An Improved Algorithm for Finding Maximum Outerplanar Subgraphs

Gruia Calinescu^{*}

Hemanshu Kaul[†]

Bahareh Kudarzi[‡]

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Abstract

We study the NP-complete MAXIMUM OUTERPLANAR SUBGRAPH problem. The previous best known approximation ratio for this problem is 2/3. We propose a new approximation algorithm which improves the ratio to 7/10.

Keywords. Outerplanar graph, Maximum subgraph, Approximation algorithm, Matroid parity.

1 Introduction

A graph is *planar* if it can be drawn in the 2-dimensional plane such that no two edges meet in a point other than a common end. A planar graph is *outerplanar* if it can be drawn in a way in which every vertex lies on the boundary of the same connected region of the plane. Let G be a simple graph, and G' be a subgraph of G. We say that G' is a maximum *outerplanar subgraph* of G if it is outerplanar and there is no outerplanar subgraph G'' such that |E(G'')| > |E(G')|. Given a graph G, finding an outerplanar subgraph of G with the maximum number of edges is called the MAXIMUM OUTERPLANAR SUBGRAPH problem.

While outerplanar graphs have been investigated in-depth for their applications [18] and theoretical properties [26, 13, 25, 16, 38, 19, 7, 29], the problem of finding large outerplanar subgraphs of a graph has not been studied as much.

Most problems which are NP-hard on arbitrary graphs become polynomial on outerplanar graphs [15, 1]. An in-depth exploration of practical algorithms for MAXIMUM OUTERPLANAR SUBGRAPH was done by Poranen [33]. His main experimental result is that simulated annealing with initial solution taken from the greedy triangular cactus approximation algorithm (which we discuss below) yields the best known heuristic for the MAXIMUM OUTERPLANAR SUBGRAPH problem.

^{*}Department of Computer Science, Illinois Institute of Technology, Chicago, IL 60616. E-mail: calinescu@iit.edu

[†]Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616. E-mail: kaul@iit.edu

[‡]Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616. E-mail: bkudarzi@hawk.iit.edu

Maximum Outerplanar Subgraph problem is NP-hard, [42, 15]. Hence, instead of solving the problem exactly, we use polynomial-time approximation algorithms. An algorithm's approximation ratio is the worst-case ratio between the result obtained by the algorithm and the optimal solution. For maximization problems, the approximation ratio is smaller than 1, and the closer we are to 1, the better.

Previous Work. By constructing a spanning tree of a given graph (recall, a tree is an outerplanar graph), one obtains an approximation ratio for the MAXIMUM OUTERPLANAR SUBGRAPH problem of 1/2 [9]. The approximation ratio was improved by using the concept of *triangular cactus*, discussed in Section 2 and depicted in Figure 1. The greedy triangular cactus approximation algorithm from [3] results in a non-trivial approximation ratio of 7/12. Furthermore, [3] used the fact that computing a triangular cactus with maximum number of triangles can be done in polynomial time based on MATROID PARITY [23, 14, 39, 8, 30] to obtain a 2/3 approximation ratio.

Utilizing an outerplanarity testing algorithm [2, 37, 27, 36, 41], a greedy technique to approximate the maximum outerplanar subgraph involves adding edges to an outerplanar subgraph as long as the subgraph stays outerplanar. Cimikowski and Coppersmith [9] showed that after constructing the spanning tree, using this greedy technique does not improve the approximation ratio of 1/2. Poranen [33] claimed the same for the 7/12 ratio obtained by the greedy triangular cactus. There are examples that this holds for the 2/3 approximation as well. All these bounds assume that the greedy technique adds the worst possible edge if it has a choice. In practice, this greedy technique is effective [32, 33, 34].

Related Work. The MAXIMUM INDUCED OUTERPLANAR SUBGRAPH problem is the task of finding the size of the largest subset of vertices in a graph that induces an outerplanar subgraph. This problem is known to be NP-hard [42] and does not admit good approximations [24, 12]. Morgan and Farr [28] presented an efficient algorithm that finds an induced outerplanar subgraph with at least 3n/(d + 5/3) vertices for graphs of n vertices with maximum degree at most d. Donkers et al. [10] study the OUTERPLANAR DELETION problem, in which one wants to remove at most k vertices from a graph to make it outerplanar, and showed that this problem is fixed-parameter tractable.

Another related problem is MAXIMUM PLANAR SUBGRAPH, in which one wants to find a planar subgraph of the input graph with the maximum number of edges. This problem is also known to be NP-hard [22], and it has many applications (see e.g. [21, 35, 33, 34]). For this problem, constructing a spanning tree of a given graph gives an approximation ratio of 1/3, the greedy triangular cactus approximation algorithm has ratio 7/18, and the best known approximation is 4/9 and is obtained by computing a triangular cactus with maximum number of triangles [3].

MAXIMUM WEIGHT OUTERPLANAR SUBGRAPH, in which one wants to find an outerplanar subgraph with the maximum weight of the input edge-weighted graph, admits an approximation ratio of 1/2 by considering the maximum weight spanning tree. This has been improved by [5] to 7/12, and by combining Theorem 29 of [5] with the recent algorithm for WEIGHTED MATROID PARITY [17], one immediately obtains a (2/3)-approximation algorithm. A (2/3)-approximation was also claimed in the Master thesis of Osipov [31].

In this paper, we present,

Theorem 1. There exists a polynomial-time (7/10)-approximation algorithm for the MAXI-MUM OUTERPLANAR SUBGRAPH problem. Our algorithm has a greedy phase of adding appropriate induced 4-cycles after the main phase of the (2/3)-approximation of [3]. While the algorithm is very simple (except for the MATROID PARITY part already used by the previous best work), the tight analysis we provide is nice and elementary, and may have wider applications.

Applying a greedy method after matching methods (MATROID PARITY is an extension of graph matching) is new to us; applying matching methods after greedy methods was done before in [20], while [11] combines local improvement with matching.

In Section 2 we introduce further definitions and notation. Our algorithm and the proof of Theorem 1 appears in Section 3. In Section 4 we discuss some limitations of our method and conclude with some open questions.

2 Preliminaries

In this paper all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [40] for terminology and notation.

Given a graph G = (V, E), $V' \subseteq V$, and $E' \subseteq E$, we denote by G[V'], the *induced* subgraph of G with vertex set given by V', and we denote by G[E'] = (V, E'), the *spanning* subgraph of G with edge set given by E'. A *triangle* in a graph is C_3 , a cycle of length three and a *square* in a graph is an induced C_4 , a cycle of length four.

A plane graph is defined as an embedding of an planar graph with a mapping from every vertex to a point in the 2-dimensional plane, and from every edge to a curve on the plane, such that the extreme points of each curve are the points mapped from its end vertices, and all curves are disjoint except on their extreme points. The plane graph divides the plane into a set of connected regions, called faces. Each face is bounded by a closed walk called the boundary of the face. The *outer face* is the unbounded region outside the whole embedding. An *outerplane graph* is a plane graph where every vertex is mapped to a point on the boundary of the outer face. Any other face is called an *inner face* of the outerplane graph.

An *outeredge* in an outerplane graph is an edge which is in the boundary of the outer face and an *inneredge* in an outerplane graph is an edge which is not an outeredge. The boundary of the outer face of a biconnected outerplane graph is a cycle. An *inner triangle* is a triangle that is the boundary of an inner face. An *inner square* is a square that is the boundary of an inner face.

Let H be a biconnected outerplane graph and uv be an edge of H and $w \in V(H) \setminus \{u, v\}$. Consider the path P from u to v using only outeredges of H that does not contain w and let $H_1 = H[V(P)]$. We say that the subgraph H_1 of H is *split from* w by uv. As an example, in Figure 2, H_1 is split from y by xz. Note that if uv is an outeredge, then H_1 will be the graph $(\{u, v\}, \{uv\})$.

A triangular cactus is a graph whose cycles (if any) are triangles and such that all edges appear in some cycle. A triangular cactus in a graph G is a subgraph of G which is a triangular cactus. A triangular cactus in a graph G is maximum if it has maximum number of triangles. A square-triangular cactus is a graph whose cycles (if any) are triangles or squares and such that all edges appear in some cycle. A square-triangular cactus in a graph G is a subgraph of G which is a square-triangular cactus. A square-triangular structure is a graph whose cycles (if any) are triangles or squares. Note that every square-triangular cactus is a square-triangular structure, but not vice versa. See Figure 1.



Figure 1: Figure (a) is a triangular cactus. Figure (b) is a square-triangular cactus. Figure (c) is a square-triangular structure.

3 The Approximation Algorithm

In this section we describe our algorithm, which we call STS for square-triangular structure, and give its approximation ratio analysis.

Input to the algorithm below is a graph G.

Algorithm STS

The algorithm has three phases, as numbered below:

- 1. Find M_0 a maximum triangular cactus in G.
- 2. Starting with $E_1 = E(M_0)$, repeatedly (as long as possible) find a square S whose vertices are in different components of $G[E_1]$, and add the edges of S to E_1 . Let $M_1 := G[E_1]$.
- 3. Starting with $E_2 = E(M_1)$, repeatedly (as long as possible) find an edge e in G whose endpoints are in different components of $G[E_2]$, and add e to E_2 .

Output $G[E_2]$.

Algorithm STS produces a square-triangular structure in the given graph G. This is indeed an outerplanar graph.

The (2/3)-approximation algorithm of [3] has only two phases corresponding to our Phase 1 and Phase 3.

Phase 1 of the algorithm can be implemented to run in polynomial time as explained in [3]. This is done by an immediate reduction to the Graphic Matroid Parity problem, for which [23, 14, 39, 8, 30] provide polynomial-time algorithms.

Phase 2 can be implemented in time $O(n^4)$ as we try all possible subsets of four vertices to see if they form a square that can be added. Within the same time bounds, we also maintain the connected components of $G[E_1]$ - these components need to be updated only O(n) times. To complete the proof of Theorem 1, it only remains to show the approximation ratio of 7/10.

3.1 Approximation Ratio Analysis

To establish the approximation ratio, we first prove a lemma about the structure of biconnected outerplanar graphs. This is a generalization of Lemma 3.1 of [3].

Lemma 2. Let *H* be a biconnected outerplane graph. Suppose that *H* has *t* inner triangles. Then, the following holds:

- 1. If t is even, then for every outeredge xy in H, there is a triangular cactus C in H with at least $\frac{t}{2}$ triangles such that x and y are in different components of C.
- 2. If t is odd, then there is a triangular cactus C in H with at least $\lceil \frac{t}{2} \rceil$ triangles, and for every outeredge xy in H, there is a triangular cactus C in H with at least $\lfloor \frac{t}{2} \rfloor$ triangles such that x and y are in different components of C.

Proof. This proof uses ideas from the proof of Lemma 3.1 of [3], with many extra cases. We will prove this by induction on n + t, where n is the number of vertices of H and t is the number of inner triangles of H. Note that n is at least 3.

Fact 3. Before we prove the base cases, we note that if a graph H' consists of only one edge xy and therefore it has t = 0 inner triangles, then there is triangular cactus in H' which has $r = 0 \ge \frac{t}{2}$ triangles and the vertices x and y are in different components of the triangular cactus. The same argument applies when the graph H is biconnected with $n \ge 3$ but has zero inner triangles.

The case n + t = 3 cannot happen since if n = 3 then the biconnected H has an inner triangle.

Now let n + t = 4. By Fact 3 we can assume that H has one inner triangle uvw. Now $C = H[\{u, v, w\}]$ is a triangular cactus with r = 1 triangle and we have $r \ge \lfloor \frac{t}{2} \rfloor = 1$. For the outeredge xy, subgraph $C = \emptyset$ is a triangular cactus in H with r = 0 such that x and y are not in the same component of C. Then we have $r \ge \lfloor \frac{t}{2} \rfloor = 0$. Now assume that the statements are true for any n' and t' such that n' + t' < n + t. We will prove the statements for n + t.

In the first part of the proof, we assume that t is even.

Let f be the inner face such that its boundary contains the edge xy. Note that f is not all of H because in that case H must have less than two inner triangles. There are two cases: the boundary of f is a triangle or not.

Case 1: Suppose that the boundary of f is a triangle consisting of vertices x, y, z.

Let H_1 be the subgraph of H that is split from vertex y by edge xz and H_2 be the subgraph that is split from vertex x by edge yz. See Figure 2. $H_1 \cup H_2$ has t - 1 inner triangles. Since t is even, WLOG, let H_1 have an even number of inner triangles t_1 and H_2 have an odd number of inner triangles t_2 . By applying the induction hypothesis or Fact 3 to H_1 with outeredge xz, there exists a triangular cactus C_1 with $r_1 \geq \frac{t_1}{2}$ triangles such that vertices x and z are in different components of C_1 . By applying the induction hypothesis to H_2 , there exists a triangular cactus C_2 with $r_2 \geq \lfloor \frac{t_2}{2} \rfloor$ triangles. Since x and z are in different



Figure 2: Case 1, f is triangular face xyz

components of C_1 , we can conclude that $C_1 \cup C_2$ is a triangular cactus in which vertices x and y are in different components and has r triangles where $r = r_1 + r_2 \ge \frac{t_1}{2} + \lceil \frac{t_2}{2} \rceil = \frac{t}{2}$.



Figure 3: An illustration of Case 2, where straight segments are inneredges of H.

Case 2: Suppose that the boundary of f is not a triangle. If t = 0, using Fact 3 we are done. From now on in this case we assume that t > 0. Consider all of the inneredges of fand name them as $e_1, e_2, ..., e_k$. Note that since t > 0 so k > 0. For each i = 1, ..., k, assume that $e_i = v_{i_1}v_{i_2}$ and let vertex $v_i \in V(f) \setminus \{v_{i_1}, v_{i_2}\}$. Then, let H_i be the subgraph which is split from vertex v'_i by edge e_i . See Figure 3 for an example. Assume that H_i has t_i inner triangles.

For graph H_i , we have $t_i + n_i < t + n$ for each $i \in [k]$.

If all t_i are even, by applying induction hypothesis to each H_i , we get that there exists a triangular cactus C_i in which v_{i_1} and v_{i_2} are in different components of C_i with $r_i \geq \frac{t_i}{2}$ triangles. Now consider the subgraph $C = \bigcup_{i=1}^k C_i$ which is a triangular cactus, because the vertices v_{i_1} and v_{i_2} of the edge e_i are in different components of C_i for each i, and therefore, in different components of C. Hence, x and y cannot be in the same component of C.

If some t_i is odd, then there must be another t_l which is odd. By applying induction on



Figure 4: $t \ge 1$ is odd, H has at least one inner triangle, uvw.

 H_j , we get that there is a triangular cactus C_j with $r_j \ge \lfloor \frac{t_j}{2} \rfloor$ triangles. Then by applying induction on H_l with outeredge (for H_l) e_l , we get that, there is a triangular cactus C_l which has $r_l \ge \lfloor \frac{t_l}{2} \rfloor$ triangles such that v_{l_1} and v_{l_2} are not in the same component of C_l . For $i \in [k] \setminus \{j, l\}$, by applying induction to H_i there is a triangular cactus C_i with at least $\frac{t_i}{2}$ triangles. Now consider subgraph $C = \bigcup_{i=1}^k C_i$ which is a triangular cactus because its cycles can only be triangles and C has $r = \sum_{i=1}^k r_i$ triangles and $r \ge \sum_{i \in [k] \setminus \{j, l\}} \frac{t_i}{2} + \lfloor \frac{t_j}{2} \rfloor + \lfloor \frac{t_j}{2} \rfloor = \frac{t}{2}$. Note that x and y are not in the same component of C since v_{l_1} and v_{l_2} are not in the same component of C_l .

This completes the first part of the proof when t is even.

For the second part of the proof, we consider that t is odd. We will first prove that there exists a triangular cactus C with $r \ge \lfloor \frac{t}{2} \rfloor$ triangles.

Since $t \ge 1$ is odd, then H has at least one inner triangle, uvw. Let H_1 be the subgraph of H that is split from vertex v by edge uw, H_2 be the subgraph of H that is split from vertex u by edge vw, and H_3 be the subgraph of H that is split from vertex w by edge uv. See Figure 4. These three subgraphs together have exactly t - 1 inner triangles of H. Let H_1 , H_2 , and H_3 have t_1 , t_2 and t_3 inner triangles, respectively. Because t - 1 is even, there are two possibilities: either, each of these three subgraph has even number of inner triangles, or exactly two subgraphs have odd number of inner triangles.

For the first possibility, we apply the induction hypothesis or Fact 3 to H_1 with outeredge uw, H_2 with outeredge vw, and H_3 with outeredge uv. For H_1 , there is a triangular cactus C_1 with $r_1 \geq \frac{t_1}{2}$ triangles such that u and w are in different components of C_1 . Similarly for H_2 and H_3 , there is a triangular cactus C_2 with $r_2 \geq \frac{t_2}{2}$ triangles such that v and w are in different components of C_2 and there is a triangular cactus C_3 with $r_3 \geq \frac{t_3}{2}$ triangles such that u and v are in different components of C_2 and there is a triangular cactus C_3 with $r_3 \geq \frac{t_3}{2}$ triangles such that u and v are in different components of C_3 , respectively. We claim that $C = \bigcup_{i=1}^3 C_i \cup H[\{u, v, w\}]$ is a triangular cactus. Indeed, since u, v, and w are in different components of each cactus (or they are not in the cacti at all) by putting them together u, v, w are still in different components allowing us to add the triangle uvw and maintain the property of being a triangular cactus. Obviously C has $r = r_1 + r_2 + r_3 + 1$ triangles and $t_1 + t_2 + t_3 = t - 1$ which is even, so $r \geq \frac{t_1}{2} + \frac{t_2}{2} + \frac{t_3}{2} + 1 = \lceil \frac{t}{2} \rceil$.

For the second possibility, WLOG, assume that t_1 is even while t_2 and t_3 are odd. We apply the induction hypothesis or Fact 3 to H_1 with outeredge uw which tells there is a triangular cactus C_1 with $r_1 \ge \frac{t_1}{2}$ triangles such that u and w are in different components of C_1 . Applying the induction hypothesis to H_2 and H_3 , there is a cactus in H_i with $r_i \ge \lfloor \frac{t_i}{2} \rfloor$ triangles, for i = 1, 2. Because u and w are in two different components of C_1 , the subgraph $C = \bigcup_{i=1}^3 C_i$ is a triangular cactus with $r = r_1 + r_2 + r_3$ triangles and $r \ge \frac{t_1}{2} + \lceil \frac{t_2}{2} \rceil + \lceil \frac{t_3}{2} \rceil = \lceil \frac{t}{2} \rceil$.

Finally, we consider the situation t is odd and xy be an outeredge. We have proved that there is a triangular cactus C with $r \ge \lfloor \frac{t}{2} \rfloor$ triangles. If x and y are in different components of C we are done. If x and y are in the same component of C we can remove one of the triangles in that component such that x and y are not in the same component any more. Let C' be the subgraph after removing the triangle. C' is a triangular cactus with at least $\lfloor \frac{t}{2} \rfloor - 1 = \lfloor \frac{t}{2} \rfloor$ triangles. This concludes the proof of Lemma 2.

We need the following consequence of the lemma.

Corollary 4. Let H be an outerplane graph with t inner triangles. Then there exists a triangular cactus in H with $\lfloor t/2 \rfloor$ triangles.

Proof. Assume that H has k components, $H_1, H_2, ..., H_k$. Suppose that H_i has t_i inner triangles and l_i blocks, $B_{i1}, ..., B_{il_i}$. Using Lemma 2, every B_{ij} with t_{ij} inner triangles has a triangular cactus C_{ij} as subgraph with at least $\lceil \frac{t_{ij}}{2} \rceil$ triangles. The subgraph $C_i = \bigcup_{j=1}^{l_i} C_{ij}$ is a triangular cactus in H_i since the union does not add any new cycle. C_i has at least $\sum_{j=1}^{l_i} \lceil \frac{t_{ij}}{2} \rceil \ge \lceil \frac{t_i}{2} \rceil$ triangles, since $t_i = \sum_{j=1}^{l_i} t_{ij}$. Now consider subgraph $C = \bigcup_{i=1}^k C_i$, which is a triangular cactus in H and it has at least $\sum_{i=1}^k \lceil \frac{t_i}{2} \rceil \ge \lceil t/2 \rceil$ triangles as we have $t = \sum_{i=1}^k t_i$.

We are now ready to establish the approximation ratio.

Theorem 5. The approximation ratio of Algorithm STS is $\frac{7}{10}$.

Proof. First we show that the approximation ratio is at most 7/10. Take a triangulated outerplanar graph with vertex set Q and q vertices such that q is odd. For each outeredge uv of Q, add two new vertices a_{uv} and b_{uv} and the three edges ua_{uv} , $a_{uv}b_{uv}$, and $b_{uv}v$ in the outer face of Q such that it remains outerplanar. Name this new graph H, it has 3q vertices, and (2q-3) + 3q = 5q - 3 edges. Figure 5 presents a drawing of H with q = 7. Consider H as the input to the algorithm. Note that H is the optimum solution. Phase 1 of the algorithm finds a triangular cactus M_0 with $\lfloor q/2 \rfloor$ triangles connecting all the vertices of Q. One can easily check that no square can be added to M_0 in Phase 2 of the algorithm. So the algorithm's output has $3q - 1 + \lfloor q/2 \rfloor$ edges, as every triangle of M_0 adds one more edge compared to a spanning tree. By letting $q \to \infty$ the ratio of number of edges in output divided by the number of edges in the optimum converges to $\frac{7}{10}$.

Let OPT denote a maximum outerplanar subgraph of the input graph G and let opt denote the number of edges of OPT (opt is the value of the objective). We assume that the input graph is connected (or else we can run the algorithm on every connected component) and therefore OPT is also connected. Also, we fix an embedding of OPT, so that it becomes an outerplane graph. Let t and s denote respectively the number of inner triangles and inner squares in OPT. Let r and c be respectively the number of triangles and squares in the algorithm's output.

From Corollary 4 and the fact that Phase I finds a cactus with maximum number of triangles in the input graph:

$$t \le 2r. \tag{1}$$



Figure 5: An example of the graph H used to upper bound the approximation ratio of the algorithm.

We need the following claim.

Claim 6.

$$4t + s \le 10r + 6c \tag{2}$$

Proof. Let M_1 (from the pseudocode of the algorithm) have q non-trivial connected components A_1, A_2, \ldots, A_q . Let r_i, c_i be the number of triangles and squares in A_i , respectively. This implies that the number of vertices in the component A_i is exactly $3c_i + 2r_i + 1$. We have $r = \sum_{j=1}^{q} r_j$ and $c = \sum_{j=1}^{q} c_j$.

Let $B_i = OPT[V(A_i)]$ and let B_i^j , $j = 1, 2, ..., j_i$ be the connected components of B_i , and let $b_i^j = |V(B_i^j)|$. If $b_i^j > 1$, let $B_i^j(l)$, for $l = 1, 2, ..., l_i^j$, be the blocks of B_i^j , and let $b_i^j(l) = |V(B_i^j(l))|$. Note that $b_i^j(l) \ge 2$. We have

$$\sum_{l=1}^{l_i^j} b_i^j(l) = b_i^j + (l_i^j - 1).$$
(3)

Notice that all the inner faces of all the B_i are faces of some block $B_i^j(l)$. Consider an inner square S of OPT. At the end of Phase 2 of the algorithm, S will satisfy exactly one of the following conditions.

- 1. S is the boundary of an inner face of some $B_i^j(l)$ (we say that S is of Type I)
- 2. S is not of Type I and has one edge that is also an edge of some $B_i^j(l)$ (we say that S is of Type II)
- 3. S is not of Type I or II and is such that two non-consecutive vertices belong to the same B_i (we say that S is of Type III).

If none of these three conditions holds, then S can be added to E_1 in Phase 2 of the algorithm, and becomes a inner square of Type I. See Figure 6. Let s'_1 , s'_2 , and s'_3 be the number of inner squares of OPT of types I, II, and III respectively. Then $s = s'_1 + s'_2 + s'_3$.

No triangle T can be added to E_1 while keeping a square-triangular cactus, since otherwise M_0 would not be a triangular cactus with maximum number of triangles. Using this, we classify the inner triangles of OPT into Type I and Type II,

- 1. T is the boundary of an inner face of some $B_i^j(l)$ (we say that T is of Type I)
- 2. T is not of Type I and has at least one edge that is also an edge of some $B_i^j(l)$ (we say that T is of Type II).



Figure 6: An example of the different type of inner squares. Filled black circles are vertices in B_1 and empty circles are vertices not in B_1 . Here we have two connected components of B_1 : B_1^1 and B_1^2 . Furthermore, B_1^1 has three blocks: $B_1^1(1)$ with 8 vertices, $B_1^1(2)$ with only two vertices, and $B_1^1(3)$, with three vertices. Similarly, B_1^2 has two blocks: $B_1^2(1)$ and $B_1^2(2)$.

Let t'_1 and t'_2 be the number of inner triangles of OPT of types I and II respectively. Then $t = t'_1 + t'_2$.

Block $B_i^j(l)$ has at most $b_i^j(l) - 2$ inner faces. We obtain:

$$t_1' + s_1' \le \sum_{i,j,l} (b_i^j(l) - 2).$$
(4)

We associate with each inner triangle (inner square, respectively) of OPT of Type II an edge of some $B_i^j(l)$, precisely the edge that is both in $B_i^j(l)$ and in the inner triangle (inner square, respectively).

If $b_i^j(l) > 2$, there are exactly $b_i^j(l)$ edges on the outer face of $B_i^j(l)$. Any such edge e can be associated with at most one inner triangle or inner square of Type II, as we argue below. For illustration look at edge e in Figure 6. Every edge of OPT participates in at most two inner faces of OPT. On one side of e lies an inner face of $B_i^j(l)$ and this is also an inner face of OPT, based on the following fact.

Fact 7. Let H be a outerplane graph. Let H' be an induced subgraph of H. Then, an inner face of H' is also an inner face of H.

Thus one of the at most two inner triangles or inner squares of OPT that could be associated with e is in fact an inner triangle or an inner square of Type I, and not of Type II.

Inneredges of $B_i^j(l)$ are not associated with any inner triangles or inner squares of Type II, since the boundary of two faces of OPT on both sides of such an inneredge of $B_i^j(l)$ are of Type I, based on the reasoning above.

If $b_i^j(l) = 2$, there exists exactly one edge in $B_i^j(l)$ and this edge can be associated with at most two inner triangles or inner squares of Type II, since every edge of *OPT* participates in at most two inner faces of *OPT*. As an illustration, please look at $B_1^2(2)$ in Figure 6.

Based on these arguments, we conclude:

$$t_2' + s_2' \le \sum_{i,j,l} b_i^j(l).$$
(5)

We continue by counting the inner squares of Type III of OPT. Let S be such a square with vertices u, v, w, x in clockwise order. We must have that some B_i contains two nonconsecutive of these four vertices, and does not contain the other two, or else S would be of Type I or Type II. WLOG, let $u, w \in V(B_i)$ and $x, v \notin V(B_i)$. Note that u and w are not adjacent in OPT, since there is no edge inside this square, and an edge outside this square will make either x or v not on the outer face of OPT, contradicting outerplanarity. By the same reasoning, OPT does not have a path from u to w that does not pass through either xor v. So u and w are in different components of B_i . We associate with S the pair of vertices u, w (recall that uw is not an edge in OPT). We claim that the total number of inner squares of Type III of OPT whose corresponding pair of vertices are in different components of B_i is at most $j_i - 1$ (recall that j_i is the number of components of B_i), as we argue in the next paragraph.

Let F_i denote the set of all edges uw for each pair of vertices u, w of B_i corresponding to some square of Type III. *OPT* with F_i added is still outerplanar, and cannot contain a path from u to w (except for the edge uw) that does not go through the corresponding v or x. Since v and x are not added to B_i , if we add F_i to B_i , then this new graph $(V(B_i), E(B_i) \cup F_i)$ does not contain any cycle which is not in B_i . We conclude that $|F_i| \leq j_i - 1$.

Thus, it follows that

$$s'_3 \le \sum_{i=1}^q (j_i - 1).$$
 (6)

By adding up Equation (5) and Equation (4) we get that

$$t_1' + s_1' + t_2' + s_2' \le \sum_{i=1}^q \sum_{j \in [j_i], b_i^j > 1} \sum_{l=1}^{l_i^j} (2b_i^j(l) - 2).$$

$$\tag{7}$$

Using Equation (3), we have that $\sum_{l=1}^{l_i^j} (2b_i^j(l) - 2) = 2b_i^j - 2$. Therefore, Inequality (7) simplifies to

$$t_1' + s_1' + t_2' + s_2' \le \sum_{i=1}^q \sum_{j \in [j_i], b_i^j > 1} (2b_i^j - 2),$$

Since for $b_i^j = 1$ we have $2b_i^j - 2 = 0$ we can add them to the sum without changing the amount. Hence,

$$t'_1 + s'_1 + t'_2 + s'_2 \le \sum_{i=1}^q \sum_{j \in [j_i]} (2b_i^j - 2).$$

Note that since B_i^j are components of B_i and the set of vertices of B_i is the same as the set of vertices of A_i , we will have $\sum_{j \in [j_i]} b_i^j = |V(B_i)| = |V(A_i)|$ which, recall, is equal to $3c_i + 2r_i + 1$. Hence,

$$t'_1 + s'_1 + t'_2 + s'_2 \le \sum_{i=1}^q (2(3c_i + 2r_i + 1) - 2j_i).$$

Now by adding Equation (6) to equation above we get that

$$t'_1 + s'_1 + t'_2 + s'_2 + s'_3 \le \sum_{i=1}^q (2(3c_i + 2r_i + 1) - 2j_i + j_i - 1).$$

Recall that $t = t'_1 + t'_2$ and $s = s'_1 + s'_2 + s'_3$, hence,

$$t + s \le \sum_{i} (2(3c_i + 2r_i) - j_i + 1) \le \sum_{i} (4r_i + 6c_i) = 4r + 6c.$$

Multiplying Equation (1) by 3 and adding it to equation above, we obtain Equation (2):

$$4t + s \le 10r + 6c.$$

This concludes the proof of Claim 6

We continue with the proof of Theorem 5.

Let *OPT* have blocks H_1, H_2, \ldots, H_k , and let $n_j = |V(H_j)|$ and $m_j = |E(H_j)|$. Recall that *OPT* is connected. Then $n := |V| = |V(G)| = |V(OPT)| = \left(\sum_{j=1}^k n_j\right) - (k-1)$. We also have that $opt = |E(OPT)| = \sum_{j=1}^k m_j$. Let f_i^j be the number of internal faces of length i in H_j , and t_j and s_j be respectively f_3^j and f_4^j . We have that $n_j = 2 + t_j + 2s_j + \sum_{i \ge 5} (i-2)f_i^j$, and $m_j = 1 + 2t_j + 3s_j + \sum_{i \ge 5} (i-1)f_i^j$.

Note that Phase 3 makes the output subgraph connected. The algorithm outputs a solution with (n-1) + r + c edges, as every triangle or square increases the cyclomatic number¹ of the output by 1. We have

$$n - 1 + r + c = r + c - k + \sum_{j=1}^{k} n_j$$

= $r + c - k + \sum_{j=1}^{k} \left(2 + t_j + 2s_j + \sum_{i \ge 5} (i - 2)f_i^j \right)$
= $r + c + k + t + 2s + \sum_{j=1}^{k} \sum_{i \ge 5} (i - 2)f_i^j$,

and

$$opt = \sum_{j=1}^{k} \left(1 + 2t_j + 3s_j + \sum_{i \ge 5} (i-1)f_i^j \right) = k + 2t + 3s + \sum_{j=1}^{k} \sum_{i \ge 5} (i-1)f_i^j.$$

For $i \ge 5$, we get that $10(i-2) \ge 7(i-1)$. By Inequality (2), we have $10r + 10c \ge 4t + s$ and we also have $10k \ge 7k$. Hence, one can easily check that

$$10\left(r+c+k+t+2s+\sum_{j=1}^{k}\sum_{i\geq 5}(i-2)f_{i}^{j}\right)\geq 7\left(k+2t+3s+\sum_{j=1}^{k}\sum_{i\geq 5}(i-1)f_{i}^{j}\right).$$

Therefore we showed that $\frac{n-1+r+c}{opt} \geq \frac{7}{10}$. This finishes the proof of Theorem 5.

4 Concluding Remarks

In this paper we presented an algorithm which improved the ratio of MAXIMUM OUTER-PLANAR SUBGRAPH to 7/10. It is natural to ask if this algorithm can be improved. Some obvious modifications to the algorithm do not help.

Observe that adding pentagons or larger outerplanar graphs after Phase 2 of Algorithm STS will not make the ratio better than 7/10. The example used in the first part of the proof of Theorem 5 illustrates this. See Figure 5. There are no pentagons or any other structure

¹the cyclomatic number (also called circuit rank, cycle rank, or nullity) of an undirected graph is the minimum number of edges that must be removed from the graph to break all its cycles, making it into a tree or forest.



Figure 7: The vertices of H_1 are solid, and the newly added vertices of H_2 are the empty small circles.

in the optimum that have their vertices in different components of the graph M_1 from the algorithm.

We believe that the greedy technique of adding edges to an outerplanar subgraph as long as the subgraph stays outerplanar does not improve the ratio provided that the worst possible edge is chosen.

An outerplanar k-restricted structure is a simple graph whose blocks are outerplanar and each has at most k vertices. In this paper, outerplanar 4-restricted structures are used by our approximation algorithm. The outerplanar k-restricted ratio is the infimum, over simple outerplanar graphs H, of the ratio of the number of edges in a maximum k-restricted structure subgraph of H to the number edges of H. It is proved in [4] that, as k tends to infinity, the outerplanar k-restricted ratio tends to 1. This could be useful in improving the approximation ratio for MAXIMUM OUTERPLANAR SUBGRAPH, although we do not know if there is a polynomial time algorithm for finding an outerplanar 4-restricted structure with maximum number of edges that is a subgraph of the input graph.

A diamond is a cycle of size four with exactly one chord and is an outerplanar graph. Note that a diamond is not a square-triangular structure. A diamond has five edges and four vertices, therefore, it is a better structure than a square since it has more edges and also a better structure than two triangles since it has fewer vertices. Since K_4 is not outerplanar, the only blocks that an outerplanar 4-restricted structure can have are bridges, triangles, squares and diamonds.

One way to use diamonds is to add them greedily (as long as the four vertices are in different connected components), followed by triangles as in [6]. We could also add squares and pentagons after this. And after this, one employs Phase 3 of Algorithm STS. This approach does not lead to an approximation ratio better than 2/3, as the following series of examples shows. The optimum solutions H_k are taken from Theorem 4 of [4]. and H_2 is illustrated in Figure 7. H_0 is a triangle. To obtain H_k from H_{k-1} duplicate and then

subdivide every outeredge. Thus H_k has $3 \cdot 2^k$ vertices and is triangulated with $2 \cdot 3 \cdot 2^k - 3$ edges. We also have a separate diamond cactus D_k with d diamonds that includes all the vertices of H_k that came from H_{k-1} , plus another one vertex. Here, $d = 2^{k-1}$, as one diamond structure with d diamonds has 1 + 3d vertices. Given an input that has both the edges of H_k and of D_k , the algorithm can select D_k by greedily adding diamonds, after which it goes directly to Phase 3 as the vertices of H_k that are not in D_k form an independent set and therefore cannot make any triangle, square, pentagon, or larger block that one can greedily add to D_k . The output of this greedy algorithm has $3 \cdot 2^k - 1 + 2d$ edges, as every diamond increases the cyclomatic number by 2. Recall that $d = 2^{k-1}$ and that the optimum has $2 \cdot 3 \cdot 2^k - 3$ edges. The ratio of the output to optimum converges to 2/3 as k goes to infinity.

The question of how large an approximation ratio for the MAXIMUM OUTERPLANAR SUBGRAPH problem can be achieved remains open. Is there a linear-time algorithm with approximation ratio $1/2 + \epsilon$? Is there an approximation algorithm for the MAXIMUM WEIGHT OUTERPLANAR SUBGRAPH problem with a ratio better than 2/3?

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