# On the distance-edge-monitoring numbers of graphs * 

Chengxu Yang $\stackrel{\dagger}{\text {, Ralf Klasing, }}{ }^{\ddagger}$ Yaping Mao, ${ }^{\S}$ Xingchao Deng $\mathbb{I}$


#### Abstract

Foucaud et al. [Discrete Appl. Math. 319 (2022), 424-438] recently introduced and initiated the study of a new graph-theoretic concept in the area of network monitoring. For a set $M$ of vertices and an edge $e$ of a graph $G$, let $P(M, e)$ be the set of pairs $(x, y)$ with a vertex $x$ of $M$ and a vertex $y$ of $V(G)$ such that $d_{G}(x, y) \neq d_{G-e}(x, y)$. For a vertex $x$, let $E M(x)$ be the set of edges $e$ such that there exists a vertex $v$ in $G$ with $(x, v) \in P(\{x\}, e)$. A set $M$ of vertices of a graph $G$ is distance-edge-monitoring set if every edge $e$ of $G$ is monitored by some vertex of $M$, that is, the set $P(M, e)$ is nonempty. The distance-edge-monitoring number of a graph $G$, denoted by $\operatorname{dem}(G)$, is defined as the smallest size of distance-edge-monitoring sets of $G$. The vertices of $M$ represent distance probes in a network modeled by $G$; when the edge $e$ fails, the distance from $x$ to $y$ increases, and thus we are able to detect the failure. It turns out that not only we can detect it, but we can even correctly locate the failing edge. In this paper, we continue the study of distance-edge-monitoring sets. In particular, we give upper and lower bounds of $P(M, e), \operatorname{EM}(x), \operatorname{dem}(G)$, respectively, and extremal graphs attaining the bounds are characterized. We also characterize the graphs with $\operatorname{dem}(G)=3$.


Keywords: Distance; Metric dimension; Distance-edge-monitoring set.
AMS subject classification 2020: 05C12; 11J83; 35A30; 51K05.

## 1 Introduction

Foucaud et al. [8] recently introduced a new concept of network monitoring using distance probes, called distance-edge-monitoring. Networks are naturally modeled by finite undirected simple connected graphs, whose vertices represent computers and whose edges represent connections between them. We wish to be able to monitor the network in the sense that when a connection (an edge) fails, we can detect this failure. We will select a (hopefully) small set of vertices of the network, that will be called probes. At any given moment, a probe of the network can measure its graph distance to any other vertex of the network. The goal is that, whenever some edge of the network fails, one of the measured distances changes, and thus the probes are able to detect the failure of any edge. Probes that measure

[^0]distances in graphs are present in real-life networks, for instance this is useful in the fundamental task of routing [7, 9]. They are also frequently used for problems concerning network verification [1, 3, 5].

We will now present the formal definition of the concept of distance-edge-monitoring sets, as introduced by Foucaud et al. [8]. Graphs considered are finite, undirected and simple. Let $G=(V, E)$ be a graph with vertex set $V$ and edge set $E$, respectively. We denote by $d_{G}(x, y)$ the distance between two vertices $x$ and $y$ in a graph $G$. For an edge $e$ of $G$, we denote by $G-e$ the graph obtained by deleting $e$ from $G$.

Definition 1. For a set $M$ of vertices and an edge e of a graph $G$, let $P(M, e)$ be the set of pairs $(x, y)$ with a vertex $x$ of $M$ and a vertex $y$ of $V(G)$ such that $d_{G}(x, y) \neq d_{G-e}(x, y)$. In other words, $e$ belongs to all shortest paths between $x$ and $y$ in $G$.

Definition 2. For a vertex $x$, let $E M(x)$ be the set of edges e such that there exists a vertex $v$ in $G$ with $(x, v) \in P(\{x\}, e)$, that is $E M(x)=\left\{e \mid e \in E(G)\right.$ and $\exists v \in V(G)$ such that $\left.d_{G}(x, v) \neq d_{G-e}(x, v)\right\}$, or $E M(x)=\{e \mid e \in E(G)$ and $P(\{x\}, e) \neq \emptyset\}$. If $e \in E M(x)$, we say that $e$ is monitored by $x$.

Definition 3. $A$ set $M$ of vertices of a graph $G$ is distance-edge-monitoring set if every edge e of $G$ is monitored by some vertex of $M$, that is, the set $P(M, e)$ is nonempty. Equivalently, $\bigcup_{x \in M} E M(x)=$ $E(G)$.

One may wonder about the existence of such an edge detection set $M$. The answer is affirmative. If we take $M=V(G)$, then

$$
E(G) \subseteq \bigcup_{x \in V(G)} N(x) \subseteq \bigcup_{x \in V(G)} E M(x)
$$

Therefore, we consider the smallest cardinality of $M$ and give the following parameter.
Definition 4. The distance-edge-monitoring number $\operatorname{dem}(G)$ of a graph $G$ is defined as the smallest size of a distance-edge-monitoring set of $G$, that is

$$
\operatorname{dem}(G)=\min \left\{|M| \mid \bigcup_{x \in M} E M(x)=E(G)\right\}
$$

The vertices of $M$ represent distance probes in a network modeled by $G$, distance-edge-monitoring sets are very effective in network fault tolerance testing. For example, a distance-edge-monitoring set can detect a failing edge, and it can correctly locate the failing edge by distance from $x$ to $y$, because the distance from $x$ to $y$ will increases when the edge $e$ fails. Concepts related to distance-edge-monitoring sets have been considered e.g. in [1, 2, 3, 4, 4, 10, 11, 12, 13, 14, 15, 16]. A detailed discussion of these concepts can be found in [8].

Foucaud et al. [8] introduced and initiated the study of distance-edge-monitoring sets. They showed that for a nontrivial connected graph $G$ of order $n, 1 \leq \operatorname{dem}(G) \leq n-1$ with $\operatorname{dem}(G)=1$ if and only if $G$ is a tree, and $\operatorname{dem}(G)=n-1$ if and only if it is a complete graph. They derived the exact value of dem for grids, hypercubes, and complete bipartite graphs. Then, they related dem to other standard graph parameters. They showed that $\operatorname{dem}(G)$ is lower-bounded by the arboricity of the graph, and upper-bounded by its vertex cover number. It is also upper-bounded by twice its feedback edge set
number. Moreover, they characterized connected graphs $G$ with $\operatorname{dem}(G)=2$. Then, they showed that determining $\operatorname{dem}(G)$ for an input graph $G$ is an NP-complete problem, even for apex graphs. There exists a polynomial-time logarithmic-factor approximation algorithm, however it is NP-hard to compute an asymptotically better approximation, even for bipartite graphs of small diameter and for bipartite subcubic graphs. For such instances, the problem is also unlikely to be fixed parameter tractable when parameterized by the solution size.

In this paper, we continue the study of distance-edge-monitoring sets. In particular, we give upper and lower bounds of $P(M, e), E M(x), \operatorname{dem}(G)$, respectively, and extremal graphs attaining the bounds are characterized. We also characterize the graphs with $\operatorname{dem}(G)=3$.

## 2 Preliminaries

Graphs considered are finite, undirected and simple. Let $G=(V, E)$ be a graph with vertex set $V$ and edge set $E$, respectively. The neighborhood set of a vertex $v \in V(G)$ is $N_{G}(v)=\{u \in V(G) \mid u v \in$ $E(G)\}$. Let $N_{G}[v]=N_{G}(v) \cup\{v\}$. The degree of a vertex $v$ in $G$ is denoted by $d(v)=\left|N_{G}(v)\right| . \delta(G)$, $\Delta(G)$ is the minimum, maximum degree of the graph $G$, respectively. For a vertex subset $S \subseteq V(G)$, the subgraph induced by $S$ in $G$ is denoted by $G[S]$ and similarly $G[V \backslash S]$ for $G \backslash S$ or $G-S . v^{k+}$ is a vertex $v$ whose degree is at least $k$. In a graph $G$, a vertex is a core vertex if it is $v^{3+}$. A path with all internal vertices of degree 2 and whose end-vertices are core vertices is called a core path (note that we allow the two end-vertices to be equal, but all other vertices must be distinct). A core path that is a cycle (that is, both end-vertices are equal) is a core cycle. The base graph $G_{b}$ of a graph $G$ is the graph obtained from $G$ by iteratively removing vertices of degree 1 . Clearly, $\operatorname{dem}(G)=\operatorname{dem}\left(G_{b}\right)$.

Foucaud et al. [8] showed that $1 \leq \operatorname{dem}(G) \leq n-1$ for any $G$ with order $n$, and characterized graphs with $\operatorname{dem}(G)=1,2, n-1$.

Theorem 2.1. [8] Let $G$ be a connected graph with at least one edge. Then $\operatorname{dem}(G)=1$ if and only if $G$ is a tree.

For two vertices $u, v$ of a graph $G$ and two non-negative integers $i, j$, we denote by $B_{i, j}(u, v)$ the set of vertices at distance $i$ from $u$ and distance $j$ from $v$ in $G$.

Theorem 2.2. [8] Let $G$ be a connected graph with at least one cycle, and let $G_{b}$ be the base graph of $G$. Then, $\operatorname{dem}(G)=2$ if and only if there are two vertices $u, v$ in $G_{b}$ such that all of the following conditions (1)-(4) hold in $G_{b}$ :
(1) for all $i, j \in\{0,1,2, \cdots\}, B_{i, j}(u, v)$ is an independent set.
(2) for all $i, j \in\{0,1,2, \cdots\}$, every vertex $x$ in $B_{i, j}(u, v)$ has at most one neighbor in each of the four sets $B_{i-1, j}(u, v) \cup B_{i-1, j-1}(u, v), B_{i-1, j}(u, v) \cup B_{i-1, j+1}(u, v), B_{i, j-1}(u, v) \cup B_{i-1, j-1}(u, v)$ and $B_{i, j-1}(u, v) \cup B_{i+1, j-1}(u, v)$.
(3) for all $i, j \in\{1,2, \cdots\}$, there is no 4-vertex path zxyz' with $z \in B_{i-1, a}(u, v), z^{\prime} \in B_{a^{\prime}, j}(u, v)$, $x \in B_{i, j}(u, v), y \in B_{i-1, j+1}(u, v), a \in\{j-1, j+1\}, a^{\prime} \in\{i-2, i\}$.
(4) for all $i, j \in\{1,2, \cdots\}, x \in B_{i, j}(u, v)$ has neighbors in at most two sets among $B_{i-1, j+1}(u, v)$, $B_{i-1, j-1}(u, v), B_{i+1, j-1}(u, v)$.

Theorem 2.3. [8] $\operatorname{dem}(G)=n-1$ if and only if $G$ is the complete graph of order $n$.

## 3 Results for $P(M, e)$

For the parameter $P(M, e)$, we have the following monotonicity property.
Proposition 3.1. For two vertex sets $M_{1}, M_{2}$ and an edge $e$ of a graph $G$, if $M_{1} \subset M_{2}$, then $P\left(M_{1}, e\right) \subset P\left(M_{2}, e\right)$.

Proof. For any $(x, y) \in P\left(M_{1}, e\right)$ with $x \in M_{1}$ and $y \in V(G)$, we have $d_{G}(x, y) \neq d_{G-e}(x, y)$. Since $M_{1} \subset M_{2}$, it follows that $x \in M_{2}$. Since $d_{G}(x, y) \neq d_{G-e}(x, y)$, we have $(x, y) \in P\left(M_{2}, e\right)$, and so $P\left(M_{1}, e\right) \subset P\left(M_{2}, e\right)$.

From Proposition 3.1, one may think $P\left(M_{1}, e\right) \nsubseteq P\left(M_{2}, e\right)$ if $M_{1} \nsubseteq M_{2}$.
Proposition 3.2. For two vertex sets $M_{1}, M_{2}$ and an edge e of a graph $G$, if $P\left(M_{1} \cap M_{2}, e\right) \neq \emptyset$, then $M_{1} \cap M_{2}=\emptyset$ if and only if $P\left(M_{1}, e\right) \cap P\left(M_{2}, e\right)=\emptyset$.

Proof. If $M_{1} \cap M_{2}=\emptyset$, then it follows from the definition of $P(M, e)$ that $P\left(M_{1}, e\right) \cap P\left(M_{2}, e\right)=\emptyset$. Conversely, we suppose that $P\left(M_{1}, e\right) \cap P\left(M_{2}, e\right)=\emptyset$. Assume that $M_{1} \cap M_{2} \neq \emptyset$. Let $M_{1} \cap M_{2}=M$. Clearly, $M \subset M_{1}$ and $M \subset M_{2}$. From Proposition 3.1, we have $P(M, e) \subset P\left(M_{1}, e\right)$ and $P(M, e) \subset$ $P\left(M_{2}, e\right)$, and hence $P(M, e) \subseteq P\left(M_{1}, e\right) \cap P\left(M_{2}, e\right)$. Obviously, $P\left(M_{1}, e\right) \cap P\left(M_{2}, e\right) \subseteq P(M, e)$ and hence $P\left(M_{1}, e\right) \cap P\left(M_{2}, e\right)=P(M, e)$. Since $P(M, e) \neq \emptyset$, it follows that $P\left(M_{1}, e\right) \cap P\left(M_{2}, e\right) \neq \emptyset$, a contradiction. So, we have $M_{1} \cap M_{2}=\emptyset$.

### 3.1 Upper and lower bounds

The following observation is immediate.
Observation 3.1. [8] Let $M$ be a distance-edge-monitoring set of a graph $G$. Then, for any two distinct edges $e_{1}$ and $e_{2}$ in $G$, we have $P\left(M, e_{1}\right) \neq P\left(M, e_{2}\right)$.

For any graph $G$ with order $n$, if $|M|=1$, then we have the following observation.
Observation 3.2. Let $G$ be a graph with order $n$, and $v \in V(G)$. Then

$$
0 \leq|P(\{v\}, u w)| \leq n-1 .
$$

Moreover, the bounds are sharp.
In terms of order of a graph $G$, we can derive the following upper and lower bounds.
Proposition 3.3. Let $G$ be a graph of order n. For a vertex set $M$ and an edge e of a graph $G$, we have

$$
0 \leq|P(M, e)| \leq n(n-1) .
$$

Moreover, the bounds are sharp.
Proof. Clearly, $|P(M, e)| \geq 0$. From Proposition 3.1, we have $P(M, e) \subset P(V(G), e)$. Let $M=V(G)$. Then the number of ordered pairs is $n(n-1)$ in $G$, and hence $|P(M, e)| \leq n(n-1)$, as desired.

To show the sharpness of the bounds in Proposition 3.3, we consider the following examples.

Example 1. For any graph $H$, let $G=K_{n} \vee H$. Let $M=V\left(K_{n}\right)$ and $e \in E(H)$. If $x, y \in M$, then $d_{G}(x, y)=d_{G-e}(x, y)=1$, and so $(x, y) \notin P(M, e)$. If $x \in V\left(K_{n}\right)$ and $y \in V(H)$, then $d_{G}(x, y)=$ $d_{G-e}(x, y)=1$, and hence $(x, y) \notin P(M, e)$. Clearly, $P(M, e)=\emptyset$, and hence $|P(M, e)|=0$. If $G=K_{2}$, then $|P(M, e)|=n(n-1)$, which means that the bounds in Proposition 3.3 are sharp.

The double star $S(n, m)$ for integers $n \geq m \geq 0$ is the graph obtained from the union of two stars $K_{1, n}$ and $K_{1, m}$ by adding the edge $e$ between their centers.

Proposition 3.4. Let $G$ be a graph of order $n$ with a cut edge e. For any vertex set $M$, we have

$$
2(n-1) \leq|P(M, e)| \leq 2\lfloor n / 2\rfloor\lceil n / 2\rceil .
$$

Moreover, the bounds are sharp.
Proof. Let $G_{1}, G_{2}$ be the two components of $G \backslash e$, and let $\left|V\left(G_{1}\right)\right|=n_{1}$ and $\left|V\left(G_{2}\right)\right|=n_{2}$. For any $x \in V\left(G_{1}\right)$ and $y \in V\left(G_{2}\right)$, since $e$ is cut edge, it follows that $d_{G}(x, y) \neq d_{G-e}(x, y)$. If $M=V(G)$, then $P(M, e)=\left\{(x, y),(y, x) \mid x \in V\left(G_{1}\right)\right.$ and $\left.y \in V\left(G_{2}\right)\right\}$, and hence $|P(M, e)|=2\left|V\left(G_{1}\right)\right|\left|V\left(G_{2}\right)\right|=$ $2 n_{1} n_{2}=2 n_{1}\left(n-n_{1}\right) \leq 2\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil$, and so $|P(M, e)| \leq 2\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil$. Since $|P(M, e)|=2 n_{1}\left(n-n_{1}\right) \geq$ $2(n-1)$, it follows that $|P(M, e)| \geq 2(n-1)$.

Example 2. Let $G$ be the double star $S(\lfloor n / 2\rfloor-1,\lceil n / 2\rceil-1)$. If $M=V(G)$, then $d_{G}(x, y) \neq$ $d_{G-e}(x, y)$ for any $x \in V\left(K_{1,\lfloor n / 2\rfloor-1}\right)$ and $y \in V\left(K_{1,\lfloor n / 2\rfloor-1}\right)$. Then $(x, y),(y, x) \in P(M, e)$, and hence $|P(M, e)| \geq 2\lfloor n / 2\rfloor\lceil n / 2\rceil$. From Proposition [3.4, we have $|P(M, e)| \leq 2\lfloor n / 2\rfloor\lceil n / 2\rceil$ and hence $|P(M, e)|=2\lfloor n / 2\rfloor\lceil n / 2\rceil$.

In fact, we can characterize the graphs attaining the lower bounds in Proposition 3.3,
Proposition 3.5. Let $G$ be a graph with uv $\in E(G)$ and $M \subset V(G)$. Then $|P(M, u v)|=0$ if and only if one of the following conditions holds.
(i) $M=\emptyset$;
(ii) $d_{G}(x, u)=d_{G}(x, v)$ for any $x \in M$.
(iii) for any $x \in M$ and $d_{G}(x, u)=d_{G}(x, v)+1$, we have $d_{G-u v}(x, u)=d_{G}(x, u)$.

Proof. Suppose that $|P(M, u v)|=0$. Since

$$
P(M, u v)=\left\{(x, y) \mid d_{G}(x, y) \neq d_{G-u v}(x, y), x \in M, y \in V(G)\right\}=\emptyset,
$$

it follows that $M=\emptyset$ or there exists a vertex set $M \in V(G)$ and an edge $u v \in E(G)$ such that $d_{G}(x, y)=d_{G-u v}(x, y)$ for any $x \in M$ and $y \in V(G)$. For the fixed $x$, if $y=u$ and $y=v$, then we only need to consider the path from $x$ to $y$ through $u v$, and hence $d_{G}(x, u)=d_{G-u v}(x, u)$ and $d_{G}(x, v)=d_{G-u v}(x, v)$. Clearly, we have $\left|d_{G}(x, v)-d_{G}(x, u)\right| \leq 1$. Without loss of generality, let $d_{G}(x, u) \geq d_{G}(x, v)$. For any $x \in M$, if $d_{G}(x, u)=d_{G}(x, v)$, then $(i i)$ is true.

Claim 1. If $d_{G}(x, u)=d_{G}(x, v)+1$, then $d_{G-u v}(x, u)=d_{G}(x, u)$.
Proof. Assume, to the contrary, that $d_{G-u v}(x, u)>d_{G}(x, u)$. For $u \in V(G)$, we have $d_{G-u v}(x, u) \neq$ $d_{G}(x, u)$, and hence $(x, u) \in P(M, u v)=\emptyset$, a contradiction.

Conversely, if $M=\emptyset$, then $|P(M, u v)|=0$. For any $x \in M$, suppose that $d_{G}(x, u)=d_{G}(x, v)$, then $d_{G}(x, y)=d_{G-u v}(x, y)$ for any $y \in V(G)$, and hence $(x, y) \notin P(M, u v)$. For any $x \in M$, if $d_{G}(x, u)=d_{G}(x, v)+1$ then $d_{G-u v}(x, u)=d_{G}(x, u)$ and hence $d_{G}(x, y)=d_{G-u v}(x, y)$ for any $y \in V(G)$. It follows that $(x, y) \notin P(M, u v)$. From the definition of $P(M, e)$, we have $P(M, e)=\emptyset$, and hence $|P(M, e)|=0$.

In fact, we can characterize the graphs attaining the upper bounds in Proposition 3.4.
Proposition 3.6. Let $G$ be a graph with a cut edge $v_{1} v_{2} \in E(G)$ and $M=V(G)$. Then $\left|P\left(M, v_{1} v_{2}\right)\right|=$ $2\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil$ if and only if there are two vertex disjoint subgraphs $G_{1}$ and $G_{2}$ with $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $\left|\left|V\left(G_{1}\right)\right|-\left|V\left(G_{2}\right)\right|\right| \leq 1$, where $v_{i} \in V\left(G_{i}\right), i=1,2$. In addition, $G_{1}$ and $G_{2}$ is connected by a bridge edge $v_{1} v_{2}$.

Proof. Suppose that $\left|P\left(M, v_{1} v_{2}\right)\right|=2\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil$. Since $M=V(G)$, it follows that there are two induced subgraphs $G_{1}$ and $G_{2}$ with $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$, where $v_{i} \in V\left(G_{i}\right), i=1,2$. Note that $v_{1} v_{2}$ is a cut edge of $G$.

Claim 2. If $x, y \in V\left(G_{i}\right)$, then $(x, y) \notin P(M, e)$ and $(y, x) \notin P(M, e)$, where $i=1,2$.

Proof. Assume, to the contrary, that $x, y \in V\left(G_{i}\right)$ and $(x, y) \in P(M, e)$, where $i=1,2$. Then there exists a shortest path from $x$ to $y$ such that $d_{G}(x, y) \neq d_{G-v_{1} v_{2}}(x, y)$, where $v_{i} \in V\left(G_{i}\right), i=1,2$. Since $v_{1} v_{2}$ is a cut edge, it follows that $d_{G}(x, y)=d_{G-v_{1} v_{2}}(x, y)$, and hence $(x, y) \notin P(M, e)$, a contradiction.

By Claim 2, we only consider that $x \in V\left(G_{i}\right)$ and $y \in V(G)-V\left(G_{i}\right)(i=1,2)$. Since $v_{1} v_{2}$ is a cut edge, it follows that $d_{G}(x, y) \neq d_{G-v_{1} v_{2}}(x, y)$, and hence $(x, y) \in P(M, e)$. It follows that $|P(M, e)|=2\left|V\left(G_{1}\right)\right|\left|V\left(G_{2}\right)\right|=2\left|V\left(G_{1}\right)\right|\left(n-\left|V\left(G_{1}\right)\right|\right) \leq 2\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil$, where the equality holds if and only $\left|V\left(G_{1}\right)\right|=\left\lfloor\frac{n}{2}\right\rfloor$ or $\left|V\left(G_{1}\right)\right|=\left\lceil\frac{n}{2}\right\rceil$, and hence $\left|\left|V\left(G_{1}\right)\right|-\left|V\left(G_{2}\right)\right|\right| \leq 1$.

Conversely, we suppose that there are two vertex disjoint subgraphs $G_{1}$ and $G_{2}$ with $V(G)=$ $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $\left|\left|V\left(G_{1}\right)\right|-\left|V\left(G_{2}\right)\right|\right| \leq 1$, where $v_{i} \in V\left(G_{i}\right), i=1,2$. Then $G_{1}$ and $G_{2}$ are connected by a bridge edge, and hence $|P(M, e)|=|P(V(G), e)|=2\left|V\left(G_{1}\right)\right|\left|V\left(G_{2}\right)\right|=2\lfloor n / 2\rfloor\lceil n / 2\rceil$, as desired.

For $|P(M, e)|$, we give some results for some special graphs.
Lemma 3.1. Let $K_{n}$ be a complete graph, and let $M \subseteq V\left(K_{n}\right)$. Then

$$
P(M, u v)= \begin{cases}\{(u, v),(v, u)\} & \text { if } u, v \in M, \\ \{(u, v)\} & \text { if } u \in M \text { and } v \notin M, \\ \{(v, u)\} & \text { if } v \in M \text { and } u \notin M, \\ \emptyset, & \text { if } u, v \notin M,\end{cases}
$$

where $u v \in E\left(K_{n}\right)$.

Proof. Let $V\left(K_{n}\right)=\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$. For any edge $u v$, if $u \in M$ and $v \notin M$, then $P(M, u v)=$ $\left\{(x, y) \mid x \in M, y \in V(G)\right.$ and $\left.d_{G}(x, y) \neq d_{G-u v}(x, y)\right\}$. Since $d_{K_{n}}(u, v)=1$ and $d_{K_{n}-u v}(u, v)=2$, we have $(u, v) \in P(M, x y)$. The result follows for $u \in M$ and $v \notin M$. Similarly, if $u, v \in M$, then $P(M, e)=\{(u, v),(v, u)\}$. Suppose that $u \notin M$ and $v \notin M$. Let $P_{x, y}$ be the shortest path from $x \in M$ to $y \in V(G)$, and hence there is no the shortest path $P_{x, y}$ such that $u v \notin E\left(P_{x, y}\right)$, and hence $P(M, u v)=\emptyset$.

The following corollary is immediate.
Proposition 3.7. Let $K_{n}$ be a complete graph, and let $M \subseteq V\left(K_{n}\right)$. Then

$$
0 \leq\left|P_{G}(M, u v)\right| \leq 2,
$$

where $u v \in E\left(K_{n}\right)$. Furthermore, $\left|P_{G}(M, u v)\right|=0$ if and only if $u, v \notin M ;\left|P_{G}(M, u v)\right|=2$ if and only if $u, v \in M ;\left|P_{G}(M, u v)\right|=1$ for otherwise.

Proof. For any $u v \in E(G)$ and $M \in V(G)$, if $u, v \notin M$, then it follows from Lemma 3.1 that $P_{G}(M, u v)=\emptyset$, and hence $\left|P_{G}(M, u v)\right|=0$. If $u, v \in M$, then it follows from Lemma 3.1 that $P_{G}(M, u v)=\{(u, v),(v, u)\}$, and so $\left|P_{G}(M, u v)\right|=2$. Similarly, for other case, we have $\left|P_{G}(M, u v)\right|=$ 1.

## 4 Results for $E M(x)$

For $E M(x)$, we can observe some basic properties of distance-edge-monitoring sets. Obviously, for any bridge edge $e \in E(G)$, the edge $e \in E M(x)$, which is given by Foucaud et al. in [8], see Theorem 4.1.

Theorem 4.1. [8] Let $G$ be a connected graph and let e be a bridge edge of $G$. For any vertex $x$ of $G$, we have $e \in E M(x)$.

The following corollary is immediate.
Corollary 4.2. For a vertex $v$ of a tree $T$, we have $E M(v)=E(T)$.
Proof. For a vertex $v$ of a tree $T$, we have $E M(v) \subset E(T)$. Since any edge $e \in E(T)$ is a bridge edge of $T$, it follows from Theorem 4.1 that $e \in E M(v)$ for any vertex $v \in V(T)$, and hence $E(T) \subset$ $E M(v)$.

Theorem 4.3. [8] Let $G$ be a connected graph with a vertex $x$ of $G$. The following two conditions are equivalent:
(1) $E M(x)$ is the set of edges incident with $x$.
(2) For $y \in V(G)-N_{G}[x]$, there exist two shortest paths from $x$ to $y$ sharing at most one edge.

Now, let's investigate the edges of $E M(x)$ in $G$. Firstly, we introduced the following result, which is given in Foucaud el al. [8].

Theorem 4.4. 8] Let $G$ be a connected graph with a vertex $x$ of $G$ and for any $y \in N(x)$, then, we have $x y \in E M(x)$.

By Theorem 4.4, we can obtain a lower bound on $E M(x)$ for any graph $G$ with minimum degree $\delta$, the description is as follows.

Corollary 4.5. Let $G$ be a connected graph. For any $x \in V(G)$, we have

$$
|E M(x)| \geq\left|N_{G}(x)\right| \geq \delta(G),
$$

with equality if and only if $G$ is a regular graph such that there exist two shortest paths from $u$ to $x$ sharing at most one edge, where $u \in V(G)-N_{G}[x]$. For example, a balanced complete bipartite graph $K_{n, n}$.

Theorem 4.6. 8] For a vertex $x$ of a graph $G$, the set of edges $E M(x)$ induces a forest.
For a graph $G$ and a vertex $x \in V(G)$, one can derive the edge set $E M(x)$ from $G$ by Algorithm 1 . This algorithm is based on the breadth-first spanning tree algorithm. In the process of finding breadthfirst spanning trees, we delete some edges that cannot be monitored by vertex $x$, and obtain the edge set $E M(x)$ when the algorithm terminates. The time complexity of the breadth-first search tree algorithm is $O(|V(G)|+|E(G)|)$. In Algorithm 1 , we only add the steps of deleting specific edges and checking neighbor vertex shown in Lines 17-26.

```
Algorithm 1 The algorithm of finding an edge set \(E M(x)\) in \(G\)
Input: a graph \(G\) and a vertex \(x \in V(G)\);
Output: A edge set \(E M(x)\) in \(G\);
    for each vertx \(u \in V(G)-\{x\}\) do
        colour \([u] \leftarrow\) White
        \(\mathrm{d}[\mathrm{u}] \leftarrow \infty\)
    \(E M(x) \leftarrow E(G)\)
    \(\mathrm{d}[\mathrm{x}] \leftarrow 0\)
    \(\mathrm{Q} \leftarrow \emptyset\)
    Enqueue \([Q, x]\)
    while \(Q \neq \emptyset\) do
        \(u \leftarrow\) Dequeue \([Q]\)
        \(N^{\prime}[u] \leftarrow \emptyset\)
        for each vertx \(v \in \operatorname{Adj}[u]\) do
            if colour \([\mathrm{v}] \leftarrow\) White then
                    \(N^{\prime}[u] \leftarrow N^{\prime}[u] \cup\{v\}\)
```

colour $[\mathrm{v}] \leftarrow$ Gray
$d[v] \leftarrow d[u]+1$
Enqueue $[\mathrm{Q}, \mathrm{v}]$
for $v_{i}, v_{j} \in N^{\prime}[u]$ do
if $v_{i} v_{j} \in E(G)$ then $E M(x)=E M(x)-v_{i} v_{j}$
Dv $\leftarrow \emptyset$
for each vertx $\left.v_{o} \in \operatorname{Adj} j v\right]$ do
if colour $\left[v_{o}\right]=$ Gray then $D_{v} \leftarrow D_{v} \cup\left\{v_{o}\right\}$
if $\left|D_{v}\right| \geq 1$ then for $v_{o} \in D_{v}$ do

$$
E M(x)=E M(x)-v v_{o}
$$

colour $[\mathrm{u}] \leftarrow$ DarkGary
28: return $E M(x)$

We now give upper and lower bounds on $E M(x)$ in terms of the order $n$.
Proposition 4.1. Let $G$ be a connected graph with $|V(G)| \geq 2$. For any $v \in V(G)$, we have

$$
1 \leq|E M(v)| \leq|V(G)|-1 .
$$

Moreover, the bounds are sharp.

Proof. For any vertex $v \in V(G)$, it follows from Theorem 4.6 that the set of edges $E M(x)$ induces a forest $F$ in $G$, and hence $|E M(v)| \leq|E(F)| \leq|E(T)|=|V(G)|-1$, where $T$ is a spanning tree of $G$. Since $G$ is a connected graph, it follows from Corollary 4.5 that $|E M(x)| \geq \delta(G) \geq 1$, and hence $|E M(v)| \geq 1$.

Given a vertex $x$ of a graph $G$ and an integer $i$, let $N_{i}(x)$ denote the set of vertices at distance $i$ of $x$ in $G$. Is there a way to quickly determine whether $e \in E M(v)$ or $e \notin E M(v)$ ? Foucaud et al. [8] gave the following characterization about edge $u v$ in $E M(x)$.

Theorem 4.7. [8 Let $x$ be a vertex of a connected graph $G$. Then, $u v \in E M(x)$ if and only if $u \in N_{i}(x)$ and $v$ is the only neighbor of $u$ in $N_{i-1}(x)$, for some integer $i$.

The following results are immediate from Theorem 4.7. These results show that it is easy to determine $e \notin E M(v)$ for $v \in V(G)$.

Corollary 4.8. Let $G$ be a connected graph, and $x \in V(G)$. Let $\mathcal{P}_{x, y}$ denote the set of shortest paths from $x$ to $y$. Suppose that uv is an edge of $G_{b}$ satisfying one of the following conditions.
(1) there exists an odd cycle $C_{2 k+1}$ containing the vertices $x^{\prime}, u, v$ such that $V\left(\mathcal{P}_{x, x^{\prime}}\right) \cap V\left(C_{2 k+1}\right)=x^{\prime}$ and $d_{G}\left(x^{\prime}, u\right)=d_{G}\left(x^{\prime}, v\right)=k$.
(2) there exists an even cycle $C_{2 k}$ containing the vertices $x^{\prime}, u, v$ such that $V\left(\mathcal{P}_{x, x^{\prime}}\right) \cap V\left(C_{2 k}\right)=x^{\prime}$, $d_{G}\left(x^{\prime}, u\right)=k-1$ and $d_{G}\left(x^{\prime}, v\right)=k$.

Then $u v \notin E M(x)$.
Proof. Since $d_{G}\left(x^{\prime}, u\right)=d_{G}\left(x^{\prime}, v\right)=k$, it follows that $d_{G}\left(x^{\prime}, u\right)=d_{G-u v}\left(x^{\prime}, u\right)=k$ and $d_{G}\left(x^{\prime}, v\right)=$ $d_{G-u v}\left(x^{\prime}, v\right)=k$. Since $V\left(\mathcal{P}_{x, x^{\prime}}\right) \cap V\left(C_{2 k+1}\right)=x^{\prime}$, it follows that $d_{G}(x, u)=d_{G}\left(x, x^{\prime}\right)+d_{G}\left(x^{\prime}, u\right)$ and $d_{G}(x, v)=d_{G}\left(x, x^{\prime}\right)+d_{G}\left(x^{\prime}, v\right)$, and so $d_{G}(x, u)=d_{G-u v}(x, u)$ and $d_{G}(x, v)=d_{G-u v}(x, v)$. Clearly, $u v \notin E M(x)$, and so (1) holds. From Theorem4.7, the results are immediate, and hence (2) holds.

Theorem 4.9. For any $k(1 \leq k \leq n-1)$, there exists a graph of order $n$ and a vertex $v \in V(G)$ such that $|E M(v)|=k$.

Proof. Let $F_{1}$ be a graph of order $k$ and $F_{2}$ be a graph obtained from $F_{1}$ by adding a new vertex $v$ and then adding all edges from $v$ to $V\left(F_{1}\right)$. Let $H$ be a graph obtained from $F_{2}$ and a graph $F_{3}$ of order $n-k-1$ such that there are at least two edges from each vertex in $F_{3}$ to $V\left(F_{1}\right)$.

From Corollary 4.5, we have $|E M(v)| \geq\left|N_{G}(v)\right|=k$. To show $|E M(v)| \leq k$, it suffices to prove that $E M(v)=E_{H}\left[v, V\left(F_{1}\right)\right]$. Clearly, $E_{H}\left[v, V\left(F_{1}\right)\right] \subseteq E M(v)$. We need to prove that $E M(v) \subseteq$ $E_{H}\left[v, V\left(F_{1}\right)\right]$, that is, $E M(v) \cap\left(E(H) \backslash E_{H}\left[v, V\left(F_{1}\right)\right]\right)=\emptyset$. It suffices to show that for any $x y \in$ $E(H) \backslash E_{H}\left[v, V\left(F_{1}\right)\right]$, we have $d_{G}(v, x)=d_{H-x y}(v, x)$ or $d_{H}(v, y)=d_{H-x y}(v, y)$. Note that $E(H) \backslash$ $E_{H}\left[v, V\left(F_{1}\right)\right]=E\left(F_{1}\right) \cup E\left(F_{3}\right) \cup E_{H}\left[V\left(F_{1}\right), V\left(F_{3}\right)\right]$. If $x y \in E\left(F_{1}\right)$, then $d_{H}(v, x)=d_{H}(v, y)=1$, and it follows from Corollary 4.8 (1) that $x y \notin E M(v)$. If $x y \in E\left(F_{3}\right)$, then $d_{H}(v, x)=d_{H}(v, y)=2$, and it follows from Corollary 4.8 (1) that $x y \notin E M(v)$. Suppose that $x y \in E_{H}\left[V\left(F_{1}\right), V\left(F_{3}\right)\right]$. Without loss of generality, let $x \in V\left(F_{1}\right)$ and $y \in V\left(F_{3}\right)$. Since there are at least two edges from $y$ to $V\left(F_{1}\right)$, it follows that there exists a vertex $z \in V\left(F_{1}\right)$ such that $z y \in E(H)$. Then $d_{H}(v, z)=d_{H}(v, x)=2$ and $d_{H}(v, y)=3$. From Corollary $4.8(2)$, we have $x y \notin E M(v)$. From the above argument, $|E M(v)| \leq k$, and hence $|E M(v)|=k$.

Graphs with small values of $|E M(v)|$ can be characterized in the following.
Theorem 4.10. For a connected graph $G$ and $v \in V(G)$, we have $|E M(v)|=1$ if and only if $G=K_{2}$.
Proof. If $|E M(v)|=1$, then it follows from Corollary 4.5 that $d_{G}(v) \leq 1$. Since $G$ is connected, it follows that $d_{G}(v) \geq 1$ and hence $d_{G}(v)=1$. Let $u$ be the vertex such that $v u \in E(G)$.

Claim 3. $d_{G}(u)=1$.
Proof. Assume, to the contrary, that $d_{G}(u) \geq 2$. For any vertex $y \in N_{G}(u)-v$, we have $y \in N_{2}(v)$, and hence $d_{G}(y, v)=2$, and so $N_{1}(v)=\{u\}$. From Theorem 4.7, $u y \in E M(v)$, and hence $|E M(v)| \geq 2$, a contradiction.

From Claim 3, we have $d_{G}(u)=1$. Since $G$ is connected, it follows that $G=K_{2}$.
Conversely, let $G=K_{2}$. For any $v \in V\left(K_{2}\right)$, we have $|E M(v)|=\{u v\}$, and hence $|E M(v)|=1$.
We now define a new graph $A_{d}(d \geq 3)$ such that the eccentricity of $v$ in $A_{d}$ is $d$ and all of the following conditions are true.

For each $i(2 \leq i \leq d)$, let $B_{i}$ be a graph such that $\left|B_{i}\right| \geq 2$ for $2 \leq i \leq d-1$.
$V\left(A_{d}\right)=\left\{v, u_{1}, u_{2}\right\} \cup\left(\bigcup_{2 \leq i \leq d} V\left(B_{i}\right)\right)$, where $B_{1}$ is a graph with vertex set $\left\{u_{1}, u_{2}\right\}$.
$E\left(A_{d}\right)=\left\{v u_{1}, v u_{2}\right\} \cup\left(\bigcup_{2 \leq i \leq d} E_{A_{d}}\left(B_{i}\right)\right) \cup\left(\bigcup_{2 \leq i \leq d} E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right.$ with $\left|E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right| \geq 2$, where $v^{i} \in V\left(B_{i}\right)$ for $2 \leq i \leq d$.

Note that for each vertex in $B_{i}$, there are at least two edges from this vertex to $B_{i-1}$, where $2 \leq i \leq d$.

For $d=2$, let $D$ be a graph of order $n-3, D_{1}(n)$ be a graph with $V\left(D_{1}(n)\right)=\left\{v, u_{1}, u_{2}\right\} \cup V(D)$ and $E\left(D_{1}(n)\right)=\left\{u_{1} w, u_{2} w \mid w \in V(D)\right\} \cup\left\{u_{1} v, u_{2} v, u v\right\} \cup E(D)$, and $D_{2}(n)$ be a graph with $V\left(D_{2}(n)\right)=$ $\left\{v, u_{1}, u_{2}\right\} \cup V(D)$ and $E\left(D_{2}(n)\right)=\left\{u_{1} w, u_{2} w \mid w \in V(D)\right\} \cup\left\{u_{1} v, u_{2} v\right\} \cup E(D)$.

Theorem 4.11. Let $G$ be connected graph with at least 3 vertices. Then there exists a vertex $v \in V(G)$ such that $|E M(v)|=2$ if and only if $=D_{1}(n)$ or $G=D_{2}(n)$ or $G=A_{d}$ for $d \geq 3$.

Proof. Suppose that $G=D_{1}(n)$ or $G=D_{2}(n)$. Then there is a vertex $v \in V(G)$. Let $d$ be the eccentricity of $v$ in $G$. For $w \in V(D)$, the subgraph induced by the vertices in $\left\{w, u_{1}, u_{2}, v\right\}$ is an even cycle $C_{4}$, and hence $d_{G}\left(v, u_{1}\right)=1$ and $d_{G}(v, w)=2$. It follows from Corollary 4.8 that $w u_{1} \notin E M(v)$. Similarly, we have $w u_{2} \notin E M(v)$. If $u_{1} u_{2} \in E(G)$, then the subgraph induced by the vertices in $\left\{u_{1}, u_{2}, v\right\}$ is a 3-cycle, and hence $d_{G}\left(v, u_{1}\right)=1$ and $d_{G}\left(v, u_{2}\right)=1$. From Corollary 4.8, we have $u_{1} u_{2} \notin E M(v)$. Similarly, we have $d_{G}\left(v, w_{i}\right)=2$ and $d_{G}\left(v, w_{j}\right)=2$ for $w_{i} w_{j} \in E(D)$. From Corollary 4.8. we have $w_{i} w_{j} \notin E M(v)$, and hence $|E M(v)|=\left\{u_{1} v, u_{1} v\right\}$, and so $|E M(v)|=2$.

Suppose that $G=A_{d}$, where $d \geq 3$. Note that $d$ is the eccentricity of $v$ in $G$. Then

$$
E\left(A_{d}\right)=\left\{v u_{1}, v u_{2}\right\} \cup\left(\bigcup_{2 \leq i \leq d} E_{A_{d}}\left(B_{i}\right)\right) \cup\left(\bigcup_{2 \leq i \leq d} E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right)
$$

with $\left|E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right| \geq 2$, where $v^{i} \in V\left(B_{i}\right)$ for $2 \leq i \leq d$. Since $d_{G}\left(v, u_{i s}\right)=i$ and $d_{G}\left(v, u_{i t}\right)=i$ for any $u_{i s} u_{i t} \in E\left(B_{i}\right)$, it follows from Corollary 4.8 that $u_{i s} u_{i t} \notin E M(v)$. Similarly, let $\mathcal{C}_{i}=$
$E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]$ with $\left|E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right| \geq 2$ ，where $v^{i} \in V\left(B_{i}\right)$ for $2 \leq i \leq d$ ．If $y x \in \mathcal{C}_{i}$ ，then $x \in N_{i-1}(v), y \in N_{i}(i)$ and there exists a vertex $x_{1} \in N_{i-1}(v)$ such that $y x_{1} \in E(G)$ ．From Corollary 4．8．we have $y x \notin E M(v)$ ，and so $|E M(v)|=\left\{u_{1} v, u_{1} v\right\}$ ，and hence $|E M(v)|=2$ ．

Conversely，if $|E M(v)|=2$ ，then it follows from Corollary 4．5that $d_{G}(v) \leq 2$ ．If $d_{G}(v)=1$ ，without loss of generality，let $u v \in E(G)$ and $y \in N_{G}(u)$ ，then $u y \in E M(v)$ ，and hence $\left|N_{G}(u)-v\right|=1$ ， and so $G \cong P_{3}$ ，and hence $G \cong B_{2}(3)$ ．Suppose that $d_{G}(v)=2$ ．Without loss of generality，let $N_{G}(v)=\left\{u_{1}, u_{2}\right\}$ ．Suppose that $n=3$ ．If $u_{1} u_{2} \notin E(G)$ ，then $G=D_{2}(3)$ ．If $u_{1} u_{2} \in E(G)$ ，then the subgraph induced by the vertices in $\left\{v, u_{1}, u_{2}\right\}$ is a 3－cycle，and hence $d_{G}\left(v, u_{1}\right)=d_{G}\left(v, u_{2}\right)$ ．From Corollary 4．8，we have $u_{1} u_{2} \notin E M(v)$ ，and hence $G=D_{1}(3)$ ．

Suppose that $n \geq 4$ ．Since $|E M(v)|=2$ ，it follows that $\left\{v u_{1}, v u_{2}\right\} \subseteq E M(v)$ ，and hence $e \notin E M(v)$ for any $e \in E(G)-\left\{v u_{1}, v u_{2}\right\}$ ．

Claim 4．For any $i \geq 2, y \in N_{i}(v)$ and $x \in N_{i-1}(v)$ ，if $y x \in E(G)$ ，then there exists a vertex $x_{1} \in N_{i-1}(v)$ with $y x_{1} \in E(G)$ ．

Proof．Assume，to the contrary，that there exists no $x_{1} \in N_{i-1}$ such that $y x_{1} \notin E(G)$ ．Then $d_{G}(v, y)=$ $i$ but $d_{G-y x}(v, y) \geq i+1$ ，and so $y x \in E M(v)$ ，and hence $|E M(v)| \geq 3$ ，a contradiction．

If $d=2$ ，then for any $w \in V(G)-\left\{v, u_{1}, u_{2}\right\}$ ，it follows from Claim $⿴ 囗 十$ that if $w \in N_{2}(v)$ and $w u_{1} \in E(G)$ ，then $w u_{2} \in E(G)$ ．For any $w_{s}, w_{t} \in N_{2}(v)$ ，we assume that $w_{s} w_{t} \in E(G)$ ．Since $d_{G}\left(v, w_{s}\right)=2$ and $d_{G}\left(v, w_{t}\right)=2$ ，it follows from Corollary 4.8 that $w_{s} w_{t} \notin E M(v)$ ，and hence $G=B_{1}(n)$ or $G=B_{2}(n)$ ．

If $d \geq 3$ ，then
$V\left(G^{d *}\right)=\left\{v, u_{1}, u_{2}\right\} \cup\left\{u_{i j} \mid 2 \leq i \leq d, 1 \leq j \leq t_{d}\right\}=\left\{v, u_{1}, u_{2}\right\} \cup\left\{u_{21}, \ldots, u_{2 t_{2}}\right\} \cup \cdots \cup\left\{u_{d 1}, \ldots, u_{d t_{d}}\right\}$,
where $v \in N_{0}(v), u_{1}, u_{2} \in N_{1}(v), u_{21}, \ldots u_{2 t_{2}} \in N_{2}(v), \ldots u_{d 1}, \ldots u_{d t_{d}} \in N_{d}(v)$ and $\sum_{i=2}^{i=d} t_{s}=n-3$ ．
By Claim 4，if $y \in N_{i}(v)$ and $y x \in E(G)$ ，then there exists a vertex $x_{1} \in N_{i-1}(v)$ and $x_{1} \neq x$ such that $y x_{1} \in E(G)$ ，and hence $y x \in E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]$ with $\left|E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right| \geq 2$ ，where $v^{i} \in V\left(B_{i}\right)$ for $2 \leq i \leq d$ ．

For any $u_{i s}, u_{i t} \in B_{i}(v)$ and $u_{i s} u_{i t} \in E\left(B_{i}\right)$ ，since $d_{G}\left(v, u_{i s}\right)=i$ and $d_{G}\left(v, u_{i t}\right)=i$ ，it follows from Corollary 4.8 that $u_{i s} u_{i t} \notin E M(v)$ ，and hence $B_{i}$ is a graph with order at least 2，and so

$$
E\left(A_{d}\right)=\left\{v u_{1}, v u_{2}\right\} \cup\left(\bigcup_{2 \leq i \leq d} E_{A_{d}}\left(B_{i}\right)\right) \cup\left(\bigcup_{2 \leq i \leq d} E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right)
$$

with $\left|E_{A_{d}}\left[v^{i}, V\left(B_{i-1}\right)\right]\right| \geq 2$ ，where $v^{i} \in V\left(B_{i}\right)$ for $2 \leq i \leq d$ ．Therefore，$G=A_{d}$ ．
Theorem 4．12．Let $G$ be a connected graph of order $n$ ．Then there exists a vertex $v \in V(G)$ such that $|E M(v)|=n-1$ if and only if for any $w \in V(G)$ ，there are no $w_{1}, w_{2} \in N_{G}(w)$ such that $d_{G}\left(w_{1}, v\right)=d_{G}\left(w_{2}, v\right)=d_{G}(w, v)-1$.

Proof．Suppose that $|E M(v)|=n-1$ ．Since $G$ is a connected graph of order $n$ ，it follows from Theorem 4.6 that $E M(v)$ forms a spanning tree of $G$ ．

Claim 5．For any vertex $w \in V(G)$ ，there exists a vertex $w_{i} \in N_{d_{G}(v, w)-1}(v)$ with $w_{i} w \in E M(v)$ ．

Proof. Assume, to the contrary, that there is no $w_{i} \in N_{d_{G}(v, w)-1}(v)$ with $w_{i} w \in E M(v)$. it follows that $E M(v)$ is disconnected, which contradicts to the fact that the subgraph induced by the edges in $E M(v)$ is connected.

By Claim㺃, for any vertex $w \in V(G)$, there exists a vertex $w_{i} \in N_{d_{G}(v, w)-1}(v)$ with $w_{i} w \in E M(v)$. From Theorem 4.7, $w_{i}$ is the unique neighbor of $w$ in $N_{d_{G}(v, w)-1}(v)$, and hence for any $w \in V(G)$, there are no two vertices $w_{1}, w_{2} \in N_{G}(w)$ such that $d_{G}\left(w_{1}, v\right)=d_{G}\left(w_{2}, v\right)=d_{G}(w, v)-1$.

Conversely, we suppose that for any $w \in V(G)$, there are no $w_{1}, w_{2} \in N_{G}(w)$ such that $d_{G}\left(w_{1}, v\right)=$ $d_{G}\left(w_{2}, v\right)=d_{G}(w, v)-1$. Since $G$ is connected, it follows that there is only one vertex $w_{i} \in$ $N_{d_{G}(v, w)-1}(v)$. From Theorem4.7, we have $w_{i} w \in E M(v)$, and hence $|E M(v)|=n-1$.

The existence of $\operatorname{dem}(G)$ is obvious, because $V(G)$ is always a distance-edge-monitoring set. Thus, the definition of $\operatorname{dem}(G)$ is meaningful. The arboricity $\operatorname{arb}(G)$ of a graph $G$ is the smallest number of sets into which $E(G)$ can be partitioned and such that each set induces a forest. The clique number $\omega(G)$ of $G$ is the order of a largest clique in $G$.

Theorem 4.13. [8 For any graph $G$ of order $n$ and size $m$, we have $\operatorname{dem}(G) \geq \operatorname{arb}(G)$, and thus $\operatorname{dem}(G) \geq \frac{m}{n-1}$ and $\operatorname{dem}(G) \geq \frac{\omega(G)}{2}$.

We next see that distance-edge-monitoring sets are relaxations of vertex covers. A vertex set $M$ is called a vertex cover of $G$ if every edge of $G$ has one of its endpoints in $M$. The minimum cardinality of a vertex cover $M$ in $G$ is the vertex covering number of $G$, denoted by $\beta(G)$.

Theorem 4.14. 8] In any graph $G$ of order $n$, any vertex cover of $G$ is a distance-edge-monitoring set, and thus $\operatorname{dem}(G) \leq \beta(G)$.

An independent set is a set of vertices of $G$ such that no two vertices are adjacent. The largest cardinality of an independent set is the independence number of $G$, denoted by $\alpha(G)$.

The following well-known theorem was introduced by Gallaí in 1959.
Theorem 4.15 (Gallaí Theorem). [6] In any graph $G$ of order $n$, we have

$$
\beta(G)+\alpha(G)=n
$$

Corollary 4.16. For a graph $G$ with order $n$, we have

$$
\operatorname{dem}(G) \leq n-\alpha(G)
$$

Moreover, the bound is sharp.
Proof. From Theorem4.15, we have $\beta(G)=n-\alpha(G)$. From Theorem 4.14, we have $\operatorname{dem}(G) \leq \beta(G)$, and hence $\operatorname{dem}(G) \leq n-\alpha(G)$, as desired. For a complete graph $G=K_{n}$ or complete bipartite graph $G=K_{m, n}$, we have $\operatorname{dem}(G)=n-\alpha(G)$.

Theorem 4.17. [8] For any graph $G$, we have $\beta(G) \leq \operatorname{dem}\left(G \vee K_{1}\right) \leq \beta(G)+1$. Moreover, if $G$ has radius at least 4 , then $\beta(G)=\operatorname{dem}\left(G \vee K_{1}\right)$.

Similarly to the proof of Theorem 4.17, we can obtain the following result.

Corollary 4.18. For any graph $G$ and integer $m$, we have

$$
\beta(G) \leq \operatorname{dem}\left(G \vee m K_{1}\right) \leq \beta(G)+m
$$

Moreover, the bounds are sharp.
Proof. For any graph $G$ and integer $m$, we have $\operatorname{dem}\left(G \vee m K_{1}\right) \leq \beta\left(G \vee m K_{1}\right)$ by Theorem 4.14. Clearly, $\beta\left(G \vee m K_{1}\right) \leq \beta(G)+m$, and hence $\operatorname{dem}\left(G \vee m K_{1}\right) \leq \beta(G)+m$. It suffices to show that an edge monitoring set $M$ of $G \vee m K_{1}$ also is cover set of $G$. Without loss of generality, suppose that $V\left(m K_{1}\right)=\left\{w_{1}, \cdots, w_{m}\right\}$. If there exists an edge $u v \in E(G)$ with $u, v \notin M$, then $u v$ is monitored by $M \cap V(G)$ in $G \vee m K_{1}$. For any $x \in M$, we have $d_{G}(x, u) \in\{1,2\}$. Similarly, $d_{G}(x, v) \in\{1,2\}$. By Corollary 4.8, we have $d_{G}(x, v) \neq d_{G}(x, u)$. Without loss of generality, let $d_{G}(x, v)=1$ and $d_{G}(x, u)=2$, and hence $x w_{i} v$ is a shortest path from $x$ to $v$. From Corollary 4.8, $u v$ is not monitored by $M$, a contraction. Then $x \in M$ or $y \in M$, and hence $\beta(G) \leq \operatorname{dem}\left(G \vee m K_{1}\right)$. By Theorem 4.17, if $G$ has radius at least 4 and $m=1$, then $\beta(G)=\operatorname{dem}\left(G \vee K_{1}\right)$. If $m=1$ and $G=K_{n}$, then $\operatorname{dem}\left(K_{n} \vee K_{1}\right)=\beta\left(K_{n}\right)+1=n$, and hence the bound is sharp.

Proposition 4.2. For any $r$-regular graph $G$ of order $n \geq 5$, we have

$$
\frac{r n}{2 n-2} \leq \operatorname{dem}(G) \leq n-1
$$

Moreover, the bounds are sharp.
Proof. For any $r$-regular graph graph $G$ of order $n$, since $e(G)=\frac{r n}{2}$, it follows from Theorem 4.13 that $\operatorname{dem}(G) \geq \frac{m}{n-1}$, and hence $\operatorname{dem}(G) \geq \frac{r n}{2 n-2}$. From Theorem4.14, we have $\operatorname{dem}(G) \leq n-1$. From Theorem 2.3, if $r=1$ and $n=2$, then $\operatorname{dem}\left(K_{2}\right)=1$, and hence the lower bound is tight.

## 5 Graphs with distance-edge-monitoring number three

For three vertices $u, v, w$ of a graph $G$ and non-negative integers $i, j, k$, let $B_{i, j, k}$ be the set of vertices at distance $i$ from $u$ and distance $j$ from $v$ and distance $k$ from $w$ in $G$, respectively.

Lemma 5.1. Let $G$ be a graph with $u, v, w \in V(G)$, and $i, j, k$ be three non-negative integers such that $B_{i, j, k} \neq \emptyset$. If $x \in B_{i, j, k}, x y \in E(G)$, and

$$
T=\left\{\left(i^{\prime}, j^{\prime}, k^{\prime}\right) \mid i^{\prime} \in\{i-1, i, i+1\}, j^{\prime} \in\{j-1, j, j+1\}, k^{\prime} \in\{k-1, k, k+1\}\right\}
$$

then $y \in B_{i^{\prime}, j^{\prime}, k^{\prime}}$, where $\left(i^{\prime}, j^{\prime}, k^{\prime}\right) \in T$.
Proof. Since $x \in B_{i, j, k}$ and $x y \in E(G)$, it follows that $d_{G}(x, u)=i, d_{G}(x, v)=j$ and $d_{G}(x, w)=k$. We have the following claim.

Claim 6. $d_{G}(y, u) \in\{i-1, i, i+1\}$.
Proof. Assume, to the contrary, that $d_{G}(y, u) \leq i-2$ or $d_{G}(y, u) \geq i+2$. If $d_{G}(y, u) \leq i-2$, then $d_{G}(x, u) \leq d_{G}(u, y)+d_{G}(y, x)=d_{G}(u, y)+1 \leq i-1$, which contradicts to the fact that $d_{G}(x, u)=i$. If $d_{G}(y, u) \geq i+2$, then $i+2 \leq d_{G}(y, u) \leq d_{G}(u, x)+d_{G}(x, y)=i+1$, a contradiction.

From Claim 6, we have $d_{G}(y, u) \in\{i-1, i, i+1\}$. Similarly, $d_{G}(y, v) \in\{j-1, j, j+1\}$ and $d_{G}(y, w) \in\{k-1, k, k+1\}$.

Theorem 5.1. For a graph $G, \operatorname{dem}(G)=3$ if and only if there exists three vertices $u, v, w$ in $G_{b}$ such that all of the following conditions (1)-(8) hold in $G_{b}$ :
(1) For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}, B_{i, j, k}$ is an independent set.
(2) For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x y, x y^{\prime} \in E\left(G_{b}\right)$, if $y \in V\left(B_{i^{\prime}, j^{\prime}, k^{\prime}}\right)$, then $y^{\prime} \notin$ $V\left(B_{i^{\prime}, j^{\prime}, k^{\prime}}\right)$, where $i^{\prime} \in\{i-1, i\}, j^{\prime} \in\{j-1, j\}$, and $k^{\prime} \in\{k-1, k\}$.
(3) For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x y, x y^{\prime} \in E\left(B_{i, j, k}\right)$, if $y \in B_{i_{1}, j_{1}, k_{1}}$, then $y^{\prime} \notin$ $B_{i_{2}, j_{2}, k_{2}}$, where $\left(i_{1}, j_{1}, k_{1}\right)$ and $\left(i_{2}, j_{2}, k_{2}\right)$ satisfy all the following conditions:
(3.1) if $\left(i_{1}, j_{1}, k_{1}\right)=(i, j-1, k)$, then $\left(i_{2}, j_{2}, k_{2}\right) \notin\left\{\left(i_{2}, j-1, k_{2}\right) \mid i_{2} \in\{i-1, i, i+1\}, k_{2} \in\right.$ $\{k-1, k, k+1\}\}$.
(3.2) if $\left(i_{1}, j_{1}, k_{1}\right)=(i-1, j-1, k-1)$, then $\left(i_{2}, j_{2}, k_{2}\right) \notin\left\{\left(i_{2}, j_{2}, k_{2}\right) \mid i_{2} \in\{i-1, i\}, j_{2} \in\right.$ $\left.\{j-1, j\}, k_{2} \in\{k-1, k\}\right\}$.
(3.3) if $\left(i_{1}, j_{1}, k_{1}\right)=(i-1, j+1, k-1)$, then $\left(i_{2}, j_{2}, k_{2}\right) \notin\{(i-1, j, k-1),(i-1, j, k),(i-$ $1, j, k-1),(i-1, j, k-1),(i, j, k-1)\}$.
(3.4) if $\left(i_{1}, j_{1}, k_{1}\right)=(i, j-1, k-1)$, then $\left(i_{2}, j_{2}, k_{2}\right) \notin\{(i-1, j-1, k-1),(i, j-1, k-$ 1), $(i, j, k-1),(i, j-1, k),(i+1, j-1, k-1)\}$.
(3.5) if $\left(i_{1}, j_{1}, k_{1}\right)=(i, j-1, k+1)$, then $\left(i_{2}, j_{2}, k_{2}\right) \notin\{(i, j-1, k),(i, j-1, k)\}$.
(4) For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$, there is no 4-path satisfying the following conditions.
(4.1) $z_{1} x y z_{2}$ is the 4-path with $x \in B_{i, j, k}$, and $y \in B_{i-1, j+1, k+1}, z_{1} \in B_{i-1, a, b}$, and $z_{2} \in$ $B_{c, j, k}$, where $a \in\{j-1, j+1\}, b \in\{k-1, k+1\}, c \in\{i-2, i\}$.
(4.2) 4-vertex path $z_{1} x y z_{2}$ with $z_{1} \in B_{i-1, a, k-1}, z_{2} \in B_{c, j, b}, x \in B_{i, j, k}$, and $y \in B_{i-1, j+1, k+1}$, where $a \in\{j-1, j+1\}, b \in\{k-2, k\}, c \in\{i-2, i\}$.
(4.3) 4-vertex path $z_{2} x y z_{3}$ with $x=B_{i, j, k}, y=B_{i, j-1, k+1}$ and

$$
\begin{aligned}
z_{2} \in & B_{i-1, j-1, k-1} \cup B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i, j-1, k-1} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j-1, k} \\
& \cup B_{i+1, j-1, k+1}, \\
z_{3} \in & B_{i-1, j-2, k} \cup B_{i-1, j-1, k} \cup B_{i-1, j, k} \cup B_{i, j-2, k} \cup B_{i, j, k} \cup B_{i+1, j-2, k} \\
& \cup B_{i+1, j-1, k} \cup B_{i+1, j, k} .
\end{aligned}
$$

(5) For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x \in B_{i, j, k}, x$ has at most two neighbors in two of $B_{i-1, j-1, k-1}, B_{i+1, j-1, k-1}(u, v, w), B_{i-1, j+1, k^{\prime}}$, where $k^{\prime} \in\{k-1, k, k+1\}$.
(6) For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x \in B_{i, j, k}$, there is no 4 -star $K_{1,4}$ with edge set $E\left(K_{1,4}\right)=\left\{y x, z_{1} x, z_{2} x, z_{3} x\right\}$ such that $y \in B_{i-1, j-1, k-1}$,

$$
\begin{aligned}
& z_{1} \in B_{i-1, j-1, k+1} \cup B_{i-1, j, k+1} \cup B_{i-1, j+1, k-1} \cup B_{i-1, j+1, k} \cup B_{i-1, j+1, k+1}, \\
& z_{2} \in B_{i-1, j-1, k+1} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k+1}, \\
& z_{3} \in B_{i-1, j+1, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j, k-1} \cup B_{i+1, j+1, k-1},
\end{aligned}
$$

(7) There is a no $P_{4}^{+}$satisfying the following conditions:
(7.1) $V\left(P_{4}^{+}\right)=\left\{z_{1}, z_{2}, x, y, z_{3}\right\}$ and $E\left(P_{4}^{+}\right)=\left\{z_{1} x, z_{3} x, x y, y z_{2}\right\}$ such that $x=B_{i, j, k}, y=$ $B_{i-1, j+1, k-1}$, and
$z_{1} \in B_{i-1, j-1, k-1} \cup B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i-1, j, k+1} \cup B_{i-1, j+1, k-1} \cup B_{i-1, j+1, k} \cup B_{i-1, j+1, k+1}$, $z_{2} \in B_{i-2, j, k-2} \cup B_{i-2, j, k-1} \cup B_{i-2, j, k} \cup B_{i-1, j, k-2} \cup B_{i-1, j, k} \cup B_{i, j, k-2} \cup B_{i, j, k-1} \cup B_{i, j, k}$, $z_{3} \in B_{i-1, j-1, k-1} \cup B_{i-1, j+1, k-1} \cup B_{i, j-1, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j, k-1} \cup B_{i+1, j+1, k-1}$.
(7.2) $V\left(P_{4}^{+}\right)=\left\{z_{2}, z_{3}, x, y, z_{1}\right\}$ and $E\left(P_{4}^{+}\right)=\left\{z_{2} x, z_{3} x, x y, y z_{1}\right\}$ such that $x=B_{i, j, k}, y=$ $B_{i+1, j-1, k-1}$, and

$$
\left.\begin{array}{l}
z_{1} \in B_{i, j-2, k-2} \cup B_{i, j-2, k-1} \cup B_{i, j-2, k} \cup B_{i, j-1, k-2} \cup B_{i, j-1, k} \cup B_{i, j, k-2} \cup B_{i, j, k-1}, \\
z_{2} \in B_{i-1, j-1, k-1} \cup B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i, j-1, k-1} \cup B_{i, j-1, k} \cup B_{i+1, j-1, k+1} \cup B_{i+1, j-1, k-1} \\
\\
\cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k+1}, \\
z_{3} \in
\end{array} B_{i-1, j-1, k-1} \cup B_{i-1, j, k-1} \cup B_{i-2, j+1, k-1} \cup B_{i, j-1, k-1} \cup B_{i, j, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j-1, k-1}\right)
$$

(8) There is no 3-star $K_{1,3}$ with edge set $E\left(K_{1,3}\right)=\left\{x y, x z_{1}, x z_{2}\right\}$ such that $x=B_{i, j, k}, y=B_{i, j-1, k-1}$, and

$$
\begin{aligned}
& z_{2} \in B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k+1} \\
& z_{3} \in B_{i-1, j, k-1} \cup B_{i-1, j+1, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j, k-1} \cup B_{i+1, j+1, k-1}
\end{aligned}
$$

Proof. Assume that $\operatorname{dem}(G)=3$. Then $\operatorname{dem}\left(G_{b}\right)=3$. Let $\{u, v, w\}$ be a distance-edgemonitoring set of $G_{b}$.

Claim 1. $B_{i, j, k}$ is an independent set.

Proof. Assume, to the contrary, that $B_{i, j, k}$ is not an independent set. Let $x, y \in B_{i, j, k}(u, v, w)$. Then $d_{G}(x, u)=d_{G}(y, u)=i, d_{G}(x, v)=d_{G}(y, v)=j$, and $d_{G}(x, w)=d_{G}(y, w)=k$, and hence $x y$ can not be monitored by $u, v, w$ by Theorem 4.7, a contradiction.

From Claim 11, (1) holds.
For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x \in B_{i, j, k}$ with $x y, x y^{\prime} \in E\left(G_{b}\right)$, we assume that $y \in B_{i^{\prime}, j^{\prime}, k^{\prime}}$ and $\left(i^{\prime}, j^{\prime}, k^{\prime}\right) \neq(i, j, k)$. Then we have the following claim.

Claim 2. $y^{\prime} \notin B_{i^{\prime}, j^{\prime}, k^{\prime}}$ for $i^{\prime} \in\{i-1, i\}, j^{\prime} \in\{j-1, j\}$, and $k^{\prime} \in\{k-1, k\}$.
Proof. Assume, to the contrary, that $y^{\prime} \in B_{i^{\prime}, j^{\prime}, k^{\prime}}$. We first suppose that $y^{\prime} \in B_{i-1, j, k}$ (The case that $y, y^{\prime} \in B_{i, j-1, k}$ or $B_{i, j, k-1}$ is symmetric). Then $d_{G}(y, u)=d_{G}\left(y^{\prime}, u\right)=i-1$. From Theorem4.7, $x y$ can not be monitored by $u$. Since $x \in B_{i, j, k}(u, v, w)$ and $y \in B_{i-1, j, k}$, it follows that $d_{G}(y, v)=d_{G}(x, v)=$ $j$. From Theorem 4.7, $x y$ can not be monitored by $v$. Similarly, since $d_{G}(y, w)=d_{G}(x, w)=k$, it follows that $x y$ can not be monitored by $w$ by Theorem4.7, a contradiction.

Next, we suppose that $y, y^{\prime} \in B_{i-1, j-1, k}$ (The case that $y, y^{\prime} \in B_{i, j-1, k-1}$ or $B_{i-1, j, k-1}$ is symmetric). Since $d_{G}(y, u)=d_{G}\left(y^{\prime}, u\right)=i-1$, it follows from Theorem 4.7 that $x y$ can not be monitored by $u$. Similarly, $d_{G}(y, v)=d_{G}\left(y^{\prime}, v\right)=j-1$, xy can not be monitored by $v$. in addition, $d_{G}(y, w)=d_{G}(x, w)=k$, and hence $x y$ is not monitored by $w$, according to Theorem 4.7. So, $x y$ is not monitored by $u, v, w$, a contradiction.

Finally, if $y, y^{\prime} \in B_{i-1, j-1, k-1}$, it follows that $d_{G}(y, u)=d_{G}\left(y^{\prime}, u\right)=i-1, d_{G}(y, v)=d_{G}\left(y^{\prime}, v\right)=$ $j-1$, similarly, $d_{G}(y, w)=d_{G}\left(y^{\prime}, w\right)=k-1$, by Theorem 4.7, $x y$ is not monitored by $u, v, w$, a contradiction.

From Claim 2, (2) holds.
For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x y, x y^{\prime} \in E\left(G_{b}\right)$, we suppose that $y \in B_{i, j-1, k}$. Then we have the following claim.

Claim 3. $y^{\prime} \notin B_{i_{2}, j_{2}, k_{2}}$ for $\left(i_{2}, j_{2}, k_{2}\right) \in\left\{\left(i_{2}, j-1, k_{2}\right) \mid i_{2} \in\{i-1, i, i+1\}, k_{2} \in\{k-1, k, k+1\}\right\}$.

Proof. Assume, to the contrary, that $y^{\prime} \in B_{i_{2}, j_{2}, k_{2}}$. Since $x \in B_{i, j, k}$ and $y, y^{\prime}$ are both neighbors of $x$ and $y \in B_{i, j-1, k}$, it follows that $d_{G}(y, u)=d_{G}(x, u)=i$ and $d_{G}(y, w)=d_{G}(x, w)=k$. From Theorem 4.7, $x y$ is not monitored by $u$ and $w$. Since $y^{\prime} \in B_{i_{2}, j-1, k_{2}}$ and $y \in B_{i, j-1, k}$, it follows that $d_{G}\left(y^{\prime}, v\right)=d_{G}(y, v)=j-1$, and hence $x y$ is not monitored by $v$, a contradiction.

By Claim 3, (3.1) holds. By the same method, we can prove that (3.2)-(3.6) all hold.
Claim 4. For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$, there is no 4 -vertex path $z_{1} x y z_{2}$ such that $x \in B_{i, j, k}$, $y \in B_{i-1, j+1, k+1}, z_{1} \in B_{i-1, a, b}$, and $z_{2} \in B_{c, j, k}$, where $a \in\{j-1, j+1\}, b \in\{k-1, k+1\}, c \in\{i-2, i\}$.

Proof. Assume, to the contrary, that there is a 4-path satisfying the conditions of this claim. Then $d_{G}(y, u)=d_{G}\left(z_{1}, u\right)=i-1, d_{G}(x, v)=d_{G}\left(z_{2}, v\right)=j$ and $d_{G}(x, w)=d_{G}\left(z_{2}, w\right)=k$, and from Theorem 4.7, $x y$ can not be monitored by $u, v, w$, respectively, a contradiction.

By Claim [4. (4.1) holds. Similarly, the conditions (4.2) and (4.3) can be easily proved.
Claim 5. For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x \in B_{i, j, k}, x$ has at most two neighbors in two of $B_{i-1, j-1, k-1}, B_{i+1, j-1, k-1}(u, v, w), B_{i-1, j+1, k^{\prime}}$, where $k^{\prime} \in\{k-1, k, k+1\}$.

Proof. Assume, to the contrary, that $x \in B_{i, j, k}$ has three neighbors $y, y^{\prime}, y^{\prime \prime}$ such that $y \in B_{i-1, j-1, k-1}$, $y^{\prime} \in B_{i+1, j-1, k-1}(u, v, w), y^{\prime \prime} \in B_{i-1, j^{\prime}, k^{\prime}}$ where $k^{\prime} \in\{k-1, k, k+1\}$ and $j^{\prime} \in\{j-1, j, j+1\}$. Since $y \in B_{i-1, j-1, k-1}$ and $y^{\prime} \in B_{i+1, j-1, k-1}$, it follows that $d_{G}(y, v)=d_{G}\left(y^{\prime}, v\right)=j-1$ and $d_{G}(y, w)=$ $d_{G}\left(y^{\prime}, w\right)=k-1$. From Theorem 4.7, $x y$ is not monitored by $v, w$. Since $y \in B_{i-1, j-1, k-1}$ and $y^{\prime \prime} \in B_{i-1, j^{\prime}, k^{\prime}}$, it follows that $d_{G}(y, u)=d_{G}\left(y^{\prime \prime}, u\right)=i-1$, and hence $x y$ is not monitored by $u$, and so $x y$ is not monitored by $u, v, w$, a contradiction.

From Claim 5, (5) holds.

Claim 6. For any $i, j, k \in\{0,1,2, \ldots, \operatorname{diam}(G)\}$ and any $x \in B_{i, j, k}$, there is no 4-star $K_{1,4}$ with edge set $E\left(K_{1,4}\right)=\left\{y x, z_{1} x, z_{2} x, z_{3} x\right\}$ such that $y \in B_{i-1, j-1, k-1}$,

$$
\begin{aligned}
& z_{1} \in B_{i-1, j-1, k+1} \cup B_{i-1, j, k+1} \cup B_{i-1, j+1, k-1} \cup B_{i-1, j+1, k} \cup B_{i-1, j+1, k+1}, \\
& z_{2} \in B_{i-1, j-1, k+1} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k+1}, \\
& z_{3} \in B_{i-1, j+1, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j, k-1} \cup B_{i+1, j+1, k-1},
\end{aligned}
$$

Proof. Assume, to the contrary, that $x \in B_{i, j, k}$ has four neighbors $y, z_{1}, z_{2}, z_{3}$ satisfying the conditions of this claim. Then $d_{G}(y, u)=d_{G}\left(z_{1}, u\right)=i-1$. From Theorem 4.7, $x y$ can not be monitored by $u$. Similarly, since $d_{G}(y, v)=d_{G}\left(z_{2}, v\right)=j-1$, it follows from Theorem4.7 that $x y$ can not be monitored by $v$. Similarly, since $d_{G}(y, w)=d_{G}\left(z_{3}, w\right)=k-1$, it follows that $x y$ can not be monitored by $w$, a contradiction.

From Claim 6, (6) holds.
Claim 7. There is no $P_{4}^{+}$with vertex set $\left\{z_{1}, z_{2}, x, y, z_{3}\right\}$ and edge set $\left\{z_{1} x, z_{2} x, x y, y z_{3}\right\}$ such that $x \in B_{i, j, k}, y \in B_{i-1, j+1, k-1}$, and

$$
\begin{aligned}
& z_{1} \in B_{i-1, j-1, k-1} \cup B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i-1, j, k+1} \cup B_{i-1, j+1, k-1} \cup B_{i-1, j+1, k} \cup B_{i-1, j+1, k+1}, \\
& z_{2} \in B_{i-2, j, k-2} \cup B_{i-2, j, k-1} \cup B_{i-2, j, k} \cup B_{i-1, j, k-2} \cup B_{i-1, j, k} \cup B_{i-1, j, k-1} \cup B_{i, j, k-2} \cup B_{i, j, k} \cup B_{i, j, k-1} \\
& z_{3} \in B_{i-1, j-1, k-1} \cup B_{i-1, j+1, k-1} \cup B_{i, j-1, k-1} \cup B_{i, j, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j, k-1} \\
& \\
& \quad \cup B_{i+1, j+1, k-1} .
\end{aligned}
$$

Proof. Assume, to the contrary, that there is $P_{4}^{+}$satisfying the conditions of this claim. Since $d_{G}(y, u)=d_{G}\left(z_{1}, u\right)=i-1, d_{G}(x, v)=d_{G}\left(z_{2}, v\right)=j$ and $d_{G}(y, w)=d_{G}\left(z_{3}, w\right)=k-1$, it follows from Theorem 4.7 that $x y$ can not be monitored by $u, v, w$, respectively, a contradiction.

From Claim 7, (7.1) holds. Similarly, we can prove that (7.2) holds.
Claim 8. There is no 3-star $K_{1,3}$ with vertex set $\left\{z_{1}, z_{2}, x, y\right\}$ and edge set $\left\{x y, x z_{1}, x z_{2}\right\}$ such that $x=B_{i, j, k}, y=B_{i, j-1, k-1}$, and

$$
\begin{aligned}
& z_{1} \in B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i, j-1, k} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k+1}, \\
& z_{2} \in B_{i-1, j-1, k-1} \cup B_{i-1, j, k-1} \cup B_{i-1, j+1, k-1} \cup B_{i, j, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j, k-1} \cup B_{i+1, j+1, k-1}
\end{aligned}
$$

Proof. Assume, to the contrary, that there is a $K_{1,3}$ such that $V\left(K_{1,3}\right)=\left\{z_{1}, z_{2}, x, y\right\}$ and $E\left(K_{1,3}\right)=$ $\left\{x y, x z_{1}, x z_{2}\right\}$. Since $d_{G}(x, u)=d_{G}(y, u)=i, d_{G}(y, v)=d_{G}\left(z_{1}, v\right)=j-1$ and $d_{G}(y, w)=d_{G}\left(z_{2}, w\right)=$ $k-1$, it follows from Theorem 4.7 that $x y$ is not monitored by $u, v, w$, a contradiction.

From Claim 8, (8) holds.
Conversely, we assume that there exists three vertices $u, v, w$ in $G_{b}$ such that all of the conditions (1)-(8) holds in $G_{b}$. It suffices to prove that $\{u, v, w\}$ is a distance-edge-monitoring set in $G_{b}$, and hence $\operatorname{dem}(G)=3$. Let $x y$ be any edge of $G$ with $x \in B_{i, j, k}$. Since (1) holds, it follows that $y \notin B_{i, j, k}$. Then we have the following cases:

Case 1. $y \in B_{i, j-1, k}$ or $y \in B_{i-1, j, k}$ or $y \in B_{i, j, k-1}$ or $y \in B_{i, j+1, k}$ or $y \in B_{i, j, k+1}$ or $y \in B_{i+1, j, k}$.

For $x \in B_{i, j, k}$ and $y \in B_{i, j-1, k}$, we assume that $x y$ can not be monitored by $\{u, v, w\}$. Then there is a path $P_{j}$ of length $j$ from $x$ to $v$ such that $x y \notin E\left(P_{j}\right)$. Let $z_{2}$ be the neighbor of $x$ in $P_{j}$. From Lemma [5.1, we have $z_{2} \in B_{i^{\prime}, j-1, k^{\prime}}$, where $i^{\prime} \in\{i-1, i, i+1\}$ and $k^{\prime} \in\{k-1, k, k+1\}$, which contradicts to the condition (3.1). Suppose that $x \in B_{i, j, k}$ and $y \in B_{i, j-1, k}$. Then $x y$ can be monitored by $\{u, v, w\}$. Similarly, the edges $x y$ can be also monitored by $\{u, v, w\}$, where $y \in B_{i-1, j, k}$ or $y \in B_{i, j-1, k}$ or $y \in B_{i, j, k-1}$ or $y \in B_{i, j+1, k+1}$ or $y \in B_{i+1, j, k+1}$ or $y \in B_{i+1, j+1, k}$.

Case 2. $y \in B_{i-1, j-1, k-1}$ or $y \in B_{i+1, j+1, k+1}$.
For $x \in B_{i, j, k}$ and $y \in B_{i-1, j-1, k-1}$, we assume that $x y$ is not monitored by $\{u, v, w\}$. Then there exists a path $P_{i}, P_{j}, P_{k}$ of length $i, j, k$ from $x$ to $u, v, w$ such that $x y \notin E\left(P_{i}\right)$ and $x y \notin$ $E\left(P_{j}\right)$ and $x y \notin E\left(P_{k}\right)$, respectively. Let $z_{1}, z_{2}, z_{3}$ be the neighbors of $x$ in $P_{i}, P_{j}, P_{k}$, respectively. Then $z_{1} \in B_{i-1, j^{\prime}, k^{\prime}}$ and $z_{2} \in B_{i^{\prime \prime}, j-1, k^{\prime \prime}}$ and $z_{3} \in B_{i^{\prime \prime \prime}, j^{\prime \prime \prime}, k-1}$. where $i^{\prime \prime}, i^{\prime \prime \prime} \in\{i-1, i, i+1\}$, $j^{\prime}, j^{\prime \prime \prime} \in\{j-1, j, j+1\}$ and $k^{\prime}, k^{\prime \prime} \in\{k-1, k, k+1\}$. Since $x \in B_{i, j, k}$ and $y \in B_{i-1, j-1, k-1}$, it follows from the conditions (2) and (3.2) that for any $z_{i} \in N(x)(1 \leq i \leq 3)$, it follows that $z_{1} \notin\left\{B_{i-1, j-1, k-1}, B_{i-1, j-1, k}, B_{i-1, j, k-1}, B_{i-1, j, k}\right\}, z_{2} \notin\left\{B_{i-1, j-1, k-1}, B_{i-1, j-1, k}, B_{i, j-1, k-1}, B_{i, j-1, k}\right.$ $\}, z_{3} \notin\left\{B_{i-1, j-1, k-1}, B_{i-1, j, k-1}, B_{i, j-1, k-1}, B_{i, j, k-1}\right\}$, and hence there is a 4 -star with edge set $\left\{y x, z_{1} x, z_{2} x, z_{3} x\right\}$ such that $y \in B_{i-1, j-1, k-1}$,

$$
\begin{aligned}
& z_{1} \in B_{i-1, j-1, k+1} \cup B_{i-1, j, k+1} \cup B_{i-1, j+1, k-1} \cup B_{i-1, j+1, k} \cup B_{i-1, j+1, k+1}, \\
& z_{2} \in B_{i-1, j-1, k+1} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k+1}, \\
& z_{3} \in B_{i-1, j+1, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j, k-1} \cup B_{i+1, j+1, k-1},
\end{aligned}
$$

which contradicting to the condition (6).
So, $x y$ can be monitored by $\{u, v, w\}$. Similarly, the edges $x y$ can be also monitored by $\{u, v, w\}$, where $y \in B_{i+1, j+1, k+1}$.

Case 3. $y \in B_{i-1, j+1, k-1}$ or $y \in B_{i+1, j-1, k-1}$ or $y \in B_{i-1, j-1, k+1}$ or $y \in B_{i+1, j-1, k+1}$ or $y \in$ $B_{i-1, j+1, k+1}$ or $y \in B_{i+1, j+1, k-1}$.

For $x \in B_{i, j, k}$ and $y \in B_{i-1, j+1, k-1}$, we assume that $x y$ is not monitored by $\{u, v, w\}$. Then there exists a path $P_{i}$ of length $i$ from $x$ to $u$ such that $x y \notin E\left(P_{i}\right)$, and there exist two paths $P_{j+1}, P_{k}$ of length $j+1, k$ from $y$ to $v, w$ such that $x y \notin E\left(P_{j+1}\right) \cup E\left(P_{k}\right)$, respectively. Let $z_{1}, z_{3}$ be the neighbors of $x$ on the $P_{i}, P_{k}$, respectively. In addition, let $z_{2}$ be the neighbors of $y$ on the $P_{j+1}$.

Thus, there is a 5 -vertex graph $P_{4}^{+}$with $z_{1} \in B_{i-1, a, b}, z_{2} \in B_{a^{\prime}, j, b^{\prime}}, x \in B_{i, j, k}, y \in B_{i-1, j+1, k+1}$, $z_{3} \in B_{a^{\prime \prime}, b^{\prime \prime}, k-1}, a, b^{\prime \prime} \in\{j-1, j, j+1\}, b \in\{k-1, k, k+1\}, a^{\prime} \in\{i-2, i-1, i\}, b^{\prime} \in\{k-2, k-1, k\}$, $a^{\prime \prime} \in\{i-1, i, i+1\}$.

Since $y \in B_{i-1, j+1, k-1}$, it follows from the condition (2) and (3.3) that for any $z_{i} \in N(x)(1 \leq i \leq$ 3), we have $z_{1} \notin\left\{B_{i-1, j, k-1}, B_{i-1, j, k}\right\}, z_{2} \notin\left\{B_{i-1, j, k-1}\right\}, z_{3} \notin\left\{B_{i-1, j, k-1}, B_{i, j, k-1}\right\}$. Furthermore, we have

$$
\begin{aligned}
& z_{1} \in B_{i-1, j-1, k-1} \cup B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i-1, j, k+1} \cup B_{i-1, j+1, k-1} \cup B_{i-1, j+1, k} \cup B_{i-1, j+1, k+1}, \\
& z_{2} \in B_{i-2, j, k-2} \cup B_{i-2, j, k-1} \cup B_{i-2, j, k} \cup B_{i-1, j, k-2} \cup B_{i-1, j, k} \cup B_{i, j, k-2} \cup B_{i, j, k-1} \cup B_{i, j, k} . \\
& z_{3} \in B_{i-1, j-1, k-1} \cup B_{i-1, j+1, k-1} \cup B_{i, j-1, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j, k-1} . \\
& \quad \cup B_{i+1, j+1, k-1} .
\end{aligned}
$$

which contradicts to the condition (7.1).
Case 4. $y \in B_{i, j-1, k-1}$ or $y \in B_{i-1, j-1, k}$ or $y \in B_{i-1, j, k-1}$ or $y \in B_{i, j+1, k+1}$ or $y \in B_{i+1, j+1, k}$ or $y \in B_{i+1, j, k+1}$.

For $x \in B_{i, j, k}$ and $y \in B_{i-1, j+1, k-1}$, we assume that $x y$ is not monitored by $\{u, v, w\}$. Then there is a path of length $j$ from $x$ to $v$, say $P_{j}$, such that $x y \notin E\left(P_{j}\right)$. Similarly, there is a path of length $k$, say $P_{k}$, from $y$ to $w$ such that $x y \notin E\left(P_{k}\right)$. Let $z_{2}, z_{3}$ be the neighbors of $x$ on the $P_{j}, P_{k}$, respectively. Then there is a 3 -star $K_{1,3}$ with edge set $\left\{x y, x z_{2}, x z_{3}\right\}$ such that $x \in B_{i, j, k}, y \in B_{i, j-1, k-1}, z_{2} \in B_{a, j-1, c}$, $z_{3} \in B_{a^{\prime}, b^{\prime}, k-1}$, where $a, a^{\prime} \in\{i-1, i, i+1\}, c \in\{k-1, k, k+1\}, b^{\prime} \in\{j-1, j, j+1\}$. If $y \in B_{i, j-1, k-1}$, then for any $z_{i} \in N(x)(2 \leq i \leq 3)$, it follows from the conditions (2) and (3.4), that

$$
\begin{aligned}
& z_{2} \notin B_{i-1, j-1, k-1} \cup B_{i, j-1, k-1} \cup B_{i, j-1, k} \cup B_{i+1, j-1, k-1}, \\
& z_{3} \notin B_{i-1, j-1, k-1} \cup B_{i, j-1, k-1} \cup B_{i, j, k-1} \cup B_{i+1, j-1, k-1},
\end{aligned}
$$

and hence

$$
\begin{aligned}
x & \in B_{i, j, k}, y \in B_{i, j-1, k-1}, \\
z_{2} & \in B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k} \cup B_{i+1, j-1, k+1}, \\
z_{3} & \in B_{i-1, j, k-1} \cup B_{i-1, j+1, k-1} \cup B_{i, j+1, k-1} \cup B_{i+1, j, k-1} \cup B_{i+1, j+1, k-1},
\end{aligned}
$$

which contradicts to the condition (8).
Similarly, if $x \in B_{i, j, k}$ and $y \in B_{i, j-1, k-1}$, then $x y$ can be monitored by $\{u, v, w\}$. By the same method, we can prove that the edges $x y$ can be also monitored by $\{u, v, w\}$, where $y \in B_{i-1, j-1, k}$ or $y \in B_{i-1, j, k-1}$ or $y \in B_{i, j+1, k+1}$ or $y \in B_{i+1, j+1, k}$ or $y \in B_{i+1, j, k+1}$.

Case 5. $y \in B_{i, j-1, k+1}$ or $y \in B_{i-1, j, k+1}$ or $y \in B_{i, j+1, k-1}$ or $y \in B_{i+1, j-1, k}$ or $y \in B_{i+1, j, k-1}$ or $y \in B_{i-1, j+1, k}$.

For $x \in B_{i, j, k}$ and $y \in B_{i, j-1, k+1}$, we assume that $x y$ is not monitored by $\{u, v, w\}$. Then there is a path of length $k+1$ from $y$ to $w$, say $P_{k+1}$, such that $x y \notin E\left(P_{k+1}\right)$, and there is a path of length $j$ from $x$ to $v$, say $P_{j}$, such that $x y \notin E\left(P_{j}\right)$. Let $z_{3}, z_{2}$ be the neighbors of $y, x$ on the $P_{k+1}, P_{j}$, respectively. Thus, there is a 4-path $P_{4}$ with $V\left(P_{4}\right)=\left\{z_{2}, x, y, z_{3}\right\}$ and $E\left(P_{4}\right)=\left\{z_{2} x, x y, y z_{3}\right\}$ such that $x \in B_{i, j, k}$, $y \in B_{i, j-1, k+1}, z_{2} \in B_{a, j-1, b}$, and $z_{3} \in B_{a^{\prime}, b^{\prime}, k}$, where $a, a^{\prime} \in\{i-1, i, i+1\}, b \in\{k-1, k, k+1\}$, $b^{\prime} \in\{j-2, j-1, j\}$.

From the conditions (2) and (3.5), we have $y \in B_{i, j-1, k+1}$, and for any $z_{i} \in N(x)(2 \leq i \leq 3)$, we have $z_{3} \notin\left\{B_{i, j-1, k},\right\}, z_{2} \notin\left\{B_{i, j-1, k}\right\}$, and hence $x=B_{i, j, k}, y=B_{i, j-1, k+1}$,

$$
\begin{aligned}
z_{2} \in & B_{i-1, j-1, k-1} \cup B_{i-1, j-1, k} \cup B_{i-1, j-1, k+1} \cup B_{i, j-1, k-1} \cup B_{i, j-1, k+1} \cup B_{i+1, j-1, k-1} \cup B_{i+1, j-1, k} \\
& \cup B_{i+1, j-1, k+1}, \\
z_{3} \in & B_{i-1, j-2, k} \cup B_{i-1, j-1, k} \cup B_{i-1, j, k} \cup B_{i, j-2, k} \cup B_{i, j, k} \cup B_{i+1, j-2, k} \\
& \cup B_{i+1, j-1, k} \cup B_{i+1, j, k} .
\end{aligned}
$$

which contradicts the condition (4.3). If $x \in B_{i, j, k}$ and $y \in B_{i, j-1, k+1}$, then $x y$ can be monitored by $\{u, v, w\}$. Similarly, the edges $x y$ can be also monitored by $\{u, v, w\}$, where $y \in B_{i-1, j, k+1}$ or $y \in B_{i, j+1, k-1}$ or $y \in B_{i+1, j-1, k}$ or $y \in B_{i+1, j, k-1}$ or $y \in B_{i-1, j+1, k}$.

If $x \in B_{i, j, k}$, from it follows Lemma 5.1, that $y \in T$, where

$$
T=\left\{B_{i^{\prime}, j^{\prime}, k^{\prime}} \mid i^{\prime} \in\{i-1, i, i+1\}, j^{\prime} \in\{j-1, j, j+1\}, k^{\prime} \in\{k-1, k, k+1\}\right\} .
$$

From the above cases, the vertex set $B_{i, j, k}(u, v, w)$ has the arbitrariness. Then the $x y$ in $E\left(G_{b}\right)$ can be monitored by $\{u, v, w\}$, and hence $\{u, v, w\}$ is a distance-edge-monitoring set in $G_{b}$, and so $\operatorname{dem}(G)=3$.

## 6 Conclusion

In this paper, we have continued the study of distance-edge-monitoring sets, a new graph parameter recently introduced by Foucaud et al. [8], which is useful in the area of network monitoring. In particular, we have given upper and lower bounds on the parameters $P(M, e), E M(x)$, $\operatorname{dem}(G)$, respectively, and extremal graphs attaining the bounds were characterized. We also characterized the graphs with $\operatorname{dem}(G)=3$.

For future work, it would be interesting to study distance-edge monitoring sets in further standard graph classes, including pyramids, Sierpińki-type graphs, circulant graphs, graph products, or line graphs. In addition, characterizing the graphs with $\operatorname{dem}(G)=n-2$ would be of interest, as well as clarifying further the relation of the parameter $\operatorname{dem}(G)$ to other standard graph parameters, such as arboricity, vertex cover number and feedback edge set number.

## References

[1] E. Bampas, D. Bilò, G. Drovandi, L. Gualà, R. Klasing and G. Proietti. Network verification via routing table queries. Journal of Computer and System Sciences 81(1):234-248, 2015.
[2] J. Baste, F. Beggas, H. Kheddouci and I. Sau. On the parameterized complexity of the edge monitoring problem. Information Processing Letters 121:39-44, 2017.
[3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalák and L. S. Ram. Network discovery and verification. IEEE Journal on Selected Areas in Communications 24(12):21682181, 2006.
[4] Y. Bejerano and R. Rastogi. Robust monitoring of link delays and faults in IP networks. IEEE/ACM Transactions on Networking 14(5):1092-1103, 2006.
[5] D. Bilò, T. Erlebach, M. Mihalák and P. Widmayer. Discovery of network properties with all-shortest-paths queries. Theoretical Computer Science 411(14-15):1626-1637, 2010.
[6] G. Chartrand, L. Lesniak and P. Zhang. Graphs \& digraphs. Chapman \& Hall, London, 2015.
[7] L. Dall'Asta, J. I. Alvarez-Hamelin, A. Barrat, A. Vázquez and A. Vespignani. Exploring networks with traceroute-like probes: Theory and simulations. Theoretical Computer Science 355(1):6-24, 2006.
[8] F. Foucaud, S.-S. Kao, R. Klasing, M. Miller and J. Ryan. Monitoring the edges of a graph using distances. Discrete Applied Mathematics 319:424-438, 2022.
[9] R. Govindan and H. Tangmunarunkit. Heuristics for Internet map discovery. In: Proc. 19th IEEE International Conference on Computer Communications (INFOCOM 2000), pp. 1371-1380, 2000.
[10] F. Harary and R. A. Melter. On the metric dimension of a graph. Ars Combinatoria 2:191-195, 1976.
[11] A. Kelenc, D. Kuziak, A. Taranenko and I. G. Yero. Mixed metric dimension of graphs. Applied Mathematics and Computation 314:429-438, 2017.
[12] A. Kelenc, N. Tratnik and I. G. Yero. Uniquely identifying the edges of a graph: The edge metric dimension. Discrete Applied Mathematics 251:204-220, 2018.
[13] P. Manuel, S. Klavžar, A. Xavier, A. Arokiaraj and E. Thomas. Strong edge geodetic problem in networks. Open Mathematics 15:1225-1235, 2017.
[14] O. R. Oellermann and J. Peters-Fransen. The strong metric dimension of graphs and digraphs. Discrete Applied Mathematics 155:356-364, 2007.
[15] A. Sebő and E. Tannier. On metric generators of graphs. Mathematics of Operations Research 29(2):383-393, 2004.
[16] P. J. Slater. Leaves of trees. Congressus Numerantium 14:549-559, 1975.


[^0]:    *Supported by the National Science Foundation of China (Nos. 12061059, 11601254, 11551001, 11161037, 61763041, 11661068, and 11461054), the Qinghai Key Laboratory of Internet of Things Project (2017-ZJ-Y21) and Science \& Technology development Fund of Tianjin Education Commission for Higher Education, China (2019KJ090).
    ${ }^{\dagger}$ School of Computer, Qinghai Normal University, Xining, Qinghai 810008, China. cxuyang@aliyun.com
    ${ }^{\ddagger}$ Corresponding author: Université de Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR 5800, Talence, France. ralf.klasing@labri.fr
    ${ }^{\S}$ Academy of Plateau Science and Sustainability, Xining, Qinghai 810008, China. maoyaping@ymail.com
    ${ }^{\text {I }}$ School of Mathematical Science, Tianjin Normal University, Tianjin, 300387, China. dengyuqiu1980@126.com

