Decomposition of planar graphs with forbidden configurations

Lingxi Li^{*} Huajing Lu[†] Tao Wang[‡] Xuding Zhu[§]

Abstract

A (d, h)-decomposition of a graph G is an ordered pair (D, H) such that H is a subgraph of G of maximum degree at most h and D is an acyclic orientation of G - E(H) with maximum out-degree at most d. In this paper, we prove that for $l \in \{5, 6, 7, 8, 9\}$, every planar graph without 4- and l-cycles is (2, 1)-decomposable. As a consequence, for every planar graph G without 4- and l-cycles, there exists a matching M, such that G - M is 3-DP-colorable and has Alon-Tarsi number at most 3. In particular, G is 1-defective 3-DP-colorable, 1-defective 3-paintable and 1-defective 3-choosable. These strengthen the results in [Discrete Appl. Math. 157 (2) (2009) 433–436] and [Discrete Math. 343 (2020) 111797].

Keywords: decomposition; list coloring; defective coloring; Alon-Tarsi number; DP-coloring

1 Introduction

A proper k-coloring of a graph G is a mapping $\phi: V(G) \to [k]$ such that $\phi(u) \neq \phi(v)$, whenever $uv \in E(G)$, where and herein after, $[k] = \{1, 2, \dots, k\}$. The least integer k such that G admits a proper k-coloring is the chromatic number $\chi(G)$ of G. Let h be a non-negative integer. An h-defective k-coloring of G is a mapping $\phi: V(G) \to [k]$ such that each color class induces a subgraph of maximum degree at most h. In particular, a 0-defective coloring is a proper coloring of G.

A k-list assignment of G is a mapping L that assigns a list L(v) of k colors to each vertex v in G. An h-defective L-coloring of G is an h-defective coloring ψ of G such that $\psi(v) \in L(v)$ for all $v \in V(G)$. A graph G is h-defective k-choosable if G admits an h-defective L-coloring for each k-list assignment L. In particular, if G is 0-defective k-choosable, then we call it k-choosable. The choice number ch(G) is the smallest integer k such that G is k-choosable.

Cowen, Cowen, and Woodall [2] proved that every outerplanar graph is 2-defective 2-colorable, and every planar graph is 2-defective 3-colorable. Eaton and Hull [6], and independently, Škrekovski [12] proved that every outerplanar graph is 2-defective 2-choosable, and every planar graph is 2-defective 3-choosable. Cushing and Kierstead [3] proved that every planar graph is 1-defective 4-choosable. Let $\mathcal{G}_{4,l}$ be the family of planar graphs which contain no 4-cycles and no *l*-cycles. Lih et al. [10] proved that for each $l \in \{5, 6, 7\}$, every graph $G \in \mathcal{G}_{4,l}$ is 1-defective 3-choosable. Dong and Xu [4] proved that for each $l \in \{8, 9\}$, every graph $G \in \mathcal{G}_{4,l}$ is 1-defective 3-choosable.

Note that a graph being *h*-defective *k*-choosable means that for every *k*-list assignment *L* of *G*, there exists a subgraph *H* (depending on *L*) of *G* with $\Delta(H) \leq h$ such that G - E(H) is *L*-colorable. The subgraph *H* may be different for different *L*. As a strengthening of the above results, the following problem is studied in the literature: For $(h, k) \in \{(2, 3), (1, 4)\}$, is it true that every planar graph *G* has a subgraph of maximum

^{*}School of Mathematics and Statistics, Henan University, Kaifeng, 475004, P. R. China

[†]College of Basic Science, Ningbo University of Finance and Economics, Ningbo, 315000, P. R. China

[‡]Center for Applied Mathematics, Henan University, Kaifeng, 475004, P. R. China. Email: wangtao@henu.edu.cn

[§]School of Mathematical Sciences, Zhejiang Normal University, Jinhua, 321004, P. R. China. This research is supported by Grants: NSFC 11971438, U20A2068.



Fig. 1: Forbidden configurations in (1) and (2) of Theorem 1.1.

degree h such that G - E(H) is k-choosable? For $l \in \{5, 6, 7, 8, 9\}$, is it true that every graph $G \in \mathcal{G}_{4,l}$ has a matching M such that G - M is 3-choosable?

It turns out that for the first question, the answer is negative for (h, k) = (2, 3), and positive for (h, k) = (1, 4). It was proved in [8] that there exists a planar graph G such that for any subgraph H of G of maximum degree 3, G - E(H) is not 3-choosable, and proved in [7] that every planar graph G has a matching M such that G - M is 4-choosable. For the second question, for $l \in \{5, 6, 7\}$, it was shown in [11] every graph $G \in \mathcal{G}_{4,l}$ has a matching M such that G - M is 3-choosable.

Indeed, stronger results were proved in [7, 11]. The results concern two other graph parameters: The Alon-Tarsi number AT(G) of G and the paint number $\chi_P(G)$ of G. The reader is referred to [7] for the definitions. We just note here that for any graph G, $ch(G) \leq \chi_P(G) \leq AT(G)$, and the differences $\chi_P(G) - ch(G)$ and $AT(G) - \chi_P(G)$ can be arbitrarily large. It was proved in [7] that every planar graph G has a matching Msuch that $AT(G - M) \leq 4$, and proved in [11] that for $l \in \{5, 6, 7\}$, every graph $G \in \mathcal{G}_{4,l}$ has a matching Msuch that $AT(G - M) \leq 3$.

In this paper, we consider further strengthening of the results concerning graphs in $\mathcal{G}_{4,l}$ for $l \in \{5, 6, 7, 8, 9\}$. (Note that the result in [11] does not cover the cases for l = 8 and 9). We strengthen the above results in two aspects: a larger class of graphs with a stronger property.

Given two non-negative integers d, h and a graph G, a (d, h)-decomposition of G is a pair (D, H) such that H is a subgraph of G of maximum degree at most h and D is an acyclic orientation of G - E(H)with maximum out-degree at most d. We say G is (d, h)-decomposable if G has a (d, h)-decomposition. Cho et al. [1] proved that every planar graph is (4, 1)-decomposable, (3, 2)-decomposable and (2, 6)-decomposable. Note that a graph H which has an acyclic orientation of maximum out-degree at most d if and only if His d-degenerate, i.e., the vertices of H can be linearly ordered so that each vertex has at most d backward neighbors. It is well-known and easy to see that d-degenerate graphs not only have choice number, paint number, Alon-Tarsi number and DP-chromatic number at most d + 1, there is a linear time algorithm that creates the above mentioned linear ordering and the corresponding coloring is easily obtained by using a greedy coloring algorithm. The reader is referred to [5] for the definition of DP-chromatic number $\chi_{DP}(G)$ of a graph G. We just mention here that $ch(G) \leq \chi_{DP}(G)$, and there are graphs G for which $\chi_{DP}(G)$ are larger than each of AT(G) and $\chi_P(G)$, there are also graphs G for which $\chi_{DP}(G)$ are smaller than each of AT(G) and $\chi_P(G)$ [9]. This paper proves the following result:

Theorem 1.1. Assume G is a plane graph. Then G is (2, 1)-decomposable if one of the following holds:

- (1) G has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2.
- (2) G has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3.
- (3) $G \in \mathcal{G}_{4,9}$.

Note that if $G \in \mathcal{G}_{4,l}$ for some $l \in \{5, 6, 7\}$, then G has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2, and if $G \in \mathcal{G}_{4,8}$, then G has no subgraph isomorphic to any configuration in Fig. 1 and



Fig. 2: Forbidden configurations in (1) of Theorem 1.1.



Fig. 3: Forbidden configurations in (2) of Theorem 1.1.

Fig. 3. Consequently, for $l \in \{5, 6, 7, 8, 9\}$, all graphs $G \in \mathcal{G}_{4,l}$ are (2, 1)-decomposable.

All graphs in this paper are finite and simple. For a plane graph G, we use V(G), E(G) and F(G) to denote the vertex set, edge set and face set of G, respectively. For any element $x \in V(G) \cup F(G)$, the degree of x is denoted by d(x). A vertex v in G is called a k-vertex, or k^+ -vertex, or k^- -vertex, if d(v) = k, or $d(v) \geq k$, or $d(v) \leq k$, respectively. Analogously, one can define k-face, k^+ -face, and k^- -face. An n-face $[x_1x_2 \ldots x_n]$ is a (d_1, d_2, \ldots, d_n) -face if $d(x_i) = d_i$ for $1 \leq i \leq n$. Let D be an orientation of a graph G, we use $d_D^+(v)$ and $d_D^-(v)$ to denote the out-degree and in-degree of a vertex v in D, respectively. Let $\Delta^+(D)$ denote the maximum out-degree of vertices in D. Two cycles (or faces) are *adjacent* if they have at least one common edge. Two cycles (or faces) are *normally adjacent* if they intersect in exactly two vertices. Let G be a plane graph and xy be a given boundary edge of G. A vertex $v \neq x, y$ is called a *normal vertex*. A vertex v is special if v is a 5⁺-vertex or $v \in \{x, y\}$. A face is *internal* if it is not the outer face f_0 . A face is *special* if it is an internal 7⁺-face or the outer face f_0 . A normal vertex v is *minor* if d(v) = 3 and it is incident with an internal 4⁻-face. A good 5-face is an internal 5-face adjacent to at least one internal 3-face. An edge contained in a triangle is a *triangular edge*. Note that in all three cases, there are no adjacent triangles. So every triangular edge is contained in a unique triangle.

2 Proof of Theorem 1.1

For the purpose of using induction, we prove the following result. Assume G is a plane graph and e = xy is a boundary edge of G. A nice decomposition of (G, e) is a pair (D, M) such that M is a matching and D is an acyclic orientation of G - M with $d_D^+(x) = d_D^+(y) = 0$ and $\Delta^+(D) \leq 2$. Note that in a nice decomposition (D, M) of (G, e), since $d_D^+(x) = d_D^+(y) = 0$, we conclude that $e = xy \in M$.

Theorem 2.1. If G is a plane graph satisfying the condition of Theorem 1.1 and e is a boundary edge of G, then (G, e) has a nice decomposition.

Assume Theorem 2.1 is not true and G is a counterexample with minimum number of vertices. We shall

derive a sequence of properties of G that lead to a contradiction. It is obvious that G is connected, for otherwise we can consider each component of G separately.

Lemma 2.2. G is 2-connected.

Proof. Assume to the contrary that G has a cut-vertex x'. Let $G = H_1 \cup H_2$, $V(H_1 \cap H_2) = \{x'\}$ and $e = xy \in E(H_1)$. Let e' = x'y' be a boundary edge of H_2 . By the minimality of G, there is a nice decomposition (D_1, M_1) of (H_1, e) and a nice decomposition (D_2, M_2) of (H_2, e') . Let $M = (M_1 \cup M_2) \setminus \{x'y'\}$ and $D = D_1 \cup D_2 \cup \{x'y'\}$. It is straightforward to verify that (D, M) is a nice decomposition of (G, e).

Lemma 2.3. For any $v \in V(G) \setminus \{x, y\}, d(v) \ge 3$.

Proof. Assume $v \in V(G) \setminus \{x, y\}$ and $d(v) \leq 2$. By the minimality of G, there exists a nice decomposition (D, M) of (G - v, e). Let D' be obtained from D by orienting edges incident with v as out-going edges from v. Then (D', M) is a nice decomposition of (G, e).

Lemma 2.4. If u and v are two adjacent 3-vertices, then $\{u, v\} \cap \{x, y\} \neq \emptyset$.

Proof. Suppose that u and v are two adjacent 3-vertices with $\{u, v\} \cap \{x, y\} = \emptyset$. By the minimality of G, there is a nice decomposition (D, M) of $(G - \{u, v\}, e)$. Let $M' = M \cup \{uv\}$, and D' be obtained from D by orienting the other edges incident with u, v as out-going edges from u, v. Then (D', M') is a nice decomposition of (G, e).

For an internal face f, let t_f be the number of incident normal 3-vertices and let s_f be the number of adjacent internal 3-faces. Note that each 3-vertex of f is incident with at most one 3-face adjacent to f. Thus we have the following corollary.

Corollary 2.5. For any internal face $f, t_f \leq d(f)/2$ and $t_f + s_f \leq d(f)$.

The following four lemmas first appeared in [11], although the hypotheses and some definitions are slightly different. For the completeness of this paper, we include the short proofs with illustration figures.



Fig. 4: (a) A bad 5-cycle and an adjacent triangle. (b) For the proof of Lemma 2.6. Here and in figures below, a solid triangle represents a 3-vertex, a solid square represents a 4-vertex, a thick line represents an edge in the matching M.

A 5-cycle $[u_1u_2u_3u_4u_5]$ is a bad 5-cycle if it is adjacent to a triangle $[u_1u_5u_6]$ with $u_i \notin \{x, y\}$, where $1 \le i \le 6$, and $d(u_1) = d(u_3) = 3$, and $d(u_2) = d(u_4) = d(u_5) = d(u_6) = 4$, as depicted in Fig. 4(a).

Lemma 2.6 (Lemma 5.2 in [11]). There are no bad 5-cycles in G.

Proof of Lemma 2.6. Assume $C = [u_1u_2u_3u_4u_5]$ is a bad 5-cycle and $T = [u_1u_5u_6]$ is a triangle adjacent to C, where $d(u_1) = d(u_3) = 3$ and $d(u_i) = 4$ for $i \in \{2, 4, 5, 6\}$, as depicted in Fig. 4(a). A nice decomposition of $G - \{u_1, u_2, \ldots, u_6\}$ is extended to a nice decomposition as in Fig. 4(b).

A triangle T is minor if T is a (3, 4, 4)-triangle and $T \cap \{x, y\} = \emptyset$. A triangle chain in G is a subgraph of $G - \{x, y\}$ consisting of vertices $w_1, w_2, \ldots, w_{k+1}, u_1, u_2, \ldots, u_k$ in which $[w_i w_{i+1} u_i]$ is a (4, 4, 4)-cycle for $1 \le i \le k$, as depicted in Fig. 5. We denote T_i the triangle $[w_i w_{i+1} u_i]$ and denote such a triangle chain by $T_1 T_2 \ldots T_k$. If a triangle T has exactly one common vertex with a triangle chain $T_1 T_2 \ldots T_k$ and the common vertex is in T_1 , then we say T intersects the triangle chain $T_1 T_2 \ldots T_k$.



Fig. 5: A triangle chain.



Fig. 6: (a) The configuration in Lemma 2.7. (b) For the proof of Lemma 2.7.

Lemma 2.7 (Lemma 2.10 in [11]). If a minor triangle T_0 intersects a triangle chain $T_1T_2...T_k$, then every 3-vertex adjacent to a vertex in T_k belongs to $\{x, y\} \cup V(T_0)$.

The k = 0 case of the above lemma asserts that every 3-vertex adjacent to a vertex in T_0 belongs to $\{x, y\}$.

Proof of Lemma 2.7. Assume G has a minor triangle $T_0 = [w_0w_1u_0]$ intersecting a triangle chain $T_1T_2...T_k$; and $z \notin \{x, y\} \cup V(T_0)$ is a 3-vertex adjacent to a vertex in T_k , as depicted in Fig. 6(a). A nice decomposition of $G - (\bigcup_{i=0}^k V(T_i) \cup \{z\})$ is extended to a nice decomposition of G as in Fig. 6(b).

Lemma 2.8 (Lemma 2.11 in [11]). If a minor triangle T_0 intersects a triangle chain $T_1T_2...T_k$, then the distance between T_k and another minor triangle is at least two.



Fig. 7: (a) The configuration in Lemma 2.8. (b) For the proof of Lemma 2.8.



Fig. 8: (a) The configuration in Lemma 2.9. (b) For the proof of Lemma 2.9.

Proof of Lemma 2.8. Assume to the contrary that $T_1T_2...T_k$ with $T_i = [w_iw_{i+1}u_i]$, $1 \le i \le k$, is a triangle chain that intersects a minor triangle $T_0 = [w_0w_1u_0]$, and the distance between T_k and another minor triangle $T'_0 = [zz_1z_2]$ with $d(z_1) = 3$ is less than 2. By Lemma 2.7, we may assume $w_{k+1}z$ is a (4, 4)-edge connecting T_k and T'_0 , as depicted in Fig. 7(a). A nice decomposition of $G - (\bigcup_{i=0}^k V(T_i) \cup V(T'_0))$ is extended to a nice decomposition of G as in Fig. 7(b).

Lemma 2.9 (Lemma 3.1 in [11]). Assume that f is a 6-face adjacent to five 3-faces, and none of the vertices on these 3-faces is in $\{x, y\}$. If f is incident with a 3-vertex, then there is at least one 5⁺-vertex on these five 3-faces.

Proof of Lemma 2.9. Let $f = [v_1v_2v_3v_4v_5v_6]$ be a 6-face, v_1 be a 3-vertex and $T_i = [v_iv_{i+1}u_i]$, $1 \le i \le 5$, be the five 3-faces. Assume to the contrary that there is no 5⁺-vertex on T_i . By Lemma 2.7, we may assume all v_{i+1} and u_i are 4-vertices for $1 \le i \le 5$, as depicted in Fig. 8(a). A nice decomposition of $G - (\bigcup_{i=1}^5 V(T_i))$ is extended to a nice decomposition of G as in Fig. 8(b).

The above lemmas present some reducible configurations. We use standard discharging method to prove that there must be some reducible configurations in a minimum counterexample, which leads to a contradiction.

First, we define an initial charge function by $\mu(x) = d(x) - 4$, $\mu(y) = d(y) - 4$, $\mu(f_0) = d(f_0) + 4$, and $\mu(v) = d(v) - 4$ for each vertex $v \in V(G) \setminus \{x, y\}$, $\mu(f) = d(f) - 4$ for each face f other than f_0 . By Euler's formula and handshaking theorem, we obtain that the sum of all the initial charges is zero, i.e.,

$$(d(x) - 4) + (d(y) - 4) + (d(f_0) + 4) + \sum_{v \neq x, y} (d(v) - 4) + \sum_{f \neq f_0} (d(f) - 4) = 0.$$

Next, we design some discharging rules to redistribute the charges, such that the sum of the final charges is not zero, which leads to a contradiction.

Discharging Rules

- **R1.** Every internal 3-face f receives $\frac{1}{3}$ from each adjacent face.
- **R2.** Assume v is a normal 3-vertex. If v is incident with an internal 4⁻-face, then it receives $\frac{1}{2}$ from each of the other two incident faces. Otherwise it receives $\frac{1}{3}$ from each incident face.
- **R3.** Let v be a normal 5-vertex. Then v sends $\frac{1}{6}$ to each incident 4^+ -face. If v is incident with a 3-face g = [uvw], then v sends $\frac{1}{6}$ to the other face g' incident with uw. Moreover, if v is incident with three consecutive faces f_1, f_2, f_3 and f_1, f_3 are 3-faces, then v sends an extra $\frac{1}{6}$ to f_2 .
- **R4.** Let v be a normal 6⁺-vertex. Then v sends $\frac{1}{3}$ to each incident 4⁺-face. If v is incident with a 3-face g = [uvw], then v sends $\frac{1}{3}$ to the other face g' incident with uw.
- **R5.** Let v be a vertex in $\{x, y\}$. Then it sends $\frac{1}{3}$ to every incident internal 4⁺-face. If v is incident with a 3-face g = [uvw], then v sends $\frac{1}{3}$ to the other face g' incident with uw.
- **R6.** f_0 sends $\frac{1}{3}$ to each adjacent 4⁺-face.
- **R7.** In Case 2 (i.e., G has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3), every internal 5-face receives $\frac{1}{6}$ from adjacent internal 6⁺-faces via each common edge.
- **R8.** In Case 3 (i.e., $G \in \mathcal{G}_{4,9}$), every good 5-face receives $\frac{1}{3}$ from adjacent internal 7⁺-faces via each common edge.

For $z \in V(G) \cup F(G)$, let $\mu'(z)$ be the final charge of z. In the remainder of this paper, we prove that $\sum_{z \in V(G) \cup F(G)} \mu'(z) > 0$, which contradicts the fact that $\sum_{z \in V(G) \cup F(G)} \mu'(z) = \sum_{z \in V(G) \cup F(G)} \mu(z) = 0$.

Note that R7 only applies to Case 2 and R8 only applies to Case 3. Moreover, R7 and R8 only involve 5^+ -faces.

It follows from R5 that for $v \in \{x, y\}$

$$\mu'(v) \ge \mu(v) - (d(v) - 1) \times \frac{1}{3} = \frac{2d(v) - 11}{3} \ge -\frac{7}{3}.$$

Note that f_0 sends $\frac{1}{3}$ to each adjacent internal face by R1 and R6, and sends at most $\frac{1}{2}$ to each incident normal 3-vertex by R2. It follows from Lemma 2.4 that f_0 is incident with at most $\frac{d(f_0)}{2}$ normal 3-vertices. Then

$$\mu'(f_0) \ge \mu(f_0) - \frac{d(f_0)}{2} \times \frac{1}{2} - d(f_0) \times \frac{1}{3} \ge \frac{5d(f_0)}{12} + 4 \ge \frac{21}{4}.$$

Hence, $\mu'(x) + \mu'(y) + \mu'(f_0) > 0.$

Assume v is a normal 3-vertex. If v is incident with an internal 4⁻-face, then the other two incident faces are 5⁺-faces or the outer face f_0 . Hence $\mu'(v) = \mu(v) + 2 \times \frac{1}{2} = 0$. Otherwise each face incident with v is a 5⁺-face or f_0 , and $\mu'(v) = \mu(v) + 3 \times \frac{1}{3} = 0$ by R2.

If v is a normal 4-vertex, then $\mu'(v) = \mu(v) = 0$. If v is a normal 5-vertex, then it is incident with at most two 3-faces, and then $\mu'(v) \ge \mu(v) - 5 \times \frac{1}{6} - \frac{1}{6} = 0$ by R3. If v is a normal 6⁺-vertex, then $\mu'(v) = \mu(v) - d(v) \times \frac{1}{3} = \frac{2(d(v)-6)}{3} \ge 0$ by R4.

If f is an internal 3-face, then it receives $\frac{1}{3}$ via each incident edge, and $\mu'(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by R1. If f is an internal 4-face, then $\mu'(f) \ge \mu(f) = 0$.

It remains to show that $\mu'(f) \ge 0$ for internal 5⁺-faces f.

In the remainder of the paper, we consider the three cases separately in three subsections.

2.1 G has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2

Assume that $f = [v_1v_2v_3v_4v_5]$ is an internal 5-face. By Corollary 2.5, $t_f \leq 2$. If f is not adjacent to any internal 3-face, then $\mu'(f) \geq \mu(f) - 2 \times \frac{1}{2} = 0$ by R2. So we may assume that f is adjacent to at least one internal 3-face. Since the configurations Fig. 2(a)-2(d) are forbidden, f is adjacent to exactly one internal 3-face f^* and no 4-faces. If $t_f \leq 1$, then $\mu'(f) \geq \mu(f) - \frac{1}{3} - \frac{1}{2} > 0$ by R1 and R2. Assume $t_f = 2$ and $f^* = [uv_1v_2]$ is an internal 3-face. If there are some special vertices in $\{u, v_1, v_2, \ldots, v_5\}$, then f receives at least $\frac{1}{6}$ from special vertices, and then $\mu'(f) \geq \mu(f) - \frac{1}{3} - (\frac{1}{3} + \frac{1}{2}) + \frac{1}{6} = 0$ by R1, R2, R3, R4 and R5. So we may assume that none of $\{u, v_1, v_2, \ldots, v_5\}$ is a special vertex. It follows that f is incident with two 3-vertices and three 4-vertices. If neither v_1 nor v_2 is a 3-vertex, then $\mu'(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{1}{3} = 0$ by R1 and R2. Without loss of generality, assume that $d(v_2) = 3$ and $d(v_1) = d(v_3) = d(u) = 4$. If $d(v_4) = 3$ and $d(v_5) = 4$, then it contradicts Lemma 2.6. If $d(v_4) = 4$ and $d(v_5) = 3$, then it contradicts Lemma 2.7.

Assume that $f = [v_1v_2v_3v_4v_5v_6]$ is an internal 6-face. By Corollary 2.5, $t_f \leq 3$.

• $t_f = 3$. Without loss of generality, assume that v_1, v_3 and v_5 are normal 3-vertices.

By Corollary 2.5, $s_f \leq 3$. If $s_f \leq 1$, then $\mu'(f) \geq \mu(f) - \frac{1}{3} - 3 \times \frac{1}{2} > 0$ by R1 and R2.

Assume that $s_f = 2$. By symmetry, assume that one of the adjacent internal 3-face is $[v_1v_2u]$. By Lemma 2.7, one vertex in $\{u, v_2\}$ is a special vertex. Thus, $\mu'(f) \ge \mu(f) - 2 \times \frac{1}{3} - 3 \times \frac{1}{2} + \frac{1}{6} = 0$ by R1, R2, R3, R4 and R5.

Assume that $s_f = 3$.

(i) $v_i v_{i+1}$ is incident with an internal 3-face $[v_i v_{i+1} u_i]$ for $i \in \{1, 3, 5\}$. For each $i \in \{1, 3, 5\}$, by Lemma 2.7, there is a special vertex in $\{u_i, v_{i+1}\}$. Thus f receives at least $\frac{1}{6}$ from $\{u_i, v_{i+1}\}$ by R3, R4 and R5. Hence, $\mu'(f) \ge \mu(f) - 3 \times \frac{1}{3} - 3 \times \frac{1}{2} + 3 \times \frac{1}{6} = 0$ by R1, R2, R3, R4 and R5.

(ii) $v_i v_{i+1}$ is incident with an internal 3-face $[v_i v_{i+1} u_i]$ for $i \in \{1, 2, 5\}$. If v_2 is a special vertex, then f receives $\frac{1}{3}$ from v_2 . Otherwise, v_2 is a normal 4-vertex. By Lemma 2.7, both u_1 and u_2 are special vertices. Then f receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from u_1 and u_2 by R3, R4 and R5. In any way, f receives at least $\frac{1}{3}$ from $\{u_1, u_2, v_2\}$. On the other hand, one of u_5 and v_6 is also a special vertex, and f receives at least $\frac{1}{6}$ from $\{u_5, v_6\}$ by R3, R4 and R5. Thus, $\mu'(f) \ge \mu(f) - 3 \times \frac{1}{3} - 3 \times \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$.

• $t_f = 2$. By Corollary 2.5, $s_f \le 4$. If $s_f \le 3$, then $\mu'(f) \ge \mu(f) - 3 \times \frac{1}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R2.

Assume $s_f = 4$. We claim that f will receive at least $\frac{1}{3}$ from vertices. If f is incident with a 2-vertex, then the 2-vertex must be in $\{x, y\}$, and f receives at least $\frac{1}{3}$ from incident 2-vertices by R5. So we may assume that f is not incident with any 2-vertex. By symmetry, it suffices to consider five cases.

(1) The four adjacent internal 3-faces are $[v_i v_{i+1} u_i]$ for $1 \le i \le 4$. Thus, the two normal 3-vertices must be v_1 and v_5 . If one of v_2, v_3 and v_4 is a special vertex, then f receives $\frac{1}{3}$ from it by R3, R4 and R5. So we may assume that v_2, v_3 and v_4 are normal 4-vertices. By Lemma 2.7 and Lemma 2.8, there are at least two special vertices in $\{u_1, u_2, u_3, u_4\}$, thus f receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from these vertices by R3, R4 and R5.



Fig. 9: Two adjacent 5-faces. The solid vertex is a 2-vertex in G.

(2) The four adjacent internal 3-faces are $[v_i v_{i+1} u_i]$ for $i \in \{1, 2, 3, 5\}$, while v_1 and v_4 are normal 3-vertices. Similarly, if v_2 or v_3 is a special vertex, then f receives at least $\frac{1}{3}$ from it. So we may assume that v_2 and v_3 are normal 4-vertices. By Lemma 2.7 and Lemma 2.8, there are at least two special vertices in $\{u_1, u_2, u_3\}$, thus f receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from these vertices by R3, R4 and R5.

(3) The four adjacent internal 3-faces are $[v_iv_{i+1}u_i]$ for $i \in \{1, 2, 3, 5\}$, while v_1 and v_5 are normal 3-vertices. By Lemma 2.7, u_5 or v_6 is a special vertex; one of $\{v_2, v_3, v_4, u_1, u_2, u_3\}$ is a special vertex. Thus, f receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from these vertices by R3, R4 and R5.

(4) The four adjacent internal 3-faces are $[v_i v_{i+1} u_i]$ for $i \in \{1, 2, 4, 5\}$, while v_1 and v_3 are normal 3-vertices. If v_2 is a special vertex, then f receives $\frac{1}{3}$ from it by R3, R4 and R5. Otherwise, v_2 is a normal 4-vertex. By Lemma 2.7, each of u_1 and u_2 is a special vertex, thus f receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from $\{u_1, u_2\}$ by R3, R4 and R5.

(5) The four adjacent internal 3-faces are $[v_i v_{i+1} u_i]$ for $i \in \{1, 2, 4, 5\}$, while v_1 and v_4 are normal 3-vertices. By Lemma 2.7, there is at least one special vertex in $\{u_1, u_2, v_2, v_3\}$, and there is at least one special vertex in $\{u_4, u_5, v_5, v_6\}$. Thus, f receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from these vertices by R3, R4 and R5.

To sum up, f always receives at least $\frac{1}{3}$ from some vertices in the above five cases. Therefore, $\mu'(f) \ge \mu(f) - 4 \times \frac{1}{3} - 2 \times \frac{1}{2} + \frac{1}{3} = 0$ by R1 and R2.

• $t_f = 1$. By Corollary 2.5, $s_f \leq 5$. If $s_f \leq 4$, then $\mu'(f) \geq \mu(f) - \frac{1}{2} - 4 \times \frac{1}{3} > 0$. Assume that $s_f = 5$ and for $1 \leq i \leq 5$, $[v_i v_{i+1} u_i]$ is an internal 3-face. Let $X = \{v_1, \ldots, v_6, u_1, \ldots, u_5\}$. By Lemma 2.9, there is a special vertex in X. Therefore, f receives at least $\frac{1}{6}$ from the special vertices in X, and $\mu'(f) \geq \mu(f) - \frac{1}{2} - 5 \times \frac{1}{3} + \frac{1}{6} = 0$ by R3, R4 and R5.

• $t_f = 0$. Then f sends nothing to incident vertices, and $\mu'(f) \ge \mu(f) - 6 \times \frac{1}{3} = 0$.

If f is an internal 7⁺-face, then f sends out charges by R1 and R2. As $t_f + s_f \leq d(f)$, we have

$$\mu'(f) \ge \mu(f) - \frac{s_f}{3} - \frac{t_f}{2} \ge \frac{2}{3}d(f) - 4 - \frac{t_f}{6} \ge \frac{7}{12}d(f) - 4 > 0.$$

This completes the proof of Case 1 of Theorem 1.1.

2.2 G has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3

Lemma 2.10 below follows easily from the fact that configurations in Fig. 1 and Fig. 3 are forbidden.

Lemma 2.10. If two 5-faces have two consecutive common edges on their boundaries, then one of the 5-face is the outer face f_0 (see Fig. 9).

Now we calculate the final charge of internal 5^+ -faces.

Assume f is an internal d-face. If f is incident with a 2-vertex, then the 2-vertex belongs to $\{x, y\}$, and f is adjacent to at most d-2 internal faces. By R5, f receives $\frac{1}{3}$ from each of x and y. By R6, f receives $\frac{1}{3}$ via each common edge with the outer face f_0 . By R1 and R7, f sends at most $\frac{1}{3}$ to each adjacent internal face. By R2,

f sends at most $\frac{1}{2}$ to each incident normal 3-vertex. Thus, $\mu'(f) \ge d - 4 + 2 \times \frac{1}{3} + 2 \times \frac{1}{3} - (d-2) \times \frac{1}{3} - \lfloor \frac{d}{2} \rfloor \times \frac{1}{2} \ge \frac{5d-24}{12} > 0.$

Assume that f is not incident with any 2-vertex. By Lemma 2.10, there are no adjacent internal 5-faces. By Lemma 2.4, f is adjacent to at most $d - t_f$ internal 5-faces.

■ d = 5. Assume that $f = [v_1v_2v_3v_4v_5]$. Since adjacent triangles and a triangle normally adjacent to a 7-cycle are forbidden, $s_f \leq 2$. By Corollary 2.5, $t_f \leq 2$. It follows that f is incident with at most two minor 3-vertices.

If $s_f = 0$, then $\mu'(f) \ge \mu(f) - 2 \times \frac{1}{2} = 0$ by R2.

Assume $s_f \ge 1$. Since Fig. 1 and Fig. 3(c) are forbidden, f is not adjacent to any 4-face. It follows that every face adjacent to f is a 3-face or a 6⁺-face. Thus, f is adjacent to at least three 6⁺-faces (the number of adjacent 6⁺-faces is counted by the number of common edges). If f is incident with at most one minor 3-vertex, then $\mu'(f) \ge 5 - 4 - 2 \times \frac{1}{3} - (\frac{1}{2} + \frac{1}{3}) + 3 \times \frac{1}{6} = 0$ by R1, R2 and R7. Assume f is incident with exactly two minor 3-vertices. That is $t_f = 2$ and $s_f = 2$. By symmetry, we have three subcases to consider:

- f is adjacent to two internal 3-faces $[v_1v_2u_1]$, $[v_3v_4u_3]$, and v_1, v_3 are minor 3-vertices.
- f is adjacent to two internal 3-faces $[v_1v_2u_1]$, $[v_3v_4u_3]$, and v_1, v_4 are minor 3-vertices.
- f is adjacent to two internal 3-faces $[v_1v_2u_1]$, $[v_2v_3u_2]$, and v_1, v_3 are minor 3-vertices.

By Lemma 2.7 and Lemma 2.8, the two 3-faces are incident with at least one special vertex. By R3, R4 and R5, f receives at least $\frac{1}{6}$ from these special vertices. Hence, $\mu'(f) \ge 5 - 4 + \frac{1}{6} + 3 \times \frac{1}{6} - 2 \times \frac{1}{3} - 2 \times \frac{1}{2} = 0$.

■ d = 6. Assume that $f = [v_1 v_2 v_3 v_4 v_5 v_6]$. If $s_f = 0$, then it sends at most $\frac{1}{2}$ to each incident normal 3-vertex, and sends $\frac{1}{6}$ to each adjacent 5-face, thus $\mu'(f) \ge 6 - 4 - t_f \times \frac{1}{2} - (6 - t_f) \times \frac{1}{6} = 1 - \frac{t_f}{3} \ge 0$ by R2 and R7.

Suppose that f is adjacent to an internal 3-face. Then they are normally adjacent. Since the configurations in Fig. 1 and Fig. 3(c) are forbidden, $s_f = 1$. By Corollary 2.5, $t_f \leq 3$. If $t_f \leq 2$, then $\mu'(f) \geq 6 - 4 - \frac{1}{3} - t_f \times \frac{1}{2} - (6 - t_f) \times \frac{1}{6} = \frac{2 - t_f}{3} \geq 0$ by R1, R2 and R7.

Assume $t_f = 3$ and the 3-face is $[uv_1v_2]$. By Lemma 2.4, we may assume v_1, v_3 and v_5 are the three normal 3-vertices. By Lemma 2.7, there is a special vertex in $\{u, v_2\}$, thus f receives at least $\frac{1}{6}$ from $\{u, v_2\}$. Since the configurations in Fig. 1 and Fig. 3 are all forbidden, v_5 cannot be incident with an internal 4⁻-face. Thus, f is incident with at most two minor 3-vertices, which implies that $\mu'(f) \ge 6-4-(2\times\frac{1}{2}+\frac{1}{3})-\frac{1}{3}-(6-3)\times\frac{1}{6}+\frac{1}{6}=0$.

■ d = 7. Let f be a 7-face. As Fig. 3(c) is forbidden, $s_f = 0$. By Corollary 2.5, $t_f \leq 3$. By R2, f sends at most $\frac{1}{2}$ to each incident normal 3-vertex. By R7, f sends $\frac{1}{6}$ to each adjacent internal 5-face. Hence, $\mu'(f) \geq 7 - 4 - t_f \times \frac{1}{2} - (7 - t_f) \times \frac{1}{6} = \frac{11 - 2t_f}{6} > 0$.

■ $d \ge 8$. Let f be a 8⁺-face. Then f sends at most $\frac{1}{2}$ to each incident normal 3-vertex, and $\frac{1}{3}$ to each adjacent internal 3-face, and $\frac{1}{6}$ to each adjacent internal 5-face. Combining with Corollary 2.5, we have that

$$\mu'(f) \ge d - 4 - t_f \times \frac{1}{2} - s_f \times \frac{1}{3} - (d - s_f) \times \frac{1}{6} = \frac{5}{6}d - \frac{1}{2}t_f - \frac{1}{6}s_f - 4 \ge \frac{d}{2} - 4 \ge 0.$$

This completes the proof of Case 2.

2.3 $G \in \mathcal{G}_{4,9}$

Lemma 2.11. A 5-cycle contains at most three triangular edges.

Proof. Assume $[x_1x_2x_3x_4x_5]$ is a 5-cycle, and $[x_1x_2x_6], [x_2x_3x_7], [x_3x_4x_8]$ and $[x_4x_5x_9]$ are four triangles. Since there is no 4-cycle in G, x_1, x_2, \ldots, x_9 are nine distinct vertices. Thus, $[x_1x_6x_2x_7x_3x_8x_4x_9x_5]$ is a 9-cycle, a contradiction.



Fig. 10: Some local structures around 5-face.

Lemma 2.12. Let $f = [x_1x_2x_3x_4x_5]$ and $g = [x_5x_1uvw]$ be two adjacent 5-faces. If $d(x_1) \ge 3$ and $d(x_5) \ge 3$, then f and g are normally adjacent, and neither x_2x_3 nor x_3x_4 is adjacent to a 3-face. Moreover, if x_1x_2 is incident with a 3-face, then x_1 is a 3-vertex and the 3-face is $[x_1x_2u]$.

Proof. Since $d(x_1) \ge 3$ and $d(x_5) \ge 3$, we have that $x_2 \ne u$ and $x_4 \ne w$. Since G has no 4-cycle, $x_1, x_2, \ldots, x_5, u, v, w$ are distinct. Therefore, f and g are normally adjacent.

By the symmetry of x_2x_3 and x_3x_4 , suppose that x_2x_3 is incident with a 3-face $[x_2x_3x_7]$. Since there are no 4-cycles in G, x_7 is not incident with f or g. Thus, $[x_5x_4x_3x_7x_2x_1uvw]$ is a 9-cycle, a contradiction. Hence, neither x_2x_3 nor x_3x_4 is incident with a 3-face.

Let x_1x_2 be incident with a 3-face $[x_1x_2x_6]$. Since f has no chord, $x_6 \notin \{x_3, x_4, x_5, v, w\}$. If $x_6 \neq u$, then $[x_5x_4x_3x_2x_6x_1uvw]$ is a 9-cycle, a contradiction. Thus $x_6 = u$ and x_1 is a 3-vertex.

Lemma 2.13. Let $f = [x_1x_2x_3x_4x_5]$ and $g = [x_5x_1upqw]$ be two adjacent faces. If $d(x_1) \ge 3$ and $d(x_5) \ge 3$, then $\{u, w\} \cap \{x_1, \dots, x_5\} = \emptyset$, while $\{p, q\} \cap \{x_2, x_3, x_4\} = \{p\} = \{x_2\}$ or $\{p, q\} \cap \{x_2, x_3, x_4\} = \{q\} = \{x_4\}$.

Proof. Since G has no 9-cycle, $\{x_2, x_3, x_4\} \cap \{u, p, q, w\} \neq \emptyset$. For $d(x_1) \geq 3$ and $d(x_5) \geq 3$, we have that $x_2 \neq u$ and $x_4 \neq w$. Note that there are no 4-cycles, it follows that $\{x_2, x_3, x_4\} \cap \{u, w\} = \emptyset$, $x_3 \notin \{p, q\}$, $x_4 \neq p$ and $x_2 \neq q$. Therefore, $\{p, q\} \cap \{x_2, x_4\} = \{p\} = \{x_2\}$ or $\{p, q\} \cap \{x_2, x_4\} = \{q\} = \{x_4\}$.

Lemma 2.14. Let $f = [x_1x_2x_3x_4x_5]$ be a 5-face adjacent to two 3-faces, that are either $[x_1x_2x_6]$ and $[x_2x_3x_7]$, or $[x_1x_2x_6]$ and $[x_3x_4x_8]$ (see Fig. 10(a) and Fig. 10(b)). If $d(x_1) = 3$, $d(x_5) \ge 3$ and $d(x_6) \ge 3$, and $x_5x_1x_6$ is incident with a 6⁻-face g, then g is a 6-face $[x_5x_1x_6uvw]$, where $\{u, w\} \cap \{x_1, x_2, \ldots, x_8\} = \emptyset$, $v = x_4$ and $d(x_4) \ge 4$ ($d(x_4) \ge 5$ for the case of Fig. 10(b)).

Proof. We only consider the case of Fig. 10(a) here, the case of Fig. 10(b) is quite similar. Suppose that $g = [x_5x_1x_6u \dots w]$. Since $d(x_5) \ge 3$ and $d(x_6) \ge 3$, x_1, x_2, x_6, u are four distinct vertices, and x_1, x_4, x_5, w are four distinct vertices. As there is no 4-cycle in $G, x_1, x_2, \dots, x_7, u, w$ are distinct. It follows that g must be a 5- or 6-face. If g is a 5-face, then $g = [x_5x_1x_6uw]$ and $[x_5x_4x_3x_7x_2x_1x_6uw]$ is a 9-cycle, a contradiction. Let $g = [x_5x_1x_6uw]$ be a 6-face. If $v \notin \{x_2, x_3, x_4\}$, then $[uvwx_5x_4x_3x_2x_1x_6]$ is a 9-cycle, a contradiction. If $v = x_2$, then $[ux_6x_1x_2]$ is a 4-cycle, a contradiction. If $v = x_3$, then $[ux_6x_2x_3]$ is a 4-cycle, a contradiction.

Lemma 2.15. Let $f = [x_1x_2x_3x_4x_5]$ be a 5-face adjacent to two 3-faces $[x_1x_2x_6]$ and $[x_3x_4x_8]$. If $d(x_2) = 3$, $d(x_3) \ge 4$ and $d(x_6) \ge 3$, then $x_3x_2x_6$ is incident with a 7⁺-face.

Proof. Suppose that $x_3x_2x_6$ is incident with a face $g = [x_3x_2x_6u \dots w]$. Since $d(x_3) \ge 4$ and $d(x_6) \ge 3$, we have that x_2, x_3, x_4, x_8, w are five distinct vertices, and x_1, x_2, x_6, u are four distinct vertices. Since there are no 4-cycles, we have that $x_1, x_2, \dots, x_6, x_8, u, w$ are distinct. It follows that g must be a 5⁺-face. If g is a 5-face, then $g = [x_3x_2x_6uw]$ and $[x_3x_8x_4x_5x_1x_2x_6uw]$ is a 9-cycle, a contradiction. Let g be a 6-face $[x_3x_2x_6uvw]$. If $v \notin \{x_1, x_4, x_5\}$, then $[uvwx_3x_4x_5x_1x_2x_6]$ is a 9-cycle, a contradiction. If $v = x_1$, then $[ux_6x_2x_1]$ is a 4-cycle, a contradiction. If $v = x_4$, then $[wx_3x_8x_4]$ is a 4-cycle, a contradiction. If $v = x_5$, then $[ux_6x_1x_5]$ is a 4-cycle, a contradiction. Therefore, $x_3x_2x_6$ is incident with a 7⁺-face.

Lemma 2.16. Let $f = [x_1x_2x_3...]$ be a 7⁺-face. If x_2 is a normal 3-vertex, then at most one of x_1x_2 and x_2x_3 is incident with a good 5-face.

Proof. Suppose to the contrary that x_1x_2 is incident with a good 5-face $g_1 = [x_1x_2v_3v_4v_5]$ and x_2x_3 is incident with a good 5-face $g_2 = [x_3x_2v_3u_4u_5]$. Note that g_1 and g_2 are all internal faces. By Lemma 2.3, v_3 cannot be a 2-vertex. By Lemma 2.12, g_1 and g_2 are normally adjacent. Moreover, v_3 is a 3-vertex, and $g_3 = [v_3v_4u_4]$ is an internal 3-face. It is observed that g_1, g_2 and g_3 are all internal faces. It follows that v_3 does not belong to $\{x, y\}$, but this contradicts Lemma 2.4.

Let $\tau(\to f)$ be the number of charges that f receives from other elements.

Claim 1. If f is an internal 5-face and $s_f = 1$, then $\tau(\to f) \ge \frac{1}{3}$.

Proof. Let $f = [v_1v_2v_3v_4v_5]$ be an internal 5-face, and let $[v_1v_2v_6]$ be an internal 3-face. Since f has no chord, v_1, v_2, \ldots, v_6 are six distinct vertices. If $v_i \in \{x, y\}$ for any $1 \le i \le 6$, then v_i sends $\frac{1}{3}$ to f by R5, we are done. Assume $\{v_1, v_2, \ldots, v_6\} \cap \{x, y\} = \emptyset$. By Lemma 2.3, $d(v_i) \ge 3$ for $1 \le i \le 6$.

Next, we show that f is adjacent to a special face. By the hypothesis, neither v_3v_4 nor v_4v_5 is incident with an internal 4⁻-face. By Lemma 2.12, neither v_3v_4 nor v_4v_5 is incident with a 5-face. If v_3v_4 or v_4v_5 is incident with an internal 7⁺-face or f_0 , we are done. So we may assume that each of v_3v_4 and v_4v_5 is incident with an internal 6-face. By Lemma 2.13, v_3v_4 is incident with a 6-face $[v_3v_4upv_2w]$. If $[v_2v_3w]$ bounds a 3-face, then d(w) = 2 and v_2v_3 is incident with the outer face $[v_2v_3w]$, we are done. Hence, we can assume that v_2v_3 is not incident with a 3-face. By Lemma 2.12, v_2v_3 cannot be incident with a 5-face. Since there are no 9-cycles, v_2v_3 cannot be incident with a 6-face. Hence, v_2v_3 is incident with a 7⁺-face. Therefore, f is adjacent to at least one special face in any case. By R6 and R8, f receives $\frac{1}{3}$ from each adjacent special face, thus $\tau(\rightarrow f) \geq \frac{1}{3}$.

Claim 2. Let f be an internal 5-face and $s_f = 2$. If f is incident with one minor 3-vertex, then $\tau(\to f) \ge \frac{1}{3}$.

Proof. Assume that $f = [x_1x_2x_3x_4x_5]$. If x or y is incident with f or one of the adjacent 3-faces, then it sends at least $\frac{1}{3}$ to f by R5. So we may assume that neither x nor y is incident with f or the adjacent 3-faces. Now we show that f is adjacent to at least one 7⁺-face sending $\frac{1}{3}$ to f by R6 and R8.

Case 1. Let $[x_1x_2x_6]$ and $[x_2x_3x_7]$ be internal 3-faces, and let x_1 be a minor 3-vertex. By Lemma 2.3 and Lemma 2.4, $d(x_5) \ge 4$ and $d(x_6) \ge 4$. By Lemma 2.14, if $x_5x_1x_6$ is incident with a 6⁻-face, then $[x_4x_5w]$ is a triangle but it does not bound a 3-face, thus x_4x_5 is incident with a 7⁺-face. Hence, either $x_5x_1x_6$ or x_4x_5 is incident with a 7⁺-face.

Case 2. Let $[x_1x_2x_6]$ and $[x_3x_4x_8]$ be internal 3-faces, and let x_1 be a minor 3-vertex. By Lemma 2.3, Lemma 2.4 and Lemma 2.14, we also get that either $x_5x_1x_6$ or x_4x_5 is incident with a 7⁺-face.

Case 3. Let $[x_1x_2x_6]$ and $[x_3x_4x_8]$ be internal 3-faces, and let x_2 be a minor 3-vertex. By Lemma 2.3 and Lemma 2.4, $d(x_3) \ge 4$ and $d(x_6) \ge 4$. By Lemma 2.15, x_2x_3 is incident with a 7⁺-face.

Claim 3. Let f be an internal 5-face and $s_f \ge 2$. If f is incident with two minor 3-vertices, then $\tau(\rightarrow f) \ge 1$.

Proof. Assume $f = [x_1x_2x_3x_4x_5]$. If x_i is a 2-vertex, then $x_i \in \{x, y\}$ and $x_{i-1}x_ix_{i+1}$ is incident with the outer face f_0 . By R5, f receives $\frac{1}{3}$ from each of x and y. By R6, f receives $\frac{1}{3}$ via each of $x_{i-1}x_i$ and x_ix_{i+1} . Thus, $\tau(\rightarrow f) \ge 2 \times \frac{1}{3} + 2 \times \frac{1}{3} > 1$. So we may assume that $d(x_i) \ge 3$ for any $1 \le i \le 5$. Denote the adjacent face incident with x_ix_{i+1} by g_i for $i \in \{1, 2, 3, 4, 5\}$.

Case 1. Let $[x_1x_2x_6]$ and $[x_2x_3x_7]$ be internal 3-faces, and let x_1 and x_3 be minor 3-vertices. Suppose that x_6 is a 2-vertex. It follows that $\{x_2, x_6\} = \{x, y\}$ and $g_5 = f_0$. By R5, f receives $\frac{1}{3}$ from each of x_2 and x_6 . By R6, f receives at least $\frac{1}{3}$ from the outer face f_0 . Thus, $\tau(\to f) \ge 3 \times \frac{1}{3} = 1$.

So we may assume that $d(x_6) \ge 3$, and by symmetry, $d(x_7) \ge 3$. Firstly, we claim that f receives at least $\frac{1}{3}$ from $\{x_2, x_6, x_7\}$. If x_2 is a special vertex, then f receives $\frac{1}{3}$ from x_2 by R3, R4 and R5. So we may assume that x_2 is a normal 4-vertex. It follows from Lemma 2.7 that both x_6 and x_7 are special vertices. By R3, R4 and R5, f receives at least $\frac{1}{6} \times 2 = \frac{1}{3}$ from x_6 and x_7 .

Next, we show that f is adjacent to at least two special faces. Since f receives at least $2 \times \frac{1}{3} = \frac{2}{3}$ from adjacent special faces by R6 and R8, we are done. By Lemma 2.14, we get that both g_3 and g_5 are 6^+ -faces, and g_3 , g_5 cannot be 6-face simultaneously. If both g_3 and g_5 are 7^+ -faces, then we are done. By symmetry, assume that g_5 is a 6-face and g_3 is a 7^+ -face. It follows that g_4 is the outer 3-face or a 7^+ -face. That is, g_3 and g_4 are the special faces, we are done.

Case 2. Let $[x_1x_2x_6]$ and $[x_3x_4x_8]$ be internal 3-faces, and let x_1 and x_4 be minor 3-vertices. Similar to Case 1, we may assume that $d(x_6) \ge 3$ and $d(x_8) \ge 3$. Note that x_1 and x_4 are 3-vertices. Since there are no 4-cycles, neither g_4 nor g_5 is a 4⁻-face. By Lemma 2.12, neither g_4 nor g_5 is a 5-face. By Lemma 2.13 and Lemma 2.14, neither g_4 nor g_5 is a 6-face. So both g_4 and g_5 are 7⁺-faces. Thus, f receives at least $\frac{1}{3} \times 2 = \frac{2}{3}$ from these 7⁺-faces. Next we show that f will receive at least $\frac{1}{3}$ from others.

If g_2 is a 7⁺-face, then we are done. By Lemma 2.13, g_2 cannot be a 6-face. Assume g_2 is a 5-face. By Lemma 2.12, $d(x_2) = d(x_3) = 3$. By Lemma 2.4, we have that $\{x_2, x_3\} = \{x, y\}$. By R5, f receives $\frac{1}{3}$ from each of x_2 and x_3 , we are done. It is clear that g_2 cannot be a 4-face. Suppose that g_2 is a 3-face $[x_2x_3x_7]$. If there is one special vertex in $\{x_2, x_3\}$, then we are done by R3, R4 and R5. So we may assume that both x_2 and x_3 are normal 4-vertices. By Lemma 2.7 and Lemma 2.8, at least two of x_6, x_7 and x_8 are special vertices, thus f receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from these special vertices, we are done.

Case 3. Let $[x_1x_2x_6]$ and $[x_3x_4x_8]$ be internal 3-faces, and let x_1 and x_3 be minor 3-vertices. Similar to Case 1, assume $d(x_6) \ge 3$ and $d(x_8) \ge 3$. By Lemma 2.7, one of $\{x_2, x_6\}$ is a special vertex. By R3, R4 and R5, f receives at least $\frac{1}{6}$ from $\{x_2, x_6\}$.

Since there are no 4-cycles, we have that g_2 cannot be a 4⁻-face. Suppose that g_2 is a 5-face. By Lemma 2.12, we have that $d(x_2) = d(x_3) = 3$. By Lemma 2.4, x_2 belongs to $\{x, y\}$. As a consequence, $\{x_2, x_6\} = \{x, y\}$ and g_2 is the outer face f_0 . By R5 and R6, $\tau(\rightarrow f) \ge 2 \times \frac{1}{3} + \frac{1}{3} = 1$, we are done. By Lemma 2.13, g_2 cannot be a 6-face. Thus, we may assume that g_2 is a 7⁺-face. By R8, f receives $\frac{1}{3}$ from g_2 .

Next we show that f receives at least $\frac{1}{2}$ from others. By Lemma 2.12 and Lemma 2.13, g_4 cannot be a 5- or 6-face. Thus, g_4 is a 3- or 7⁺-face. Suppose that g_4 is a 3-face $[x_4x_5x_9]$. If x_9 is a 2-vertex, then $\{x, y\} \subset \{x_4, x_5, x_9\}$, and then f receives at least $2 \times \frac{1}{3} > \frac{1}{2}$ from x and y by R5. So we may assume that $d(x_9) \ge 3$. By Lemma 2.12 and Lemma 2.13, g_5 is a 7⁺-face sending $\frac{1}{3}$ to f. By Lemma 2.7, there is a special vertex in $\{x_4, x_5, x_8, x_9\}$ sending at least $\frac{1}{6}$ to f. Thus, f receives at least $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ from g_5 and the special vertex. Suppose that g_4 is a 7⁺-face. If g_5 is also a 7⁺-face, then f receives at least $2 \times \frac{1}{3} > \frac{1}{2}$ from g_4 and g_5 , we are done. So we may assume that g_5 is a 6⁻-face. By Lemma 2.14, $d(x_4) \ge 5$. By R3, R4 and R5, f receives at least $\frac{1}{6}$ from x_4 . Therefore, f still receives at least $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ from g_4 and x_4 .

Claim 4. Let f be an internal 5-face and $s_f = 2$. If $t_f = 2$, and exactly one of the two normal 3-vertices is minor, then $\tau(\to f) \ge \frac{1}{2}$.

Proof. Assume $f = [x_1x_2x_3x_4x_5]$. By the definition of normal 3-vertex and minor 3-vertex, we only need to consider two cases.

Case 1. Let $[x_1x_2x_6]$ and $[x_2x_3x_7]$ be internal 3-faces, and let x_1 and x_4 be normal 3-vertices. If x_5 or x_6 is a 2-vertex, then x_5 or x_6 belongs to $\{x, y\}$. It follows that x_1x_5 is incident with the outer face f_0 . By R5, f receives at least $\frac{1}{3}$ from $\{x, y\}$. By R6, f receives $\frac{1}{3}$ from the outer face f_0 . Thus, $\tau(\to f) \ge 2 \times \frac{1}{3} > \frac{1}{2}$. So we may assume that $d(x_5) \ge 3$ and $d(x_6) \ge 3$. Note that x_4 is a 3-vertex. By Lemma 2.14, x_1x_5 cannot be incident with a 6⁻-face. That is, x_1x_5 is incident with a 7⁺-face which sends $\frac{1}{3}$ to f. On the other hand, by Lemma 2.7, one vertex in $\{x_2, x_3, x_6, x_7\}$ is a special vertex which sends at least $\frac{1}{6}$ to f. Thus, $\tau(\to f) \ge \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

Case 2. Let $[x_1x_2x_6]$ and $[x_3x_4x_8]$ be internal 3-faces, and let x_2 and x_5 be normal 3-vertices. If x_6 is a 2-vertex, then $x_6 \in \{x, y\}$. Since x_2 is a normal vertex, $\{x, y\} = \{x_1, x_6\}$. Thus, f receives $\frac{1}{3}$ from each of x_1 and x_6 by R5, and thus $\tau(\to f) \ge \frac{1}{3} + \frac{1}{3} \ge \frac{1}{2}$. Assume $d(x_6) \ge 3$. By Lemma 2.7, at least one of x_1 and x_6 is a special vertex. By R3, R4 and R5, f receives at least $\frac{1}{6}$ from these special vertices. If x_3 is a 3-vertex, then $x_3 \in \{x, y\}$ by Lemma 2.4. By R5, f receives $\frac{1}{3}$ from x_3 . Thus, $\tau(\to f) \ge \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$. So we may assume that $d(x_3) \ge 4$. By Lemma 2.15, x_2x_3 is incident with a 7⁺-face. By R8, f receives $\frac{1}{3}$ from each adjacent 7⁺-face. Thus, $\tau(\to f) \ge \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$.

Claim 5. If f is an internal 5-face and $s_f = 3$, then $\tau(\to f) \geq \frac{2}{3}$.

Proof. Assume $f = [x_1x_2x_3x_4x_5]$. According to symmetry, we only need to consider two cases.

Case 1. Let $[x_1x_2x_6]$, $[x_2x_3x_7]$ and $[x_4x_5x_9]$ be internal 3-faces. Assume $d(x_6) = 2$. By Lemma 2.4, $\{x, y\} = \{x_1, x_6\}$ or $\{x, y\} = \{x_2, x_6\}$. By R5, f receives $\frac{1}{3}$ from each of x and y, thus $\tau(\to f) \ge 2 \times \frac{1}{3} = \frac{2}{3}$. So we may assume that $d(x_6) \ge 3$. Similarly, we can assume that $d(x_7) \ge 3$ and $d(x_9) \ge 3$. It is clear that neither x_1x_5 nor x_3x_4 is incident with a 4⁻-face. By Lemma 2.12, neither x_1x_5 nor x_3x_4 is incident with a 5-face. By Lemma 2.13, neither x_1x_5 nor x_3x_4 is incident with a 6-face. Hence, f is adjacent to two 7⁺-faces. By R6 and R8, $\tau(\to f) \ge 2 \times \frac{1}{3} = \frac{2}{3}$.

Case 2. Let $[x_1x_2x_6]$, $[x_2x_3x_7]$ and $[x_3x_4x_8]$ be internal 3-faces. If $d(x_i) = 2$ for $i \in \{5, 6, 7, 8\}$, then $x_i \in \{x, y\}$ by Lemma 2.3. Since x and y are adjacent, we have that $\{x, y\} \subset \{x_1, x_2, \ldots, x_8\}$. By R5, f receives $\frac{1}{3}$ from each of x and y, thus $\tau(\to f) \ge 2 \times \frac{1}{3} = \frac{2}{3}$. So we may assume that x_5, x_6, x_7 and x_8 are all 3^+ -vertices. It is clear that neither x_4x_5 nor x_1x_5 is contained in a 4⁻-face. By Lemma 2.12, neither x_1x_5 nor x_4x_5 is incident with a 5-face. Recall that x_6 is a 3⁺-vertex and x_4x_5 is not contained in a triangle. By Lemma 2.13, x_1x_5 cannot be incident with a 6-face. Hence, x_1x_5 is incident with a 7⁺-face. By symmetry, x_4x_5 is also incident with a 7⁺-face. By R6 and R8, $\tau(\to f) \ge 2 \times \frac{1}{3} = \frac{2}{3}$.

Now we calculate the final charge of internal 5⁺-faces. Let $f = [v_1 v_2 \dots v_d]$ be an internal *d*-face for $d \ge 5$. By Lemma 2.2, every face in *G* is bounded by a cycle. Since there are no 9-cycles, $d \ne 9$.

If v_i is a 2-vertex, then $v_i \in \{x, y\}$ and $v_{i-1}v_iv_{i+1}$ is incident with the outer face f_0 . Thus, f is adjacent to at most d-2 internal faces. By Corollary 2.5, $t_f \leq \frac{d}{2}$. By R1 and R8, f sends at most $\frac{1}{3}$ to each adjacent internal face. By R2, f sends at most $\frac{1}{2}$ to each incident normal 3-vertex. By R5, f receives $\frac{1}{3}$ from each of x and y. By R6, f receives $\frac{1}{3}$ via each of $v_{i-1}v_i$ and v_iv_{i+1} . Hence, $\mu'(f) \geq d-4+4 \times \frac{1}{3} - (d-2) \times \frac{1}{3} - \frac{d}{2} \times \frac{1}{2} > 0$.

So we may assume that there is no 2-vertex incident with f.

• d = 5.

By Corollary 2.5 and Lemma 2.11, $t_f \leq 2$ and $s_f \leq 3$. If $s_f = 0$, then $\mu'(f) \geq 5 - 4 - 2 \times \frac{1}{3} > 0$ by R2. If $s_f = 1$, then f is incident with at most one minor 3-vertex. By Claim 1, R1 and R2, $\mu'(f) \geq 5 - 4 + \frac{1}{3} - \frac{1}{3} - (\frac{1}{2} + \frac{1}{3}) > 0$.

Assume $s_f = 2$. If $t_f = 0$, then $\mu'(f) \ge 5 - 4 - 2 \times \frac{1}{3} > 0$ by R1. Let $t_f = 1$. If the normal 3-vertex is not minor, then $\mu'(f) \ge 5 - 4 - 2 \times \frac{1}{3} - \frac{1}{3} = 0$ by R1 and R2. If the normal 3-vertex is

minor, then $\mu'(f) \ge 5 - 4 + \frac{1}{3} - 2 \times \frac{1}{3} - \frac{1}{2} > 0$ by Claim 2, R1 and R2. Let $t_f = 2$. It is observed that f is incident with at least one minor 3-vertex. If f is incident with exactly one minor 3-vertex, then $\mu'(f) \ge 5 - 4 + \frac{1}{2} - 2 \times \frac{1}{3} - (\frac{1}{2} + \frac{1}{3}) = 0$ by Claim 4, R1 and R2. The other situation, f is incident with exactly two minor 3-vertices. Thus, $\mu'(f) \ge 5 - 4 + 1 - 2 \times \frac{1}{3} - 2 \times \frac{1}{2} > 0$ by Claim 3, R1 and R2.

Assume $s_f = 3$. If $t_f = 0$, then $\mu'(f) \ge 5 - 4 - 3 \times \frac{1}{3} = 0$ by R1. If $t_f = 1$, then $\mu'(f) \ge 5 - 4 + \frac{2}{3} - 3 \times \frac{1}{3} - \frac{1}{2} > 0$ by Claim 5, R1 and R2. If $t_f = 2$, then it is incident with two minor 3-vertices, and then $\mu'(f) \ge 5 - 4 + 1 - 3 \times \frac{1}{3} - 2 \times \frac{1}{2} = 0$ by Claim 3, R1 and R2.

• d = 6.

Note that there are no 4-cycle in G. If f is adjacent to a 3-face, then it must be normally adjacent to the 3-face. Since there are no 9-cycles in G, f is adjacent to at most two 3-faces. It follows that f is incident with at most two minor 3-vertices. By R1 and R2, $\mu'(f) \ge 6 - 4 - 2 \times \frac{1}{3} - (2 \times \frac{1}{2} + \frac{1}{3}) = 0$.

• d = 7.

If f is adjacent to a 3-face, then it must be normally adjacent to the 3-face. Otherwise, there is a 4-cycle in G. Since there are no 9-cycles in G, f is adjacent to at most one 3-face. It follows that f is incident with at most one minor 3-vertex. By Corollary 2.5, $t_f \leq 3$. If $t_f = 3$, then f is adjacent to at most four good 5-faces by Lemma 2.4 and Lemma 2.16, and then $\mu'(f) \geq 7 - 4 - (1+4) \times \frac{1}{3} - (\frac{1}{2} + 2 \times \frac{1}{3}) > 0$ by R1, R2 and R8. If $t_f = 2$, then f is adjacent to at most five good 5-faces by Lemma 2.4 and Lemma 2.16, and then $\mu'(f) \geq 7 - 4 - (1+5) \times \frac{1}{3} - (\frac{1}{2} + \frac{1}{3}) > 0$ by R1, R2 and R8. If $t_f = 1$, then f is adjacent to at most six good 5-faces by Lemma 2.4 and Lemma 2.16, and then $\mu'(f) \geq 7 - 4 - (1+6) \times \frac{1}{3} - \frac{1}{2} > 0$ by R1, R2 and R8. If $t_f = 0$, then $\mu'(f) \geq 7 - 4 - 7 \times \frac{1}{3} > 0$ by R1 and R8.

• d = 8.

Similar to the above cases, if f is adjacent to a 3-face, then it must be normally adjacent to the 3-face. Since there are no 9-cycles, f is not adjacent to any 3-face. Thus, f is not incident with any minor 3-vertex. By R2 and R8, $\mu'(f) \ge 8 - 4 - 8 \times \frac{1}{3} - 4 \times \frac{1}{3} = 0$.

• $d \ge 10$.

By R1 and R8, f sends at most $\frac{1}{3}$ via each incident edge. It follows that $\mu'(f) \ge d - 4 - d \times \frac{1}{3} - \frac{d}{2} \times \frac{1}{2} > 0$. This completes the proof of Theorem 2.1.

References

- E.-K. Cho, I. Choi, R. Kim, B. Park, T. Shan and X. Zhu, Decomposing planar graphs into graphs with degree restrictions, J. Graph Theory 101 (2) (2022) 165–181.
- [2] L. J. Cowen, R. H. Cowen and D. R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, J. Graph Theory 10 (2) (1986) 187–195.
- [3] W. Cushing and H. A. Kierstead, Planar graphs are 1-relaxed, 4-choosable, European J. Combin. 31 (5) (2010) 1385–1397.
- [4] W. Dong and B. Xu, A note on list improper coloring of plane graphs, Discrete Appl. Math. 157 (2) (2009) 433–436.
- [5] Z. Dvořák and L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8, J. Combin. Theory Ser. B 129 (2018) 38–54.
- [6] N. Eaton and T. Hull, Defective list colorings of planar graphs, Bull. Inst. Combin. Appl. 25 (1999) 79–87.

- [7] J. Grytczuk and X. Zhu, The Alon-Tarsi number of a planar graph minus a matching, J. Combin. Theory Ser. B 145 (2020) 511–520.
- [8] R. Kim, S.-J. Kim and X. Zhu, The Alon-Tarsi number of subgraphs of a planar graph, arXiv:1906.01506, http://arxiv.org/abs/1906.01506v1.
- [9] S.-J. Kim, A. V. Kostochka, X. Li and X. Zhu, On-line DP-coloring of graphs, Discrete Appl. Math. 285 (2020) 443–453.
- [10] K.-W. Lih, Z. Song, W. Wang and K. Zhang, A note on list improper coloring planar graphs, Appl. Math. Lett. 14 (3) (2001) 269–273.
- [11] H. Lu and X. Zhu, The Alon-Tarsi number of planar graphs without cycles of lengths 4 and l, Discrete Math. 343 (5) (2020) 111797.
- [12] R. Škrekovski, List improper colourings of planar graphs, Combin. Probab. Comput. 8 (3) (1999) 293– 299.