Remarks on odd colorings of graphs

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Abstract

A proper vertex coloring φ of graph G is said to be odd if for each non-isolated vertex $x \in V(G)$ there exists a color c such that $\varphi^{-1}(c) \cap N(x)$ is odd-sized. The minimum number of colors in any odd coloring of G, denoted $\chi_o(G)$, is the odd chromatic number. Odd colorings were recently introduced in [M. Petruševski, R. Škrekovski: Colorings with neighborhood parity condition]. Here we discuss various basic properties of this new graph parameter, characterize acyclic graphs and hypercubes in terms of odd chromatic number, establish several upper bounds in regard to degenericity or maximum degree, and pose several questions and problems.

Keywords: neighborhood, proper coloring, odd coloring, odd chromatic number.

1 Introduction

All considered graphs are simple, finite and undirected. We follow [5] for any terminology and notation not defined here. A k-(vertex-)coloring of a graph G is an assignment φ : $V(G) \to \{1, \ldots, k\}$. A coloring φ is said to be proper if every color class is an independent subset of the vertex set of G. A hypergraph $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is a generalization of a graph, its (hyper-)edges are subsets of $V(\mathcal{H})$ of arbitrary positive size. There are various notions of (vertex-)coloring of hypergraphs, which when restricted to graphs coincide with proper graph coloring. One such notion was introduced by Cheilaris et al. [8]. An odd coloring of hypergraph \mathcal{H} is a coloring such that for every edge $e \in \mathcal{E}(\mathcal{H})$ there is a color e0 with an odd number of vertices of e1 colored by e2. Particular features of the same notion notion (under the name weak-parity coloring) have been considered by Fabrici and Göring [9] (in regard to face-hypergraphs of planar graphs) and also by Bunde et al. [6] (in regard to coloring of graphs with respect to paths, i.e., path-hypergraphs).

Of recent interest are also edge-colorings of graphs with certain parity condition required at the vertices. We refer the reader to [4, 12, 13, 17] for edge-colorings which require an odd number of occurrences of every color that appears at a vertex; edge-colorings with the

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weaker assumption that at least one color per vertex has an odd number of occurrences are studied in [16]. The natural generalizations obtained by assigning a parity signature to each vertex and asking that the obtained parity condition is fulfilled by every (resp. some) color appearing at a vertex are considered in [14]. The survey [18] deals with analogous covering aspects.

In this paper we study certain aspects of odd colorings for graphs with respect to (open) neighborhoods, that is, the colorings of graph G such that for every non-isolated vertex x there is a color that occurs an odd number of times in the neighborhood $N_G(x)$. Our focus is on colorings that are at the same time proper.

As defined in [20], a proper coloring of a graph G is odd if in the open neighborhood N(v) of every non-isolated vertex v a color appears an odd number of times. Denote by $\chi_o(G)$ the minimum number of colors in any odd coloring of G, and call this the odd chromatic number of G. Note that the obvious inequality $\chi(G) \leq \chi_o(G)$ becomes an equality whenever the graph G is odd, that is, if it has only odd vertex degrees. On the other hand, the mentioned inequality may also be strict. In fact, the ratio $\chi_o(G)/\chi(G)$ can acquire arbitrarily high values. Indeed, consider a non-empty graph H and let G be the graph obtained from H by subdividing every edge in E(H) once; that is, let G be the complete subdivision of H. Then $\chi_o(G) \geq \chi(H)$ whereas $\chi(G) = 2$. In particular, for every $n \geq 2$, the complete subdivision of K_n has odd chromatic number $\geq n$ and (ordinary) chromatic number 2.

Given a graph G, let L be a function which assigns to each vertex v of G a set L(v), called the list of v. An odd coloring c of G such that $c(v) \in L(v)$ for all $v \in V(G)$ is called an odd list coloring of G with respect to L, or an odd L-coloring, and we say that G is odd L-colorable. A graph G is said to be odd k-list-colorable if it has an odd list coloring whenever all the lists are of length k. Every graph is clearly odd n-list colorable, where n is the order of G. The smallest value of k for which G is odd k-list colorable is called the odd list chromatic number (or odd choice number) of G, denoted $\operatorname{ch}_o(G)$. Note that the obvious inequality $\chi_o(G) \leq \operatorname{ch}_o(G)$ can be strict. In fact, the ratio $\operatorname{ch}_o(G)/\chi_o(G)$ can be arbitrarily large. Indeed, for any odd positive integer n it holds that $\operatorname{ch}_o(K_{n,n^n}) = n+1$ whereas $\chi_o(K_{n,n^n}) = 2$.

The paper is organized as follows. The next section collects several basic observations for the odd chromatic number. In Section 3 we characterize acyclic graphs and hypercubes in terms of the same graph parameter. This is followed by a section on general upper bounds. At the end comes a short Section 5, where we briefly convey several ideas for possible further work on the topic of odd colorings of graphs.

2 Basic properties of the odd chromatic number

Here we compare the behavior of the odd chromatic number to the behavior of the (ordinary) chromatic number in regard to standard graph notions and concepts. The followings observations are rather easy, but nevertheless their further improvements will give new interesting results and certainly will help for better understanding of this new concept.

Complexity. We start this section with a brief discussion concerning the complexity of determining the odd chromatic number.

Observation 1. The problem of determining $\chi_o(G)$ for a graph G is NP-hard.

Proof. Since determining the chromatic number of a graph is an NP-hard problem, the same holds for the odd chromatic number. Reduction is obvious, to a given graph H attach a leaf to every vertex of positive even degree, and denote the resulting odd graph by G. Obviously, $\chi(H) = \chi(G) = \chi_o(G)$.

Note that graphs with chromatic number ≤ 2 are precisely bipartite graphs, but regarding odd colorings a graph without isolated vertices has odd chromatic number 2 if and only if it is bipartite with all vertices of odd degree. Also notice that there is no non-empty graph with odd chromatic number exactly 1.

Bridges. Supposing a given graph has a non-trivial bridge, the usual chromatic number is the maximum of the chromatic numbers of both sides. Regarding the odd chromatic number, things are slightly different.

Observation 2. Suppose G has a non-trivial bridge e = uv with G_1 and G_2 being the two parts of G - e. Then

$$\chi_o(G) \le \max\{3, \chi_o(G_1), \chi_o(G_2)\}.$$
(1)

Proof. We may assume $\chi_o(G_2) \ge \chi_o(G_1)$ and that u is in G_1 and v is in G_2 . Take an odd coloring of G_1 with the colors $1, \ldots, \chi_o(G_1)$ such that the color of u is 1. Also, take an odd coloring of G_2 with the colors $1, 2, \ldots, \chi_o(G_2)$ such that v gets color 2.

If we have used at least three colors for the coloring of G_2 , then we can easily achieve that the color of v is also distinct from a color that appears an odd number of times in $N_{G_1}(u)$ and that the color of u is distinct from a color that appears an odd number of times in $N_{G_2}(v)$. And then so the constructed coloring of G is odd G and uses $\max\{\chi_o(G_1), \chi_o(G_2)\} \geq 3$ colors.

Otherwise, $\chi_o(G_1) = \chi_o(G_2) = 2$. Observe that then it is not possible to odd color G with just two colors. But in this case we simply alter the initial coloring of G_2 by recoloring all vertices of color 2 by 3 and all vertices of color 1 by 2. The modified coloring becomes odd for G.

The above bound is obviously tight for many pairs of graphs G_1, G_2 where at least one of them has odd chromatic number ≥ 3 . In the case $\chi_o(G_1) = \chi_o(G_2) = 2$, the above bound is tight always whenever both G_1 and G_2 are bipartite odd graphs. Also observe that it may happen that $\chi_o(G) \leq \min\{\chi_o(G_1), \chi_o(G_2)\}$, it happens for example when G_1 and G_2 are cycles of particular order.

Introducing/removing a vertex. By introducing a new vertex to an existing graph, the (ordinary) chromatic number either stays the same or increases by 1. There is a similar upper bound for the odd chromatic number of a graph in terms of the odd chromatic number of a vertex-deleted subgraph.

Observation 3. Let G be a graph without isolated vertices. Introduce a new vertex v connected to some/all vertices of G to obtain a graph H. Then, $\chi_o(H) \leq \chi_o(G) + 2$.

Proof. Consider an odd coloring of G with $\chi_o(G)$ colors. Choose a neighbor w of v and give w and v a pair of fresh new colors. So v has the color of w uniquely in N(v), and w has the color of v uniquely in N(w), and all neighbors of w have the color of w appearing uniquely in their neighborhoods.

The above bound can be attained for P_3 with the new vertex v being adjacent to the ends of the path to obtain C_5 . In view of Observation 3, by introducing a new vertex the odd chromatic number cannot increase by much (at most by 2).

As already observed in [20], the parameter χ_o is not monotonic in regard to the subgraph relation. One naturally wonders whether by removing a vertex the odd chromatic number can increase significantly.

Observation 4. With the notation of Observation 3, the difference $\chi_o(G) - \chi_o(H)$ can be arbitrarily large.

Proof. Let G be a bipartite graph of odd order and without isolated vertices. Note that G has an odd number of vertices with even degree. Add a new vertex and connect it by an edge to every vertex in G of even degree. The obtained graph H is 3-colorable and has only odd vertex degrees. Hence $\chi_o(H) \leq 3$. In spite of this, $\chi_o(G)$ can be arbitrarily large. For example, take G to be the graph obtained by subdividing once every edge of K_n , where $n \equiv 1$ or $2 \pmod{4}$; thus G is bipartite of order $\binom{n+1}{2} \equiv 1 \pmod{2}$ and $\chi_o(G) = n$.

Disjoint union. Here we briefly state a partition property concerning the behavior of the odd chromatic number when V(G) admits a representation as a disjoint union $A \cup B$, where the induced subgraphs on A and B are *isolate-free*, that is, are free from isolated vertices. (On the existence of such partitions already in 2-connected graphs see e.g. [11].)

Observation 5. Let $V(G) = A \cup B$ is a disjoint union, where the induced subgraphs G[A] and G[B] are isolate-free. Then

$$\chi_o(G) \le \chi_o(G[A]) + \chi_o(G[B]). \tag{2}$$

Proof. Color the vertices in A with a set of $\chi_o(G[A])$ colors so as to obtain an (optimal) odd coloring of G[A]. Using a disjoint set of $\chi_o(G[B])$ colors, color the vertices in B so as to produce an odd coloring of G[B]. Since A and B are isolate-free, the constructed coloring of G is odd and we have used $\chi_o(G[A]) + \chi_o(G[B])$ colors in total.

Notice that the bound is sharp, e.g. it is achieved by C_4 . From Observation 5 we immediately have the following for the join operation $G \vee H$ when G and H are isolate-free:

Observation 6. If G and H are nontrivial connected graphs then

$$\chi(G) + \chi(H) \le \chi_o(G \lor H) \le \chi_o(G) + \chi_o(H). \tag{3}$$

Proof. Consider V(G) and V(H) as the parts in the partition of $V(G \vee H)$.

Cartesian product. Next we consider the behavior of the odd chromatic number under taking cartesian products. Regarding the usual coloring of cartesian product graphs, there is a beautiful relation, see [21, 3], which claims

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}. \tag{4}$$

The analogous equality for the odd chromatic number is not true in general. For example, $K_2 \square Q_3 = Q_4$ and $\chi_o(K_2) = \chi_o(Q_3) = 2$ whereas $\chi_o(Q_4) = 4$ (cf. Theorem 3.2 in the next section). For the odd chromatic number of a cartesian product we have the following much weaker bound.

Observation 7. If G, H are connected nontrivial graphs, then

$$\chi_o(G \square H) \le \min\{\chi(G) \cdot \chi_o(H), \chi_o(G) \cdot \chi(H)\} \le \chi_o(G) \cdot \chi_o(H). \tag{5}$$

Proof. By symmetry, it suffices to prove that $\chi_o(G \square H) \leq \chi(G) \cdot \chi_o(H)$. Let g be a proper coloring of G with $\chi(G)$ colors, and let h be an odd coloring of H with $\chi_o(H)$ colors that are all distinct from those of G. To any vertex $v = (u, w) \in G \square H$, we assign the pair f(v) = f(u, w) = (g(u), h(w)). We prove that f is an odd coloring of $G \square H$ with $\chi(G) \cdot \chi_o(H)$ colors.

Consider the neighborhood N(v). It is comprised of vertices of two kinds: either of the form (u, w') where $w' \in N_H(w)$, or of the form (u', w) where $u' \in N_G(u)$. Since the colorings g and h are proper, the set of colors used for the vertices of the first kind is disjoint from the set of colors used for the vertices of the second kind. So it suffices to observe that a color has an odd number of occurrences on the vertices of the first kind, as h is an odd coloring of H. Thus f is indeed an odd coloring of $G \square H$ with $\chi(G) \cdot \chi_o(H)$ colors.

The above bound is attained for $G = H = K_2$. For an infinite family of graphs G such that $\chi_o(G \square K_2) = \chi_o(G) \cdot \chi_o(K_2)$ see Theorem 3.2 and the after remark in the next section. We are optimistic regarding the following problem.

Problem 2.1. Improve the bounds of (5) to a bound comparable to the right side of (4).

Domination. We have already mentioned in the introduction that the ratio $\chi_o(G)/\chi(G)$ can be arbitrarily large. Consequently, so can the difference $\chi_o(G) - \chi(G)$. Nevertheless, by using the monotonicity of the (ordinary) chromatic number, we can easily show that $\chi_o(G) - \chi(G)$ never exceeds the total domination number $\gamma_t(G)$.

Observation 8. Every graph G satisfies

$$\gamma_o(G) \le \gamma_t(G) + \gamma(G). \tag{6}$$

Proof. Color each vertex from an optimal total dominating set S with a new color. To the remaining graph G-S apply a proper coloring by new $\chi(G-S) \leq \chi(G)$ colors. This way we use at most $\gamma_t(G) + \chi(G)$ colors to properly color G and each vertex of G sees some color uniquely, namely a color of a vertex of S by which it is dominated.

In fact, the total domination number $\gamma_t(G)$ in (6) can be replaced by the so-called even domination number, $\gamma_e(G)$, defined as follows. We say that $S \subseteq V(G)$ is an even dominating set, if for every vertex v of positive even degree in G, $N_G(v) \cap S \neq \emptyset$. Note that if $v \in S$ is of positive even degree in G then there is another vertex in S (not necessarily of even degree) that dominates it. Let $\gamma_e(G)$ be the minimum size of an even dominating set. Observe that $\gamma_e(G) \leq \gamma_t(G)$ and also $\gamma_e(G) \leq n_e(G)$, where $n_e(G)$ is the number of non-isolated vertices of even degree in G. To our knowledge this domination number was not defined before and it can be of independent interest.

Observation 9. Every graph G satisfies

$$\chi_o(G) \le \gamma_e(G) + \chi(G) \le \min\{n_e(G) + \chi(G), \gamma_t(G) + \chi(G)\}. \tag{7}$$

Proof. Let S be a total even dominating set of size $\gamma_e(G)$ and color all the vertices of S with distinct colors. Consider H = G - S. Since $\chi(H) \leq \chi(G)$, properly color H with at most $\chi(G)$ colors. The constructed coloring of G is odd. Indeed, properness is clear. As for odd occurrence of a color in each non-empty vertex neighborhood, we need to care only for (non-isolated) vertices v of even degree. Such a vertex v is dominated by a vertex from S, which has a unique color. Altogether we use at most $\gamma_e(G) + \chi(G)$ colors. Now the second inequality follows by the remarks stated in front of this observation.

The obtained bound is achieved for even stars $K_{1,2m}$ as $\gamma_e(K_{1,2m}) = 1$ (while $\gamma_t(K_{1,2m}) = 2$) and the bound $\chi_o(K_{1,2m}) = \gamma_e(K_{1,2m}) + \chi(K_{1,2m}) = 3$ is sharp. Also for odd graphs G we have $\gamma_e(G) = 0$ and $\chi_o(G) = \chi(G)$.

3 Characterizations in terms of χ_o

We begin this section by observing that paths and cycles are easily characterized in terms of their odd chromatic number. It is readily checked that for paths holds:

$$\chi_o(P_n) = \begin{cases} n & \text{if } n \le 2; \\ 3 & \text{if } n \ge 3. \end{cases}$$
 (8)

And similarly, for cycles we have:

$$\chi_o(C_n) = \begin{cases} 3 & \text{if } 3 \mid n; \\ 4 & \text{if } 3 \nmid n \text{ and } n \neq 5; \\ 5 & \text{if } n = 5. \end{cases}$$
(9)

Let us now characterize trees.

Proposition 3.1. If T is a non-trivial tree then

$$\chi_o(T) = \begin{cases} 2 & \text{if } T \text{ is odd}; \\ 3 & \text{otherwise}. \end{cases}$$

Proof. Note that T is odd if and only if $\chi_o(T) = \chi(T) = 2$, since T is non-trivial and bipartite. It remains to prove that every tree admits an odd 3-coloring. This is easily seen to hold true if T is a star. For general trees we induct on the order n(T). Let T be a tree that is not a star. Consider a maximum path P in T. Let v and w be, respectively, the second and third vertex of a traversal of P. Then the set $S = N_T(v) \setminus \{w\}$ consists of leaves (i.e. v is an internal leaf of T). By induction, there exists an odd 3-coloring of T - S. We may assume that v and w are colored by 1 and 2, respectively. Extend to T by using the color 3 for S.

Remark. The proof of Proposition 3.1 can be easily modified into proving that the odd list chromatic number of any tree equals its odd chromatic number. Namely, it amounts to observing the following: if all but one leaf of a star is already colored in a way that the obtained partial coloring is odd, then this coloring extends to the remaining vertex provided that we have a choice of three colors for it.

Next we give an analogous characterization for hypercubes. It turns out that the value $\chi_o(Q_n)$ depends solely on the parity of n. Our findings are summarized in the following.

Theorem 3.2. If Q_n is the n-dimensional cube then

$$\chi_o(Q_n) = \begin{cases} 2 & \text{if } n \text{ is odd}; \\ 4 & \text{if } n \text{ is even}. \end{cases}$$

Proof. The case of odd n is rather trivial. Indeed, as then Q_n is odd and bipartite, it holds that $\chi_o(Q_n) = \chi(Q_n) = 2$. Assuming n is even, the equality $Q_n = Q_{n-1} \square K_2$ and Observation 7 give $\chi_o(Q_n) \le \chi_o(Q_{n-1}) \cdot \chi_o(K_2) = 4$. We proceed to show that $\chi_o(Q_n) = 4$. Let us look at Q_n as if comprised of two copies of Q_{n-1} with a perfect matching between them, and call any pair of end-vertices of a matching edge 'allies'; we shall denote the ally of a vertex x by \bar{x} .

Arguing by contradiction, suppose that an odd 3-coloring c of Q_n exists (still assuming n is even). Note that in each vertex neighborhood $N_{Q_n}(v)$ each of the two colors $\neq c(v)$ occurs an odd number of times. Consider the inherited proper 3-coloring of the first copy of Q_{n-1} . Let v be an arbitrary vertex from this copy, and let it be assigned with the color 1 whereas the ally \bar{v} of v is colored by 2. Let U be the intersection of $N_{Q_{n-1}}(v)$ with the color class $[2] = c^{-1}(2)$. Thus U is an even-sized set (possibly empty). Similarly, denote by W the intersection of $N_{Q_{n-1}}(v)$ with the color class [3], hence W is an odd-sized set. Finally, let V be comprised of all second-neighbors of v within the first copy of Q_{n-1} that are colored by 1. Before proceeding further, let us recall the following important property of any hypercube: every pair of vertices in N(v) have a unique neighbor in the second neighborhood of v, and conversely every vertex z in the second neighborhood of v is adjacent to exactly two vertices in N(v).

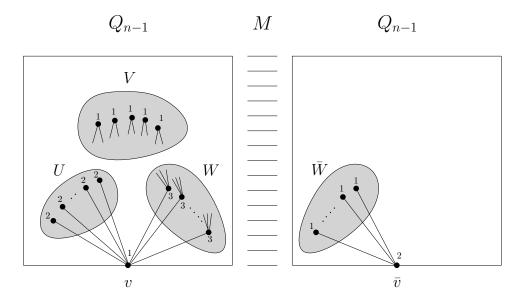


Figure 1: The sets U, V and W. Every vertex from V has two edges going down to $U \cup W$. Every edge from W has an odd number of edges going up to V.

Notice that for every $w \in W$, its ally \bar{w} is adjacent with \bar{v} , implying that \bar{w} is colored by 1. This forces that V is not empty as for each vertex w there are already two neighbors of color 1 (which are v and \bar{w}). Consequently, in the bipartite graph G = [V, W] induced by $V \cup W$, the degree of every w is odd. Thus the size $e(G) = e([V \cup W])$ can be expressed as a sum of an odd number of odd summands, hence it is an odd number. Observe that for every $z \in V$,

there are exactly two edges going from z to $U \cup W$. This gives a partition of V into three (possibly empty) subsets V_{22}, V_{23}, V_{33} , where V_{22} (resp. V_{33}) collects those z's that neighbor with two members of U (resp. W) and V_{23} is comprised of all z's having one neighbor in U and another one in W. Clearly, every vertex of V_{22} is isolated in G, every vertex of V_{23} is a leaf in G, and every vertex of V_{33} has degree 2 in G. Hence counting edges in G from the V side we get $|V_{23}| + 2|V_{33}| = e(G) \equiv 1 \pmod{2}$, and we conclude that $|V_{23}| \equiv 1 \pmod{2}$. So, $U \neq \emptyset$.

Now let us count the number N of 4-cycles lying within the first copy of Q_{n-1} that pass through v, have two vertices colored by 1, one vertex colored by 2 and one vertex colored by 3. Since every $z \in V_{23}$ belongs to exactly one such 4-cycle, we conclude that $N = |V_{23}| \equiv 1 \pmod{2}$. On the other hand, for every vertex u in U and every vertex w in W, there is a unique neighbor z in the second neighborhood of v and this z must be colored 1, hence z is a member of V_{23} . By counting pairs (u, w) in (U, W) that create (1, 2, 3, 1)-colored 4-cycles passing through v, we conclude that there are precisely |U||W| = N such pairs. However, as $|U| \equiv 0 \pmod{2}$, we have $|U||W| \equiv 0 \pmod{2}$, which contradicts the already established $N \equiv 1 \pmod{2}$.

Remark. Recall that N(v) (resp. N[v]) denotes the open (resp. closed) neighborhood of a vertex v in G. Similarly, $N_2(v)$ is the second neighborhood of v. Let us call a bipartite graph G nice if for a vertex v in G holds: (i) every vertex in N[v] is of odd degree; (ii) for every $u, w \in N(v), |N(u) \cap N(w)| = 2$; and (iii) for every $z \in N_2(v), |N(z) \cap N(v)| = 2$. The proof of Theorem 3.2 applies verbatim to the following more general result: If G is a nice graph, then $\chi_o(G \square K_2) = 4$.

4 Upper bounds on the odd chromatic number

In this final section we give and conjecture several upper bounds for the odd chromatic number. In the first third we confine to planar graphs and revisit a conjecture recently posed in [20]. The second third deals with certain degenericity aspects. Finally, in the last third we discuss bounding the odd chromatic numbers by a linear function of the maximum degree.

Planar graphs. The following has been proven in [20].

Theorem 4.1. If G is a connected planar graph, then $\chi_o(G) \leq 9$.

Also there was posed the following conjecture.

Conjecture 4.2. For every planar graph G it holds that $\chi_o(G) \leq 5$.

We prove here this conjecture for outerplanar graphs. Note that the reduction of 2-vertices in the proof bellow is taken from [20].

Proposition 4.3. For every outerplanar graph G it holds that $\chi_o(G) \leq 5$.

Proof. Suppose G is a minimal counter-example. By Proposition 3.1, we may assume that G has a cycle, and so it is of order at least 3. Let $v \in V(G)$ be a vertex of minimum degree. If d(v) = 1, then take an odd 5-coloring of G - v by minimality. This coloring extends to an odd 5-coloring of G since at most two colors are forbidden at v: the color of its only neighbor z and possibly another color that is unique in regard to having an odd number of occurrences in $N_{G-v}(z)$. So we may assume that v is of degree 2, and say $N_G(v) = \{x, y\}$.

Construct G' by removing v from G plus connecting x and y if they are not already adjacent. By minimality, G' admits an odd 5-coloring c. Say c(x) = 1 and c(y) = 2, and let color(s) 1' and 2', respectively, have odd number of occurrences in $N_{G'}(x)$ and $N_{G'}(y)$. If there are more possibilities for 1', then choose $1' \neq 2$; do similarly for 2' in regard to 1. Extend the coloring to G by using for v a color different from 1, 2, 1', 2'. The properness of the coloring is clearly preserved. As for the oddness concerning neighborhoods, v is fine because $1 \neq 2$. If $1' \neq 2$ then x is fine since 1' remains to be odd in $N_G(x)$. Contrarily, if 1' = 2 then c(v) is odd in $N_G(x)$. Similarly, the vertex y is also fine. The obtained contradiction proves our point.

As first support to Conjecture 4.2, it was shown Theorem 4.1, whose supplied proof used the discharging technique without evoking the Four Color Theorem [2, 19]. On the other hand, by making use of the Four Color Theorem, the following has been recently shown (see Theorem 4 in [1]).

Theorem 4.4 (Aashtab et al., 2020). Let G be a connected planar graph of even order. Then V(G) partitions into at most 4 sets such that each partite set induces an odd forest.

From Theorem 4.4 and Observation 5 it immediately follows that

Corollary 4.5. If G is a connected planar graph of even order, then $\chi_o(G) \leq 8$.

Proof. Every odd forest F is isolate-free and has $\chi_o(F) = \chi(F) = 2$. By Observation 5, $\chi_o(G) \le 2 + 2 + 2 + 2 = 8$.

In regard to planar graphs of odd order, we have the following.

Corollary 4.6. If G is a connected planar graph of odd order that has a vertex of degree 2 or any odd degree, then $\chi_o(G) \leq 8$.

Proof. Let v be a vertex with degree in $\{1, 2, 3, 5, 7, \ldots\}$. If d(v) = 2, then we deal as it is already explained in details in the proof of Proposition 4.3: delete v and add an edge between its neighbors if they are not already adjacent; take an odd 8-coloring c of G' by Corollary 4.5, and extend c to G as we have at least four available colors for v.

And, if v is a vertex of odd degree, then attach to it a new leaf w. The obtained graph G' is connected and of even order, hence we can apply Corollary 4.5 and use an odd coloring c of G'. Now observe that c is also an odd coloring of G as therein v is of odd degree and hence the oddness condition at v is satisfied a priori.

Notice that if one is able to somehow reduce vertices of degree 4, then it would yield the conclusion that every planar graph G satisfies $\chi_o(G) \leq 8$. For the time being, we can only reduce them by introducing a new color, say 9, at one 4-vertex. This gives an alternative proof of Theorem 4.1. The succinct proof below is only several lines long, but on other hand it relies on the Four Color Theorem.

Proof of Theorem 4.1 If the graph G is of even order or it has an odd vertex or a vertex of degree 2, then we are done by Corollaries 4.5 and 4.6. Assume all degrees in G are even ≥ 4 . As G is planar, it must have a vertex v of degree 4. Attach a leaf w to v to get G^* which is planar, connected and of even order. Hence $\chi_o(G^*) \leq 8$, and we may consider an odd 8-coloring c of G^* . Say c(w) = 1, c(v) = 2 and let 2^* be a color with an odd number of occurrences in $N_{G^*}(v)$. If $1 \neq 2^*$, then simply remove w and conclude that $\chi_o(G) \leq 8$. So

we may assume $1 = 2^*$ is the only 'odd color' in' the neighborhood of v. Choose $u \in N_G(v)$ and recolor it by setting c(u) = 9. Remove w. The resulting coloring c of G is clearly a proper coloring. All neighbors of u (including v) have the color 9 as an odd color in their respective neighborhoods. As for the vertex u, it has an 'odd color' (induced from G^*) in its neighborhood because no neighbor of it was deleted or changed color.

Degenericity. Let us consider briefly certain aspects of bounding the odd chromatic number of a graph in regard to its degenericity. Start by noting that the graph G obtained by subdividing once every edge of K_n , is a 2-degenerate bipartite graph having $\chi_o(G) \geq n$. Hence a natural question is when does a k-degenerate graph have its odd chromatic number bounded by a linear function of k, for example, $\chi_o(G) \leq 2k + 1$.

Recall the classical notion of a k-tree: it is a graph formed by starting with a copy of K_{k+1} , and then repeatedly adding vertices in such a way that each added vertex v has exactly k neighbors U so that, together, the k+1 vertices formed by v and U form a clique. (Thus 1-trees are the same as trees.) Motivated by this constructive definition of k-trees, we introduce a similar notion. Namely, we say that a connected graph G is a half k-tree if G is established in the following way:

- The initial stage of G is any connected graph H on p vertices, where $2 \le p \le k+1$ and such that $\chi(H) \ge \lfloor p/2 \rfloor + 1$. (In particular, H may contain a clique of order at least $\lfloor p/2 \rfloor + 1$.)
- At each intermediate stage, add a new vertex v, adjacent to at most k vertices such that the subgraph of the current G induced by N(v) has chromatic number at least $\lfloor d(v)/2 \rfloor + 1$, i.e. $\chi(G[N(v)]) \geq \lfloor d(v)/2 \rfloor + 1$. (In particular, N(v) may contain a clique of cardinality at least $\lfloor d(v)/2 \rfloor + 1$.)

Observe that k-trees form a subclass of the newly defined class of half k-trees. The reason for calling it 'half k-tree' is an obvious adoption inspired by k-trees and the fact the for a half k-tree, N(v) induces a subgraph of chromatic number at least $\frac{d(v)}{2} + 1$ (here is where the adjective 'half' comes from whereas in a k-tree this neighborhood gives a clique).

Proposition 4.7. If G is a half k-tree, then $\chi_o(G) \leq 2k+1$.

Proof. The proof goes by induction on the order n of G. For the initial stage H, as $V(H) \le k+1$, we have $\chi_o(H) \le k+1 < 2k+1$. Assume induction holds for n-1 and let G be a connected half k-tree graph on n vertices. Let v be the last vertex in the process of creating G. Then $d(v) = t \le k$ and $\chi(G[N(v)]) \ge \lfloor d(v)/2 \rfloor + 1$. Consider H = G - v. By definition H is a half k-tree; in particular H is connected (by definition, how these graphs are constructed) and by induction $\chi_o(H) \le 2k+1$.

Now consider the neighbors of v, say u_1, \ldots, u_t , where clearly $1 \le t \le k$ and distinguish between the following two cases.

Case 1: t is odd. Then at least one of the colors on the vertices in N(v) appears an odd number of times. Since there are t neighbors of v, all the neighbors forbid at most t colors for v. Since each of the neighbors has at least one 'odd color', such odd colors (one from each neighbor of v) may forbid (if unique) another t colors for v. Hence at most $2t \le 2k$ colors are forbidden, and thus we have an available color for v. As t is odd, the coloring is indeed odd, proving that $\chi_o(G) \le 2k + 1$ holds in this case.

Case 2: t is even. Set t = 2q. By definition, $\chi(G[N(v)]) \ge q + 1$ and at least q + 1 vertices of N(v) receives distinct colors in the coloring of H. The rest q - 1 vertices in N(v) give at least two vertices whose colors appear exactly once in N(v). So N(v) already has an odd coloring. Now as v has $t \le k$ neighbors and each such neighbor and an odd color related to this neighbor in H may forbid at most 2 colors, we conclude that at most $2t \le 2k$ colors are forbidden for v. Hence we have a free color for v that extends the odd coloring of H to an odd coloring of H by using at most H colors.

A graph is maximal k-degenerate if each induced subgraph has a vertex of degree at most k and adding any new edge to the graph violates this condition. Such a graph can be constructed in the following way [7]: starting from the complete graph K_{k+1} , in each subsequent step add a vertex adjacent to precisely k vertices. The sequence of vertices v_n, \ldots, v_{k+2} that are added sequentially to K_{k+1} is called the *elimination order*.

Proposition 4.8. Let G be a maximal k-degenerate graph where k is odd. Then $\chi_o(G) \leq 2k+1$.

Proof. The proof goes by induction on n. For n = k+1, $G = K_{k+1}$ and we have $\chi_o(G) = \chi(G) = k+1 \le 2k+1$.

Now, assuming that the claim holds for n, we prove it for n+1. In order to do so, let v be the last vertex in the elimination order. Then d(v) = k. Consider H = G - v. Clearly H is maximal k-degenerate (this is a hereditary property). Since G is k-connected, H is maximal k-degenerate k-connected (hereditary for $n \ge k+1$, see [7, Proposition 2]) and $\chi_o(H) \le 2k+1$ by induction. Consider N(v) in G. Since there are k vertices in N(v) and k is odd, one of the colors must appear an odd number of times. Also v has at most 2k forbidden colors imposed by its neighbors and their 'odd color' classes. So there is an available color for v which completes an odd coloring of G with at most 2k+1 colors.

Proposition 4.8 supplies a linear upper bound (of 2k + 1) for the odd chromatic number of maximal k-degenerate graphs in case k is odd. It is of interest to obtain an upper bound depending only on k, in particular a linear upper bound, for the odd chromatic number of maximal k-degenerate graphs in case k is even.

Maximum degree. We start this final portion by showing that, excepting C_5 , the odd list chromatic number of every other connected graph is at most twice its maximum degree.

Theorem 4.9. Let G be a connected graph of maximum degree $\Delta \geq 1$. Every vertex $v \in V(G)$ is assigned with a color list L(v) of size 2Δ . Then G admits an odd coloring φ such that $\varphi(v) \in L(v)$ for each v, unless all lists are the same and $G = C_5$.

Proof. First we deal with the case when $\Delta \leq 2$. If $\Delta = 1$ then $G = K_2$ and the assertion follows immediately as the number of vertices equals the length of each list. Assume $\Delta = 2$. Thus G is either a path or a cycle and every vertex is assigned with a 4-list of colors. In case G is a path, a straightforward induction shows that actually 3-lists as enough (see also the Remark after Proposition 3.1). Indeed, simply remove a leaf u, take an odd coloring of G - u that is compliant with the 3-lists assignment, delete from L(u) the colors used for its neighbor and second-neighbor along G, and then color u by the remaining color in that list.

So let G be a cycle. If all lists are the same, then (9) applies. Consider the situation when not all lists are the same. Then there are adjacent vertices u, v such that $L(u) \neq L(v)$. Pick a color from $L(u) \setminus L(v)$, use it on u and remove this color from all remaining lists. As shown

in the previous paragraph, the path $G - \{u, v\}$ admits an odd coloring that is compliant with the resulting lists assignment (as every list is of length at least 3). Remove from L(v) the colors used in N(v) or $N_2(v)$. At least one color remains on the list since the color of u is not in L(v), and we may use it for v.

Now we consider the case of $\Delta \geq 3$. Choose a vertex v of odd degree, if possible, and otherwise of arbitrary degree d. Consider a breadth-first search tree rooted at v, and assign indices in decreasing order $n, n-1, \ldots, 1$ to the vertices as we reach them. Thus we obtain an ordering v_1, v_2, \ldots, v_n of V(G) such that: $v_n = v$, every v_i with i < n has a higher-indexed neighbor, and $N(v_n) = \{v_{n-1}, \ldots, v_{n-d}\}$, where d is the degree of v.

We color the vertices in the order v_1, v_2, \ldots, v_n as follows. First we pick a color $c \in L(v_{n-1})$ and remove it from all lists. Denoting $L'(v_i) = L(v_i) \setminus \{c\}$, we have that $|L'(v_i)| \ge 2\Delta - 1$. For a pair of adjacent vertices v_i and v_j with j < i, we say that v_i is the terminal neighbor of v_j if v_i is the highest-indexed neighbor of v_j . For $1 \le i \le n-1$ and vertices $v_1, v_2, \ldots, v_{i-1}$ already colored, remove from $L'(v_i)$ the colors used on the lower-indexed neighbors of v_i ; additionally, if for any such v_j the vertex v_i happens to be its terminal neighbor and a unique color, say c_j , occurs an odd number of times in $N_{G[v_1,v_2,\ldots,v_{i-1}]}(v_j)$ then remove the color c_j from $L'(v_i)$ as well. Noting that at least one color remains in $L'(v_i)$ (as we have removed at most $2(\Delta - 1)$ colors), we have an available color that we use for v_i . We are left with assigning a color to v_n . Similarly to above, for any neighbor v_j of v_n such that a unique color c_j occurs an odd number of times in $N_{G[v_1,v_2,\ldots,v_{n-1}]}(v_j)$, remove c_j from $L'(v_n)$. Also, remove from $L'(v_n)$ the colors of $v_{n-d}, v_{n-d+1}, \ldots, v_{n-2}$. We have thus removed at most $2\Delta - 1 = \Delta + (\Delta - 1)$ colors from $L'(v_n)$. There are two possibilities.

Case 1: A color occurs an odd number of times in $N(v_n)$. If $c \in L(v_n)$, then color v_n with c and we are done. Contrarily, as $c \notin L(v_n)$, at least one color has remained in $L'(v_n)$ and we use such an available color for v_n . Recolor v_{n-1} with c, and we are done.

Case 2: No color occurs an odd number of times in $N(v_n)$. Then d and Δ are even, and at most d/2 colors are used for $v_{n-d}, v_{n-d+1}, \ldots, v_{n-1}$. Therefore, as we are assuming $\Delta \geq 3$, we actually have $\Delta \geq 4$. Recolor v_{n-1} with c. Note that at most $\Delta + d/2$ colors have been removed from $L'(v_n)$. Thus, as $(3/2)\Delta \leq 2\Delta - 2$ holds whenever $\Delta \geq 4$, a color has remained in $L'(v_n)$. We use such a color for v_n and we are done.

An immediate consequence of Theorem 4.9 is the following upper bound on the odd chromatic number.

Corollary 4.10. For every connected graph $G \neq C_5$, of maximum degree Δ it holds

$$\chi_o(G) \leq 2\Delta$$
.

By (9), the established bound of 2Δ colors is achieved for every cycle ($\neq C_5$) of length not divisible by 3. We are not aware of any such examples when $\Delta \geq 3$. Our next result sheds light on this issue in the case $\Delta = 3$.

Proposition 4.11. If G is a connected graph of maximum degree $\Delta = 3$, then $\chi_o(G) \leq 4$.

Proof. Consider a minimum counter-example G. It is not an odd graph, for otherwise $\chi_o(G) = \chi(G) \le 4$. So there is a 2-vertex in G. Moreover, there must exist a 2-vertex that is adjacent to a 3-vertex. Say v is the former, w the latter, and let u be the other neighbor of v. Delete v and add an edge uw unless u, w are already adjacent. If the obtained smaller graph G' admits

an odd 4-coloring, then such a coloring extends to v by forbidding the colors used for u and w, and a possible third color in regard to odd occurrence in $N_{G'}(u)$ (in case $d_G(u) = 2$). Hence $\chi_o(G') \geq 5$. Since G' is subcubic, the minimality choice of G implies that G' is of maximum degree 2 and that u, w were adjacent in G. In view of the equalities (8) and (9), we conclude that $G' = C_5$. However, then it is readily checked that $\chi_o(G) = 3$, a contradiction.

5 Further work

If G is odd then obviously $\chi_o(G) = \chi(G)$. It might be interesting to look more closely into the class of graphs for which the odd and the ordinary chromatic numbers coincide. Perhaps these graphs exist in abundance, and it is NP-hard to characterize them.

In view of Theorem 3.2, one naturally wonders about odd list colorability aspects of hypercubes.

Question 5.1. Is every hypercube odd 4-list colorable?

Even more so,

Question 5.2. Does the equality $\chi_o(Q_n) = \operatorname{ch}_o(Q_n)$ hold for every n?

Recall that for any graph G, \overline{G} denotes the complement of G, that is, the graph defined on the vertex set of G so that an edge belongs to \overline{G} if and only if it does not belong to G. Nordhaus and Gaddum [15] studied the chromatic number in a graph and in its complement together. They proved sharp lower and upper bounds on the sum and on the product of $\chi(G)$ and $\chi(\overline{G})$ in terms of the order n of G. For example, they showed that $\chi(G) + \chi(\overline{G}) \leq n + 1$. Since then, any bound on the sum and/or the product of an invariant in a graph G and the same invariant in the complement \overline{G} of G is called a Nordhaus-Gaddum type inequality or relation.

Problem 5.3. Let $\mathcal{G}(n)$ denote the class of graphs of order n. Given a positive integer n, determine (sharp) bounds for $\chi_o(G) + \chi_o(\overline{G})$ and $\chi_o(G) \cdot \chi_o(\overline{G})$, where G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs.

Let us share some of our initial thoughts on this matter. By Nordhaus-Gaddum Theorem, we know $\chi(G) \cdot \chi(\overline{G}) \geq n$, and since $\chi_o(G) \geq \chi(G)$ it follows that $\chi_o(G) \cdot \chi_o(\overline{G}) \geq n$. This bound is attained by taking $G = K_n$.

By (6), we have $\chi_o(G) \leq \chi(G) + \gamma_t(G)$. Consequently, $\chi_o(G) + \chi_o(\overline{G}) \leq (\chi(G) + \chi(\overline{G})) + (\gamma_t(G) + \gamma_t(\overline{G}))$. As already mentioned, the first summand is bounded from above by n+1. In regard to the second summand, we make use of Proposition 1.5 in [10], which reads: If G is a graph of order n and minimum degree $\delta \geq 1$ then $\gamma_t(G) \leq \frac{n(1+\ln\delta)}{\delta}$. Consequently, for $\min\{\delta(G), \delta(\overline{G})\} \geq 10$ we have $\chi_o(G) + \chi_o(\overline{G}) \leq \frac{5}{3}n$, and already for $\min\{\delta(G), \delta(\overline{G})\} \geq 6$ we get $\chi_o(G) + \chi_o(\overline{G}) \leq 1.94n$. Furthermore, if $\min\{\delta(G), \delta(\overline{G})\} \rightarrow \infty$ with n, then we are bounded by n + o(n). So for almost all graphs G of order n, the sum $\chi_o(G) + \chi_o(\overline{G})$ is at most n + o(n). We are rather optimistic and believe that the following may hold.

Conjecture 5.4. For every graph $G \neq C_5$ of order n it holds that $\chi_o(G) + \chi_o(\overline{G}) \leq n + 3$.

If true, the conjectured bound is best possible for every $n \geq 6$. Indeed, the graph $G = K_{n-5} \vee C_5$ satisfies that $\chi_o(G) = n - 2$ and $\chi_o(\overline{G}) = 5$. (The latter equality follows from the fact that G can be obtained from K_n by removing the edges of a 5-cycle.)

In view of Proposition 4.11, we are tempted to end this paper with the following.

Conjecture 5.5. If G is a connected graph of maximum degree $\Delta \geq 3$, then $\chi_o(G) \leq \Delta + 1$.

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