $a(D - u) + c(D - u) \leq 18$ for each vertex $u \in V(D)$. So we have

$$\sum_{p \leq 18} x_p = 5. \quad (4)$$

Let $Y_i = \{u : i \in C(D - u)\}$ for each $i \in C(D)$ and let $y_i = |Y_i|$. Since each color appears in at least 3 induced subdigraphs of order 4, we have $y_i \geq 3$. Note that D has 5 induced

subdigraphs of order 4, every arc of D belongs to exactly 3 of such induced subdigraphs and every color $i \in C(D)$ belongs to exactly y_i of them. So we have

$$\sum_{p \leq 18} px_p = 3a(D) + \sum_{i \in C(D)} y_i = 3a(D) + 3c(D) + \sum_{i \in C(D)} (y_i - 3) \geq 84 + \sum_{i \in C(D)} (y_i - 3). \quad (5)$$

By (5) $- 16 \times (4)$ we can get

$$\sum_{i \in C(D)} (y_i - 3) \leq 2x_{18} + x_{17} - 4.$$

Case 1. $x_{18} = 0$.

In this case, since $x_{17} \leq 5$, we have $0 \leq \sum_{i \in C(D)} (y_i - 3) \leq 1$. This means that either $y_i = 3$ for all $i \in C(D)$ or there is only one color j such that $y_j = 4$.

If $y_i = 3$ for all $i \in C(D)$, then every triangle in D must be a rainbow triangle. This implies that D contains no directed triangles. So $a(D) \leq \lfloor \frac{5^2}{2} \rfloor = 12$. Thus

$$28 \leq a(D) + c(D) \leq 2a(D) \leq 24,$$

a contradiction. If there is only one color j such that $y_j = 4$. Then let u be the only vertex in D such that $j \notin C(D - u)$. Then $D - u$ contains no directed triangle. Thus $a(D - u) + c(D - u) \leq 2a(D - u) \leq 2 \lfloor \frac{4^2}{2} \rfloor = 16$. So $d^s(u) + d(u) \geq 12$. Note that $d^s(u) + d(u) \leq 2d(u) - a(D^j) + 1$. So

$$a(D^j) \leq 2d(u) - 11. \quad (6)$$

On the other hand, let D' be an arc-colored digraph such that $V(D') = V(D)$ and $A(D') = (A(D) \setminus A(D^j)) \cup \{e\}$. Here e is an arc from D^j . Then we have $28 - a(D^j) + 1 = a(D') + c(D') \leq 2a(D') \leq 2 \lfloor \frac{5^2}{2} \rfloor$. Thus

$$a(D^j) \geq 5. \quad (7)$$

Combine (6) and (7). We have $d(u) \geq 8$. Note that $d(u) \leq 8$. We have $d(u) = 8$, $a(D^j) = 5$ and there must be a vertex $v \in V(D - u)$ such that $C(uv) = C(vu) = j$. Let D'' be an arc-colored digraph such that $V(D'') = V(D)$ and $A(D'') = (A(D) \setminus A(D^j)) \cup \{uv, vu\}$. Then each triangle in D'' must be a rainbow triangle. So D'' contains no triangles. We have

$$a(D) - a(D^j) + 2 = a(D'') \leq \lfloor \frac{5^2}{2} \rfloor.$$

Thus $a(D) \leq 15$. So $c(D) \geq 13 = \lfloor \frac{5^2}{2} \rfloor + 1$, which implies that D contains a rainbow triangle, a contradiction.

Case 2. $x_{18} \geq 1$.

In this case, there is a vertex $u \in V(D)$ such that $a(D - u) + c(D - u) = 18$ and $d(u) + d^s(u) \geq 10$. By Lemma 4, we can see that $D - u \cong \overleftrightarrow{K}_4$ and $D - u \in \mathcal{G}_4$. If $D \cong \overleftrightarrow{K}_5$, then we obtain a rainbow triangle by Theorem 1, a contradiction. So $d(u) \leq 7$ and $d^s(u) \geq 3$. By Lemma 1, we can assume that $CN^-(u) \cap C^s(u) = \emptyset$. Then $d^s(u) \leq 4$. Let the two monochromatic cycles in $D - u$ are $xyzwx$ and $wzyxw$ with colors α and β , respectively. Assume that $C(ux), C(uy)$ and $C(uz)$ are three distinct colors in $C^s(u)$. If $yu \in A(D)$, then consider triangles $uxyu$ and $uzyu$, we get $\alpha = C(yu) = \beta$, a contradiction. So $yu \notin A(D)$. Similarly, we can get $xu \notin A(D)$, $zu \notin A(D)$, $wu \notin A(D)$. So $d(u) \leq 4$, and thus $d(u) + d^s(u) \leq 8$, a contradiction.

The proof is complete. \square

To prove Theorem 4, we need the following famous theorem of Moon [17]:

Theorem 5 (Moon's theorem). *Let T be a strongly connected tournament on $n \geq 3$ vertices. Then each vertex of T is contained in a cycle of length k for all $k \in [3, n]$. In particular, a tournament is hamiltonian if and only if it is strongly connected.*

Proof of Theorem 4. By induction on n . For $n = 3$, since D is strongly connected, we can see that D is a directed triangle. If $c(D) \geq \frac{n(n-1)}{2} - n + 3 = 3$, then all arcs of D have distinct colors. So D is a rainbow triangle.

Suppose that every arc-colored strongly connected tournament D' of order $n - 1$ with $c(D') \geq \frac{(n-1)(n-2)}{2} - (n-1) + 3$ contains a rainbow triangle for $n \geq 4$. Now we consider an arc-colored strongly connected tournament D of order n . Since D is strongly connected, by Moon's theorem, D contains a directed $(n - 1)$ -cycle C . Let v be the vertex not in C . Then $D - v$ contains a hamiltonian cycle C . Thus, $D - v$ is strongly connected. If $c(D) \geq \frac{n(n-1)}{2} - n + 3$ and D contains no rainbow triangles, then $D - v$ contains no rainbow triangles either, and hence $c(D - v) \leq \frac{(n-1)(n-2)}{2} - (n-1) + 2$. So we have

$$d^s(v) \geq \frac{n(n-1)}{2} - n + 3 - \left(\frac{(n-1)(n-2)}{2} - (n-1) + 2 \right) = n - 1.$$

This implies that $CN(v) \cap C(D - v) = \emptyset$ and every two different arcs incident to v have distinct colors. Since D is strongly connected, there exists an arc from $N^+(v)$ to $N^-(v)$. Assume that $wu \in A(D)$, where $w \in N^+(v)$ and $u \in N^-(v)$, then $vwuv$ is a directed triangle. Since $wu \in A(D - v)$ and vw, uv are two different arcs incident to v , we can see that $vwuv$ is a rainbow triangle, a contradiction.

The proof is complete. \square

3 Concluding remarks

By Lemmas 3 and 4 in Theorem 3, we proved that for $n = 3, 4$, if $a(D) + c(D) = a(\overleftrightarrow{K}_n) + f(\overleftrightarrow{K}_n) - 1$ and D contains no rainbow triangles, then $D \cong \overleftrightarrow{K}_n$. We conjecture that this is true for all $n \geq 5$.

Conjecture 1. Let D be an arc-colored digraph of order $n \geq 5$ without containing rainbow triangles. If $a(D) + c(D) = n(n - 1) + \lfloor \frac{n^2}{4} \rfloor + 1$, then $D \cong \overleftrightarrow{K}_n$.

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