

Chain method for panchromatic colorings of hypergraphs

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Abstract. We deal with an extremal problem concerning panchromatic colorings of hypergraphs. A vertex r -coloring of a hypergraph H is *panchromatic* if every edge meets every color. We prove that for every $r < \sqrt[3]{\frac{n}{100 \ln n}}$, every n -uniform hypergraph H with $|E(H)| \leq \frac{1}{20r^2} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(\frac{r}{r-1}\right)^{n-1}$ has a panchromatic coloring with r colors.

Keywords: panchromatic coloring, property B, proper coloring, uniform hypergraph.

1 Introduction and related work

We study colorings of uniform hypergraphs. Let us recall some definitions.

A *vertex r -coloring* of a hypergraph $H = (V, E)$ is a mapping from the vertex set V to a set of r colors. An r -coloring of H is *panchromatic* if each edge has at least one vertex of each color.

The first sufficient condition on the existence of a panchromatic coloring of a hypergraph was obtained in 1975 by Erdős and Lovász [8]. They proved that if every edge of an n -uniform hypergraph intersects at most

$$\frac{r^{n-1}}{4(r-1)^n} \tag{1}$$

other edges then the hypergraph has a panchromatic coloring with r colors.

The next generalization of the problem was formulated in 2002 by Kostochka [11], who posed the following question: *What is the minimum possible number of edges in an n -uniform hypergraph that does not admit a panchromatic coloring with r colors?* He denoted this number by $p(n, r)$.

Following closely behind this problem is a related one: a hypergraph $H = (V, E)$ has property B if there is a coloring of V by 2 colors so that no edge $f \in E$ is monochromatic. Erdős and Hajnal [7] (1961) proposed to find the value $m(n)$ equal to the minimum possible number of edges in a n -uniform hypergraph without property B . Erdős [6] (1963–1964) found bounds $\Omega(2^n) \leq m(n) = O(2^n n^2)$ and Radhakrishnan and Srinivasan [13] (2000) proved $m(n) \geq \Omega(2^n (n/\ln n)^{1/2})$. Clearly, $m(n) = p(n, 2)$.

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We return to the panchromatic coloring. Kostochka [11] has found connections between $p(n, r)$ and minimum possible number of vertices in a k -partite graph with list chromatic number greater than r . Using results of Erdős, Rubin and Taylor [9] and also Alon's result [2] Kostochka [11] proved the existence of constants c_1 and c_2 that for every large n and fixed r :

$$\frac{e^{c_1 \frac{n}{r}}}{r} \leq p(n, r) \leq r e^{c_2 \frac{n}{r}}. \quad (2)$$

In 2010, bounds (2) were considerably improved in the paper of Shabanov [15]:

$$\begin{aligned} p(n, r) &\geq \frac{\sqrt{21} - 3}{4r} \left(\frac{n}{(r-1)^2 \ln n} \right)^{1/3} \left(\frac{r}{r-1} \right)^n, \quad \text{for all } r < n, \\ p(n, r) &\leq \frac{1}{r} \left(\frac{r}{r-1} \right)^n e(\ln r) \frac{n^2}{2(r-1)} \varphi_1, \quad \text{when } r = o(\sqrt{n}), \\ p(n, r) &\leq \frac{1}{r} \left(\frac{r}{r-1} \right)^n e(\ln r) n^{3/2} \varphi_2, \quad \text{when } n = o(r^2), \end{aligned}$$

where φ_1, φ_2 some functions of n and $r(n)$, tending to one at $n \rightarrow \infty$.

In 2012, Rozovskaya and Shabanov [14] improved Shabanov's lower bound by proving that for $r < n/(2 \ln n)$

$$\frac{1}{2r^2} \left(\frac{n}{\ln n} \right)^{1/2} \left(\frac{r}{r-1} \right)^n \leq p(n, r) \leq c_2 n^2 \left(\frac{r}{r-1} \right)^n \ln r. \quad (3)$$

Further research was conducted by Cherkashin [3] in 2018. In his work, Cherkashin introduced the auxiliary value $p'(n, r)$, which is numerically equal to the minimum number of edges in the class of n -uniform hypergraphs $H = (V, E)$, in which any subset of vertices $V' \subset V$ with $|V'| \geq \lceil \frac{r-1}{r} |V| \rceil$ must contain an edge. Analyzing the value $p'(n, r)$ and using Sidorenko's [16] estimates on the Turan numbers, Cherkashin proved that for $n \geq 2, r \geq 2$

$$p(n, r) \leq c \frac{n^2 \ln r}{r} \left(\frac{r}{r-1} \right)^n.$$

Cherkashin also proved that for $r \leq c \frac{n}{\ln n}$

$$p(n, r) \geq c \max \left(\frac{n^{1/4}}{r\sqrt{r}}, \frac{1}{\sqrt{n}} \right) \left(\frac{r}{r-1} \right)^n. \quad (4)$$

And repeating the ideas of Gebauer [10] Cherkashin constructed an example of a hypergraph that has few edges and does not admit a panchromatic coloring in r colors. The reader is referred to the survey [4] for the detailed history of panchromatic colorings.

It is thus natural to consider the local case. Formally, *the degree of an edge A* is the number of hyperedges intersecting A . Let $d(n, r)$ be the minimum possible value of the maximum edge

degree in an n -uniform hypergraph that does not admit panchromatic coloring with r colors. Then, the Erdős and Lovász result (1) can be easily translated into following form:

$$d(n, r) \geq \frac{r^{n-1}}{4(r-1)^n}. \quad (5)$$

However, the bound (5) appeared not to be sharp. The restriction on $d(n, r)$ have been improved by Rozovskaya and Shabanov [14]. In their work they achieved that

$$d(n, r) > \frac{\sqrt{11} - 3}{4r(r-1)} \left(\frac{n}{\ln n}\right)^{1/2} \left(\frac{r}{r-1}\right)^n, \quad \text{when } r \leq n/(2 \ln n). \quad (6)$$

2 Our results

The main result of our paper improves the estimate (3) as follows.

Theorem 1. *Suppose $r \leq \sqrt[3]{\frac{n}{100 \ln n}}$. Then we have*

$$p(n, r) \geq \frac{1}{20r^2} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(\frac{r}{r-1}\right)^n. \quad (7)$$

Corollary 1. *There is an absolute constant C so that for every $n > 2$ and $\ln n < r < \sqrt[3]{\frac{n}{100 \ln n}}$*

$$p(n, r) \geq \frac{Cn}{r^2 \ln n} \cdot e^{\frac{n}{r} + \frac{n}{2r^2}}.$$

We refine the bound (6) as follows.

Theorem 2. *For every $2 < r < \sqrt[3]{\frac{n}{100 \ln n}}$*

$$d(n, r) \geq \frac{1}{40r^3} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(\frac{r}{r-1}\right)^n. \quad (8)$$

2.1 Methods

In the work, we propose a new idea based on the Pluhar ordered chain method [12]. In the case of panchromatic coloring, the resulting structure is no longer a real ordered chain, but rather an intricate "snake ball". Nevertheless, with the help of probabilistic analysis, we managed to obtain a strong lower bound.

The rest of this paper is organised as follows. The next section describes a coloring algorithm. Section 4 is devoted to the detailed analysis of the algorithm. In Section 5 we collect some inequalities that will be subsequently useful. The last two sections contain proofs of Theorems 1 and 2.

3 The coloring algorithm

We may and will assume that $r \geq 3$, because case $r = 2$ corresponds to the case $m(n)$. Let $H = (V, E)$ be an n -uniform hypergraph with less than $\frac{1}{20r^2} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(\frac{r}{r-1}\right)^n$ edges and let $r < \sqrt[3]{\frac{n}{100 \ln n}}$. We will show that H has a panchromatic coloring with r colors.

We define a special random order on the set V of vertices of hypergraph H using a mapping $\sigma : V \rightarrow [0, 1]$, where $\sigma(v), v \in V$ - i.i.d. with uniform distribution on $[0, 1]$. The value $\sigma(v)$ we will call the *weight* of the vertex v . Reorder the vertices so that $\sigma(v_1) < \dots < \sigma(v_{|V|})$. Put

$$p = \left(\frac{r-1}{r}\right) \frac{(r-1)^2 \ln\left(\frac{n}{\ln n}\right)}{n}. \quad (9)$$

We divide the unit interval $[0, 1)$ into subintervals $\Delta_1, \delta_1, \Delta_2, \delta_2, \dots, \Delta_r$ as on the Figure 1, i.e.

$$\Delta_i = \left[(i-1) \left(\frac{1-p}{r} + \frac{p}{r-1} \right), i \cdot \frac{1-p}{r} + (i-1) \cdot \frac{p}{r-1} \right), \quad i = 1, \dots, r;$$

$$\delta_i = \left[i \cdot \frac{1-p}{r} + (i-1) \cdot \frac{p}{r-1}, i \left(\frac{1-p}{r} + \frac{p}{r-1} \right) \right), \quad i = 1, \dots, r-1.$$

The length of each large subinterval Δ_i is equal to $\frac{1-p}{r}$ and every small subinterval δ_i has length equal to $\frac{p}{r-1}$. Since $p < \frac{1}{100r}$ under the given assumptions on r , we can see that the intervals $\Delta_1, \dots, \Delta_r$ are each wider than the intervals $\delta_1, \dots, \delta_{r-1}$. A vertex v is said to belong to a subinterval $[c, d)$, if $\sigma(v) \in [c, d)$. We note that the same division of the segment $[0, 1]$ has already been used by the first author for proving some bounds on proper colorings [1].

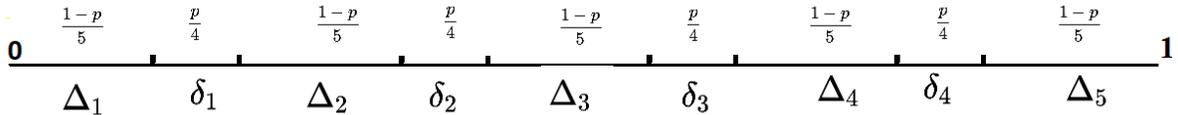


Figure 1: Partition of $[0, 1)$ into $\Delta_1, \delta_1, \Delta_2, \delta_2, \dots, \Delta_5$ when $r = 5$.

We color the vertices of hypergraph H according to the following algorithm, which consists of two steps.

1. First, each $v \in \Delta_i$ is colored with color i for every $i \in [r]$.
2. Then, moving with the growth of σ , we color a vertex $v \in \delta_i$ with color i if there exists an edge $e, v \in e$ such that e does not have color i in the current coloring. Otherwise we color v with color $i + 1$.

4 Analysis of the algorithm

4.1 Short edge

We say that an edge A is *short* if $A \cap (\Delta_i \cup \delta_i) = \emptyset$ or $A \cap (\Delta_{i+1} \cup \delta_i) = \emptyset$ for some $i \in [r-1]$. The probability of this event for fixed edge A and fixed i is at most $2 \left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1}\right)\right)^n$. Summing up this upper bound over all edges and $i \in [r-1]^n$ we get

$$2(r-1)|E| \left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1}\right)\right)^n \leq \frac{2(r-1)}{20r^2} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(\frac{r}{r-1}\right)^n \cdot \left(\frac{r-1}{r} - \frac{p}{r(r-1)}\right)^n \leq \frac{1}{10r} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(1 - \frac{p}{(r-1)^2}\right)^n \leq \frac{1}{10r}.$$

Hence, we conclude that the expected number of short edges is less than $1/10r$, hence with probability at least $1 - 1/10r$ there is no short edge.

4.2 Snake ball

Suppose our algorithm fails to produce a panchromatic r -coloring and there is no short edges. Let A be an edge, which does not contain some color i .

Now we have two possibilities:

- $i < r$, in this situation edge A is disjoint from the interval $\Delta_i \cup \delta_i$, which means that A is short, a contradiction.
- $i = r$.

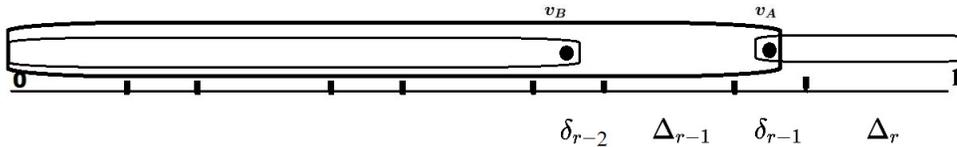


Figure 2: Edges A and B in a snake ball.

Edge A is not short, so $A \cap (\delta_{r-1} \cup \Delta_r) \neq \emptyset$. Since A does not contain color r we have $A \cap \Delta_r = \emptyset$. Denote v_A the last vertex of $A \cap \delta_{r-1}$. We note that v_A could receive color $r-1$ only if at the moment of coloring v_A there was an edge B without color $r-1$ and v_A was the first vertex of $B \cap \delta_{r-1}$. In this situation we say that the pair (A, B) is *conflicting in δ_{r-1}* and the vertex v_A is *dangerous vertex in δ_{r-1}* .

Again, edge B is not short and did not contain color $r-1$ at the moment of coloring v_A , so $B \cap (\delta_{r-2} \cup \Delta_{r-1}) \neq \emptyset$ and $B \cap \Delta_{r-1} = \emptyset$. For v_B , the last vertex of $B \cap \delta_{r-2}$, there exists an edge C , which at the moment of coloring v_B was without color $r-2$ and v_B was the first vertex of $C \cap \delta_{r-2}$. We get (B, C) is conflicting pair in δ_{r-2} and v_B is *dangerous vertex in δ_{r-2}* .

Repeating the above arguments, we obtain a construction called *snake ball*. It is an edge sequence $H' = (C_1 = A, C_2 = B, \dots, C_r)$ such that consecutive edges (C_i, C_{i+1}) form conflicting pairs in δ_{r-i} .

Summarizing the above, we can say that

Claim 1. *If for injective $\sigma : V \rightarrow [0;1)$ there are neither snack balls nor short edges then Algorithm 1 produces a panchromatic r -coloring.*

Lemma 1. *Let $H' = (C_1, \dots, C_r)$ be an ordered r -tuple of edges in the hypergraph H . Then the probability of the event that H' forms a snake ball and all the edges C_1, \dots, C_r are not short does not exceed*

$$\left(\frac{p}{r-1}\right)^{r-1} \left(\frac{r-1}{r}\right)^{(n-2)r} \prod_{v \in H': s(v) \geq 2} \frac{(1 - s(v) \frac{1-p}{r})}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right) s(v)\right)} \prod_{i=1}^{r-1} |C_i \cap C_{i+1}|,$$

where $s(v)$ is the number of edges of H' that contain vertex v .

Before we present the proof of this lemma, we introduce some facts and give the basic scheme of the proof. Note that if $v \in C_i$ then $\sigma(v) \notin \Delta_{r-i+1}$. Furthermore, for each v its weight $\sigma(v)$ belongs to the subintervals of total length at most

$$1 - s(v) \frac{1-p}{r}. \quad (10)$$

The scheme of the proof is following:

- fix vertex $v_j \in C_j \cap C_{j+1}$ and its weight $\sigma(v_j)$ for all $j = 1, \dots, r-1$. Assuming that v_j is the dangerous vertex in δ_{r-j} calculate conditional probability given weights of dangerous vertices.
- sum up (integrate) the previous probability over all possible values of weights, using that $\sigma(v_j) \in \delta_{r-j}$, as this is needed for H' to be a snake ball.
- Finally, sum over all choices of v_1, \dots, v_{r-1} .

Proof. Fix dangerous vertex $v_j \in C_j \cap C_{j+1}$ for each $j = 1, \dots, r-1$. Put $[\alpha_j, \beta_j) = \delta_j$, $\beta_j - \alpha_j = p/(r-1)$ and $y_j = \beta_{r-j} - \sigma(v_j)$. Recall that $0 \leq y_j \leq p/(r-1)$.

Fix for a moment variables y_1, \dots, y_{r-1} . Then, for $v \in C_i$ with $s(v) = 1$ its weight $\sigma(v)$ belongs to the subinterval of total length at most

$$1 - \left(\frac{1-p}{r} + y_{i+1} + \frac{p}{r-1} - y_i\right) \quad \text{if } i \in [2, r-1].$$

And similarly, $1 - \left(\frac{1-p}{r} + y_1\right)$ for $i = 1$ and $1 - \left(\frac{1-p}{r} + \frac{p}{r-1} - y_{r-1}\right)$ for $i = r$.

Now we are ready to give an upper bound for the probability of the event that “ H' forms a snake ball”, conditional on the value taken by y_1, \dots, y_{r-1} :

$$\left(1 - \left(\frac{1-p}{r} + y_1\right)\right)^{n-1} \cdot \left(1 - \left(\frac{1-p}{r} + y_2 + \frac{p}{r-1} - y_1\right)\right)^{n-2} \cdot \dots \cdot \quad (11)$$

$$\cdot \left(1 - \left(\frac{1-p}{r} + y_{r-1} + \frac{p}{r-1} - y_{r-2}\right)\right)^{n-2} \cdot \left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1} - y_{r-1}\right)\right)^{n-1} \cdot \quad (12)$$

$$\cdot \prod_{v \in H': s(v) \geq 2} \frac{\left(1 - s(v) \frac{1-p}{r}\right)}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^{s(v)}}. \quad (13)$$

Here we estimated as if all the rest of the vertices have $s(v) = 1$ (factors (11) and factor (12)), and then using (10), edited for vertices with $s(v) > 1$ by multiplying by $1 - s(v) \frac{1-p}{r}$ and divided by $\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^{s(v)}$. The factor $\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)$ is obviously no more than any factor for $s(v) = 1$, so we get a correct upper bound.

Taking out factor $\left(\frac{r-1}{r}\right)^{(n-2)r+2}$ in the above equation and using estimate $(1+y)^s \leq \exp\{ys\}$, we get the following upper bound on product of (11) and (12):

$$\begin{aligned} & \left(\frac{r-1}{r}\right)^{r(n-2)+2} \exp\left(\frac{(n-1)p}{r-1} - \frac{(n-2)p}{r-1} - \frac{p}{(r-1)^2} - \frac{ry_1}{r-1} + \frac{ry_{r-1}}{r-1}\right) \leq \\ & \left(\frac{r-1}{r}\right)^{r(n-2)+2} \exp\left(\frac{p(r-2)}{(r-1)^2} + \frac{ry_{r-1}}{r-1}\right) \leq \left(\frac{r-1}{r}\right)^{r(n-2)+2} \exp\left(\frac{p(r-2)}{(r-1)^2} + \frac{rp}{(r-1)^2}\right) = \\ & \left(\frac{r-1}{r}\right)^{r(n-2)+2} \exp\left(\frac{2p}{r-1}\right) < \left(\frac{r-1}{r}\right)^{r(n-2)}. \end{aligned}$$

To obtain the final estimate, we have to integrate over the weights y_1, y_2, \dots, y_{r-1} (factor $(p/(r-1))^{r-1}$) and sum up over all possible choices for the v_1, \dots, v_{r-1} (factor $\prod_{i=1}^{r-1} |C_i \cap C_{i+1}|$). \square

5 Auxiliary calculations

Under the assumptions of Theorem 1 we will formulate and prove three auxiliary lemmas needed to prove Theorem 1. In particular, in Lemma 2, we replace product of pairwise intersections on their sum $\sum_{i < j} |C_i \cap C_j|$ and in Lemma 4, we will use double-counting for estimating the sum $\sum_{i < j} |C_i \cap C_j|$, which can be large with n , by special bounded terms.

Lemma 2. *Let $H' = (C_1, \dots, C_r)$ be an ordered r -tuple of edges in the hypergraph H . Then*

$$\sum_{\pi \in S_r} |C_{i_1} \cap C_{i_2}| |C_{i_2} \cap C_{i_3}| \cdot \dots \cdot |C_{i_{r-1}} \cap C_{i_r}| \leq \left(\frac{2 \sum_{i < j} |C_i \cap C_j| + r}{r}\right)^r, \quad (14)$$

where S_r denotes all permutations $\pi = (i_1, \dots, i_r)$ of $(1, 2, \dots, r)$.

Proof. Denote the cardinality of the edge intersection $|C_i \cap C_j|$ by $x_{i,j}$. Then, we have to prove that

$$\sum_{\pi \in S_r} x_{i_1, i_2} x_{i_2, i_3} \cdots x_{i_{r-1}, i_r} \leq \left(\frac{2 \sum_{i < j} x_{i,j} + r}{r} \right)^r.$$

First, we will show that

$$\sum_{\pi \in S_r} x_{i_1, i_2} x_{i_2, i_3} \cdots x_{i_{r-1}, i_r} \leq (x_{1,2} + \cdots + x_{1,r} + 1) \cdots (x_{r,1} + \cdots + x_{r,r-1} + 1). \quad (15)$$

Let us call $(x_{i_1} + \cdots + x_{i_r} + 1)$ from (15) the *bracket number* i . We define a mapping f between elements from the left-hand side of (15) and ordered sets that are obtained after performing the multiplication in (15).

Let $f : x_{i_1, i_2} x_{i_2, i_3} \cdots x_{i_{r-1}, i_r} \mapsto x_{1, t_1} x_{2, t_2} \cdots x_{r, t_r}$, where $x_{1, t_1} x_{2, t_2} \cdots x_{r, t_r}$ is the product of the following r elements: x_{i_{r-1}, i_r} from the bracket number i_{r-1} , $x_{i_{r-2}, i_{r-1}}$ from the bracket number i_{r-2} and so forth, finally we take the factor 1 from the unused bracket. For example,

$x_{5,6} x_{6,1} x_{1,4} x_{4,3} x_{3,2}$ is mapped to $x_{1,4} \cdot 1 \cdot x_{3,2} \cdot x_{4,3} \cdot x_{5,6} \cdot x_{6,1}$.

We note that f is an injection. Indeed, for each $x_{1, t_1} x_{2, t_2} \cdots x_{r, t_r}$ there exists at most one sequence $x_{i_1, i_2} x_{i_2, i_3} \cdots x_{i_{r-1}, i_r}$, with $i_1 \neq i_2 \neq \cdots \neq i_r$, such as $f(x_{i_1, i_2} x_{i_2, i_3} \cdots x_{i_{r-1}, i_r}) = x_{1, t_1} \cdots x_{r, t_r}$.

So, since f does not change the product and f is an injection we get that the right-hand side of (15) is not less than the left-hand side.

Finally, by the inequality on the arithmetic-geometric means and by $x_{i,j} = x_{j,i}$

$$(x_{1,2} + \cdots + x_{1,r} + 1) \cdots (x_{r,1} + \cdots + x_{r,r-1} + 1) \leq \left(\frac{2 \sum_{i < j} x_{i,j} + r}{r} \right)^r.$$

□

Lemma 3. For all $s \in \{2, \dots, r-1\}$

$$\frac{\left(1 - s \frac{1-p}{r}\right)}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^s} \leq e^{-\frac{s^2}{20r^2}}. \quad (16)$$

Proof. First prove the case $s \geq 3$.

$$\frac{\left(1 - \frac{s(1-p)}{r}\right)}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^s} = \frac{\left(1 - \frac{s(1-p)}{r}\right)}{\left(1 - \left(\frac{1-p}{r}\right)\right)^s \left(1 - \frac{2pr}{(r-1)(r-1+p)}\right)^s} \leq \frac{\left(1 - \frac{s(1-p)}{r}\right) \left(1 + \frac{1-p}{r-1+p}\right)^s}{\left(1 - \frac{2pr}{(r-1)^2}\right)^s}. \quad (17)$$

Now we deal with factors in (17) separately:

$$\begin{aligned} \left(1 + \frac{1-p}{r-1+p}\right)^s &\leq \left(1 + \frac{1-p}{r-1}\right)^s = |\text{Apply Taylor's formula with Lagrange Remainder}|= \\ &1 + \frac{s(1-p)}{r-1} + \frac{s(s-1)(1-p)^2}{2(r-1)^2} + \frac{s(s-1)(s-2)(1-p)^3(1+\theta \cdot \frac{1-p}{r-1})^{s-3}}{6(r-1)^3} \leq \\ &\text{bound } (s-1)/(r-1) \text{ by } s/r, (s-1)(s-2)/(r-1)^2 \text{ by } s^2/r^2 \text{ and } (1+\theta/(r-1))^{s-3} \text{ by } e. \\ &\leq 1 + \frac{s(1-p)}{r-1} + \frac{s^2(1-p)}{2r(r-1)} + \frac{s^3(1-p)^2 e}{6r^2(r-1)}. \end{aligned}$$

Hence, the numerator of (17) does not exceed

$$\begin{aligned} \left(1 - \frac{s(1-p)}{r}\right) \left(1 + \frac{s(1-p)}{r-1} + \frac{s^2(1-p)}{2r(r-1)} + \frac{s^3(1-p)^2}{2r^2(r-1)}\right) &< 1 - \frac{s^2(1-p)}{r(r-1)}(1-p-1/2) + \\ \frac{s(1-p)}{r(r-1)} &= 1 - \frac{s^2(1-p)}{r(r-1)}(1/2 - 1/s - p) < 1 - \frac{s^2(1/6 - p)(1-p)}{r^2} < 1 - \frac{s^2}{7r^2} \leq \exp\left\{-\frac{s^2}{7r^2}\right\}. \end{aligned}$$

Using bounds $1/(1-x) < 1+2x$ for $x < 1/2$ and estimating $pr < 1/100$, which follows from restrictions on r , we finally get

$$\begin{aligned} \frac{\left(1 - s\frac{1-p}{r}\right)}{\left(1 - \left(\frac{1-p}{r} + \frac{2pr}{r-1}\right)\right)^s} &\leq \exp\left\{-\frac{s^2}{7r^2}\right\} \left(1 - \frac{2pr}{(r-1)^2}\right)^{-s} < \exp\left\{-\frac{s^2}{7r^2}\right\} \left(1 + \frac{4pr}{(r-1)^2}\right)^s \leq \\ \exp\left\{\frac{4pr s}{(r-1)^2} - \frac{s^2}{7r^2}\right\} &\leq \exp\left\{\frac{s}{25(r-1)^2} - \frac{s^2}{7r^2}\right\} < \exp\left\{\frac{4}{25s} \cdot \frac{s^2}{r^2} - \frac{s^2}{7r^2}\right\} < \exp\left\{-\frac{s^2}{20r^2}\right\}. \end{aligned}$$

Consider the case $s = 2$.

$$\begin{aligned} \frac{1 - 2(1-p)/r}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^2} &\leq \frac{1 - 2(1-p)/r}{1 - \frac{2(1-p)}{r} - \frac{4p}{(r-1)} + \frac{1}{2r^2}} \leq \frac{1 - 2(1-p)/r}{1 - \frac{2(1-p)}{r} - \frac{1}{4r^2} + \frac{1}{2r^2}} = 1 - \frac{1/4r^2}{1 - 2\frac{1-p}{r} + \frac{1}{4r^2}} \\ &\leq 1 - 1/4r^2 \leq \exp\{-1/4r^2\} < \exp\{-1/5r^2\}, \end{aligned}$$

where we used that $4p/(r-1) < 8p/r = 8pr/r^2 < 8/100r^2 < 1/4r^2$. \square

Lemma 4.

$$\left(\prod_{v \in H': s(v) \geq 2} \frac{\left(1 - s(v)\frac{1-p}{r}\right)}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^{s(v)}}\right) \cdot \sum_{\sigma \in S_r} |C_{i_1} \cap C_{i_2}| |C_{i_2} \cap C_{i_3}| \cdots |C_{i_{r-1}} \cap C_{i_r}| \leq 20^r r^{2r} e^{-r+1} \quad (18)$$

Proof. By Lemmas 2 and 3 the left hand side of (18) does not exceed

$$\exp\left\{-\sum_{v \in H': s(v) \geq 2} \frac{s^2(v)}{20r^2}\right\} \left(\frac{2\sum_{i < j} |C_i \cap C_j| + r}{r}\right)^r.$$

Now we will use the following double-counting: $\sum_{i < j} |C_i \cap C_j|$ is equal to $\sum_{v \in H': s(v) \geq 2} \binom{s(v)}{2} < 1/2 \sum_{v \in H': s(v) \geq 2} s^2(v)$. Hence,

$$\begin{aligned} \exp \left\{ - \sum_{v \in H': s(v) \geq 2} \frac{s^2(v)}{20r^2} \right\} \left(\frac{2 \sum_{i < j} |C_i \cap C_j| + r}{r} \right)^r &\leq \exp \left\{ - \sum_{v \in H': s(v) \geq 2} \frac{s^2(v)}{20r^2} \right\} \cdot r^r \\ \cdot \left(\frac{\sum_{v \in H': s(v) \geq 2} s^2(v) + r}{r^2} \right)^r &\leq r^r e^{-t/20} (t+1)^r \leq \frac{20^r r^{2r}}{e^{r-1}}, \end{aligned}$$

where we used $t = \sum_{v \in H': s(v) \geq 2} s^2(v)/r^2$ and observed that the expression $((t+1)^r e^{-t/20})$ is maximized when $t = 20r - 1$. \square

6 Proof of Theorem 1

We want to show that there is a positive probability that no edge is short and no tuple of edges form a snake ball.

Denote \sum^* the sum over all r -sets $J \subseteq (1, 2, \dots, |E|)$, \sum^o the sum over all ordered r -tuples (j_1, \dots, j_r) , with $\{j_1, \dots, j_r\}$ forming such a J and $\sum_{\pi \in S_r}$ denote the sum over all permutations $\pi = (i_1, \dots, i_r)$ of $(1, 2, \dots, r)$.

In Section 4.1 we already proved that the expected number of short edges does not exceed $1/(10r)$. The expected number of snake ball can be upper bounded as follows:

$$\sum^o \mathbb{P}((C_{j_1}, \dots, C_{j_r}) \text{ forms a snake ball}) = \sum^* \sum_{\pi \in S_r} \mathbb{P}((C_{i_1}, \dots, C_{i_r}) \text{ forms a snake ball}).$$

On the other hand,

$$\begin{aligned} &\sum_{\pi \in S_r} \mathbb{P}((C_{i_1}, \dots, C_{i_r}) \text{ forms a snake ball}) \\ &\leq \sum_{\pi \in S_r} \left(\frac{p}{r-1} \right)^{r-1} \left(\frac{r-1}{r} \right)^{(n-2)r} \prod_{v \in H': s(v) \geq 2} \frac{(1 - s(v) \frac{1-p}{r})}{(1 - (\frac{1-p}{r} + \frac{2p}{r-1}))^{s(v)}} |C_{i_1} \cap C_{i_2}| \dots |C_{i_{r-1}} \cap C_{i_r}| \\ &= \left(\frac{p}{r-1} \right)^{r-1} \left(\frac{r-1}{r} \right)^{(n-2)r} \prod_{v \in H': s(v) \geq 2} \frac{(1 - s(v) \frac{1-p}{r})}{(1 - (\frac{1-p}{r} + \frac{2p}{r-1}))^{s(v)}} \sum_{\pi \in S_r} |C_{i_1} \cap C_{i_2}| \dots |C_{i_{r-1}} \cap C_{i_r}| \\ &\leq \left(\frac{p}{r-1} \right)^{r-1} \left(\frac{r-1}{r} \right)^{(n-2)r} \frac{20^r r^{2r}}{e^{r-1}} \leq \left(\frac{(r-1)^2 \ln(\frac{n}{\ln n})}{rn} \right)^{r-1} \cdot \left(\frac{r-1}{r} \right)^{(n-2)r} \cdot \frac{20^r r^{2r}}{e^{r-1}}, \end{aligned}$$

where for the first inequality we used Lemma 1 and for the second Lemma 4 and in the final

inequality we took p from 9. Finally,

$$\begin{aligned} & \sum_{\pi \in S_r}^* \mathbb{P}((C_{i_1}, \dots, C_{i_r}) \text{ forms a snake ball}) \leq \\ & \binom{|E|}{r} \cdot \left(\frac{(r-1)^2 \ln(\frac{n}{\ln n})}{rn} \right)^{r-1} \cdot \left(\frac{r-1}{r} \right)^{(n-2)r} \cdot \frac{20^r r^{2r}}{e^{r-1}} \leq \\ & \frac{\left(\frac{1}{20r^2} \left(\frac{n}{\ln n} \right)^{\frac{r-1}{r}} \left(\frac{r}{r-1} \right)^n \right)^r}{r!} \cdot \left(\frac{(r-1)^2 \ln(\frac{n}{\ln n})}{rn} \right)^{r-1} \cdot \left(\frac{r-1}{r} \right)^{(n-2)r} \cdot \frac{20^r r^{2r}}{e^{r-1}} \leq \frac{1}{r} \left(\frac{r}{r-1} \right)^2. \end{aligned}$$

Since $1 - \frac{1}{10r} - \frac{1}{r} \left(\frac{r}{r-1} \right)^2 > 0$, with positive probability the Algorithm creates a panchromatic coloring with r colors, which proves Theorem 1.

Corollary 2. *There is an absolute constant c so that for every $n > 2$ and $\ln n < r < \sqrt[3]{\frac{n}{100 \ln n}}$*

$$p(n, r) \geq c \frac{n}{r^2 (\ln n)} \cdot e^{\frac{n}{r} + \frac{n}{2r^2}}.$$

Proof. By applying Taylor's formula with Peano remainder, we obtain

$$\left(1 + \frac{1}{r-1} \right) e^{-\frac{1}{r} - \frac{1}{2r^2}} = 1 + \frac{1}{3r^3} + O\left(\frac{1}{r^4}\right).$$

Thus, $\left(1 + \frac{1}{r-1} \right) > e^{\frac{1}{r} + \frac{1}{2r^2}}$. Finally, we use $\left(\frac{n}{\ln n} \right)^{-\frac{1}{r}} > \frac{1}{e}$ when $r > \ln n$ and Theorem 1. \square

7 Local variant: proof of Theorem 2

A useful parameter of H is its *maximal edge degree*

$$D := D(H) = \max_{e \in E(H)} |\{e' \in E(H) : e \cap e' \neq \emptyset\}|.$$

We show that for $3 < r < \sqrt[3]{\frac{n}{100 \ln n}}$ every n -uniform hypergraph with $D \leq \frac{1}{40r^3} \left(\frac{n}{\ln n} \right)^{\frac{r-1}{r}} \left(\frac{r}{r-1} \right)^n$ has a panchromatic coloring with r colors, which implies Theorem 2.

Let us recall Lovász Local Lemma, which shows a useful sufficient condition for simultaneously avoiding a set A_1, A_2, \dots, A_N of “bad” events:

Lemma 5 (The Local Lemma; General Case, [8]). *Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. A directed graph $\bar{D} = (V, E)$ on the set of vertices $V = \{1, 2, \dots, n\}$ is a dependency digraph for the events A_1, \dots, A_n if for each $i, 1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j : (i, j) \notin E\}$. Suppose that $\bar{D} = (V, E)$ is a dependency digraph*

for the above events and suppose there are real numbers x_1, \dots, x_n such that $0 \leq x_i < 1$ and $\mathbb{P}[A_i] \leq x_i \prod_{(i,j) \in E} (1 - x_j)$ for all $1 \leq i \leq n$. Then

$$\mathbb{P} \left[\bigwedge_{i=1}^n \overline{A_i} \right] \geq \prod_{i=1}^n (1 - x_i).$$

In particular, with positive probability, no event A_i holds.

To prove Theorem 2 we will use the following generalization of Lemma 5.

Lemma 6. *If all events have probability $\mathbb{P}(A_i) \leq \frac{1}{2}$, and for all i*

$$\sum_{j:(i,j) \in E} \mathbb{P}(A_j) \leq \frac{1}{4}, \tag{19}$$

then there is a positive probability that no A_i holds.

For the sake of completeness, we give the proof of Lemma 6 here.

Proof. Put $x_i = 2\mathbb{P}(A_i)$. Then, for all i

$$x_i \prod_{(i,j) \in E} (1 - x_j) = 2\mathbb{P}(A_i) \prod_{(i,j) \in E} (1 - 2\mathbb{P}(A_j)) \geq \mathbb{P}(A_i).$$

□

In our case the set of bad events has two types: short edges and snake balls. Let $\mathcal{Q}(C)$ be the event “edge C is short” and $\mathcal{W}(C_1, \dots, C_r)$ be the event “ (C_1, \dots, C_r) forms a snake ball and all the edges C_1, \dots, C_r are not short”. Note that $\mathcal{Q}(C)$ depends on at most on $D + 1$ events $\mathcal{Q}(C')$ and at most on $2r(D + 1)D^{r-1}$ events $\mathcal{W}(C_1, \dots, C_r)$. Similarly, $\mathcal{W}(C_1, \dots, C_r)$ depends at most on $r(D + 1)$ events $\mathcal{Q}(C')$ and at most on $2r^2(D + 1)D^{r-1}$ events $\mathcal{W}(C'_1, \dots, C'_r)$. Hence, using bounds from Sections 4.1 and 6 we get the following upper bounds:

1. if $A_i = \mathcal{W}(C_1, \dots, C_r)$:

$$\begin{aligned} \sum_{j:(i,j) \in E} \mathbb{P}(A_j) &\leq r(D + 1) \cdot 2(r - 1) \left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1} \right) \right)^n + \\ &+ 2r^2(D + 1)D^{r-1} \cdot \left(\frac{r-1}{r} \right)^{(n-2)r} \left(\frac{p}{r-1} \right)^{r-1} \frac{20^r r^{2r}}{e^{r-1}} < \frac{2r^2}{40r^3} + \frac{2r^2}{r2^r e^{r-1}} < \frac{1}{4}. \end{aligned}$$

2. if $A_i = \mathcal{Q}(C)$:

$$\begin{aligned} \sum_{j:(i,j) \in E} \mathbb{P}(A_j) &\leq (D + 1) \cdot 2(r - 1) \left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1} \right) \right)^n + \\ &+ 2r(D + 1)D^{r-1} \cdot \left(\frac{r-1}{r} \right)^{(n-2)r} \left(\frac{p}{r-1} \right)^{r-1} \frac{20^r r^{2r}}{e^{r-1}} < \frac{1}{4}. \end{aligned}$$

In both cases inequality (19) holds, completing the proof of Theorem 2.

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