# Top to random shuffles on colored permutations 

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#### Abstract

A deck of $n$ cards are shuffled by repeatedly taking off the top card, flipping it with probability $1 / 2$, and inserting it back into the deck at a random position. This process can be considered as a Markov chain on the group $B_{n}$ of signed permutations. We show that the eigenvalues of the transition probability matrix are $0,1 / n, 2 / n, \ldots,(n-1) / n, 1$ and the multiplicity of the eigenvalue $i / n$ is equal to the number of the signed permutation having exactly $i$ fixed points. We show the similar results hold also for the colored permutations. Further, we show that the mixing time of this Markov chain is $n \log n$ and exhibits cut off, same as the ordinary 'top to random' shuffles without flipping the cards. The cut off is also analyzed by using the asymptotic formula of the Stirling numbers of the second kind.


## 1 Introduction

The top to random shuffle of cards, which is a Markov chain on the symmetric group, has long been studied $[1,5]$. By modifying the arguments in [5] and

[^0][7], this paper studies the generalized top to random shuffling defined on the colored permutation group. For example, when the number of the 'colors' is two, the Markov chain can be described as follows: We take the top card of the deck of $n$ cards and before inserting it back into the deck at the random position, we flip the card with probability $1 / 2$. After repeating this procedure, we have a random configuration of the cards which can be regarded as an element of the hyperoctahedral group $B_{n}$. This generalization is similar to those of the riffle shuffle $[3,10]$. Our main aim in this paper is to show a closed formula describing the probability distribution after shuffling $k$ times in terms of (generalized) Stirling numbers, the explicit form of the eigenvalues of the transition probability matrix, and the mixing time and cut off of the Markov chain. By a colored permutation group, we mean the wreath product of a cyclic group and a symmetric group. Throughout, the symmetric group of degree $n$ is denoted by $\mathfrak{S}_{n}$, and the cyclic group of order $p$ is denoted by $C_{p}$. For positive integers $p$ and $n$, the wreath product $C_{p} \prec \mathfrak{S}_{n}$ is denoted by $G_{n, p}$. That is, the colored permutation group $G_{n, p}$ is defined by
$$
G_{n, p}=\left\{(s, \sigma) \mid s=\left(s_{1}, \ldots, s_{n}\right) \in C_{p}^{n}, \sigma \in \mathfrak{S}_{n}\right\}
$$
equipped with the following multiplication rule,
$$
(t, \tau)(s, \sigma)=(\sigma t+s, \tau \sigma)
$$
for $(t, \tau)$ and $(s, \sigma) \in G_{n, p}$, where
$$
\sigma t=\left(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(n)}\right)
$$

For example, let $s=(0, \ldots, 0, \stackrel{i}{k}, 0, \ldots, 0)$ and $\sigma=(1,2, \ldots, i)$ be a cyclic permutation in $\mathfrak{S}_{n}$. Then,
$(t, \tau)(s, \sigma)=\left(\left(t_{2}, t_{3}, \ldots, t_{i}, t_{1}+k, t_{i+1}, \ldots, t_{n}\right),(\tau(2), \tau(3), \ldots, \tau(i), \tau(1), \tau(i+1), \ldots, \tau(n))\right)$.
We interpret the multiplication by this special element $S_{i, k}=(s, \sigma)$ as follows: We have a deck of $n$ cards each numbered $\tau(1), \tau(2), \ldots, \tau(n)$ from the top to the bottom. We take the top card and change the color of the card to $t_{1}+k$ and insert it into the deck at the $i$ th place from the top. To introduce the shuffle we regard $(s, \sigma) \in G_{n, p}$ as a sequence of the pairs $\left(s_{i}, \sigma(i)\right) \in C_{p} \times[n]$, so that $(s, \sigma)$ is a word over the alphabet $C_{p} \times[n]$. Especially for $p \leq 3$, we
express $(0, k) \in C_{p} \times[n]$ simply by $k,(1, k) \in C_{p} \times[n]$ by $\bar{k}$, and $(2, k)$ by $\overline{\bar{k}}$. For example, by using this notation, $((0,1,0,2), 4123) \in G_{4,3}$ can be simply expressed as $4 \overline{1} 2 \overline{\overline{3}}$. Thus elements of $G_{n, p}$ can be considered as words over the alphabet $C_{p} \times[n]$, on which we can use the shuffle operator $\boldsymbol{w}$. Here the shuffle operator $\omega$ is defined as follows. Let $\epsilon$ be the empty word, $u, v$ be any words, and let $a, b$ be the words of length 1 . Then $m$ between two words is defined inductively by the following equations.

$$
\begin{aligned}
u \boldsymbol{ш} & =\epsilon \boldsymbol{\omega} u:=u \\
u a \boldsymbol{\omega} b & :=(u \boldsymbol{v} v) a+(u a \boldsymbol{} v) b .
\end{aligned}
$$

Define the word $W_{k, n}$ by

$$
W_{k, n}:=(0, k+1)(0, k+2) \cdots(0, n) \in\left(C_{p} \times[n]\right)^{n-k} .
$$

Then an element $\mathbf{B}_{k}$ of the group algebra $\mathbb{Q} G_{n, p}$ is defined by

$$
\mathbf{B}_{k}:= \begin{cases}i d & (k=0) \\ \sum_{\alpha \in G_{k, p}} \alpha \boldsymbol{} W_{k, n}, & (1 \leq k \leq n-1) \\ \sum_{\alpha \in G_{n, p}} \alpha & (k=n)\end{cases}
$$

$\mathbf{B}_{k}$ for $k \geq 2$ can be regarded as generalized top to random shuffle, which corresponds to taking off top $k$ cards, flipping them into any colors, and inserting back into random positions. In particular, we have

$$
\mathbf{B}_{1}=(0,1) ш W_{1, n}+(1,1) \boldsymbol{w} W_{1, n}+\cdots+(p-1,1) ш W_{1, n}
$$

For example, when $n=3$ and $p=2$, we have,

$$
\mathbf{B}_{1}=1 \boldsymbol{\omega} 23+\overline{1} \boldsymbol{\omega} 23=123+213+231+\overline{1} 23+2 \overline{1} 3+23 \overline{1}
$$

Therefore $\frac{1}{n p} \mathbf{B}_{1}$ can be regarded as a probability distribution over $G_{n, p}$, which we call the top to random shuffle over $G_{n, p}$. When $p=1$, the powers of $\left(\frac{1}{n p} \mathbf{B}_{1}\right)^{k}$ exhibit very interesting properties and have been studied extensively [5]. The main purpose of this paper is to consider the case for general $p:$ (i) to give a precise description of the distribution of the eigenvalues of the left regular representation of $\mathbf{B}_{1}$, and (ii) to derive a sharp estimate on
the distance between the distribution of $\left(\frac{1}{n p} \mathbf{B}_{1}\right)^{k}$ and the stationary distribution, and show that it exhibits the cut off phenomenon. To state our first main result we need to define the fixed points of a colored permutation. An element $(s, \sigma) \in G_{n, p}$ has a fixed point at $i$ if $s_{i}=0$ and $\sigma(i)=i$. For example $4 \overline{1} 2 \overline{\overline{3}} \in G_{4,3}$ has no fixed point and $1 \overline{2} 35 \overline{\overline{4}}$ has two fixed points at 1 and 3. A derangement in $G_{n, p}$ is a colored permutation having no fixed points. We denote the number of derangement in $G_{n, p}$ by $D_{n, p}$, which is expressed by a closed form given later. Therefore, the number of colored permutations in $G_{n, p}$ having exactly $i$ fixed points is equal to $\binom{n}{i} D_{n-i, p}$.

Theorem 1.1 Let $L: G_{n, p} \rightarrow \operatorname{GL}\left(L^{2}\left(G_{n, p}\right)\right)$ be the left regular representation of $G_{n, p}$. Then the eigenvalues of $L\left(\mathbf{B}_{1}\right)$ are $0, p, 2 p, \cdots, n p$. The multiplicity of the eigenvalue ip $(i=0,1, \cdots, n)$ is equal to the number of colored permutations having exactly $i$ fixed points.

## Remark

(1) Let $P_{n, p}$ be the transition probability matrix of the Markov chain generated by the top to random shuffle. Then the eigenvalues of $P_{n, p}$ are given by $(i p) /(n p)=i / n, i=0,1, \cdots, n$.
(2) If $p=1, P_{n, 1}$ does not have $(n-1) / n$ as an eigenvalue because of $D_{n-(n-1), 1}=0$. It is not the case for $p \geq 2$, since $D_{n-(n-1), p} \neq 0$ for $p \geq 2$.

## Example

When $n=3$ and $p=2$, we have 48 elements in $G_{n, p}$. The left regular

## representation of $\mathbf{B}_{1}$ is


#### Abstract

$\left(\begin{array}{llllllllllllllllllllllllllllllllllllllllllllll}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{array}\right)$    $\left.\begin{array}{llllllllllllllllllllllllllllllllllllllllllllllll}0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{array}\right]$ $\begin{array}{llllllllllllllllllllllllllllllllllllllllllllllll}0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\end{array}$    001000000000100000001000000010000000000001000001000000           $\left.0 \begin{array}{lllllllllllllllllllllllllllllllllllllllllllll}0 & 0\end{array}\right)$         $\left.0 \begin{array}{llllllllllllllllllllllllllllllllllllllllllll}1 & 0\end{array}\right)$   0001000001000000000010000001000000001000000000100000              $\left(\begin{array}{lllllllllllllllllllllllllllllllllllllllllllll}0 & 0\end{array}\right)$


whose characteritic polynomial $\operatorname{det}\left(x I-L\left(\mathbf{B}_{1}\right)\right)$ is

$$
(x-6) \cdot(x-4)^{3} \cdot(x-2)^{15} \cdot x^{29} .
$$

We turn to study the mixing time and cut off. We see that the mixing time is in the order of $n \log n$ (Theorem 3.1). Let $d_{T V}(\mu, \nu):=\max _{A \subset G_{n, p}} \mid \mu(A)-$ $\nu(A) \mid$ be the total variation distance between the probability distributions $\mu, \nu$ on $G_{n, p}$.

## Theorem 1.2

(1) Let $c>0$. Then we can find $f(c)>0$, s.t. for $k=\lfloor n \log n+c n\rfloor$ we have

$$
d_{T V}\left(\left(\frac{1}{n p} \mathbf{B}_{1}\right)^{k}, U\right)=f(c)+o(1), \quad n \rightarrow \infty
$$

where

$$
f(c) \leq e^{-c}+\mathcal{O}\left(e^{-2 c}\right), \quad c \rightarrow \infty
$$

(2) Suppose $\left\{c_{n}\right\}_{n}$ satisfy

$$
\log (\log (n p) \cdot(\log n+\alpha)) \leq c_{n} \ll \log n, \quad \alpha>0
$$

Let $k:=\left\lfloor n \log n-c_{n} \cdot n\right\rfloor$. Then for any $\delta>0$, we have

$$
d_{T V}\left(\left(\frac{1}{n p} \mathbf{B}_{1}\right)^{k}, U\right) \geq 1-\mathcal{O}\left(\frac{1}{n^{\alpha}}\right), \quad n \rightarrow \infty
$$

where $a_{n} \ll b_{n}$ means that $\lim _{n \rightarrow \infty} a_{n} / b_{n}=0$ and $\lfloor x\rfloor:=\max \{k \in \mathbf{Z} \mid k \leq x\}$ is the integer part of $x$.

## Remark

(1) In Theorem 1.2(2), the condition $\log (\log (n p) \cdot(\log n+\alpha)) \leq c_{n} \ll \log n$ for $\left\{c_{n}\right\}$ roughly means that $2 \log \log n+\frac{(\log p+\alpha)}{\log n}<c_{n} \ll \log n$ for large $n$.
(2) The argument using the strong stationary time in [1], [8] (eq.(6.16) and Proposition 7.14) still works for this case, but Theorem 1.2 aims to study the same problem with purely combinatoric method. The upper bound (Theorem 1.2(1)) is the same as that in [1, 8]. However, the lower bound in Theorem 1.2(2) is not good enough as in [1, 8].

The outline of this paper is as follows. In section 2, we study basic properties of $\mathbf{B}_{k}$ and derive a formula expressing the powers of $\mathbf{B}_{k}$ in terms of orthogonal idempotents, from which we can compute the eigenvalues and corresponding eigenspaces of the left regular representation of $\mathbf{B}_{1}$ explicitly. In section 3, we derive a formula computing the total variance distance beween the probability distribution of the repeated top to random shuffles and the uniform distribution. It then follows that the mixing time is in the order of $n \log n$. In section 4 , we further estimate this total variation distance using the asymptotic formula for the Stirling numbers of the second kind [9], yielding a cut off statement. In Appendix, we provide proofs for some elementary facts for completeness.

## 2 Eigenvalues and their multiplicities

We begin by studying some algebraic properties of $\mathbf{B}_{k}$ 's by which we derive the representation of the powers of $\mathbf{B}_{1}$ (Theorem 2.2). The following lemma follows from a theorem in [13] which studies more general cases. However we present its elementary proof.

Lemma 2.1 We have the following formulas. (1)

$$
\mathbf{B}_{k} \mathbf{B}_{1}= \begin{cases}p k \mathbf{B}_{k}+\mathbf{B}_{k+1} & (1 \leq k \leq n-1) \\ p n \mathbf{B}_{n} & (k=n)\end{cases}
$$

(2)

$$
\begin{aligned}
& \mathbf{B}_{k}=\mathbf{B}_{1}\left(\mathbf{B}_{1}-p \mathbf{I}\right)\left(\mathbf{B}_{1}-2 p \mathbf{I}\right) \cdots\left(\mathbf{B}_{1}-(k-1) p \mathbf{I}\right), \quad k=1,2, \cdots, n \\
& \mathbf{B}_{1}\left(\mathbf{B}_{1}-p \mathbf{I}\right)\left(\mathbf{B}_{1}-2 p \mathbf{I}\right) \cdots\left(\mathbf{B}_{1}-(\ell-1) p \mathbf{I}\right)=\mathbf{0}, \quad \ell>k .
\end{aligned}
$$

In particular, $\mathbf{B}_{1}, \mathbf{B}_{2}, \ldots, \mathbf{B}_{n}$ generate a commutative subalgebra of $\mathbb{Q} G_{n, p}$.
Proof. (1) We suppose that $k \leq n-1$. The case for $k=n$ follows similarly. Then we rewrite $\mathbf{B}_{k}=\sum_{\alpha \in G_{k, p}} \alpha \boldsymbol{\omega} W_{k, n}$ by grouping the terms by the leading letter as follows.

$$
\mathbf{B}_{k}=\sum_{t \in\left(C_{p} \times[k]\right) \cup\{(0, k+1)\}} \mathbf{C}_{k}(t),
$$

where $\mathbf{C}_{k}(t)$ is the sum of the elements in $\mathbf{B}_{k}$ whose leading letter is $t$. For example when $p=2$ and $n=4$,

$$
\mathbf{C}_{2}(\overline{2})=\overline{2} 134+\overline{2} 314+\overline{2} 341+\overline{2} \overline{1} 34+\overline{2} 3 \overline{1} 4+\overline{2} 34 \overline{1}
$$

and

$$
\mathbf{B}_{2}=\mathbf{C}_{2}(1)+\mathbf{C}_{2}(\overline{1})+\mathbf{C}_{2}(2)+\mathbf{C}_{2}(\overline{2})+\mathbf{C}_{2}(3) .
$$

By lemma 5.1, we have

$$
\mathbf{C}_{k}(t) \mathbf{B}_{1}= \begin{cases}\mathbf{B}_{k} & t \in[p] \times[k] \\ \mathbf{B}_{k+1} & t=(0, k+1)\end{cases}
$$

which yields

$$
\mathbf{B}_{k} \mathbf{B}_{1}=p k \mathbf{B}_{k}+\mathbf{B}_{k+1}
$$

(2) From the identity derived in (1) we have, inductively,
$\mathbf{B}_{k}=\mathbf{B}_{k-1}\left(\mathbf{B}_{1}-p(k-1) \mathbf{I}\right)=\mathbf{B}_{1}\left(\mathbf{B}_{1}-p \mathbf{I}\right)\left(\mathbf{B}_{1}-2 p \mathbf{I}\right) \cdots\left(\mathbf{B}_{1}-(k-1) p \mathbf{I}\right), \quad k=1,2, \cdots, n$.
Second identity in (2) follows similarly, by noting $\mathbf{0}=\mathbf{B}_{n}\left(\mathbf{B}_{1}-n p \mathbf{I}\right)$.
Let $\left[\begin{array}{l}k \\ a\end{array}\right]$ (resp. $\left\{\begin{array}{l}k \\ a\end{array}\right\}$ ), where $k, a$ are non negative integers, be the Stirling number of the first kind (resp. the second kind) defined respectively by

$$
\begin{aligned}
& {\left[\begin{array}{c}
k+1 \\
a
\end{array}\right]=k\left[\begin{array}{l}
k \\
a
\end{array}\right]+\left[\begin{array}{c}
k \\
a-1
\end{array}\right], \quad a \geq 1, \quad\left[\begin{array}{l}
0 \\
a
\end{array}\right]=1(a=0)} \\
& \left\{\begin{array}{c}
k+1 \\
a
\end{array}\right\}=a\left\{\begin{array}{l}
k \\
a
\end{array}\right\}+\left\{\begin{array}{c}
k \\
a-1
\end{array}\right\}, \quad a \geq 1, \quad\left\{\begin{array}{l}
0 \\
a
\end{array}\right\}=1(a=0)
\end{aligned}
$$

Then we can express $\mathbf{B}_{1}^{k}$ as a linear combination of $\mathbf{B}_{a}$ 's in terms of the Stirling numbers of the second kind. Since these numbers defined above are the Möbius function each other, the other way around is also possible.

Theorem 2.2

$$
\begin{align*}
& \mathbf{B}_{1}^{k}=\sum_{a=0}^{n \wedge k} p^{k-a}\left\{\begin{array}{l}
k \\
a
\end{array}\right\} \mathbf{B}_{a}, \quad k=0,1, \cdots  \tag{1}\\
& \mathbf{B}_{a}=\sum_{i=0}^{a}(-p)^{a-i}\left[\begin{array}{l}
a \\
i
\end{array}\right] \mathbf{B}_{1}^{i}, \quad a=0,1, \cdots, n . \tag{2}
\end{align*}
$$

with the convention that $\mathbf{B}_{1}^{0}=\mathbf{I}$ and $n \wedge k:=\min \{n, k\}$.
For proof, we introduce

$$
(x)_{n, p}:=x(x-p)(x-2 p) \cdots(x-(n-1) p)
$$

and show basic identities.

## Lemma 2.3

$$
\begin{align*}
& (x)_{n, p}=\sum_{k=0}^{n}(-p)^{n-k}\left[\begin{array}{l}
n \\
k
\end{array}\right] x^{k}  \tag{1}\\
& x^{n}=\sum_{k=0}^{n} p^{n-k}\left\{\begin{array}{l}
n \\
k
\end{array}\right\}(x)_{k, p} .
\end{align*}
$$

Proof. It suffices to substitute $(x)_{n, p}=p^{n}\left(\frac{x}{p}\right)_{n}$ into the following wellknown formulas.

$$
\begin{aligned}
(x)_{n} & =\sum_{k=0}^{n}(-1)^{n-k}\left[\begin{array}{l}
n \\
k
\end{array}\right] x^{k} \\
x^{n} & =\sum_{k=0}^{n}\left\{\begin{array}{l}
n \\
k
\end{array}\right\}(x)_{k} .
\end{aligned}
$$

Theorem 2.2 follows immediately from Lemma 2.3.

Proof of Theorem 2.2
By Lemma 2.1(2), we have $\mathbf{B}_{a}=\left(\mathbf{B}_{1}\right)_{a, p}$ for $a=1,2, \cdots, n$, and $\left(\mathbf{B}_{1}\right)_{a, p}=\mathbf{0}$ for $a \geq n+1$. Taking $x=\mathbf{B}_{1}$ in Lemma 2.3 yields

$$
\begin{aligned}
& \mathbf{B}_{a}=\left(\mathbf{B}_{1}\right)_{a, p}=\sum_{k=0}^{a}(-p)^{a-k}\left[\begin{array}{l}
a \\
k
\end{array}\right] \mathbf{B}_{1}^{k}, \quad a=1,2, \cdots, n \\
& \mathbf{B}_{1}^{k}=\sum_{a=0}^{k} p^{k-a}\left\{\begin{array}{l}
k \\
a
\end{array}\right\}\left(\mathbf{B}_{1}\right)_{a, p}=\sum_{a=0}^{n \wedge k} p^{k-a}\left\{\begin{array}{l}
k \\
a
\end{array}\right\} \mathbf{B}_{a}, \quad k=1,2, \cdots,
\end{aligned}
$$

Besides, we can explicitly see that they are valid also for $a=0$ and $k=0$.
Remark For given $p \in \mathbf{N},\left[\begin{array}{l}k \\ a\end{array}\right]_{p}:=p^{k-a}\left[\begin{array}{l}k \\ a\end{array}\right]$ and $\left\{\begin{array}{l}k \\ a\end{array}\right\}_{p}:=p^{k-a}\left\{\begin{array}{l}k \\ a\end{array}\right\}$ satisfy the recursion equation and the Möbius relation similar to the usual one, so that we can regard them as a p-version of the Stirling numbers.

$$
\begin{aligned}
& {\left[\begin{array}{c}
k+1 \\
a
\end{array}\right]_{p}=p k\left\{\begin{array}{l}
k \\
a
\end{array}\right\}_{p}+\left[\begin{array}{c}
k \\
a-1
\end{array}\right]_{p}, \quad\left[\begin{array}{l}
0 \\
0
\end{array}\right]_{p}=1} \\
& \left\{\begin{array}{c}
k+1 \\
a
\end{array}\right\}_{p}=p a\left\{\begin{array}{l}
k \\
a
\end{array}\right\}_{p}+\left\{\begin{array}{c}
k \\
a-1
\end{array}\right\}_{p}, \quad\left\{\begin{array}{l}
0 \\
0
\end{array}\right\}_{p}=1 \\
& \sum_{j}(-1)^{n-j}\left[\begin{array}{l}
n \\
j
\end{array}\right]_{p}\left\{\begin{array}{l}
j \\
i
\end{array}\right\}_{p}=\delta_{n, i} .
\end{aligned}
$$

However, $\left[\begin{array}{l}k \\ a\end{array}\right]_{p}$ is different from the Stirling-Frobenius cycle number of $p a$ rameter $p$ which appears in the analysis of a p-version of the riffle shuffle
[10, 11] ; For a generalized riffle shuffle (i.e., the riffle shuffle on $G_{n, p}$ ), the multiplicity of eigenvalues are equal to the Stirling-Frobenius cycle number [11].

We define the elements $\mathbf{e}_{i}$ of the group algebra $\mathbb{Q} G_{n, p}$ by

$$
\begin{equation*}
\mathbf{e}_{i}=\frac{1}{i!} \sum_{a=i}^{n} \frac{(-1)^{a-i}}{p^{a}(a-i)!} \mathbf{B}_{a} \tag{3}
\end{equation*}
$$

for $i=0,1, \ldots, n$. Then the powers of $\mathbf{B}_{1}$ are expressed in terms of $\left\{\mathbf{e}_{i}\right\}$.

## Theorem 2.4

$$
\begin{equation*}
\mathbf{B}_{1}^{k}=\sum_{i=0}^{n}(p i)^{k} \mathbf{e}_{i}, \quad k=0,1, \ldots \tag{4}
\end{equation*}
$$

Proof. We use the following idendity [7]

$$
\left\{\begin{array}{l}
k \\
a
\end{array}\right\}=\left[t^{k}\right]\left(\frac{k!}{a!}\left(e^{t}-1\right)^{a}\right)
$$

In fact, by Taylor's expansion,

$$
\frac{k!}{a!}\left(e^{t}-1\right)^{a}=\frac{k!}{a!}\left(\frac{t}{1!}+\frac{t^{2}}{2!}+\cdots\right)\left(\frac{t}{1!}+\frac{t^{2}}{2!}+\cdots\right) \cdots\left(\frac{t}{1!}+\frac{t^{2}}{2!}+\cdots\right) .
$$

Taking the coefficient of $t^{k}$ leads us to this formula:

$$
\left[t^{k}\right] \frac{k!}{a!}\left(e^{t}-1\right)^{a}=\sum_{\substack{k_{1}+\cdots+k_{a}=k \\
k_{1}, \cdots, k_{a} \geq 1}} \frac{k!}{k_{1}!k_{2}!\cdots k_{a}!} \cdot \frac{1}{a!}=\left\{\begin{array}{l}
k \\
a
\end{array}\right\} .
$$

By the binomial theorem,

$$
\left\{\begin{array}{l}
k \\
a
\end{array}\right\}=\left[t^{k}\right] \frac{k!}{a!} \sum_{i=0}^{a}\binom{a}{i}(-1)^{a-i} e^{i t}=\sum_{i=0}^{a} \frac{(-1)^{a-i}}{i!(a-i)!} \cdot i^{k} .
$$

We note that this formula is valid also for $k=0$. Using this equation in (1) and changing the order of summation yield the conclusion. We note that in (1), the summation $\sum_{a=0}^{n \wedge k}$ may be replaced by $\sum_{a=0}^{n}$, since $\left\{\begin{array}{l}k \\ a\end{array}\right\}=0$ for $a>k$.

$$
\mathbf{B}_{1}^{k}=\sum_{a=0}^{n} p^{k-a} \sum_{i=0}^{a} \frac{(-1)^{a-i}}{i!(a-i)!} \cdot i^{k} \mathbf{B}_{a}=\sum_{i=0}^{n}(i p)^{k} \sum_{a=i}^{n} \frac{1}{p^{a}} \cdot \frac{(-1)^{a-i}}{i!(a-i)!} \mathbf{B}_{a}=\sum_{i=0}^{n}(i p)^{k} \mathbf{e}_{i}
$$

Remark The argument of proof of Lemma 5.2 and eq.(4) imply that $\left\{\mathbf{e}_{i}\right\}$ is the orthogonal idempotents :

$$
\begin{equation*}
\sum_{i=0}^{n} \mathbf{e}_{i}=I, \quad \mathbf{e}_{i} \mathbf{e}_{j}=\delta_{i, j} \mathbf{e}_{i} \tag{5}
\end{equation*}
$$

The following lemma is stated in [2] which can be proved by standard inclusionexclusion principle [12].

Lemma 2.5 [2] Let $D_{n, p}$ be the number of derangements in $G_{n, p}$. Then,

$$
D_{n, p}=p^{n} n!\sum_{k=0}^{n} \frac{(-1)^{k}}{p^{k} k!}
$$

Proof of Theorem 1.1 Let $E_{i}$ be the matrices defined by

$$
E_{i}=L\left(e_{i}\right)
$$

for $i=0,1, \ldots, n$. By transforming both sides of eq. (4) by $L$, powers of $L\left(\mathbf{B}_{1}\right)$ can be represented in terms of $E_{i}$ 's.

$$
L\left(\mathbf{B}_{1}\right)^{k}=\sum_{i=0}^{n}(p i)^{k} E_{i} .
$$

Then by Lemma 5.2 in Appendix it follows that $\{(i p)\}_{i=0}^{n}$ are the eigenvalues and the range $\operatorname{Ran} E_{i}$ of $E_{i}$ (if it is nonzero) are the corresponding eigenspaces. Since each $\mathbf{B}_{a}$ contains exactly one identity permutation, we have Trace $L\left(\mathbf{B}_{a}\right)=\left|G_{n, p}\right|=p^{n} n$ !. Then we compute

Trace $E_{i}=\frac{1}{i!} \sum_{a=i}^{n} \frac{(-1)^{a-i}}{p^{a}(a-i)!} p^{n} n!=\binom{n}{i} p^{n-i}(n-i)!\sum_{b=0}^{n-i} \frac{(-1)^{b}}{p^{b} b!}=\binom{n}{i} D_{n-i, p}$ which is the number of elements of $G_{n, p}$ with $i$ fixed points.

We have analogous formulas for $\mathbf{B}_{a}$.

## Corollary 2.6

$$
\begin{equation*}
\mathbf{B}_{a}^{k}=\sum_{i=a}^{n}\left(p^{a} a!\binom{i}{a}\right)^{k} \mathbf{e}_{i}, \quad k=0,1, \ldots \tag{6}
\end{equation*}
$$

Therefore, the eigenvalues of $L\left(\mathbf{B}_{a}\right)$ are

$$
0, p^{a} a!, p^{a} a!\binom{a+1}{a}, p^{a} a!\binom{a+2}{a}, \ldots, p^{a} a!\binom{n}{a} .
$$

The multiplicity of the eigenvalue $p^{a} a!\binom{i}{a}$ is same as that of the eigenvalues of $\mathbf{B}_{1}$.

Proof. It suffices to show

$$
\mathbf{B}_{a}=\sum_{i=a}^{n} p^{a} a!\binom{i}{a} \mathbf{e}_{i}, \quad a=0,1, \cdots, n .
$$

and then use eq.(5). In order for that, we aim to express $\mathbf{B}_{a}$ in terms of $\mathbf{e}_{i}{ }^{\prime}$ 's by using eq. (3)

$$
\begin{equation*}
\mathbf{e}_{i}=\sum_{b=i}^{n} \frac{1}{p^{i} i!} \frac{(-1)^{b-i}}{p^{b-i}(b-i)!} \mathbf{B}_{b}=\sum_{b=i}^{n} \frac{1}{p^{i} i!}\left[x^{b-i}\right]\left(e^{-\frac{x}{p}}\right) \mathbf{B}_{b} \tag{7}
\end{equation*}
$$

The "reciprocal" of these coefficients are equal to

$$
i!p^{i}\left[x^{i-a}\right]\left(e^{\frac{x}{p}}\right)=i!p^{i} \frac{1}{(i-a)!}\left[x^{i-a}\right]\left(\frac{x}{p}\right)^{i-a}=p^{a} a!\binom{i}{a}
$$

which satisfy

$$
\sum_{i=a}^{b}\left(e^{\frac{x}{p}}\right)\left[x^{i-a}\right]\left(e^{-\frac{x}{p}}\right)\left[x^{b-i}\right]=\left(e^{\frac{x}{p}} \cdot e^{-\frac{x}{p}}\right)\left[x^{b-a}\right]=\delta_{a, b} .
$$

Thus, applying $\sum_{i=a}^{n} i!p^{i}\left[x^{i-a}\right]\left(e^{\frac{x}{p}}\right)$ on both sides of (7) yields

$$
\begin{aligned}
\sum_{i=a}^{n} i!p^{i}\left(e^{\frac{x}{p}}\right)\left[x^{i-a}\right] \mathbf{e}_{i} & =\sum_{i=a}^{n} i!p^{i}\left(e^{\frac{x}{p}}\right)\left[x^{i-a}\right] \sum_{b=i}^{n} \frac{1}{p^{i} i!}\left(e^{-\frac{x}{p}}\right)\left[x^{b-i}\right] \mathbf{B}_{b} \\
\sum_{i=a}^{n} p^{a} a!\binom{i}{a} \mathbf{e}_{i} & =\sum_{i=a}^{n} \sum_{b=i}^{n}\left(e^{\frac{x}{p}}\right)\left[x^{i-a}\right]\left(e^{-\frac{x}{p}}\right)\left[x^{b-i}\right] 1(0 \leq a \leq i \leq b \leq n) \mathbf{B}_{b} \\
& =\sum_{b=a}^{n} \sum_{i=a}^{b}\left(e^{\frac{x}{p}}\right)\left[x^{i-a}\right]\left(e^{-\frac{x}{p}}\right)\left[x^{b-i}\right] \mathbf{B}_{b} \\
& =\sum_{b=a}^{n} \delta_{a, b} \mathbf{B}_{b}=\mathbf{B}_{a} .
\end{aligned}
$$

Here, we introduce the generalized Stirling numbers, which arise in the Boson normal ordering problem [4].

$$
\begin{aligned}
S_{r, s}(n, k) & :=\frac{(-1)^{k}}{k!} \sum_{p=s}^{k}(-1)^{p}\binom{k}{p} \prod_{j=1}^{n}(p+(j-1)(r-s))^{\underline{s}} \\
\text { where } m^{\underline{s}} & :=m(m-1) \cdots(m-s+1) .
\end{aligned}
$$

Then we obtain $\mathbf{B}_{a}$-analogue of Theorem 2.2.

## Theorem 2.7

$$
\mathbf{B}_{a}^{k}=\sum_{b=a}^{n} p^{k a-b} S_{a, a}(k, b) \mathbf{B}_{b} .
$$

Proof. Using $p^{\underline{a}}=a!\binom{p}{a}$ in the definition, we have

$$
S_{a, a}(k, b)=\frac{(-1)^{b}}{b!} \sum_{i=a}^{b}(-1)^{i}\binom{b}{i}\left(a!\binom{i}{a}\right)^{k} .
$$

Using

$$
\mathbf{e}_{i}=\frac{1}{i!} \sum_{b=i}^{n} \frac{(-1)^{b-i}}{p^{b}(b-i)!} \mathbf{B}_{b}=\sum_{b=i}^{n} \frac{1}{p^{b}} \frac{(-1)^{b-i}}{b!}\binom{b}{i} \mathbf{B}_{b}
$$

in eq. (6) yields

$$
\begin{aligned}
\mathbf{B}_{a}^{k} & =\sum_{i=a}^{n}\left(p^{a} a!\binom{i}{a}\right)^{k} \sum_{b=i}^{n} \frac{1}{p^{b}} \frac{(-1)^{b-i}}{b!}\binom{b}{i} \mathbf{B}_{b} \\
& =\sum_{b=a}^{n} \frac{p^{k a-b}}{b!} \sum_{i=a}^{b}(-1)^{b-i}\binom{b}{i}\left(a!\binom{i}{a}\right)^{k} \mathbf{B}_{b} \\
& =\sum_{b=a}^{n} p^{k a-b} S_{a, a}(k, b) \mathbf{B}_{b} .
\end{aligned}
$$

Remark An asymptotic formula for $S_{a, a}(k, b)$ would yield a cut off statement for the shuffles corresponding to $\mathbf{B}_{a}$.

## 3 Mixing time

In this section, we consider the mixing time of the top to random shuffle on the colored permutation. Although the state space of the Markov chain is $p^{n}$ times larger than the ordinary top to random shuffle on $\mathfrak{S}_{n}$, it turns out that the mixing time does not differ significantly ; in fact, our bound on the mixing time is independent of $p$, for $p>1$. Let $\mathbf{B}=\sum_{w \in G_{n, p}} c_{w} w \in \mathbb{Q} G_{n, p}$ be an element of the group algebra $\mathbb{Q} G_{n, p}$. We define the $L^{1}$-norm $|\mathbf{B}|$ of $\mathbf{B}$ by

$$
|\mathbf{B}|=\sum_{w \in G_{n, p}}\left|c_{w}\right| .
$$

Then it can easily be confirmed

$$
\left|\mathbf{B}_{a}\right|=\binom{n}{a} p^{a} a!.
$$

and since these elements in $G_{n, p}$ consisting of $\mathbf{B}_{a-1}$ are contained by those in $\mathbf{B}_{a}$,

$$
\left|\mathbf{B}_{a}-\mathbf{B}_{a-1}\right|=\left|\mathbf{B}_{a}\right|-\left|\mathbf{B}_{a-1}\right| .
$$

A probability distribution $\mathbf{P}$ over $G_{n, p}$ can be regarded as an element of $\mathbb{R} G_{n, p}$, i.e., $\mathbf{P}$ can be expressed as

$$
\mathbf{P}=\sum_{w \in G_{n, p}} p_{w} w
$$

where $p_{w} \geq 0$ and $\sum_{w \in G_{n, p}} p_{w}=1$. Let $\mathbf{P}, \mathbf{Q} \in \mathbb{R} G_{n, p}$ be two probability distributions on $G_{n, p}$. Then $d_{\mathrm{TV}}(\mathbf{P}, \mathbf{Q})$ is equal to

$$
d_{\mathrm{TV}}(\mathbf{P}, \mathbf{Q})=\frac{1}{2}|\mathbf{P}-\mathbf{Q}| .
$$

Theorem 3.1 Let $p$ be greater than 1 and let $U$ be the uniform distribution over $G_{n, p}$, that is,

$$
U=\sum_{w \in G_{n, p}} \frac{1}{p^{n} n!} w
$$

(1) The total variation distance between the distribution of $k$-repeated top to random shuffle and the uniform distribution is bounded above by

$$
\begin{gathered}
d_{T V}\left(\left(\frac{1}{p n} \mathbf{B}_{1}\right)^{k}, U\right) \leq 1-\frac{\left\{\begin{array}{l}
k \\
n
\end{array}\right\} n!}{n^{k}} . \\
\text { (2) Let } \epsilon>0 \text {. Then } d_{T V}\left(\left(\frac{1}{p n} \mathbf{B}_{1}\right)^{k}, U\right)<\epsilon \text { for } k \geq n \log n+n \log \frac{-1}{\log (1-\varepsilon)} .
\end{gathered}
$$

To prove Theorem 3.1, we need the following lemma which gives us the TV distance between the distribution of $k$-repeated top to random shuffles and the uniform distribution.

Lemma 3.2 Let

$$
A:=\min \left\{a \left\lvert\, \frac{1}{p^{n} n!}>\frac{1}{(n p)^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\}\right.\right\} .
$$

Then

$$
d_{T V}\left(\left(\frac{1}{p n} B_{1}\right)^{k}, U\right)=\sum_{a \geq A}\left(\frac{1}{p^{n} n!}-\frac{1}{p^{k} n^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}\right)\left(\left|\mathbf{B}_{a}\right|-\left|\mathbf{B}_{a-1}\right|\right)
$$

Proof. Let

$$
\begin{aligned}
\mathbf{C}_{a} & :=\mathbf{B}_{a}-\mathbf{B}_{a-1}, \mathbf{C}_{0}:=\mathbf{B}_{0} . \\
x_{a} & :=\frac{1}{(p n)^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}, \quad a=0,1, \cdots, n .
\end{aligned}
$$

Then

$$
\begin{aligned}
\left(\frac{1}{p n} \mathbf{B}_{1}\right)^{k} & =\frac{1}{(p n)^{k}} \sum_{b=1}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\} \sum_{a=1}^{b} \mathbf{C}_{a}+\frac{1}{(p n)^{k}} \sum_{b=1}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\} \mathbf{C}_{0} \\
& =\sum_{a=1}^{n} \frac{1}{(p n)^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\} \mathbf{C}_{a}+\frac{1}{(p n)^{k}} \sum_{b=0}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\} \mathbf{C}_{0} \\
& =\sum_{a=0}^{n} x_{a} \mathbf{C}_{a}, \quad k=0,1, \cdots
\end{aligned}
$$

Where we used $\left\{\begin{array}{l}k \\ 0\end{array}\right\}=0$ for $k \neq 0$. Similarly, using $\mathbf{B}_{n}=\sum_{a=0}^{n} \mathbf{C}_{a}$ we have

$$
U=\sum_{a=0}^{n} y_{a} \mathbf{C}_{a}, \quad y_{a}:=\frac{1}{n!p^{n}}
$$

Hence

$$
d_{T V}\left(\left(\frac{1}{p n} \mathbf{B}_{1}\right)^{k}, U\right)=\frac{1}{2} \sum_{a=0}^{n}\left|x_{a}-y_{a}\right|\left|\mathbf{C}_{a}\right|=\sum_{a: y_{a}>x_{a}}\left(x_{a}-y_{a}\right)\left|\mathbf{C}_{a}\right|
$$

where the second equality follows from the fact that $\sum_{a=0}^{n} y_{a}\left|\mathbf{C}_{a}\right|=\sum_{a=0}^{n} x_{a}\left|\mathbf{C}_{a}\right|=$ 1 (Proposition 4.2 [8]). Since $x_{a}$ is monotonically decreasing and $y_{a}$ is constant, and since $A:=\min \left\{a \mid y_{a}>x_{a}\right\}$, we have

$$
\begin{aligned}
d_{T V}\left(\left(\frac{1}{p n} \mathbf{B}_{1}\right)^{k}, U\right) & =\sum_{a \geq A}\left(y_{a}-x_{a}\right)\left|\mathbf{C}_{a}\right| \\
& =\sum_{a \geq A}\left(\frac{1}{p^{n} n!}-\frac{1}{p^{k} n^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\}\right)\left(\left|\mathbf{B}_{a}\right|-\left|\mathbf{B}_{a-1}\right|\right)
\end{aligned}
$$

Proof of Theorem 3.1 (1) By Theorem 2.2, we have

$$
\left(\frac{1}{p n} \mathbf{B}_{1}\right)^{k}=\frac{1}{p^{k} n^{k}} \sum_{a=1}^{n} p^{k-a}\left\{\begin{array}{l}
k \\
a
\end{array}\right\} \mathbf{B}_{a} .
$$

Therefore, if we let $A$ be the smallest of the integers $a$ such that

$$
\frac{1}{p^{n} n!}>\frac{1}{p^{k} n^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}
$$

we have, by Lemma 3.2,

$$
\begin{aligned}
d_{\mathrm{TV}}\left(\left(\frac{1}{p n} \mathbf{B}_{1}\right)^{k}, U\right) & =\sum_{a \geq A}\left(\frac{1}{p^{n} n!}-\frac{1}{p^{k} n^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}\right)\left(\left|B_{a}\right|-\left|B_{a-1}\right|\right) \\
& \leq \sum_{a \geq A}\left(\frac{1}{p^{n} n!}-\frac{1}{p^{k} n^{k}} \cdot p^{k-n}\left\{\begin{array}{l}
k \\
n
\end{array}\right\}\right)\left(\left|\mathbf{B}_{a}\right|-\left|\mathbf{B}_{a-1}\right|\right) \\
& \leq\left(\frac{1}{p^{n} n!}-\frac{1}{p^{k} n^{k}} \cdot p^{k-n}\left\{\begin{array}{l}
k \\
n
\end{array}\right\}\right)\left|\mathbf{B}_{n}\right|=1-\frac{\left\{\begin{array}{l}
k \\
n
\end{array}\right\} n!}{n^{k}} .
\end{aligned}
$$

(2) This immediately follows from (1) above and Lemma 5.4 in Appendix.

## 4 Cut off

In this section we prove Theorem 1.2. First of all, it is easy to show the upper bound in Theorem 1.2(1). In fact, by Theorem 3.1 and Lemma 5.4 we have, for $k=\lfloor n \log n+c n\rfloor$,

$$
d_{T V}\left(\frac{1}{(n p)^{k}} B_{1}^{k}, U\right) \leq 1-\exp \left[-n e^{-\frac{k}{n}}\right](1+o(1)) \xrightarrow{n \rightarrow \infty} 1-\exp \left[-e^{-c}\right] .
$$

It then suffices to compute the Taylor's expansion of $\exp \left[-e^{-c}\right]$ to prove Theorem 1.2(1). For the lower bound, let

$$
X:=\lfloor n-\log n\rfloor .
$$

We shall divide into two cases : Case $1: A \leq X$ and Case $2: X \leq A$.

### 4.1 Case 1: $(A \leq X)$

We first substitute $\sum_{a \geq A}$ for $\sum_{a \geq X}$ in the formula in Lemma 3.2. Using $\left|B_{a}\right|=p^{a} a!\binom{n}{a}$, we have

$$
\begin{align*}
& d_{T V}\left(\frac{1}{(n p)^{k}} B_{1}^{k}, U\right) \\
\geq & \sum_{a \geq X}\left(\frac{1}{p^{n} n!}-\frac{1}{(n p)^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\}\right)\left(\left|B_{a}\right|-\left|B_{a-1}\right|\right) \\
\geq & \sum_{a \geq X}\left(\frac{1}{p^{n} n!}-\frac{1}{(n p)^{k}} \sum_{b=X}^{n} p^{k-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\}\right)\left(\left|B_{a}\right|-\left|B_{a-1}\right|\right) \\
= & \left(1-\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\}\right)\left(1-\frac{1}{(n-X+1)!\cdot p^{n-X+1}}\right)  \tag{8}\\
= & (1-C)(1-D) .
\end{align*}
$$

We aim to show $C, D=\mathcal{O}\left(n^{-\alpha}\right)$ below.

### 4.1.1 Estimate for $C$

We use the following property of the Stirling numbers of the second kind [6]: for given $k \in \mathbf{N}$, we can uniquely find $r_{k}$ such that

$$
\left\{\begin{array}{c}
k \\
1
\end{array}\right\}<\left\{\begin{array}{c}
k \\
2
\end{array}\right\}<\cdots<\left\{\begin{array}{c}
k \\
r_{k}
\end{array}\right\} \geq\left\{\begin{array}{c}
k \\
r_{k}+1
\end{array}\right\}>\cdots>\left\{\begin{array}{c}
k \\
k
\end{array}\right\}
$$

Moreover by eq.(1.6) in [9], $r_{k}$ satisfies

$$
r_{k}=\frac{k}{\log k}+\mathcal{O}\left(k(\log k)^{-\frac{3}{2}}\right)
$$

Taking $k=n \log n-n \cdot c_{n}$ yields

$$
\frac{k}{\log k}=\frac{n \log n\left(1-\frac{c_{n}}{n}\right)}{\log n+\log \log n+\log \left(1-\frac{c_{n}}{n}\right)}
$$

Thus $r_{k}$ quite likely is contained in the sum $\sum_{b=X}^{n}$ in eq.(8). Therefore we further divide into three cases, according to the large and small relationship, that is, Case (i) $X \leq n \leq r_{k}$, Case (ii) $X \leq r_{k} \leq n$, and Case (iii) $r_{k} \leq X \leq$ $n$.

Case (i) $X \leq n \leq r_{k}$ : $\quad$ Since $X \leq b \leq n$, we have $\left\{\begin{array}{l}k \\ b\end{array}\right\} \leq\left\{\begin{array}{l}k \\ n\end{array}\right\}$. Lemma 5.5 and the equation $e^{-k / n}=e^{-\log n+c}=e^{c} / n$ yield

$$
\begin{aligned}
\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} & \leq \frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
n
\end{array}\right\} \\
& \leq \exp \left[-e^{c_{n}}\right]\left(1+\mathcal{O}\left(\frac{\log n}{n^{1-\epsilon}}\right)\right) \frac{p}{p-1}\left(p^{n-X}-\frac{1}{p}\right), \quad \epsilon>0
\end{aligned}
$$

In what follows, $\epsilon>0$ is kept fixed. By the assumption $\log (\log (n p) \cdot(\log n+$ $\alpha)) \leq c_{n}$ on $\left\{c_{n}\right\}$, we have

$$
\exp \left[e^{c_{n}}\right] \geq \exp [\log (n p) \cdot(\log n+\alpha)]=(n p)^{\log n+\alpha}
$$

which leads to

$$
\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} \leq \frac{1}{(n p)^{\log n+\alpha}} \cdot \frac{p}{p-1}\left(p^{\log n}-\frac{1}{p}\right)=\mathcal{O}\left(\frac{1}{n^{\log n+\alpha}}\right)
$$

Case (ii) $X \leq r_{k} \leq n$ : We define $g(n)$ by the equation $r_{k}=: n-g(n)$. $X \leq r_{k} \leq n$ implies $0 \leq g(n) \leq \log n$. We then compute

$$
\begin{aligned}
\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} & \leq \frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
r_{k}
\end{array}\right\} \\
& =\left(\frac{r_{k}}{n}\right)^{k} \frac{n!}{r_{k}!} \cdot \frac{r_{k}!}{r_{k}^{k}}\left\{\begin{array}{c}
k \\
r_{k}
\end{array}\right\} \frac{p}{p-1}\left(p^{n-X}-\frac{1}{p}\right) \\
& =: E \cdot F \cdot G \cdot \frac{p}{p-1}\left(p^{n-X}-\frac{1}{p}\right)
\end{aligned}
$$

We shall estimate each factors $E, F, G$ below. $E$ and $F$ are easy :

$$
\begin{align*}
E & :=\left(\frac{r_{k}}{n}\right)^{k}=\left(1-\frac{g(n)}{n}\right)^{k}=\left(1-\frac{g(n)}{n}\right)^{n \log n-c_{n} \cdot n} \leq 1  \tag{9}\\
F & :=\frac{n!}{(n-g(n))!} \leq n^{g(n)} \leq n^{\log n} . \tag{10}
\end{align*}
$$

To estimate $G$, we first use Lemma 5.5 in Appendix.

$$
\begin{aligned}
G & :=\frac{r_{k}!}{r_{k}^{k}}\left\{\begin{array}{c}
k \\
r_{k}
\end{array}\right\} \leq \exp \left[-r_{k} e^{-\frac{k}{r_{k}}}\right] e^{\frac{1}{2 k}}\left(1+\mathcal{O}\left(\frac{\log n}{n^{1-\epsilon}}\right)\right) \\
& =\exp \left[-n\left(1-\frac{g(n)}{n}\right)\left(\frac{1}{n}\right)^{\frac{1}{1-\frac{g(n)}{n}}} \cdot\left(e^{c_{n}}\right)^{\frac{1}{1-\frac{g(n)}{n}}}\right]\left(1+\mathcal{O}\left(\frac{\log n}{n^{1-\epsilon}}\right)\right)
\end{aligned}
$$

Each factors in the exponential satisfy

$$
\begin{aligned}
& \left(\frac{1}{n}\right)^{\frac{1}{1-\frac{g(n)}{n}}}=\frac{1}{n}\left(1+\mathcal{O}\left(\frac{(\log n)^{2}}{n}\right)\right) \\
& \left(e^{c_{n}}\right)^{\frac{1}{1-\frac{g(n)}{n}}}=\left(1+\mathcal{O}\left(\frac{(\log n)^{2}}{n}\right)\right) e^{c_{n}}
\end{aligned}
$$

so that

$$
\begin{equation*}
G \leq \exp \left[-e^{c_{n}}\left(1+\mathcal{O}\left(\frac{(\log n)^{2}}{n}\right)\right)\right] \tag{11}
\end{equation*}
$$

By $(9,10,11)$, we have

$$
\begin{aligned}
\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} & \leq 1 \cdot n^{\log n} \cdot \exp \left[-e^{c_{n}}\left(1+\mathcal{O}\left(\frac{(\log n)^{2}}{n}\right)\right)\right] \frac{p}{p-1}\left(p^{\log n}-\frac{1}{p}\right) \\
& =(n p)^{\log n} \exp \left[-e^{c_{n}}\left(1+\mathcal{O}\left(\frac{(\log n)^{2}}{n}\right)\right)\right] \frac{p}{p-1}\left(1-\frac{1}{p^{\log n+1}}\right)
\end{aligned}
$$

Therefore the condition $\log ((\log n p)(\log n+\alpha)) \leq c_{n}$ on $\left\{c_{n}\right\}$ yields

$$
\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} \leq \frac{(n p)^{\log n}}{(n p)^{\log n+\alpha}} \cdot \exp \left[(\log n)^{2} \mathcal{O}\left(\frac{(\log n)^{2}}{n}\right)\right]=\mathcal{O}\left(\frac{1}{n^{\alpha}}\right)
$$

Case (iii) $r_{k} \leq X \leq n$ : We proceed as Case (ii) :

$$
\begin{aligned}
\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} & \leq \frac{n!p^{n}}{n^{k}}\left\{\begin{array}{c}
k \\
X
\end{array}\right\} \sum_{b=X}^{n} p^{-b} \\
& =\frac{n!}{n^{k}} \cdot \frac{X^{k}}{X!} \cdot \frac{X!}{X^{k}}\left\{\begin{array}{c}
k \\
X
\end{array}\right\} p^{n} \sum_{b=X}^{n} p^{-b} \\
& \leq\left(\frac{X}{n}\right)^{k} \frac{n!}{X!} \exp \left[-X e^{-\frac{k}{X}}\right] \cdot \frac{p^{2}}{p-1}\left(p^{\log n}-\frac{1}{p}\right) \\
& =E \cdot F \cdot G \cdot \frac{p^{2}}{p-1}\left(p^{\log n}-\frac{1}{p}\right)
\end{aligned}
$$

and aim to estimate each factors $E, F, G$. However, this is reduced to replacing $g(n)$ in Case (ii) by $\log n$, and hence a similar argument yields

$$
\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}=\mathcal{O}\left(\frac{1}{n^{\alpha}}\right)
$$

### 4.1.2 Estimate for $D$

Using $(\log n+1)!\approx n^{\log \log n}$ and $p^{\log n}=n^{\log p}$ directly leads us to the conclusion :

$$
D=\frac{1}{(\log n+1)!p^{\log n+1}}=\mathcal{O}\left(n^{-\log \log n-\log p}\right)
$$

### 4.2 Case 2: $(A \geq X)$

We first compute

$$
\begin{aligned}
d_{T V}\left(\frac{1}{(n p)^{k}} B_{1}^{k}, U\right) & =\sum_{a \geq A}\left(\frac{1}{p^{n} n!}-\frac{1}{(n p)^{k}} \sum_{b=a}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}\right)\left(\left|B_{a}\right|-\left|B_{a-1}\right|\right) \\
& \geq \sum_{a \geq A}\left(\frac{1}{p^{n} n!}-\frac{1}{(n p)^{k}} \sum_{b=A}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}\right)\left(\left|B_{a}\right|-\left|B_{a-1}\right|\right) \\
& =\left(\frac{1}{p^{n} n!}-\frac{1}{(n p)^{k}} \sum_{b=A}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}\right) \sum_{a \geq A}\left(\left|B_{a}\right|-\left|B_{a-1}\right|\right) \\
& \geq\left(\frac{1}{p^{n} n!}-\frac{1}{(n p)^{k}} \sum_{b=X}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}\right) \sum_{a \geq A}\left(\left|B_{a}\right|-\left|B_{a-1}\right|\right) \\
& =\left(1-\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\}\right)\left(1-\frac{1}{(n-A+1)!p^{n-A+1}}\right) \\
& =:(1-C)\left(1-D^{\prime}\right)
\end{aligned}
$$

where $D^{\prime}$ is equal to $D$ but $X$ is substituted by $A$. $C$ satifies the same estimate as in Case 1. In fact, we did not use the fact that $A \leq X$ to estimate $C$ in Case 1. Hence

$$
C=\frac{n!}{n^{k}} \sum_{b=X}^{n} p^{n-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}=\mathcal{O}\left(\frac{1}{n^{\alpha}}\right) .
$$

To estimate $D^{\prime}$, let $M:=n-A$. It then suffices to show $M \geq$ (Const.) $\log n$. By the definition of $A$,

$$
\frac{1}{p^{n} n!} \leq \frac{1}{(n p)^{k}} \sum_{b=A-1}^{n} p^{k-b}\left\{\begin{array}{l}
k \\
b
\end{array}\right\}
$$

so that we have

By the argument in Case 1, in any cases provided $(X-1 \leq) A-1 \leq b \leq n$, we have

$$
\frac{n!}{n^{k}}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} \leq(\text { Const. }) \frac{1}{n^{\alpha} p^{\log n+\alpha}}
$$

Hence

$$
1 \leq \frac{n!}{n^{k}} \sum_{b=A-1}^{n} p^{n-b}\left\{\begin{array}{c}
k \\
b
\end{array}\right\} \leq(\text { Const. }) \frac{p^{M}}{n^{\alpha} p^{\log n+\alpha}}
$$

Therefore $n^{\alpha} p^{\log n+\alpha} \leq($ Const. $) p^{M}$ from which we have $M \geq$ (Const.) $\log n$.

## 5 Appendix

### 5.1 Some elementary facts on the shuffle algebra

We collect two facts on the shuffle algebra which are used in this paper.
Lemma 5.1 Suppose $k \leq n-1$ and we rewrite $\mathbf{B}_{k}=\sum_{\alpha \in G_{k, p}} \alpha \boldsymbol{\omega} W_{k, n}$ by grouping the terms by the head (or top) letter as follows.

$$
\mathbf{B}_{k}=\sum_{t \in\left(C_{p} \times[k]\right) \cup\{(0, k+1)\}} \mathbf{C}_{k}(t),
$$

where $\mathbf{C}_{k}(t)$ is the sum of the elements in $\mathbf{B}_{k}$ whose leading letter is $t$. We then have

$$
\mathbf{C}_{k}(t) \mathbf{B}_{1}= \begin{cases}\mathbf{B}_{k} & t \in[p] \times[k] \\ \mathbf{B}_{k+1} & t=(0, k+1)\end{cases}
$$

Proof. We regard each $\alpha \in G_{k, p}$ as a word, so that we denote by $i(\alpha)$ the leading letter, and by $\widetilde{\alpha}$ the remaining ones : $\alpha=i(\alpha) \widetilde{\alpha}$. Then we write

$$
\begin{aligned}
\mathbf{B}_{k} & =\sum_{\alpha \in G_{k, p}}(i(\alpha) \widetilde{\alpha}) \boldsymbol{w} W_{k, n} \\
& =\sum_{\alpha \in G_{k, p}} i(\alpha)\left(\widetilde{\alpha} \boldsymbol{w} W_{k, n}\right)+\sum_{\alpha \in G_{k, p}}(0, k+1)\left(\alpha ш W_{k+1, n}\right) \\
& =\sum_{t \in\left(C_{p} \times[k]\right) \cup\{(0, k+1)\}} \mathbf{C}_{k}(t) .
\end{aligned}
$$

We note that the expression $i(\alpha)\left(\widetilde{\alpha} \omega W_{k, n}\right)$ stands for the concatenation ; e.g., $1(23+32)=123+132$.
(i) $t \in[p] \times[k]$ :

$$
C_{k}(t)=\sum_{\substack{\alpha \in G_{k, p} \\ t=i(\alpha)}} i(\alpha)\left(\widetilde{\alpha} ш W_{k, n}\right)=\sum_{\substack{\alpha \in G_{k, p} \\ t=i(\alpha)}} t\left(\widetilde{\alpha} \omega W_{k, n}\right) .
$$

For $t \in[p] \times[k]$ and $q \in[p]$, let $t_{q}$ be the $q$-shift of colors in $t: t=(s, i) \mapsto$ $t_{q}=(s+q, i)$. Applying $\mathbf{B}_{1}$ from the right is equivalent to shifting colors of the first alphabet and then inserting it randomly. Since the shuffle operator is associative, we have

$$
C_{k}(t) \mathbf{B}_{1}=\sum_{\substack{\alpha \in G_{k, p} \\ t=i(\alpha) \\ q \in[p]}} t_{q} \boldsymbol{\omega}\left(\widetilde{\alpha} \boldsymbol{\omega} W_{k}\right)=\sum_{\substack{\alpha \in G_{k}, p \\ t=i(\alpha) \\ q \in[p]}}\left(t_{q} \boldsymbol{\sim} \widetilde{\alpha}\right) \boldsymbol{\omega} W_{k}=\left(\sum_{\substack{\alpha \in G_{k}, p \\ t=i(\alpha) \\ q \in[p]}} t_{q} \boldsymbol{\sim} \widetilde{\alpha}\right) \boldsymbol{w} W_{k} .
$$

Equation $\sum_{\substack{\alpha \in G_{k, p} \\ t=i(\alpha) \\ q \in[p]}} t_{q} \omega \widetilde{\alpha}=\sum_{\beta \in G_{k, p}} \beta$ yields the result.
(ii) $t=(0, k+1)$ : we can argue similarly as in (i).

The lemma below is an elementary fact in the linear algebra, which yields the eigenvalues and the corresponding eigenspaces of a matrix.

Lemma 5.2 Suppose that $A, E_{1}, \cdots, E_{m}$ are nonzero $n \times n$ matrices satisfying

$$
\begin{aligned}
A^{k} & =\lambda_{1}^{k} E_{1}+\cdots+\lambda_{m}^{k} E_{m}, \quad k=0,1, \cdots, \\
\lambda_{i} & \neq \lambda_{j}, \quad i \neq j .
\end{aligned}
$$

Then $P(x)=\prod_{j=1}^{m}\left(x-\lambda_{j}\right)$ is the minimal polynomial of $A$ and

$$
\begin{aligned}
E_{i} E_{j} & =\delta_{i j} E_{i}, \quad i, j=1,2, \cdots, m \\
A E_{i} & =\lambda_{i} E_{i}
\end{aligned}
$$

Remark 5.3 Since $x \in \operatorname{Ran} E_{i}$ satisfies $A x=A E_{i} x=\lambda_{i} E_{i} x$, and since letting $k=0$ in the assumption implies $I=E_{1}+\cdots+E_{m}, \lambda_{1}, \cdots, \lambda_{m}$ are the eigenvalues of $A$ with Ran $E_{1}, \cdots$, Ran $E_{m}$ being the corresponding eigenspaces.

Proof. Let $Q$ be a polynomial. Then by assumption

$$
\begin{equation*}
Q(A)=Q\left(\lambda_{1}\right) E_{1}+\cdots+Q\left(\lambda_{m}\right) E_{m} . \tag{13}
\end{equation*}
$$

Since $P\left(\lambda_{i}\right)=0, i=1,2, \cdots, m$, we have $P(A)=P\left(\lambda_{1}\right) E_{1}+\cdots+P\left(\lambda_{m}\right) E_{m}=$ 0 . On the other hand, let

$$
P_{s}(x):=\prod_{j \neq s} \frac{x-\lambda_{j}}{\lambda_{s}-\lambda_{j}}=(\text { Const. }) \frac{P(x)}{x-\lambda_{s}}, \quad s=1,2, \cdots, m .
$$

Then $P_{s}\left(\lambda_{i}\right)=\left\{\begin{array}{ll}0 & (i \neq s) \\ 1 & (i=s)\end{array}\right.$ so that $P_{s}(A)=P_{s}\left(\lambda_{1}\right) E_{1}+\cdots+P_{s}\left(\lambda_{m}\right) E_{m}=$ $E_{s}(\neq 0)$ and hence $P$ is the minimal polynomial of $A$. Plugging $Q(\lambda)=$ $P_{i}(\lambda) P_{j}(\lambda)$ and $Q(\lambda)=\lambda P_{i}(\lambda)$ respectively in (13), we have

$$
\begin{aligned}
E_{i} E_{j} & =P_{i}(A) P_{j}(A)=\left(P_{i} P_{j}\right)(A) \\
& =P_{i}\left(\lambda_{1}\right) P_{j}\left(\lambda_{1}\right) E_{1}+\cdots+P_{i}\left(\lambda_{m}\right) P_{j}\left(\lambda_{m}\right) E_{m}=\delta_{i, j} E_{i} \\
A E_{i} & =Q(A)=\lambda_{1} P_{i}\left(\lambda_{1}\right) E_{1}+\cdots+\lambda_{m} P_{i}\left(\lambda_{m}\right) E_{m}=\lambda_{i} E_{i} .
\end{aligned}
$$

### 5.2 Asymptotics of Stirling numbers

The following lemma is well-known, but we provide a proof for completeness.
Lemma 5.4 Let $\lambda>0$. Then if $n$ and $k$ goes to infinity satisfying $n e^{-\frac{k}{n}} \rightarrow$ $\lambda$, we have

$$
\left\{\begin{array}{l}
k \\
n
\end{array}\right\} \frac{n!}{n^{k}} \rightarrow e^{-\lambda} .
$$

Proof. We consider putting $k$ balls uniformly at random into $n$ boxes. Then $p(k, n):=\left\{\begin{array}{l}k \\ n\end{array}\right\} \frac{n!}{n^{k}}$ is equal to the probability that no boxes are empty. We
aim to show $p(k, n) \rightarrow e^{-\lambda}$. By the inclusion-exclusion principle,

$$
\begin{equation*}
p(k, n)=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\left(1-\frac{j}{n}\right)^{k} . \tag{14}
\end{equation*}
$$

Here we use the following estimates :

$$
\begin{aligned}
& \binom{n}{j}=\frac{n(n-1) \cdots(n-j+1)}{j!}\left\{\begin{array}{l}
\leq \frac{n^{j}}{j!} \\
=\frac{n^{j}}{j!}\left(1-\mathcal{O}\left(\frac{j^{2}}{n}\right)\right) \\
0 \leq x \leq 1 / 2 \quad \Longrightarrow \quad e^{-x-x^{2}} \leq 1-x \leq e^{-x} .
\end{array} .\right.
\end{aligned}
$$

Then we have

$$
\begin{equation*}
\frac{1}{j!}\left(n e^{-\frac{k}{n}}\right)^{j} e^{-k\left(\frac{j}{n}\right)^{2}}\left(1-\mathcal{O}\left(\frac{j^{2}}{n}\right)\right) \leq\binom{ n}{j}\left(1-\frac{j}{n}\right)^{k} \leq \frac{1}{j!}\left(n e^{-\frac{k}{n}}\right)^{j} \tag{15}
\end{equation*}
$$

Now we use the fact $k=n \log n-n \cdot \log (\lambda+o(1))$ and the estimate (15) to apply the dominated convergence theorem on (14), yielding the conclusion.

We next turn to the general case.
Lemma 5.5 Suppose

$$
\begin{equation*}
\frac{n}{\sqrt{k}} \rightarrow \infty, \quad \frac{k}{n}-\log \sqrt{k} \rightarrow \infty \tag{16}
\end{equation*}
$$

(1) Then for any $\delta>0$, the following bound is valid for sufficiently large $n$.

$$
\left\{\begin{array}{l}
k \\
n
\end{array}\right\} \leq \frac{n^{k}}{n!} \exp \left[-n e^{-\frac{k}{n}}\right] e^{\frac{1}{2} e^{-\frac{k}{n}}}(1+o(1))
$$

(2) In particular, when $k=n \log n-c_{n} n, c_{n} \ll \log n$, we have

$$
\left\{\begin{array}{l}
k \\
n
\end{array}\right\} \leq \frac{n^{k}}{n!} \exp \left[-n e^{-\frac{k}{n}}\right]\left(1+\mathcal{O}\left(\frac{\log n}{n^{1-\epsilon}}\right)\right), \quad \forall \epsilon>0
$$

Proof. Lemma 5.5 follows directly from the result by Menon [9] which is stated here as Lemma 5.6 below. In fact, we have

$$
\left\{\begin{array}{l}
k \\
n
\end{array}\right\}=\frac{n^{k}}{n!} \exp \left[-n e^{-\frac{k}{n}} \cdot e^{\frac{D}{n}}\right](1+R), \quad D:=\frac{1}{2}\left(1-\frac{1}{6 n}\right)
$$

$R$ is defined in the statement of Lemma 5.6.
(1) Under the assumption (16), one has $R=o(1)$. Then it suffices to use the inequality $e^{D / n} \geq 1+D / n$ and noting that $D \leq 1 / 2$.
(2) If $k=n \log n-c n, e^{\frac{1}{2} e^{-k / n}}=e^{\frac{e^{c n}}{2 n}}=1+\mathcal{O}\left(\frac{\log n}{n}\right)$ and the error term in Lemma 5.6 satisfies $R=\mathcal{O}\left(\frac{\log n}{n^{1-\epsilon}}\right)$ for any $\epsilon>0$. $\square$

Remark We use Lemma 5.5 several times in the proof of Theorem 1.2, so that we shall check the assumption (16) is valid in all cases.
(0) $k=n \log n-c n, c \ll \log n$ :

$$
\begin{aligned}
\frac{n}{\sqrt{k}} & =\frac{n}{\sqrt{n \log n\left(1-\frac{c}{\log n}\right)}}=\sqrt{\frac{n}{\log n\left(1-\frac{c}{\log n}\right)}} \rightarrow \infty \\
\frac{k}{n}-\log \sqrt{k} & =(\log n-c)-\frac{1}{2} \log \left[n \log n\left(1-\frac{c}{\log n}\right)\right] \\
& =(\log n-c)-\frac{1}{2} \log n-\frac{1}{2} \log \log n-\frac{1}{2} \log \left(1-\frac{c}{\log n}\right) \rightarrow \infty
\end{aligned}
$$

(1) $n=r_{k}=\frac{k}{\log k}(1+o(1)):$

$$
\begin{aligned}
\frac{r_{k}}{\sqrt{k}} & =\frac{\frac{k}{\log k}(1+o(1))}{\sqrt{k}}=\frac{\sqrt{k}}{\log k}(1+o(1)) \rightarrow \infty \\
\frac{k}{r_{k}}-\log \sqrt{k} & =\frac{k}{\frac{k}{\log k}(1+o(1))}-\frac{1}{2} \log k=\frac{1}{2} \log k(1+o(1)) \rightarrow \infty
\end{aligned}
$$

(2) $n$ is replaced by $n-\log n=n(1+o(1))$ :

$$
\begin{aligned}
\frac{n}{\sqrt{k}} & =\frac{n(1+o(1))}{\sqrt{n \log n}(1+o(1))}=\sqrt{\frac{n}{\log n}}(1+o(1)) \rightarrow \infty \\
\frac{k}{n}-\log \sqrt{k} & =\frac{n \log n(1+o(1))}{n}-\frac{1}{2} \log n+o(1)=\frac{1}{2} \log n+o(1) \rightarrow \infty .
\end{aligned}
$$

Lemma 5.6 ([9], Theorem 2.2) Suppose

$$
\frac{n}{\sqrt{k}} \rightarrow \infty, \quad \frac{k}{n}-\log \sqrt{k} \rightarrow \infty
$$

Then

$$
\begin{aligned}
\left\{\begin{array}{l}
k \\
k
\end{array}\right\}= & \frac{n^{k}}{n!} \exp \left[-e^{\lambda}\right](1+R) \\
\text { where } \lambda:= & \log n-\frac{k}{n}+\frac{1}{2 n}-\frac{1}{12 n^{2}} . \\
R:= & 1+\frac{1}{n} e^{-k / n}\left(\frac{k+n}{2}-\frac{k^{2}}{8 n^{2}}+\frac{k}{3 n}+\frac{1}{24}\right) \\
& -e^{-2 k / n}\left(\frac{k+n}{2}-\frac{7 k^{2}}{8 n^{2}}-\frac{k}{4 n}-\frac{1}{8}\right)-n e^{-3 k / n}\left(\frac{3 k^{2}}{4 n^{2}}+\frac{7 k}{6 n}+\frac{7}{12}\right) \\
& +e^{-4 k / n} \frac{(k+n)^{2}}{8}+R_{1} \\
R_{1}:= & \mathcal{O}\left(\frac{k^{2}}{n^{4}} e^{-k / n}+\frac{k^{2}}{n^{3}} e^{-2 k / n}+\frac{k^{2}}{n^{2}} e^{-3 k / n}+\frac{k^{2}}{n} e^{-4 k / n}+k^{3} e^{-4 k / n}\right) .
\end{aligned}
$$

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