# Decomposition of planar graphs with forbidden configurations 

Lingxi $\mathrm{Li}^{*} \quad$ Huajing $\mathrm{Lu}^{\dagger} \quad$ Tao Wang ${ }^{\ddagger} \quad$ Xuding $\mathrm{Zhu}^{\S}$


#### Abstract

A $(d, h)$-decomposition of a graph $G$ is an ordered pair $(D, H)$ such that $H$ is a subgraph of $G$ of maximum degree at most $h$ and $D$ is an acyclic orientation of $G-E(H)$ with maximum out-degree at most $d$. In this paper, we prove that for $l \in\{5,6,7,8,9\}$, every planar graph without 4 - and $l$-cycles is $(2,1)$-decomposable. As a consequence, for every planar graph $G$ without 4 - and $l$-cycles, there exists a matching $M$, such that $G-M$ is 3-DP-colorable and has Alon-Tarsi number at most 3. In particular, $G$ is 1-defective 3-DP-colorable, 1-defective 3-paintable and 1-defective 3-choosable. These strengthen the results in [Discrete Appl. Math. 157 (2) (2009) 433-436] and [Discrete Math. 343 (2020) 111797].


Keywords: decomposition; list coloring; defective coloring; Alon-Tarsi number; DP-coloring

## 1 Introduction

A proper $k$-coloring of a graph $G$ is a mapping $\phi: V(G) \rightarrow[k]$ such that $\phi(u) \neq \phi(v)$, whenever $u v \in E(G)$, where and herein after, $[k]=\{1,2, \ldots, k\}$. The least integer $k$ such that $G$ admits a proper $k$-coloring is the chromatic number $\chi(G)$ of $G$. Let $h$ be a non-negative integer. An $h$-defective $k$-coloring of $G$ is a mapping $\phi: V(G) \rightarrow[k]$ such that each color class induces a subgraph of maximum degree at most $h$. In particular, a 0 -defective coloring is a proper coloring of $G$.

A $k$-list assignment of $G$ is a mapping $L$ that assigns a list $L(v)$ of $k$ colors to each vertex $v$ in $G$. An $h$-defective $L$-coloring of $G$ is an $h$-defective coloring $\psi$ of $G$ such that $\psi(v) \in L(v)$ for all $v \in V(G)$. A graph $G$ is $h$-defective $k$-choosable if $G$ admits an $h$-defective $L$-coloring for each $k$-list assignment $L$. In particular, if $G$ is 0 -defective $k$-choosable, then we call it $k$-choosable. The choice number $\operatorname{ch}(G)$ is the smallest integer $k$ such that $G$ is $k$-choosable.

Cowen, Cowen, and Woodall [2] proved that every outerplanar graph is 2-defective 2-colorable, and every planar graph is 2-defective 3-colorable. Eaton and Hull [6], and independently, Škrekovski [12] proved that every outerplanar graph is 2 -defective 2 -choosable, and every planar graph is 2 -defective 3 -choosable. Cushing and Kierstead [3] proved that every planar graph is 1-defective 4 -choosable. Let $\mathcal{G}_{4, l}$ be the family of planar graphs which contain no 4 -cycles and no $l$-cycles. Lih et al. [10] proved that for each $l \in\{5,6,7\}$, every graph $G \in \mathcal{G}_{4, l}$ is 1 -defective 3 -choosable. Dong and $\mathrm{Xu}[4]$ proved that for each $l \in\{8,9\}$, every graph $G \in \mathcal{G}_{4, l}$ is 1 -defective 3 -choosable.

Note that a graph being $h$-defective $k$-choosable means that for every $k$-list assignment $L$ of $G$, there exists a subgraph $H$ (depending on $L$ ) of $G$ with $\Delta(H) \leq h$ such that $G-E(H)$ is $L$-colorable. The subgraph $H$ may be different for different $L$. As a strengthening of the above results, the following problem is studied in the literature: For $(h, k) \in\{(2,3),(1,4)\}$, is it true that every planar graph $G$ has a subgraph of maximum

[^0]

Fig. 1: Forbidden configurations in (1) and (2) of Theorem 1.1.
degree $h$ such that $G-E(H)$ is $k$-choosable? For $l \in\{5,6,7,8,9\}$, is it true that every graph $G \in \mathcal{G}_{4, l}$ has a matching $M$ such that $G-M$ is 3 -choosable?

It turns out that for the first question, the answer is negative for $(h, k)=(2,3)$, and positive for $(h, k)=$ $(1,4)$. It was proved in [8] that there exists a planar graph $G$ such that for any subgraph $H$ of $G$ of maximum degree $3, G-E(H)$ is not 3 -choosable, and proved in [7] that every planar graph $G$ has a matching $M$ such that $G-M$ is 4-choosable. For the second question, for $l \in\{5,6,7\}$, it was shown in [11] every graph $G \in \mathcal{G}_{4, l}$ has a matching $M$ such that $G-M$ is 3 -choosable.

Indeed, stronger results were proved in $[7,11]$. The results concern two other graph parameters: The AlonTarsi number $A T(G)$ of $G$ and the paint number $\chi_{P}(G)$ of $G$. The reader is referred to [7] for the definitions. We just note here that for any graph $G, \operatorname{ch}(G) \leq \chi_{P}(G) \leq A T(G)$, and the differences $\chi_{P}(G)-c h(G)$ and $A T(G)-\chi_{P}(G)$ can be arbitrarily large. It was proved in [7] that every planar graph $G$ has a matching $M$ such that $A T(G-M) \leq 4$, and proved in [11] that for $l \in\{5,6,7\}$, every graph $G \in \mathcal{G}_{4, l}$ has a matching $M$ such that $A T(G-M) \leq 3$.

In this paper, we consider further strengthening of the results concerning graphs in $\mathcal{G}_{4, l}$ for $l \in\{5,6,7,8,9\}$. (Note that the result in [11] does not cover the cases for $l=8$ and 9 ). We strengthen the above results in two aspects: a larger class of graphs with a stronger property.

Given two non-negative integers $d, h$ and a graph $G$, a $(d, h)$-decomposition of $G$ is a pair $(D, H)$ such that $H$ is a subgraph of $G$ of maximum degree at most $h$ and $D$ is an acyclic orientation of $G-E(H)$ with maximum out-degree at most $d$. We say $G$ is $(d, h)$-decomposable if $G$ has a $(d, h)$-decomposition. Cho et al. [1] proved that every planar graph is (4, 1)-decomposable, (3, 2)-decomposable and (2, 6)-decomposable. Note that a graph $H$ which has an acyclic orientation of maximum out-degree at most $d$ if and only if $H$ is $d$-degenerate, i.e., the vertices of $H$ can be linearly ordered so that each vertex has at most $d$ backward neighbors. It is well-known and easy to see that $d$-degenerate graphs not only have choice number, paint number, Alon-Tarsi number and DP-chromatic number at most $d+1$, there is a linear time algorithm that creates the above mentioned linear ordering and the corresponding coloring is easily obtained by using a greedy coloring algorithm. The reader is referred to [5] for the definition of DP-chromatic number $\chi_{D P}(G)$ of a graph $G$. We just mention here that $\operatorname{ch}(G) \leq \chi_{D P}(G)$, and there are graphs $G$ for which $\chi_{D P}(G)$ are larger than each of $A T(G)$ and $\chi_{P}(G)$, there are also graphs $G$ for which $\chi_{D P}(G)$ are smaller than each of $A T(G)$ and $\chi_{P}(G)[9]$. This paper proves the following result:

Theorem 1.1. Assume $G$ is a plane graph. Then $G$ is $(2,1)$-decomposable if one of the following holds:
(1) $G$ has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2.
(2) $G$ has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3.
(3) $G \in \mathcal{G}_{4,9}$.

Note that if $G \in \mathcal{G}_{4, l}$ for some $l \in\{5,6,7\}$, then $G$ has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2, and if $G \in \mathcal{G}_{4,8}$, then $G$ has no subgraph isomorphic to any configuration in Fig. 1 and


Fig. 2: Forbidden configurations in (1) of Theorem 1.1.

(a)

(b)

(c)

(d)

(e)

Fig. 3: Forbidden configurations in (2) of Theorem 1.1.

Fig. 3. Consequently, for $l \in\{5,6,7,8,9\}$, all graphs $G \in \mathcal{G}_{4, l}$ are (2,1)-decomposable.
All graphs in this paper are finite and simple. For a plane graph $G$, we use $V(G), E(G)$ and $F(G)$ to denote the vertex set, edge set and face set of $G$, respectively. For any element $x \in V(G) \cup F(G)$, the degree of $x$ is denoted by $d(x)$. A vertex $v$ in $G$ is called a $k$-vertex, or $k^{+}$-vertex, or $k^{-}$-vertex, if $d(v)=k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. Analogously, one can define $k$-face, $k^{+}$-face, and $k^{-}$-face. An $n$-face $\left[x_{1} x_{2} \ldots x_{n}\right]$ is a $\left(d_{1}, d_{2}, \ldots, d_{n}\right)$-face if $d\left(x_{i}\right)=d_{i}$ for $1 \leq i \leq n$. Let $D$ be an orientation of a graph $G$, we use $d_{D}^{+}(v)$ and $d_{D}^{-}(v)$ to denote the out-degree and in-degree of a vertex $v$ in $D$, respectively. Let $\Delta^{+}(D)$ denote the maximum out-degree of vertices in $D$. Two cycles (or faces) are adjacent if they have at least one common edge. Two cycles (or faces) are normally adjacent if they intersect in exactly two vertices. Let $G$ be a plane graph and $x y$ be a given boundary edge of $G$. A vertex $v \neq x, y$ is called a normal vertex. A vertex $v$ is special if $v$ is a $5^{+}$-vertex or $v \in\{x, y\}$. A face is internal if it is not the outer face $f_{0}$. A face is special if it is an internal $7^{+}$-face or the outer face $f_{0}$. A normal vertex $v$ is minor if $d(v)=3$ and it is incident with an internal $4^{-}$-face. A good 5 -face is an internal 5 -face adjacent to at least one internal 3 -face. An edge contained in a triangle is a triangular edge. Note that in all three cases, there are no adjacent triangles. So every triangular edge is contained in a unique triangle.

## 2 Proof of Theorem 1.1

For the purpose of using induction, we prove the following result. Assume $G$ is a plane graph and $e=x y$ is a boundary edge of $G$. A nice decomposition of $(G, e)$ is a pair $(D, M)$ such that $M$ is a matching and $D$ is an acyclic orientation of $G-M$ with $d_{D}^{+}(x)=d_{D}^{+}(y)=0$ and $\Delta^{+}(D) \leq 2$. Note that in a nice decomposition $(D, M)$ of $(G, e)$, since $d_{D}^{+}(x)=d_{D}^{+}(y)=0$, we conclude that $e=x y \in M$.

Theorem 2.1. If $G$ is a plane graph satisfying the condition of Theorem 1.1 and $e$ is a boundary edge of $G$, then $(G, e)$ has a nice decomposition.

Assume Theorem 2.1 is not true and $G$ is a counterexample with minimum number of vertices. We shall
derive a sequence of properties of $G$ that lead to a contradiction. It is obvious that $G$ is connected, for otherwise we can consider each component of $G$ separately.

Lemma 2.2. $G$ is 2-connected.
Proof. Assume to the contrary that $G$ has a cut-vertex $x^{\prime}$. Let $G=H_{1} \cup H_{2}, V\left(H_{1} \cap H_{2}\right)=\left\{x^{\prime}\right\}$ and $e=x y \in E\left(H_{1}\right)$. Let $e^{\prime}=x^{\prime} y^{\prime}$ be a boundary edge of $H_{2}$. By the minimality of $G$, there is a nice decomposition $\left(D_{1}, M_{1}\right)$ of $\left(H_{1}, e\right)$ and a nice decomposition $\left(D_{2}, M_{2}\right)$ of $\left(H_{2}, e^{\prime}\right)$. Let $M=\left(M_{1} \cup M_{2}\right) \backslash\left\{x^{\prime} y^{\prime}\right\}$ and $D=D_{1} \cup D_{2} \cup\left\{\overleftarrow{x^{\prime} y^{\prime}}\right\}$. It is straightforward to verify that $(D, M)$ is a nice decomposition of $(G, e)$.

Lemma 2.3. For any $v \in V(G) \backslash\{x, y\}, d(v) \geq 3$.
Proof. Assume $v \in V(G) \backslash\{x, y\}$ and $d(v) \leq 2$. By the minimality of $G$, there exists a nice decomposition $(D, M)$ of $(G-v, e)$. Let $D^{\prime}$ be obtained from $D$ by orienting edges incident with $v$ as out-going edges from $v$. Then $\left(D^{\prime}, M\right)$ is a nice decomposition of $(G, e)$.

Lemma 2.4. If $u$ and $v$ are two adjacent 3 -vertices, then $\{u, v\} \cap\{x, y\} \neq \emptyset$.
Proof. Suppose that $u$ and $v$ are two adjacent 3-vertices with $\{u, v\} \cap\{x, y\}=\emptyset$. By the minimality of $G$, there is a nice decomposition $(D, M)$ of $(G-\{u, v\}, e)$. Let $M^{\prime}=M \cup\{u v\}$, and $D^{\prime}$ be obtained from $D$ by orienting the other edges incident with $u, v$ as out-going edges from $u, v$. Then $\left(D^{\prime}, M^{\prime}\right)$ is a nice decomposition of $(G, e)$.

For an internal face $f$, let $t_{f}$ be the number of incident normal 3 -vertices and let $s_{f}$ be the number of adjacent internal 3 -faces. Note that each 3 -vertex of $f$ is incident with at most one 3 -face adjacent to $f$. Thus we have the following corollary.

Corollary 2.5. For any internal face $f, t_{f} \leq d(f) / 2$ and $t_{f}+s_{f} \leq d(f)$.
The following four lemmas first appeared in [11], although the hypotheses and some definitions are slightly different. For the completeness of this paper, we include the short proofs with illustration figures.

(a)

(b)

Fig. 4: (a) A bad 5-cycle and an adjacent triangle. (b) For the proof of Lemma 2.6. Here and in figures below, a solid triangle represents a 3 -vertex, a solid square represents a 4 -vertex, a thick line represents an edge in the matching $M$.

A 5-cycle $\left[u_{1} u_{2} u_{3} u_{4} u_{5}\right]$ is a bad 5 -cycle if it is adjacent to a triangle $\left[u_{1} u_{5} u_{6}\right]$ with $u_{i} \notin\{x, y\}$, where $1 \leq i \leq 6$, and $d\left(u_{1}\right)=d\left(u_{3}\right)=3$, and $d\left(u_{2}\right)=d\left(u_{4}\right)=d\left(u_{5}\right)=d\left(u_{6}\right)=4$, as depicted in Fig. 4(a).

Lemma 2.6 (Lemma 5.2 in [11]). There are no bad 5-cycles in $G$.

Proof of Lemma 2.6. Assume $C=\left[u_{1} u_{2} u_{3} u_{4} u_{5}\right]$ is a bad 5 -cycle and $T=\left[u_{1} u_{5} u_{6}\right]$ is a triangle adjacent to $C$, where $d\left(u_{1}\right)=d\left(u_{3}\right)=3$ and $d\left(u_{i}\right)=4$ for $i \in\{2,4,5,6\}$, as depicted in Fig. 4(a). A nice decomposition of $G-\left\{u_{1}, u_{2}, \ldots, u_{6}\right\}$ is extended to a nice decomposition as in Fig. 4(b).

A triangle $T$ is minor if $T$ is a (3,4,4)-triangle and $T \cap\{x, y\}=\emptyset$. A triangle chain in $G$ is a subgraph of $G-\{x, y\}$ consisting of vertices $w_{1}, w_{2}, \ldots, w_{k+1}, u_{1}, u_{2}, \ldots, u_{k}$ in which $\left[w_{i} w_{i+1} u_{i}\right]$ is a $(4,4,4)$-cycle for $1 \leq i \leq k$, as depicted in Fig. 5 . We denote $T_{i}$ the triangle $\left[w_{i} w_{i+1} u_{i}\right]$ and denote such a triangle chain by $T_{1} T_{2} \ldots T_{k}$. If a triangle $T$ has exactly one common vertex with a triangle chain $T_{1} T_{2} \ldots T_{k}$ and the common vertex is in $T_{1}$, then we say $T$ intersects the triangle chain $T_{1} T_{2} \ldots T_{k}$.


Fig. 5: A triangle chain.


Fig. 6: (a) The configuration in Lemma 2.7. (b) For the proof of Lemma 2.7.

Lemma 2.7 (Lemma 2.10 in [11]). If a minor triangle $T_{0}$ intersects a triangle chain $T_{1} T_{2} \ldots T_{k}$, then every 3-vertex adjacent to a vertex in $T_{k}$ belongs to $\{x, y\} \cup V\left(T_{0}\right)$.

The $k=0$ case of the above lemma asserts that every 3 -vertex adjacent to a vertex in $T_{0}$ belongs to $\{x, y\}$.

Proof of Lemma 2.7. Assume $G$ has a minor triangle $T_{0}=\left[w_{0} w_{1} u_{0}\right]$ intersecting a triangle chain $T_{1} T_{2} \ldots T_{k}$, and $z \notin\{x, y\} \cup V\left(T_{0}\right)$ is a 3-vertex adjacent to a vertex in $T_{k}$, as depicted in Fig. 6(a). A nice decomposition of $G-\left(\bigcup_{i=0}^{k} V\left(T_{i}\right) \cup\{z\}\right)$ is extended to a nice decomposition of $G$ as in Fig. 6(b).

Lemma 2.8 (Lemma 2.11 in [11]). If a minor triangle $T_{0}$ intersects a triangle chain $T_{1} T_{2} \ldots T_{k}$, then the distance between $T_{k}$ and another minor triangle is at least two.


Fig. 7: (a) The configuration in Lemma 2.8. (b) For the proof of Lemma 2.8.


Fig. 8: (a) The configuration in Lemma 2.9. (b) For the proof of Lemma 2.9.

Proof of Lemma 2.8. Assume to the contrary that $T_{1} T_{2} \ldots T_{k}$ with $T_{i}=\left[w_{i} w_{i+1} u_{i}\right], 1 \leq i \leq k$, is a triangle chain that intersects a minor triangle $T_{0}=\left[w_{0} w_{1} u_{0}\right]$, and the distance between $T_{k}$ and another minor triangle $T_{0}^{\prime}=\left[z z_{1} z_{2}\right]$ with $d\left(z_{1}\right)=3$ is less than 2. By Lemma 2.7, we may assume $w_{k+1} z$ is a (4,4)-edge connecting $T_{k}$ and $T_{0}^{\prime}$, as depicted in Fig. 7(a). A nice decomposition of $G-\left(\bigcup_{i=0}^{k} V\left(T_{i}\right) \cup V\left(T_{0}^{\prime}\right)\right)$ is extended to a nice decomposition of $G$ as in Fig. 7(b).

Lemma 2.9 (Lemma 3.1 in [11]). Assume that $f$ is a 6 -face adjacent to five 3 -faces, and none of the vertices on these 3 -faces is in $\{x, y\}$. If $f$ is incident with a 3 -vertex, then there is at least one $5^{+}$-vertex on these five 3 -faces.

Proof of Lemma 2.9. Let $f=\left[v_{1} v_{2} v_{3} v_{4} v_{5} v_{6}\right]$ be a 6 -face, $v_{1}$ be a 3 -vertex and $T_{i}=\left[v_{i} v_{i+1} u_{i}\right], 1 \leq i \leq 5$, be the five 3 -faces. Assume to the contrary that there is no $5^{+}$-vertex on $T_{i}$. By Lemma 2.7, we may assume all $v_{i+1}$ and $u_{i}$ are 4 -vertices for $1 \leq i \leq 5$, as depicted in Fig. 8(a). A nice decomposition of $G-\left(\bigcup_{i=1}^{5} V\left(T_{i}\right)\right)$ is extended to a nice decomposition of $G$ as in Fig. 8(b).

The above lemmas present some reducible configurations. We use standard discharging method to prove that there must be some reducible configurations in a minimum counterexample, which leads to a contradiction.

First, we define an initial charge function by $\mu(x)=d(x)-4, \mu(y)=d(y)-4, \mu\left(f_{0}\right)=d\left(f_{0}\right)+4$, and $\mu(v)=d(v)-4$ for each vertex $v \in V(G) \backslash\{x, y\}, \mu(f)=d(f)-4$ for each face $f$ other than $f_{0}$. By Euler's formula and handshaking theorem, we obtain that the sum of all the initial charges is zero, i.e.,

$$
(d(x)-4)+(d(y)-4)+\left(d\left(f_{0}\right)+4\right)+\sum_{v \neq x, y}(d(v)-4)+\sum_{f \neq f_{0}}(d(f)-4)=0
$$

Next, we design some discharging rules to redistribute the charges, such that the sum of the final charges is not zero, which leads to a contradiction.

## Discharging Rules

R1. Every internal 3-face $f$ receives $\frac{1}{3}$ from each adjacent face.
R2. Assume $v$ is a normal 3-vertex. If $v$ is incident with an internal $4^{-}$-face, then it receives $\frac{1}{2}$ from each of the other two incident faces. Otherwise it receives $\frac{1}{3}$ from each incident face.

R3. Let $v$ be a normal 5 -vertex. Then $v$ sends $\frac{1}{6}$ to each incident $4^{+}$-face. If $v$ is incident with a 3 -face $g=[u v w]$, then $v$ sends $\frac{1}{6}$ to the other face $g^{\prime}$ incident with $u w$. Moreover, if $v$ is incident with three consecutive faces $f_{1}, f_{2}, f_{3}$ and $f_{1}, f_{3}$ are 3 -faces, then $v$ sends an extra $\frac{1}{6}$ to $f_{2}$.

R4. Let $v$ be a normal $6^{+}$-vertex. Then $v$ sends $\frac{1}{3}$ to each incident $4^{+}$-face. If $v$ is incident with a 3 -face $g=[u v w]$, then $v$ sends $\frac{1}{3}$ to the other face $g^{\prime}$ incident with $u w$.

R5. Let $v$ be a vertex in $\{x, y\}$. Then it sends $\frac{1}{3}$ to every incident internal $4^{+}$-face. If $v$ is incident with a 3 -face $g=[u v w]$, then $v$ sends $\frac{1}{3}$ to the other face $g^{\prime}$ incident with $u w$.

R6. $f_{0}$ sends $\frac{1}{3}$ to each adjacent $4^{+}$-face.
R7. In Case 2 (i.e., $G$ has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3), every internal 5 -face receives $\frac{1}{6}$ from adjacent internal $6^{+}$-faces via each common edge.

R8. In Case 3 (i.e., $G \in \mathcal{G}_{4,9}$ ), every good 5 -face receives $\frac{1}{3}$ from adjacent internal $7^{+}$-faces via each common edge.

For $z \in V(G) \cup F(G)$, let $\mu^{\prime}(z)$ be the final charge of $z$. In the remainder of this paper, we prove that $\sum_{z \in V(G) \cup F(G)} \mu^{\prime}(z)>0$, which contradicts the fact that $\sum_{z \in V(G) \cup F(G)} \mu^{\prime}(z)=\sum_{z \in V(G) \cup F(G)} \mu(z)=0$.

Note that R7 only applies to Case 2 and R8 only applies to Case 3. Moreover, R7 and R8 only involve $5^{+}$-faces.

It follows from R5 that for $v \in\{x, y\}$

$$
\mu^{\prime}(v) \geq \mu(v)-(d(v)-1) \times \frac{1}{3}=\frac{2 d(v)-11}{3} \geq-\frac{7}{3}
$$

Note that $f_{0}$ sends $\frac{1}{3}$ to each adjacent internal face by R1 and R6, and sends at most $\frac{1}{2}$ to each incident normal 3 -vertex by R2. It follows from Lemma 2.4 that $f_{0}$ is incident with at most $\frac{d\left(f_{0}\right)}{2}$ normal 3 -vertices. Then

$$
\mu^{\prime}\left(f_{0}\right) \geq \mu\left(f_{0}\right)-\frac{d\left(f_{0}\right)}{2} \times \frac{1}{2}-d\left(f_{0}\right) \times \frac{1}{3} \geq \frac{5 d\left(f_{0}\right)}{12}+4 \geq \frac{21}{4}
$$

Hence, $\mu^{\prime}(x)+\mu^{\prime}(y)+\mu^{\prime}\left(f_{0}\right)>0$.

Assume $v$ is a normal 3-vertex. If $v$ is incident with an internal $4^{-}$-face, then the other two incident faces are $5^{+}$-faces or the outer face $f_{0}$. Hence $\mu^{\prime}(v)=\mu(v)+2 \times \frac{1}{2}=0$. Otherwise each face incident with $v$ is a $5^{+}$-face or $f_{0}$, and $\mu^{\prime}(v)=\mu(v)+3 \times \frac{1}{3}=0$ by R2.

If $v$ is a normal 4 -vertex, then $\mu^{\prime}(v)=\mu(v)=0$. If $v$ is a normal 5 -vertex, then it is incident with at most two 3 -faces, and then $\mu^{\prime}(v) \geq \mu(v)-5 \times \frac{1}{6}-\frac{1}{6}=0$ by R3. If $v$ is a normal $6^{+}$-vertex, then $\mu^{\prime}(v)=\mu(v)-d(v) \times \frac{1}{3}=\frac{2(d(v)-6)}{3} \geq 0$ by R 4 .

If $f$ is an internal 3-face, then it receives $\frac{1}{3}$ via each incident edge, and $\mu^{\prime}(f)=\mu(f)+3 \times \frac{1}{3}=0$ by R1. If $f$ is an internal 4-face, then $\mu^{\prime}(f) \geq \mu(f)=0$.
It remains to show that $\mu^{\prime}(f) \geq 0$ for internal $5^{+}$-faces $f$.
In the remainder of the paper, we consider the three cases separately in three subsections.

## 2.1 $G$ has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2

Assume that $f=\left[v_{1} v_{2} v_{3} v_{4} v_{5}\right]$ is an internal 5 -face. By Corollary $2.5, t_{f} \leq 2$. If $f$ is not adjacent to any internal 3 -face, then $\mu^{\prime}(f) \geq \mu(f)-2 \times \frac{1}{2}=0$ by R2. So we may assume that $f$ is adjacent to at least one internal 3 -face. Since the configurations Fig. 2(a) $-2(\mathrm{~d})$ are forbidden, $f$ is adjacent to exactly one internal 3 -face $f^{*}$ and no 4-faces. If $t_{f} \leq 1$, then $\mu^{\prime}(f) \geq \mu(f)-\frac{1}{3}-\frac{1}{2}>0$ by R1 and R2. Assume $t_{f}=2$ and $f^{*}=\left[u v_{1} v_{2}\right]$ is an internal 3-face. If there are some special vertices in $\left\{u, v_{1}, v_{2}, \ldots, v_{5}\right\}$, then $f$ receives at least $\frac{1}{6}$ from special vertices, and then $\mu^{\prime}(f) \geq \mu(f)-\frac{1}{3}-\left(\frac{1}{3}+\frac{1}{2}\right)+\frac{1}{6}=0$ by R1, R2, R3, R4 and R5. So we may assume that none of $\left\{u, v_{1}, v_{2}, \ldots, v_{5}\right\}$ is a special vertex. It follows that $f$ is incident with two 3 -vertices and three 4 -vertices. If neither $v_{1}$ nor $v_{2}$ is a 3 -vertex, then $\mu^{\prime}(f) \geq \mu(f)-\frac{1}{3}-2 \times \frac{1}{3}=0$ by R1 and R2. Without loss of generality, assume that $d\left(v_{2}\right)=3$ and $d\left(v_{1}\right)=d\left(v_{3}\right)=d(u)=4$. If $d\left(v_{4}\right)=3$ and $d\left(v_{5}\right)=4$, then it contradicts Lemma 2.6. If $d\left(v_{4}\right)=4$ and $d\left(v_{5}\right)=3$, then it contradicts Lemma 2.7.

Assume that $f=\left[v_{1} v_{2} v_{3} v_{4} v_{5} v_{6}\right]$ is an internal 6 -face. By Corollary $2.5, t_{f} \leq 3$.

- $t_{f}=3$. Without loss of generality, assume that $v_{1}, v_{3}$ and $v_{5}$ are normal 3 -vertices.

By Corollary 2.5, $s_{f} \leq 3$. If $s_{f} \leq 1$, then $\mu^{\prime}(f) \geq \mu(f)-\frac{1}{3}-3 \times \frac{1}{2}>0$ by R1 and R2.
Assume that $s_{f}=2$. By symmetry, assume that one of the adjacent internal 3 -face is $\left[v_{1} v_{2} u\right]$. By Lemma 2.7, one vertex in $\left\{u, v_{2}\right\}$ is a special vertex. Thus, $\mu^{\prime}(f) \geq \mu(f)-2 \times \frac{1}{3}-3 \times \frac{1}{2}+\frac{1}{6}=0$ by R1, R2, R3, R4 and R5.

Assume that $s_{f}=3$.
(i) $v_{i} v_{i+1}$ is incident with an internal 3 -face $\left[v_{i} v_{i+1} u_{i}\right]$ for $i \in\{1,3,5\}$. For each $i \in\{1,3,5\}$, by Lemma 2.7, there is a special vertex in $\left\{u_{i}, v_{i+1}\right\}$. Thus $f$ receives at least $\frac{1}{6}$ from $\left\{u_{i}, v_{i+1}\right\}$ by R3, R4 and R5. Hence, $\mu^{\prime}(f) \geq \mu(f)-3 \times \frac{1}{3}-3 \times \frac{1}{2}+3 \times \frac{1}{6}=0$ by R1, R2, R3, R 4 and R5.
(ii) $v_{i} v_{i+1}$ is incident with an internal 3 -face $\left[v_{i} v_{i+1} u_{i}\right]$ for $i \in\{1,2,5\}$. If $v_{2}$ is a special vertex, then $f$ receives $\frac{1}{3}$ from $v_{2}$. Otherwise, $v_{2}$ is a normal 4 -vertex. By Lemma 2.7, both $u_{1}$ and $u_{2}$ are special vertices. Then $f$ receives at least $2 \times \frac{1}{6}=\frac{1}{3}$ from $u_{1}$ and $u_{2}$ by R3, R4 and R5. In any way, $f$ receives at least $\frac{1}{3}$ from $\left\{u_{1}, u_{2}, v_{2}\right\}$. On the other hand, one of $u_{5}$ and $v_{6}$ is also a special vertex, and $f$ receives at least $\frac{1}{6}$ from $\left\{u_{5}, v_{6}\right\}$ by R3, R4 and R5. Thus, $\mu^{\prime}(f) \geq \mu(f)-3 \times \frac{1}{3}-3 \times \frac{1}{2}+\frac{1}{3}+\frac{1}{6}=0$.

- $t_{f}=2$. By Corollary 2.5, $s_{f} \leq 4$. If $s_{f} \leq 3$, then $\mu^{\prime}(f) \geq \mu(f)-3 \times \frac{1}{3}-2 \times \frac{1}{2}=0$ by R1 and R2.

Assume $s_{f}=4$. We claim that $f$ will receive at least $\frac{1}{3}$ from vertices. If $f$ is incident with a 2 -vertex, then the 2 -vertex must be in $\{x, y\}$, and $f$ receives at least $\frac{1}{3}$ from incident 2 -vertices by R5. So we may assume that $f$ is not incident with any 2 -vertex. By symmetry, it suffices to consider five cases.
(1) The four adjacent internal 3 -faces are $\left[v_{i} v_{i+1} u_{i}\right]$ for $1 \leq i \leq 4$. Thus, the two normal 3 -vertices must be $v_{1}$ and $v_{5}$. If one of $v_{2}, v_{3}$ and $v_{4}$ is a special vertex, then $f$ receives $\frac{1}{3}$ from it by $\mathrm{R} 3, \mathrm{R} 4$ and R 5 . So we may assume that $v_{2}, v_{3}$ and $v_{4}$ are normal 4 -vertices. By Lemma 2.7 and Lemma 2.8, there are at least two special vertices in $\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}$, thus $f$ receives at least $2 \times \frac{1}{6}=\frac{1}{3}$ from these vertices by R3, R4 and R 5 .


Fig. 9: Two adjacent 5 -faces. The solid vertex is a 2 -vertex in $G$.
(2) The four adjacent internal 3 -faces are $\left[v_{i} v_{i+1} u_{i}\right]$ for $i \in\{1,2,3,5\}$, while $v_{1}$ and $v_{4}$ are normal 3vertices. Similarly, if $v_{2}$ or $v_{3}$ is a special vertex, then $f$ receives at least $\frac{1}{3}$ from it. So we may assume that $v_{2}$ and $v_{3}$ are normal 4-vertices. By Lemma 2.7 and Lemma 2.8, there are at least two special vertices in $\left\{u_{1}, u_{2}, u_{3}\right\}$, thus $f$ receives at least $2 \times \frac{1}{6}=\frac{1}{3}$ from these vertices by R3, R 4 and R 5 .
(3) The four adjacent internal 3 -faces are $\left[v_{i} v_{i+1} u_{i}\right]$ for $i \in\{1,2,3,5\}$, while $v_{1}$ and $v_{5}$ are normal 3 vertices. By Lemma 2.7, $u_{5}$ or $v_{6}$ is a special vertex; one of $\left\{v_{2}, v_{3}, v_{4}, u_{1}, u_{2}, u_{3}\right\}$ is a special vertex. Thus, $f$ receives at least $2 \times \frac{1}{6}=\frac{1}{3}$ from these vertices by R3, R 4 and R 5 .
(4) The four adjacent internal 3 -faces are $\left[v_{i} v_{i+1} u_{i}\right]$ for $i \in\{1,2,4,5\}$, while $v_{1}$ and $v_{3}$ are normal 3vertices. If $v_{2}$ is a special vertex, then $f$ receives $\frac{1}{3}$ from it by $\mathrm{R} 3, \mathrm{R} 4$ and R 5 . Otherwise, $v_{2}$ is a normal 4 -vertex. By Lemma 2.7, each of $u_{1}$ and $u_{2}$ is a special vertex, thus $f$ receives at least $2 \times \frac{1}{6}=\frac{1}{3}$ from $\left\{u_{1}, u_{2}\right\}$ by R3, R4 and R5.
(5) The four adjacent internal 3 -faces are $\left[v_{i} v_{i+1} u_{i}\right]$ for $i \in\{1,2,4,5\}$, while $v_{1}$ and $v_{4}$ are normal 3vertices. By Lemma 2.7, there is at least one special vertex in $\left\{u_{1}, u_{2}, v_{2}, v_{3}\right\}$, and there is at least one special vertex in $\left\{u_{4}, u_{5}, v_{5}, v_{6}\right\}$. Thus, $f$ receives at least $2 \times \frac{1}{6}=\frac{1}{3}$ from these vertices by R3, R4 and R5.

To sum up, $f$ always receives at least $\frac{1}{3}$ from some vertices in the above five cases. Therefore, $\mu^{\prime}(f) \geq$ $\mu(f)-4 \times \frac{1}{3}-2 \times \frac{1}{2}+\frac{1}{3}=0$ by R1 and R2.
$\bullet t_{f}=1$. By Corollary 2.5, $s_{f} \leq 5$. If $s_{f} \leq 4$, then $\mu^{\prime}(f) \geq \mu(f)-\frac{1}{2}-4 \times \frac{1}{3}>0$. Assume that $s_{f}=5$ and for $1 \leq i \leq 5,\left[v_{i} v_{i+1} u_{i}\right]$ is an internal 3 -face. Let $X=\left\{v_{1}, \ldots, v_{6}, u_{1}, \ldots, u_{5}\right\}$. By Lemma 2.9, there is a special vertex in $X$. Therefore, $f$ receives at least $\frac{1}{6}$ from the special vertices in $X$, and $\mu^{\prime}(f) \geq \mu(f)-\frac{1}{2}-5 \times \frac{1}{3}+\frac{1}{6}=0$ by R3, R4 and R5.

- $t_{f}=0$. Then $f$ sends nothing to incident vertices, and $\mu^{\prime}(f) \geq \mu(f)-6 \times \frac{1}{3}=0$.

If $f$ is an internal $7^{+}$-face, then $f$ sends out charges by R 1 and R 2 . As $t_{f}+s_{f} \leq d(f)$, we have

$$
\mu^{\prime}(f) \geq \mu(f)-\frac{s_{f}}{3}-\frac{t_{f}}{2} \geq \frac{2}{3} d(f)-4-\frac{t_{f}}{6} \geq \frac{7}{12} d(f)-4>0
$$

This completes the proof of Case 1 of Theorem 1.1.

## 2.2 $G$ has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3

Lemma 2.10 below follows easily from the fact that configurations in Fig. 1 and Fig. 3 are forbidden.
Lemma 2.10. If two 5 -faces have two consecutive common edges on their boundaries, then one of the 5 -face is the outer face $f_{0}$ (see Fig. 9).

Now we calculate the final charge of internal $5^{+}$-faces.
Assume $f$ is an internal $d$-face. If $f$ is incident with a 2 -vertex, then the 2 -vertex belongs to $\{x, y\}$, and $f$ is adjacent to at most $d-2$ internal faces. By R5, $f$ receives $\frac{1}{3}$ from each of $x$ and $y$. By R6, $f$ receives $\frac{1}{3}$ via each common edge with the outer face $f_{0}$. By R1 and R7, $f$ sends at most $\frac{1}{3}$ to each adjacent internal face. By R2,
$f$ sends at most $\frac{1}{2}$ to each incident normal 3-vertex. Thus, $\mu^{\prime}(f) \geq d-4+2 \times \frac{1}{3}+2 \times \frac{1}{3}-(d-2) \times \frac{1}{3}-\left\lfloor\frac{d}{2}\right\rfloor \times \frac{1}{2} \geq$ $\frac{5 d-24}{12}>0$.

Assume that $f$ is not incident with any 2-vertex. By Lemma 2.10, there are no adjacent internal 5 -faces. By Lemma 2.4, $f$ is adjacent to at most $d-t_{f}$ internal 5 -faces.
$\boldsymbol{d}=\mathbf{5}$. Assume that $f=\left[v_{1} v_{2} v_{3} v_{4} v_{5}\right]$. Since adjacent triangles and a triangle normally adjacent to a 7 -cycle are forbidden, $s_{f} \leq 2$. By Corollary 2.5, $t_{f} \leq 2$. It follows that $f$ is incident with at most two minor 3 -vertices.

If $s_{f}=0$, then $\mu^{\prime}(f) \geq \mu(f)-2 \times \frac{1}{2}=0$ by R 2 .
Assume $s_{f} \geq 1$. Since Fig. 1 and Fig. 3(c) are forbidden, $f$ is not adjacent to any 4 -face. It follows that every face adjacent to $f$ is a 3 -face or a $6^{+}$-face. Thus, $f$ is adjacent to at least three $6^{+}$-faces (the number of adjacent $6^{+}$-faces is counted by the number of common edges). If $f$ is incident with at most one minor 3 -vertex, then $\mu^{\prime}(f) \geq 5-4-2 \times \frac{1}{3}-\left(\frac{1}{2}+\frac{1}{3}\right)+3 \times \frac{1}{6}=0$ by R1, R2 and R7. Assume $f$ is incident with exactly two minor 3 -vertices. That is $t_{f}=2$ and $s_{f}=2$. By symmetry, we have three subcases to consider:

- $f$ is adjacent to two internal 3 -faces $\left[v_{1} v_{2} u_{1}\right],\left[v_{3} v_{4} u_{3}\right]$, and $v_{1}, v_{3}$ are minor 3 -vertices.
- $f$ is adjacent to two internal 3 -faces $\left[v_{1} v_{2} u_{1}\right]$, $\left[v_{3} v_{4} u_{3}\right]$, and $v_{1}, v_{4}$ are minor 3 -vertices.
- $f$ is adjacent to two internal 3 -faces $\left[v_{1} v_{2} u_{1}\right]$, $\left[v_{2} v_{3} u_{2}\right]$, and $v_{1}, v_{3}$ are minor 3 -vertices.

By Lemma 2.7 and Lemma 2.8, the two 3 -faces are incident with at least one special vertex. By R3, R4 and R5, $f$ receives at least $\frac{1}{6}$ from these special vertices. Hence, $\mu^{\prime}(f) \geq 5-4+\frac{1}{6}+3 \times \frac{1}{6}-2 \times \frac{1}{3}-2 \times \frac{1}{2}=0$.
$\square \boldsymbol{d}=\mathbf{6}$. Assume that $f=\left[v_{1} v_{2} v_{3} v_{4} v_{5} v_{6}\right]$. If $s_{f}=0$, then it sends at most $\frac{1}{2}$ to each incident normal 3 -vertex, and sends $\frac{1}{6}$ to each adjacent 5-face, thus $\mu^{\prime}(f) \geq 6-4-t_{f} \times \frac{1}{2}-\left(6-t_{f}\right) \times \frac{1}{6}=1-\frac{t_{f}}{3} \geq 0$ by R2 and R7.

Suppose that $f$ is adjacent to an internal 3-face. Then they are normally adjacent. Since the configurations in Fig. 1 and Fig. 3(c) are forbidden, $s_{f}=1$. By Corollary 2.5, $t_{f} \leq 3$. If $t_{f} \leq 2$, then $\mu^{\prime}(f) \geq 6-4-\frac{1}{3}-$ $t_{f} \times \frac{1}{2}-\left(6-t_{f}\right) \times \frac{1}{6}=\frac{2-t_{f}}{3} \geq 0$ by R1, R2 and R7.

Assume $t_{f}=3$ and the 3 -face is $\left[u v_{1} v_{2}\right]$. By Lemma 2.4, we may assume $v_{1}, v_{3}$ and $v_{5}$ are the three normal 3 -vertices. By Lemma 2.7, there is a special vertex in $\left\{u, v_{2}\right\}$, thus $f$ receives at least $\frac{1}{6}$ from $\left\{u, v_{2}\right\}$. Since the configurations in Fig. 1 and Fig. 3 are all forbidden, $v_{5}$ cannot be incident with an internal $4^{-}$-face. Thus, $f$ is incident with at most two minor 3 -vertices, which implies that $\mu^{\prime}(f) \geq 6-4-\left(2 \times \frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}-(6-3) \times \frac{1}{6}+\frac{1}{6}=0$.

■ d=7. Let $f$ be a 7 -face. As Fig. $3(\mathrm{c})$ is forbidden, $s_{f}=0$. By Corollary $2.5, t_{f} \leq 3$. By R2, $f$ sends at most $\frac{1}{2}$ to each incident normal 3-vertex. By R7, $f$ sends $\frac{1}{6}$ to each adjacent internal 5 -face. Hence, $\mu^{\prime}(f) \geq 7-4-t_{f} \times \frac{1}{2}-\left(7-t_{f}\right) \times \frac{1}{6}=\frac{11-2 t_{f}}{6}>0$.

- d$\geq 8$. Let $f$ be a $8^{+}$-face. Then $f$ sends at most $\frac{1}{2}$ to each incident normal 3 -vertex, and $\frac{1}{3}$ to each adjacent internal 3 -face, and $\frac{1}{6}$ to each adjacent internal 5 -face. Combining with Corollary 2.5 , we have that

$$
\mu^{\prime}(f) \geq d-4-t_{f} \times \frac{1}{2}-s_{f} \times \frac{1}{3}-\left(d-s_{f}\right) \times \frac{1}{6}=\frac{5}{6} d-\frac{1}{2} t_{f}-\frac{1}{6} s_{f}-4 \geq \frac{d}{2}-4 \geq 0
$$

This completes the proof of Case 2.

## $2.3 G \in \mathcal{G}_{4,9}$

Lemma 2.11. A 5-cycle contains at most three triangular edges.
Proof. Assume $\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$ is a 5 -cycle, and $\left[x_{1} x_{2} x_{6}\right],\left[x_{2} x_{3} x_{7}\right],\left[x_{3} x_{4} x_{8}\right]$ and $\left[x_{4} x_{5} x_{9}\right]$ are four triangles. Since there is no 4 -cycle in $G, x_{1}, x_{2}, \ldots, x_{9}$ are nine distinct vertices. Thus, $\left[x_{1} x_{6} x_{2} x_{7} x_{3} x_{8} x_{4} x_{9} x_{5}\right]$ is a 9 -cycle, a contradiction.


Fig. 10: Some local structures around 5-face.

Lemma 2.12. Let $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$ and $g=\left[x_{5} x_{1} u v w\right]$ be two adjacent 5 -faces. If $d\left(x_{1}\right) \geq 3$ and $d\left(x_{5}\right) \geq 3$, then $f$ and $g$ are normally adjacent, and neither $x_{2} x_{3}$ nor $x_{3} x_{4}$ is adjacent to a 3 -face. Moreover, if $x_{1} x_{2}$ is incident with a 3 -face, then $x_{1}$ is a 3 -vertex and the 3 -face is $\left[x_{1} x_{2} u\right]$.

Proof. Since $d\left(x_{1}\right) \geq 3$ and $d\left(x_{5}\right) \geq 3$, we have that $x_{2} \neq u$ and $x_{4} \neq w$. Since $G$ has no 4 -cycle, $x_{1}, x_{2}, \ldots, x_{5}, u, v, w$ are distinct. Therefore, $f$ and $g$ are normally adjacent.

By the symmetry of $x_{2} x_{3}$ and $x_{3} x_{4}$, suppose that $x_{2} x_{3}$ is incident with a 3 -face $\left[x_{2} x_{3} x_{7}\right]$. Since there are no 4 -cycles in $G, x_{7}$ is not incident with $f$ or $g$. Thus, $\left[x_{5} x_{4} x_{3} x_{7} x_{2} x_{1} u v w\right]$ is a 9 -cycle, a contradiction. Hence, neither $x_{2} x_{3}$ nor $x_{3} x_{4}$ is incident with a 3 -face.

Let $x_{1} x_{2}$ be incident with a 3 -face $\left[x_{1} x_{2} x_{6}\right]$. Since $f$ has no chord, $x_{6} \notin\left\{x_{3}, x_{4}, x_{5}, v, w\right\}$. If $x_{6} \neq u$, then $\left[x_{5} x_{4} x_{3} x_{2} x_{6} x_{1} u v w\right]$ is a 9 -cycle, a contradiction. Thus $x_{6}=u$ and $x_{1}$ is a 3 -vertex.

Lemma 2.13. Let $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$ and $g=\left[x_{5} x_{1} u p q w\right]$ be two adjacent faces. If $d\left(x_{1}\right) \geq 3$ and $d\left(x_{5}\right) \geq 3$, then $\{u, w\} \cap\left\{x_{1}, \ldots, x_{5}\right\}=\emptyset$, while $\{p, q\} \cap\left\{x_{2}, x_{3}, x_{4}\right\}=\{p\}=\left\{x_{2}\right\}$ or $\{p, q\} \cap\left\{x_{2}, x_{3}, x_{4}\right\}=\{q\}=\left\{x_{4}\right\}$.
Proof. Since $G$ has no 9 -cycle, $\left\{x_{2}, x_{3}, x_{4}\right\} \cap\{u, p, q, w\} \neq \emptyset$. For $d\left(x_{1}\right) \geq 3$ and $d\left(x_{5}\right) \geq 3$, we have that $x_{2} \neq u$ and $x_{4} \neq w$. Note that there are no 4 -cycles, it follows that $\left\{x_{2}, x_{3}, x_{4}\right\} \cap\{u, w\}=\emptyset, x_{3} \notin\{p, q\}$, $x_{4} \neq p$ and $x_{2} \neq q$. Therefore, $\{p, q\} \cap\left\{x_{2}, x_{4}\right\}=\{p\}=\left\{x_{2}\right\}$ or $\{p, q\} \cap\left\{x_{2}, x_{4}\right\}=\{q\}=\left\{x_{4}\right\}$.

Lemma 2.14. Let $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$ be a 5 -face adjacent to two 3 -faces, that are either $\left[x_{1} x_{2} x_{6}\right]$ and [ $\left.x_{2} x_{3} x_{7}\right]$, or $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$ (see Fig. 10(a) and Fig. 10(b)). If $d\left(x_{1}\right)=3, d\left(x_{5}\right) \geq 3$ and $d\left(x_{6}\right) \geq 3$, and $x_{5} x_{1} x_{6}$ is incident with a $6^{-}$-face $g$, then $g$ is a 6 -face $\left[x_{5} x_{1} x_{6} u v w\right]$, where $\{u, w\} \cap\left\{x_{1}, x_{2}, \ldots, x_{8}\right\}=\emptyset$, $v=x_{4}$ and $d\left(x_{4}\right) \geq 4\left(d\left(x_{4}\right) \geq 5\right.$ for the case of Fig. 10(b)).

Proof. We only consider the case of Fig. 10(a) here, the case of Fig. 10(b) is quite similar. Suppose that $g=\left[x_{5} x_{1} x_{6} u \ldots w\right]$. Since $d\left(x_{5}\right) \geq 3$ and $d\left(x_{6}\right) \geq 3, x_{1}, x_{2}, x_{6}, u$ are four distinct vertices, and $x_{1}, x_{4}, x_{5}, w$ are four distinct vertices. As there is no 4 -cycle in $G, x_{1}, x_{2}, \ldots, x_{7}, u, w$ are distinct. It follows that $g$ must be a 5 - or 6 -face. If $g$ is a 5 -face, then $g=\left[x_{5} x_{1} x_{6} u w\right]$ and $\left[x_{5} x_{4} x_{3} x_{7} x_{2} x_{1} x_{6} u w\right]$ is a 9 -cycle, a contradiction. Let $g=\left[x_{5} x_{1} x_{6} u v w\right]$ be a 6 -face. If $v \notin\left\{x_{2}, x_{3}, x_{4}\right\}$, then $\left[u v w x_{5} x_{4} x_{3} x_{2} x_{1} x_{6}\right]$ is a 9 -cycle, a contradiction. If $v=x_{2}$, then $\left[u x_{6} x_{1} x_{2}\right]$ is a 4 -cycle, a contradiction. If $v=x_{3}$, then $\left[u x_{6} x_{2} x_{3}\right]$ is a 4 -cycle, a contradiction. Hence, $v=x_{4}$ and $\left[x_{4} x_{5} w\right]$ is a triangle.

Lemma 2.15. Let $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$ be a 5 -face adjacent to two 3 -faces $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$. If $d\left(x_{2}\right)=3$, $d\left(x_{3}\right) \geq 4$ and $d\left(x_{6}\right) \geq 3$, then $x_{3} x_{2} x_{6}$ is incident with a $7^{+}$-face.

Proof. Suppose that $x_{3} x_{2} x_{6}$ is incident with a face $g=\left[x_{3} x_{2} x_{6} u \ldots w\right]$. Since $d\left(x_{3}\right) \geq 4$ and $d\left(x_{6}\right) \geq 3$, we have that $x_{2}, x_{3}, x_{4}, x_{8}, w$ are five distinct vertices, and $x_{1}, x_{2}, x_{6}, u$ are four distinct vertices. Since there are no 4 -cycles, we have that $x_{1}, x_{2}, \ldots, x_{6}, x_{8}, u, w$ are distinct. It follows that $g$ must be a $5^{+}$-face. If $g$ is a 5 -face, then $g=\left[x_{3} x_{2} x_{6} u w\right]$ and $\left[x_{3} x_{8} x_{4} x_{5} x_{1} x_{2} x_{6} u w\right]$ is a 9 -cycle, a contradiction. Let $g$ be a 6 -face $\left[x_{3} x_{2} x_{6} u v w\right]$. If $v \notin\left\{x_{1}, x_{4}, x_{5}\right\}$, then $\left[u v w x_{3} x_{4} x_{5} x_{1} x_{2} x_{6}\right]$ is a 9 -cycle, a contradiction. If $v=x_{1}$, then [ $\left.u x_{6} x_{2} x_{1}\right]$ is a 4 -cycle, a contradiction. If $v=x_{4}$, then $\left[w x_{3} x_{8} x_{4}\right]$ is a 4 -cycle, a contradiction. If $v=x_{5}$, then $\left[u x_{6} x_{1} x_{5}\right]$ is a 4 -cycle, a contradiction. Therefore, $x_{3} x_{2} x_{6}$ is incident with a $7^{+}$-face.

Lemma 2.16. Let $f=\left[x_{1} x_{2} x_{3} \ldots\right]$ be a $7^{+}$-face. If $x_{2}$ is a normal 3 -vertex, then at most one of $x_{1} x_{2}$ and $x_{2} x_{3}$ is incident with a good 5 -face.

Proof. Suppose to the contrary that $x_{1} x_{2}$ is incident with a good 5 -face $g_{1}=\left[x_{1} x_{2} v_{3} v_{4} v_{5}\right]$ and $x_{2} x_{3}$ is incident with a good 5 -face $g_{2}=\left[x_{3} x_{2} v_{3} u_{4} u_{5}\right]$. Note that $g_{1}$ and $g_{2}$ are all internal faces. By Lemma 2.3, $v_{3}$ cannot be a 2 -vertex. By Lemma 2.12, $g_{1}$ and $g_{2}$ are normally adjacent. Moreover, $v_{3}$ is a 3 -vertex, and $g_{3}=\left[v_{3} v_{4} u_{4}\right]$ is an internal 3 -face. It is observed that $g_{1}, g_{2}$ and $g_{3}$ are all internal faces. It follows that $v_{3}$ does not belong to $\{x, y\}$, but this contradicts Lemma 2.4.

Let $\tau(\rightarrow f)$ be the number of charges that $f$ receives from other elements.
Claim 1. If $f$ is an internal 5 -face and $s_{f}=1$, then $\tau(\rightarrow f) \geq \frac{1}{3}$.
Proof. Let $f=\left[v_{1} v_{2} v_{3} v_{4} v_{5}\right]$ be an internal 5 -face, and let [ $v_{1} v_{2} v_{6}$ ] be an internal 3 -face. Since $f$ has no chord, $v_{1}, v_{2}, \ldots, v_{6}$ are six distinct vertices. If $v_{i} \in\{x, y\}$ for any $1 \leq i \leq 6$, then $v_{i}$ sends $\frac{1}{3}$ to $f$ by R5, we are done. Assume $\left\{v_{1}, v_{2}, \ldots, v_{6}\right\} \cap\{x, y\}=\emptyset$. By Lemma 2.3, $d\left(v_{i}\right) \geq 3$ for $1 \leq i \leq 6$.

Next, we show that $f$ is adjacent to a special face. By the hypothesis, neither $v_{3} v_{4}$ nor $v_{4} v_{5}$ is incident with an internal $4^{-}$-face. By Lemma 2.12 , neither $v_{3} v_{4}$ nor $v_{4} v_{5}$ is incident with a 5 -face. If $v_{3} v_{4}$ or $v_{4} v_{5}$ is incident with an internal $7^{+}$-face or $f_{0}$, we are done. So we may assume that each of $v_{3} v_{4}$ and $v_{4} v_{5}$ is incident with an internal 6 -face. By Lemma $2.13, v_{3} v_{4}$ is incident with a 6 -face $\left[v_{3} v_{4} u p v_{2} w\right]$. If $\left[v_{2} v_{3} w\right]$ bounds a 3 -face, then $d(w)=2$ and $v_{2} v_{3}$ is incident with the outer face $\left[v_{2} v_{3} w\right]$, we are done. Hence, we can assume that $v_{2} v_{3}$ is not incident with a 3 -face. By Lemma 2.12, $v_{2} v_{3}$ cannot be incident with a 5 -face. Since there are no 9 -cycles, $v_{2} v_{3}$ cannot be incident with a 6 -face. Hence, $v_{2} v_{3}$ is incident with a $7^{+}$-face. Therefore, $f$ is adjacent to at least one special face in any case. By R 6 and $\mathrm{R} 8, f$ receives $\frac{1}{3}$ from each adjacent special face, thus $\tau(\rightarrow f) \geq \frac{1}{3}$.

Claim 2. Let $f$ be an internal 5 -face and $s_{f}=2$. If $f$ is incident with one minor 3 -vertex, then $\tau(\rightarrow f) \geq \frac{1}{3}$.
Proof. Assume that $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$. If $x$ or $y$ is incident with $f$ or one of the adjacent 3 -faces, then it sends at least $\frac{1}{3}$ to $f$ by R5. So we may assume that neither $x$ nor $y$ is incident with $f$ or the adjacent 3 -faces. Now we show that $f$ is adjacent to at least one $7^{+}$-face sending $\frac{1}{3}$ to $f$ by R6 and R8.

Case 1. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{2} x_{3} x_{7}\right]$ be internal 3 -faces, and let $x_{1}$ be a minor 3-vertex. By Lemma 2.3 and Lemma 2.4, $d\left(x_{5}\right) \geq 4$ and $d\left(x_{6}\right) \geq 4$. By Lemma 2.14, if $x_{5} x_{1} x_{6}$ is incident with a $6^{-}$-face, then $\left[x_{4} x_{5} w\right]$ is a triangle but it does not bound a 3 -face, thus $x_{4} x_{5}$ is incident with a $7^{+}$-face. Hence, either $x_{5} x_{1} x_{6}$ or $x_{4} x_{5}$ is incident with a $7^{+}$-face.

Case 2. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$ be internal 3 -faces, and let $x_{1}$ be a minor 3 -vertex. By Lemma 2.3, Lemma 2.4 and Lemma 2.14, we also get that either $x_{5} x_{1} x_{6}$ or $x_{4} x_{5}$ is incident with a $7^{+}$-face.

Case 3. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$ be internal 3 -faces, and let $x_{2}$ be a minor 3-vertex. By Lemma 2.3 and Lemma 2.4, $d\left(x_{3}\right) \geq 4$ and $d\left(x_{6}\right) \geq 4$. By Lemma 2.15, $x_{2} x_{3}$ is incident with a $7^{+}$-face.

Claim 3. Let $f$ be an internal 5 -face and $s_{f} \geq 2$. If $f$ is incident with two minor 3 -vertices, then $\tau(\rightarrow f) \geq 1$.

Proof. Assume $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$. If $x_{i}$ is a 2-vertex, then $x_{i} \in\{x, y\}$ and $x_{i-1} x_{i} x_{i+1}$ is incident with the outer face $f_{0}$. By R5, $f$ receives $\frac{1}{3}$ from each of $x$ and $y$. By R6, $f$ receives $\frac{1}{3}$ via each of $x_{i-1} x_{i}$ and $x_{i} x_{i+1}$. Thus, $\tau(\rightarrow f) \geq 2 \times \frac{1}{3}+2 \times \frac{1}{3}>1$. So we may assume that $d\left(x_{i}\right) \geq 3$ for any $1 \leq i \leq 5$. Denote the adjacent face incident with $x_{i} x_{i+1}$ by $g_{i}$ for $i \in\{1,2,3,4,5\}$.

Case 1. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{2} x_{3} x_{7}\right]$ be internal 3 -faces, and let $x_{1}$ and $x_{3}$ be minor 3 -vertices. Suppose that $x_{6}$ is a 2-vertex. It follows that $\left\{x_{2}, x_{6}\right\}=\{x, y\}$ and $g_{5}=f_{0}$. By R5, $f$ receives $\frac{1}{3}$ from each of $x_{2}$ and $x_{6}$. By R6, $f$ receives at least $\frac{1}{3}$ from the outer face $f_{0}$. Thus, $\tau(\rightarrow f) \geq 3 \times \frac{1}{3}=1$.

So we may assume that $d\left(x_{6}\right) \geq 3$, and by symmetry, $d\left(x_{7}\right) \geq 3$. Firstly, we claim that $f$ receives at least $\frac{1}{3}$ from $\left\{x_{2}, x_{6}, x_{7}\right\}$. If $x_{2}$ is a special vertex, then $f$ receives $\frac{1}{3}$ from $x_{2}$ by $\mathrm{R} 3, \mathrm{R} 4$ and R 5 . So we may assume that $x_{2}$ is a normal 4 -vertex. It follows from Lemma 2.7 that both $x_{6}$ and $x_{7}$ are special vertices. By R3, R4 and R5, $f$ receives at least $\frac{1}{6} \times 2=\frac{1}{3}$ from $x_{6}$ and $x_{7}$.

Next, we show that $f$ is adjacent to at least two special faces. Since $f$ receives at least $2 \times \frac{1}{3}=\frac{2}{3}$ from adjacent special faces by R6 and R8, we are done. By Lemma 2.14, we get that both $g_{3}$ and $g_{5}$ are $6^{+}$-faces, and $g_{3}, g_{5}$ cannot be 6 -face simultaneously. If both $g_{3}$ and $g_{5}$ are $7^{+}$-faces, then we are done. By symmetry, assume that $g_{5}$ is a 6 -face and $g_{3}$ is a $7^{+}$-face. It follows that $g_{4}$ is the outer 3 -face or a $7^{+}$-face. That is, $g_{3}$ and $g_{4}$ are the special faces, we are done.

Case 2. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$ be internal 3 -faces, and let $x_{1}$ and $x_{4}$ be minor 3 -vertices. Similar to Case 1 , we may assume that $d\left(x_{6}\right) \geq 3$ and $d\left(x_{8}\right) \geq 3$. Note that $x_{1}$ and $x_{4}$ are 3 -vertices. Since there are no 4 -cycles, neither $g_{4}$ nor $g_{5}$ is a $4^{-}$-face. By Lemma 2.12, neither $g_{4}$ nor $g_{5}$ is a 5 -face. By Lemma 2.13 and Lemma 2.14, neither $g_{4}$ nor $g_{5}$ is a 6 -face. So both $g_{4}$ and $g_{5}$ are $7^{+}$-faces. Thus, $f$ receives at least $\frac{1}{3} \times 2=\frac{2}{3}$ from these $7^{+}$-faces. Next we show that $f$ will receive at least $\frac{1}{3}$ from others.

If $g_{2}$ is a $7^{+}$-face, then we are done. By Lemma 2.13, $g_{2}$ cannot be a 6 -face. Assume $g_{2}$ is a 5 -face. By Lemma 2.12, $d\left(x_{2}\right)=d\left(x_{3}\right)=3$. By Lemma 2.4, we have that $\left\{x_{2}, x_{3}\right\}=\{x, y\}$. By R $5, f$ receives $\frac{1}{3}$ from each of $x_{2}$ and $x_{3}$, we are done. It is clear that $g_{2}$ cannot be a 4 -face. Suppose that $g_{2}$ is a 3 -face $\left[x_{2} x_{3} x_{7}\right]$. If there is one special vertex in $\left\{x_{2}, x_{3}\right\}$, then we are done by R3, R4 and R5. So we may assume that both $x_{2}$ and $x_{3}$ are normal 4 -vertices. By Lemma 2.7 and Lemma 2.8, at least two of $x_{6}, x_{7}$ and $x_{8}$ are special vertices, thus $f$ receives at least $2 \times \frac{1}{6}=\frac{1}{3}$ from these special vertices, we are done.

Case 3. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$ be internal 3 -faces, and let $x_{1}$ and $x_{3}$ be minor 3 -vertices. Similar to Case 1, assume $d\left(x_{6}\right) \geq 3$ and $d\left(x_{8}\right) \geq 3$. By Lemma 2.7, one of $\left\{x_{2}, x_{6}\right\}$ is a special vertex. By R3, R4 and R5, $f$ receives at least $\frac{1}{6}$ from $\left\{x_{2}, x_{6}\right\}$.

Since there are no 4 -cycles, we have that $g_{2}$ cannot be a $4^{-}$-face. Suppose that $g_{2}$ is a 5 -face. By Lemma 2.12, we have that $d\left(x_{2}\right)=d\left(x_{3}\right)=3$. By Lemma 2.4, $x_{2}$ belongs to $\{x, y\}$. As a consequence, $\left\{x_{2}, x_{6}\right\}=\{x, y\}$ and $g_{2}$ is the outer face $f_{0}$. By R5 and R6, $\tau(\rightarrow f) \geq 2 \times \frac{1}{3}+\frac{1}{3}=1$, we are done. By Lemma 2.13, $g_{2}$ cannot be a 6 -face. Thus, we may assume that $g_{2}$ is a $7^{+}$-face. By R8, $f$ receives $\frac{1}{3}$ from $g_{2}$.

Next we show that $f$ receives at least $\frac{1}{2}$ from others. By Lemma 2.12 and Lemma 2.13, $g_{4}$ cannot be a 5 - or 6 -face. Thus, $g_{4}$ is a 3 - or $7^{+}$-face. Suppose that $g_{4}$ is a 3 -face $\left[x_{4} x_{5} x_{9}\right]$. If $x_{9}$ is a 2 -vertex, then $\{x, y\} \subset\left\{x_{4}, x_{5}, x_{9}\right\}$, and then $f$ receives at least $2 \times \frac{1}{3}>\frac{1}{2}$ from $x$ and $y$ by R5. So we may assume that $d\left(x_{9}\right) \geq 3$. By Lemma 2.12 and Lemma 2.13, $g_{5}$ is a $7^{+}$-face sending $\frac{1}{3}$ to $f$. By Lemma 2.7, there is a special vertex in $\left\{x_{4}, x_{5}, x_{8}, x_{9}\right\}$ sending at least $\frac{1}{6}$ to $f$. Thus, $f$ receives at least $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$ from $g_{5}$ and the special vertex. Suppose that $g_{4}$ is a $7^{+}$-face. If $g_{5}$ is also a $7^{+}$-face, then $f$ receives at least $2 \times \frac{1}{3}>\frac{1}{2}$ from $g_{4}$ and $g_{5}$, we are done. So we may assume that $g_{5}$ is a $6^{-}$-face. By Lemma $2.14, d\left(x_{4}\right) \geq 5$. By R3, R4 and R5, $f$ receives at least $\frac{1}{6}$ from $x_{4}$. Therefore, $f$ still receives at least $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$ from $g_{4}$ and $x_{4}$.

Claim 4. Let $f$ be an internal 5 -face and $s_{f}=2$. If $t_{f}=2$, and exactly one of the two normal 3 -vertices is minor, then $\tau(\rightarrow f) \geq \frac{1}{2}$.

Proof. Assume $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$. By the definition of normal 3 -vertex and minor 3 -vertex, we only need to consider two cases.

Case 1. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{2} x_{3} x_{7}\right]$ be internal 3 -faces, and let $x_{1}$ and $x_{4}$ be normal 3 -vertices. If $x_{5}$ or $x_{6}$ is a 2-vertex, then $x_{5}$ or $x_{6}$ belongs to $\{x, y\}$. It follows that $x_{1} x_{5}$ is incident with the outer face $f_{0}$. By $\mathrm{R} 5, f$ receives at least $\frac{1}{3}$ from $\{x, y\}$. By R6, $f$ receives $\frac{1}{3}$ from the outer face $f_{0}$. Thus, $\tau(\rightarrow f) \geq 2 \times \frac{1}{3}>\frac{1}{2}$. So we may assume that $d\left(x_{5}\right) \geq 3$ and $d\left(x_{6}\right) \geq 3$. Note that $x_{4}$ is a 3 -vertex. By Lemma $2.14, x_{1} x_{5}$ cannot be incident with a $6^{-}$-face. That is, $x_{1} x_{5}$ is incident with a $7^{+}$-face which sends $\frac{1}{3}$ to $f$. On the other hand, by Lemma 2.7, one vertex in $\left\{x_{2}, x_{3}, x_{6}, x_{7}\right\}$ is a special vertex which sends at least $\frac{1}{6}$ to $f$. Thus, $\tau(\rightarrow f) \geq \frac{1}{3}+\frac{1}{6}=\frac{1}{2}$.

Case 2. Let $\left[x_{1} x_{2} x_{6}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$ be internal 3 -faces, and let $x_{2}$ and $x_{5}$ be normal 3 -vertices. If $x_{6}$ is a 2 -vertex, then $x_{6} \in\{x, y\}$. Since $x_{2}$ is a normal vertex, $\{x, y\}=\left\{x_{1}, x_{6}\right\}$. Thus, $f$ receives $\frac{1}{3}$ from each of $x_{1}$ and $x_{6}$ by R5, and thus $\tau(\rightarrow f) \geq \frac{1}{3}+\frac{1}{3} \geq \frac{1}{2}$. Assume $d\left(x_{6}\right) \geq 3$. By Lemma 2.7, at least one of $x_{1}$ and $x_{6}$ is a special vertex. By $\mathrm{R} 3, \mathrm{R} 4$ and $\mathrm{R} 5, f$ receives at least $\frac{1}{6}$ from these special vertices. If $x_{3}$ is a 3 -vertex, then $x_{3} \in\{x, y\}$ by Lemma 2.4. By R5, $f$ receives $\frac{1}{3}$ from $x_{3}$. Thus, $\tau(\rightarrow f) \geq \frac{1}{6}+\frac{1}{3}=\frac{1}{2}$. So we may assume that $d\left(x_{3}\right) \geq 4$. By Lemma 2.15, $x_{2} x_{3}$ is incident with a $7^{+}$-face. By R8, $f$ receives $\frac{1}{3}$ from each adjacent $7^{+}$-face. Thus, $\tau(\rightarrow f) \geq \frac{1}{6}+\frac{1}{3}=\frac{1}{2}$.

Claim 5. If $f$ is an internal 5 -face and $s_{f}=3$, then $\tau(\rightarrow f) \geq \frac{2}{3}$.
Proof. Assume $f=\left[x_{1} x_{2} x_{3} x_{4} x_{5}\right]$. According to symmetry, we only need to consider two cases.
Case 1. Let $\left[x_{1} x_{2} x_{6}\right],\left[x_{2} x_{3} x_{7}\right]$ and $\left[x_{4} x_{5} x_{9}\right]$ be internal 3-faces. Assume $d\left(x_{6}\right)=2$. By Lemma 2.4, $\{x, y\}=\left\{x_{1}, x_{6}\right\}$ or $\{x, y\}=\left\{x_{2}, x_{6}\right\}$. By R5, $f$ receives $\frac{1}{3}$ from each of $x$ and $y$, thus $\tau(\rightarrow f) \geq 2 \times \frac{1}{3}=\frac{2}{3}$. So we may assume that $d\left(x_{6}\right) \geq 3$. Similarly, we can assume that $d\left(x_{7}\right) \geq 3$ and $d\left(x_{9}\right) \geq 3$. It is clear that neither $x_{1} x_{5}$ nor $x_{3} x_{4}$ is incident with a $4^{-}$-face. By Lemma 2.12, neither $x_{1} x_{5}$ nor $x_{3} x_{4}$ is incident with a 5 -face. By Lemma 2.13, neither $x_{1} x_{5}$ nor $x_{3} x_{4}$ is incident with a 6 -face. Hence, $f$ is adjacent to two $7^{+}$-faces. By R6 and R8, $\tau(\rightarrow f) \geq 2 \times \frac{1}{3}=\frac{2}{3}$.

Case 2. Let $\left[x_{1} x_{2} x_{6}\right],\left[x_{2} x_{3} x_{7}\right]$ and $\left[x_{3} x_{4} x_{8}\right]$ be internal 3-faces. If $d\left(x_{i}\right)=2$ for $i \in\{5,6,7,8\}$, then $x_{i} \in\{x, y\}$ by Lemma 2.3. Since $x$ and $y$ are adjacent, we have that $\{x, y\} \subset\left\{x_{1}, x_{2}, \ldots, x_{8}\right\}$. By R $5, f$ receives $\frac{1}{3}$ from each of $x$ and $y$, thus $\tau(\rightarrow f) \geq 2 \times \frac{1}{3}=\frac{2}{3}$. So we may assume that $x_{5}, x_{6}, x_{7}$ and $x_{8}$ are all $3^{+}$-vertices. It is clear that neither $x_{4} x_{5}$ nor $x_{1} x_{5}$ is contained in a $4^{-}$-face. By Lemma 2.12, neither $x_{1} x_{5}$ nor $x_{4} x_{5}$ is incident with a 5 -face. Recall that $x_{6}$ is a $3^{+}$-vertex and $x_{4} x_{5}$ is not contained in a triangle. By Lemma 2.13, $x_{1} x_{5}$ cannot be incident with a 6 -face. Hence, $x_{1} x_{5}$ is incident with a $7^{+}$-face. By symmetry, $x_{4} x_{5}$ is also incident with a $7^{+}$-face. By R6 and R8, $\tau(\rightarrow f) \geq 2 \times \frac{1}{3}=\frac{2}{3}$.

Now we calculate the final charge of internal $5^{+}$-faces. Let $f=\left[v_{1} v_{2} \ldots v_{d}\right]$ be an internal $d$-face for $d \geq 5$. By Lemma 2.2, every face in $G$ is bounded by a cycle. Since there are no 9 -cycles, $d \neq 9$.

If $v_{i}$ is a 2-vertex, then $v_{i} \in\{x, y\}$ and $v_{i-1} v_{i} v_{i+1}$ is incident with the outer face $f_{0}$. Thus, $f$ is adjacent to at most $d-2$ internal faces. By Corollary $2.5, t_{f} \leq \frac{d}{2}$. By R1 and R8, $f$ sends at most $\frac{1}{3}$ to each adjacent internal face. By R2, $f$ sends at most $\frac{1}{2}$ to each incident normal 3 -vertex. By R5, $f$ receives $\frac{1}{3}$ from each of $x$ and $y$. By R6, $f$ receives $\frac{1}{3}$ via each of $v_{i-1} v_{i}$ and $v_{i} v_{i+1}$. Hence, $\mu^{\prime}(f) \geq d-4+4 \times \frac{1}{3}-(d-2) \times \frac{1}{3}-\frac{d}{2} \times \frac{1}{2}>0$.

So we may assume that there is no 2 -vertex incident with $f$.

- $d=5$.

By Corollary 2.5 and Lemma 2.11, $t_{f} \leq 2$ and $s_{f} \leq 3$. If $s_{f}=0$, then $\mu^{\prime}(f) \geq 5-4-2 \times \frac{1}{3}>0$ by R2.
If $s_{f}=1$, then $f$ is incident with at most one minor 3 -vertex. By Claim 1, R1 and R2, $\mu^{\prime}(f) \geq$ $5-4+\frac{1}{3}-\frac{1}{3}-\left(\frac{1}{2}+\frac{1}{3}\right)>0$.

Assume $s_{f}=2$. If $t_{f}=0$, then $\mu^{\prime}(f) \geq 5-4-2 \times \frac{1}{3}>0$ by R1. Let $t_{f}=1$. If the normal 3 -vertex is not minor, then $\mu^{\prime}(f) \geq 5-4-2 \times \frac{1}{3}-\frac{1}{3}=0$ by R1 and R2. If the normal 3 -vertex is
minor, then $\mu^{\prime}(f) \geq 5-4+\frac{1}{3}-2 \times \frac{1}{3}-\frac{1}{2}>0$ by Claim 2, R1 and R2. Let $t_{f}=2$. It is observed that $f$ is incident with at least one minor 3 -vertex. If $f$ is incident with exactly one minor 3 -vertex, then $\mu^{\prime}(f) \geq 5-4+\frac{1}{2}-2 \times \frac{1}{3}-\left(\frac{1}{2}+\frac{1}{3}\right)=0$ by Claim 4, R1 and R2. The other situation, $f$ is incident with exactly two minor 3 -vertices. Thus, $\mu^{\prime}(f) \geq 5-4+1-2 \times \frac{1}{3}-2 \times \frac{1}{2}>0$ by Claim 3, R1 and R2.

Assume $s_{f}=3$. If $t_{f}=0$, then $\mu^{\prime}(f) \geq 5-4-3 \times \frac{1}{3}=0$ by R1. If $t_{f}=1$, then $\mu^{\prime}(f) \geq 5-4+\frac{2}{3}-$ $3 \times \frac{1}{3}-\frac{1}{2}>0$ by Claim 5, R1 and R2. If $t_{f}=2$, then it is incident with two minor 3 -vertices, and then $\mu^{\prime}(f) \geq 5-4+1-3 \times \frac{1}{3}-2 \times \frac{1}{2}=0$ by Claim 3, R1 and R2.

- $d=6$.

Note that there are no 4 -cycle in $G$. If $f$ is adjacent to a 3 -face, then it must be normally adjacent to the 3 -face. Since there are no 9 -cycles in $G, f$ is adjacent to at most two 3 -faces. It follows that $f$ is incident with at most two minor 3-vertices. By R1 and R2, $\mu^{\prime}(f) \geq 6-4-2 \times \frac{1}{3}-\left(2 \times \frac{1}{2}+\frac{1}{3}\right)=0$.

- $d=7$.

If $f$ is adjacent to a 3 -face, then it must be normally adjacent to the 3 -face. Otherwise, there is a 4 -cycle in $G$. Since there are no 9 -cycles in $G, f$ is adjacent to at most one 3 -face. It follows that $f$ is incident with at most one minor 3 -vertex. By Corollary 2.5, $t_{f} \leq 3$. If $t_{f}=3$, then $f$ is adjacent to at most four good 5 -faces by Lemma 2.4 and Lemma 2.16, and then $\mu^{\prime}(f) \geq 7-4-(1+4) \times \frac{1}{3}-\left(\frac{1}{2}+2 \times \frac{1}{3}\right)>0$ by R1, R2 and R8. If $t_{f}=2$, then $f$ is adjacent to at most five good 5 -faces by Lemma 2.4 and Lemma 2.16 , and then $\mu^{\prime}(f) \geq 7-4-(1+5) \times \frac{1}{3}-\left(\frac{1}{2}+\frac{1}{3}\right)>0$ by R1, R2 and R8. If $t_{f}=1$, then $f$ is adjacent to at most six good 5-faces by Lemma 2.4 and Lemma 2.16, and then $\mu^{\prime}(f) \geq 7-4-(1+6) \times \frac{1}{3}-\frac{1}{2}>0$ by R1, R2 and R8. If $t_{f}=0$, then $\mu^{\prime}(f) \geq 7-4-7 \times \frac{1}{3}>0$ by R1 and R8.

## - $d=8$.

Similar to the above cases, if $f$ is adjacent to a 3 -face, then it must be normally adjacent to the 3 -face. Since there are no 9 -cycles, $f$ is not adjacent to any 3 -face. Thus, $f$ is not incident with any minor 3 -vertex. By R2 and R8, $\mu^{\prime}(f) \geq 8-4-8 \times \frac{1}{3}-4 \times \frac{1}{3}=0$.

- $d \geq 10$.

By R1 and R8, $f$ sends at most $\frac{1}{3}$ via each incident edge. It follows that $\mu^{\prime}(f) \geq d-4-d \times \frac{1}{3}-\frac{d}{2} \times \frac{1}{2}>0$. This completes the proof of Theorem 2.1.

## References

[1] E.-K. Cho, I. Choi, R. Kim, B. Park, T. Shan and X. Zhu, Decomposing planar graphs into graphs with degree restrictions, J. Graph Theory 101 (2) (2022) 165-181.
[2] L. J. Cowen, R. H. Cowen and D. R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, J. Graph Theory 10 (2) (1986) 187-195.
[3] W. Cushing and H. A. Kierstead, Planar graphs are 1-relaxed, 4-choosable, European J. Combin. 31 (5) (2010) 1385-1397.
[4] W. Dong and B. Xu, A note on list improper coloring of plane graphs, Discrete Appl. Math. 157 (2) (2009) 433-436.
[5] Z. Dvořák and L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8, J. Combin. Theory Ser. B 129 (2018) 38-54.
[6] N. Eaton and T. Hull, Defective list colorings of planar graphs, Bull. Inst. Combin. Appl. 25 (1999) 79-87.
[7] J. Grytczuk and X. Zhu, The Alon-Tarsi number of a planar graph minus a matching, J. Combin. Theory Ser. B 145 (2020) 511-520.
[8] R. Kim, S.-J. Kim and X. Zhu, The Alon-Tarsi number of subgraphs of a planar graph, arXiv:1906.01506, http://arxiv.org/abs/1906.01506v1.
[9] S.-J. Kim, A. V. Kostochka, X. Li and X. Zhu, On-line DP-coloring of graphs, Discrete Appl. Math. 285 (2020) 443-453.
[10] K.-W. Lih, Z. Song, W. Wang and K. Zhang, A note on list improper coloring planar graphs, Appl. Math. Lett. 14 (3) (2001) 269-273.
[11] H. Lu and X. Zhu, The Alon-Tarsi number of planar graphs without cycles of lengths 4 and $l$, Discrete Math. 343 (5) (2020) 111797.
[12] R. Škrekovski, List improper colourings of planar graphs, Combin. Probab. Comput. 8 (3) (1999) 293299.


[^0]:    *School of Mathematics and Statistics, Henan University, Kaifeng, 475004, P. R. China
    ${ }^{\dagger}$ College of Basic Science, Ningbo University of Finance and Economics, Ningbo, 315000, P. R. China
    ${ }^{\ddagger}$ Center for Applied Mathematics, Henan University, Kaifeng, 475004, P. R. China. Email: wangtao@henu.edu.cn
    ${ }^{\S}$ School of Mathematical Sciences, Zhejiang Normal University, Jinhua, 321004, P. R. China. This research is supported by Grants: NSFC 11971438, U20A2068.

