COMMUTING EULERIAN OPERATORS

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ABSTRACT. Motivated by the work of Visontai and Dey-Sivasubramanian on the gammapositivity of some polynomials, we find the commutative property of a pair of Eulerian operators. As an application, we show the bi-gamma-positivity of the descent polynomials on permutations of the multiset $\{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$, where $0 \leq a_i \leq 2$. Therefore, these descent polynomials are all alternatingly increasing, and so they are unimodal with modes in the middle.

Keywords: Eulerian operators; Eulerian polynomials; Unimodality; Gamma-positivity

1. INTRODUCTION

Let $f(x) = \sum_{i=0}^{n} f_i x^i$ be a polynomial with nonnegative coefficients. We say that f(x) is unimodal if $f_0 \leq f_1 \leq \cdots \leq f_k \geq f_{k+1} \geq \cdots \geq f_n$ for some k, where the index k is called the mode of f(x). It is well known that if f(x) with only nonpositive real zeros, then f(x) is unimodal (see [6, p. 419] for instance). If f(x) is symmetric with the center of symmetry $\lfloor n/2 \rfloor$, i.e., $f_i = f_{n-i}$ for all indices $0 \leq i \leq n$, then it can be expanded as

$$f(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1+x)^{n-2k}.$$

The polynomial f(x) is γ -positive if $\gamma_k \ge 0$ for all $0 \le k \le \lfloor n/2 \rfloor$. Clearly, γ -positivity implies symmetry and unimodality. Let $f(x, y) = \sum_{i=0}^{n} f_i x^i y^{n-i}$ be a homogeneous bivariate polynomial. We say that f(x, y) is bivariate γ -positive with the center of symmetry $\frac{n}{2}$ if f(x, y) can be written as follows:

$$f(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k (xy)^k (x+y)^{n-2k}.$$

There has been considerable recent interest in the study of the γ -positivity of polynomials, see [1, 4] for details. In particular, Brändén [4, Remark 7.3.1] noted that if f(x) is symmetric and has only real zeros, then it is γ -positive.

Let $f(x) = \sum_{i=0}^{n} f_i x^i$, where $f_n \neq 0$. Following [2, 5], there is a unique symmetric decomposition f(x) = a(x) + xb(x), where

$$a(x) = \frac{f(x) - x^{n+1}f(1/x)}{1-x}, \ b(x) = \frac{x^n f(1/x) - f(x)}{1-x}.$$

According to [15, Definition 8], the polynomial f(x) is said to be *bi-\gamma-positive* if both a(x) and b(x) are γ -positive. Thus γ -positivity is a special case of bi- γ -positivity. Following [17, Definition 2.9], the polynomial f(x) is alternatingly increasing if

$$f_0 \leqslant f_n \leqslant f_1 \leqslant f_{n-1} \leqslant \cdots \leqslant f_{\lfloor (n+1)/2 \rfloor}.$$

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Brändén and Solus [5] pointed out that f(x) is alternatingly increasing if and only if the pair of polynomials in its symmetric decomposition are both unimodal and have only nonnegative coefficients. Therefore, bi- γ -positivity is stronger than alternatingly increasing property. The alternatingly increasing property first appeared in the work of Beck and Stapledon [2]. Recently, Beck-Jochemko-McCullough [3], Brändén-Solus [5] and Solus [19] studied the alternatingly increasing property of several h^* -polynomials as well as some refined Eulerian polynomials.

A multipermutation of a multiset is a sequence of its elements. Throughout this paper, we always let $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{P}^n$. Denote by $\mathfrak{S}_{\mathbf{m}}$ the set of all multipermutations of the multiset $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$, where *i* appears m_i times. Set $m = \sum_{i=1}^n m_i$. For $\pi = \pi_1 \pi_2 \ldots \pi_m \in \mathfrak{S}_{\mathbf{m}}$, we always assume that $\pi_0 = \pi_{m+1} = 0$ (except where explicitly stated). If $i \in \{0, 1, 2, \ldots, m\}$, then π_i is called an *ascent* (resp. *descent*, *plateau*) if $\pi_i < \pi_{i+1}$ (resp. $\pi_i > \pi_{i+1}, \pi_i = \pi_{i+1}$). Let $\operatorname{asc}(\pi)$ (resp. $\operatorname{des}(\pi)$, $\operatorname{plat}(\pi)$) be the number of ascents (resp. descents, plateaux) of π . The *multiset Eulerian polynomials* $A_{\mathbf{m}}(x)$ are defined by

$$A_{\mathbf{m}}(x) = \sum_{\pi \in \mathfrak{S}_{\mathbf{m}}} x^{\operatorname{asc}(\pi)} = \sum_{\pi \in \mathfrak{S}_{\mathbf{m}}} x^{\operatorname{des}(\pi)}$$

A classical result of MacMahon [16, Vol 2, Chapter IV, p. 211] says that

$$\frac{A_{\mathbf{m}}(x)}{(1-x)^{1+m}} = \sum_{k \ge 0} \binom{k+m_1}{m_1} \binom{k+m_2}{m_2} \cdots \binom{k+m_n}{m_n} x^{k+1}.$$
 (1)

Let \mathfrak{S}_n be the set of all permutations of $\{1, 2, \ldots, n\}$. As usual, we write $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$. Denote by $A_{\pi(\mathbf{m})}(x)$ the descent polynomial on multipermutations of $\{\pi_1^{m_1}, \pi_2^{m_2}, \ldots, \pi_n^{m_n}\}$. It follows from (1) that

$$A_{\mathbf{m}}(x) = A_{\pi(\mathbf{m})}(x). \tag{2}$$

When $\mathbf{m} = (1, 1, ..., 1)$, the polynomial $A_{\mathbf{m}}(x)$ is reduced to the classical Eulerian polynomial $A_n(x)$. In other words,

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{asc}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)}.$$

Simion [18, Section 2] found that $A_{\mathbf{m}}(x)$ is real-rootedness for any \mathbf{m} . When $\mathbf{m} = (p, p, \dots, p)$, Carlitz-Hoggatt [7] showed that $A_{\mathbf{m}}(x)$ is symmetric, where p is a given positive integer. By [4, Remark 7.3.1], an immediate consequence is the following well known result.

Proposition 1. For any \mathbf{m} , the multiset Eulerian polynomials $A_{\mathbf{m}}(x)$ are all unimodal. When $\mathbf{m} = (p, p, \dots, p)$, the polynomial $A_{\mathbf{m}}(x)$ is γ -positive, and so its mode is in the middle.

Recently, there has been much work on the descent polynomials of permutations over multisets, see [11, 12, 13, 14, 21] for instance. In particular, Lin-Xu-Zhao [13] found a combinatorial interpretation for the γ -coefficients of $A_{\mathbf{m}}(x)$ via the model of weakly increasing trees, where $\mathbf{m} = (p, p, \dots, p)$. Motivated by Proposition 1, it is natural to consider the following problem.

Problem 2. For any **m**, could we characterize the location of the mode of $A_{\mathbf{m}}(x)$?

A bivariate version of the Eulerian polynomial over the symmetric group is given as follows:

$$A_n(x,y) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{asc}(\pi)} y^{\operatorname{des}(\pi)}$$

In particular, $A_n(x,1) = A_n(1,x) = A_n(x)$. Carlitz and Scoville [8] found that

$$A_{n+1}(x,y) = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)A_n(x,y), \ A_1(x,y) = xy.$$

Using the following Eulerian operator

$$T = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right),\tag{3}$$

Foata and Schützenberger [10] discovered that

$$A_n(x,y) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \gamma(n,k) (xy)^k (x+y)^{n+1-2k},$$

where $\gamma(n, k)$ are all nonnegative integers. Applying the same idea, Visontai [20] investigated the joint generating polynomial of descents and inverse descents, Dey-Sivasubramanian [9] studied the descent polynomials on permutations in the alternating group. As an illustration, we now recall a result on the Eulerian operator T, which is a slightly variant of [9, Lemma 5].

Lemma 3. Let f(x, y) be a bivariate γ -positive polynomial with the center of symmetry $\frac{n}{2}$. Then T(f(x, y)) is a bivariate γ -positive polynomial with the center of symmetry $\frac{n+1}{2}$.

Motivated by the work of Visontai [20] and Dey-Sivasubramanian [9], in this paper we introduce the following Eulerian operator

$$G = xy^2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + \frac{x^2 y^2}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + x^2 y^2 \frac{\partial^2}{\partial x \partial y}.$$
 (4)

In the next section, we prove the commutative property of the Eulerian operators T and G. In Section 3, we prove following result, which gives a partial answer to Problem 2.

Theorem 4. Let $\mathbf{m} = \{m_1, m_2, \ldots, m_n\}$, where $0 \leq m_i \leq 2$. The Eulerian polynomials $A_{\mathbf{m}}(x)$ are all bi- γ -positive, and so $A_{\mathbf{m}}(x)$ are all alternating increasing. More precisely, when $\mathbf{m} = \{1, 1, \ldots, 1\}$ or $\mathbf{m} = \{2, 2, \ldots, 2\}$, the polynomial $A_{\mathbf{m}}(x)$ is γ -positive; for the other cases, the polynomial $A_{\mathbf{m}}(x)$ can be written as a sum of two γ -positive polynomials.

In the following discussion, we always set $\mathbf{m} = \{m_1, m_2, \dots, m_n\}$, where $0 \leq m_i \leq 2$. Let

$$A_{\mathbf{m}}(x,y) = \sum_{\pi \in \mathfrak{S}_{\mathbf{m}}} x^{\operatorname{des}(\pi)} y^{m+1-\operatorname{des}(\pi)}.$$

where $m = \sum_{i=1}^{n} m_i$. Clearly, $A_{\mathbf{m}}(x, 1) = A_{\mathbf{m}}(x)$. For convenience, set $A_{\emptyset}(x, y) = x$.

Example 5. We have

$$\begin{split} A_{\{1\}}(x,y) &= xy, \ A_{\{2\}}(x,y) = xy^2, \ A_{\{1,1\}}(x,y) = xy(x+y), \\ A_{\{1,1,1\}}(x,y) &= xy(x^2+4xy+y^2), \ A_{\{1,2\}}(x,y) = A_{\{2,1\}}(x,y) = xy^2(y+2x), \\ A_{\{2,2\}}(x,y) &= xy^2(y^2+4xy+x^2), \ A_{\{2,1,2\}}(x,y) = xy^2(y^3+12xy^2+15x^2y+2x^3) \end{split}$$

2. The commutative property of Eulerian operators

Lemma 6. Let $\mathbf{m} = \{m_1, m_2, \dots, m_n\}$, where $0 \leq m_i \leq 2$. Set $\overline{\mathbf{m}} = \mathbf{m} \cup \{n+1\}$ and $\underline{\mathbf{m}} = \mathbf{m} \cup \{n+1, n+1\}$. Let T and G be the Eulerian operators defined by (3) and (4), respectively. Then we have $A_{\overline{\mathbf{m}}}(x, y) = T(A_{\mathbf{m}}(x, y))$ and $A_{\underline{\mathbf{m}}}(x, y) = G(A_{\mathbf{m}}(x, y))$.

Proof. Let $\pi \in \mathfrak{S}_{\mathbf{m}}$. We introduce a labeling of π as follows:

- (L_1) if π_i is a descent, then put a superscript label x right after it;
- (L_2) if π_i is an ascent or a plateau, then put a superscript label y right after it.

For example, for $\pi = 12125433$, the labeling of π is given by ${}^{y}1{}^{y}2{}^{x}1{}^{y}2{}^{y}5{}^{x}4{}^{x}3{}^{y}3{}^{y}$.

When we insert the letter n + 1 into π , we always get a label x just before n + 1 as well as a label y right after n + 1. This corresponds to the substitution rule of labels: $x \to xy$ or $y \to xy$. Thus the term $T(A_{\mathbf{m}}(x,y))$ gives the contribution of all $\pi' \in \mathfrak{S}_{\overline{\mathbf{m}}}$ in which the element n + 1 appears in positions j, where $0 \leq j \leq m$. Therefore, one has $A_{\overline{\mathbf{m}}}(x,y) = T(A_{\mathbf{m}}(x,y))$.

When we insert two elements n + 1 into π , we distinguish among three distinct cases:

(c₁) If the pair (n + 1)(n + 1) is inserted in a position of π , then the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \to \cdots \pi_i^y (n+1)^y (n+1)^x \pi_{i+1} \cdots,$$
$$\cdots \pi_i^y \pi_{i+1} \cdots \to \cdots \pi_i^y (n+1)^y (n+1)^x \pi_{i+1} \cdots.$$

This explains the term $xy^2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$;

(c₂) If the two n + 1 are inserted into two different positions with the same label, then the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \pi_j^x \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots ,$$
$$\cdots \pi_i^y \pi_{i+1} \cdots \pi_j^y \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots .$$

This explains the term $\frac{x^2y^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right);$

(c_3) If the two n + 1 are inserted into two different positions with different labels, then the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \pi_j^y \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots ,$$
$$\cdots \pi_i^y \pi_{i+1} \cdots \pi_j^x \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots .$$

This explains the term $x^2 y^2 \frac{\partial^2}{\partial x \partial y}$.

Therefore, the action of G on the set of labeled multipermutations in $\mathfrak{S}_{\mathbf{m}}$ gives the set of labeled multipermutations in $\mathfrak{S}_{\mathbf{m}}$. This yields $A_{\mathbf{m}}(x, y) = G(A_{\mathbf{m}}(x, y))$.

We can now present the following result.

Theorem 7. The Eulerian operators T and G are commutative, i.e., TG = GT.

Proof. Let $G = G_1 + G_2 + G_3$, where

$$G_1 = xy^2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \ G_2 = \frac{x^2y^2}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \ G_3 = x^2y^2\frac{\partial^2}{\partial x\partial y}.$$
 (5)

It is easily checked that

$$\begin{split} G_{1}T &= xy^{2} \left[(x+y) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + xy \left(\frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} + 2\frac{\partial^{2}}{\partial x \partial y} \right) \right], \\ G_{2}T &= \frac{x^{2}y^{2}}{2} \left[2y \frac{\partial^{2}}{\partial x^{2}} + 2x \frac{\partial^{2}}{\partial y^{2}} + 2(x+y) \frac{\partial^{2}}{\partial x \partial y} + xy \left(\frac{\partial^{3}}{\partial x^{3}} + \frac{\partial^{3}}{\partial y^{3}} + \frac{\partial^{3}}{\partial x^{2} \partial y} + \frac{\partial^{3}}{\partial y^{2} \partial x} \right) \right], \\ G_{3}T &= x^{2}y^{2} \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + (x+y) \frac{\partial^{2}}{\partial x \partial y} + x \frac{\partial^{2}}{\partial x^{2}} + y \frac{\partial^{2}}{\partial y^{2}} + xy \left(\frac{\partial^{3}}{\partial x^{2} \partial y} + \frac{\partial^{3}}{\partial y^{2} \partial x} \right) \right], \\ TG_{1} &= xy \left[(2xy+y^{2}) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + xy^{2} \left(\frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} + 2\frac{\partial^{2}}{\partial x \partial y} \right) \right], \\ TG_{2} &= xy \left[(xy^{2}+x^{2}y) \left(\frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} \right) + \frac{x^{2}y^{2}}{2} \left(\frac{\partial^{3}}{\partial x^{3}} + \frac{\partial^{3}}{\partial y^{3}} + \frac{\partial^{3}}{\partial x^{2} \partial y} + \frac{\partial^{3}}{\partial y^{2} \partial x} \right) \right], \\ TG_{3} &= xy \left[2(x^{2}y+xy^{2}) \frac{\partial^{2}}{\partial x \partial y} + x^{2}y^{2} \left(\frac{\partial^{3}}{\partial x^{2} \partial y} + \frac{\partial^{3}}{\partial y^{2} \partial x} \right) \right]. \end{split}$$

Thus we obtain

$$GT = TG = (xy^3 + 2x^2y^2) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + (2x^2y^3 + x^3y^2) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + (4x^2y^3 + 2x^3y^2) \frac{\partial^2}{\partial x \partial y} + \frac{x^3y^3}{2} \left(\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3}\right) + \frac{3x^3y^3}{2} \left(\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^2 \partial x}\right).$$
(6)
eletes the proof.

This completes the proof.

Example 8. Note that $A_{\{2\}}(x,y) = xy^2$. Using (6), one has

$$GT(xy^2) = TG(xy^2) = xy^2(y^3 + 12xy^2 + 15x^2y + 2x^3) = A_{\{2,1,2\}}(x,y) = A_{\{2,2,1\}}(x,y).$$

3. The proof of Theorem 4

We claim that the bivariate γ -expansions of $A_{\mathbf{m}}(x, y)$ has three types:

$$A_{\mathbf{m}}(x,y) = \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} a(m,k)(xy)^k (x+y)^{m+1-2k},$$
(7)

$$A_{\mathbf{m}}(x,y) = y \sum_{k=1}^{\lfloor m/2 \rfloor} b(m,k) (xy)^k (x+y)^{m-2k},$$
(8)

$$A_{\mathbf{m}}(x,y) = \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} c(m,k) (xy)^k (x+y)^{m+1-2k} + y \sum_{k=1}^{\lfloor m/2 \rfloor} d(m,k) (xy)^k (x+y)^{m-2k}, \quad (9)$$

where the first expansion corresponds to $\mathbf{m} = \{1, 1, \dots, 1\}$, the second expansion corresponds to $\mathbf{m} = \{2, 2, \dots, 2\}$, and the last expansion corresponds to the other cases.

As illustrated by Example 5, the claim holds for any $m \leq 4$. We proceed by induction. It suffices to distinguish among three distinct cases:

(a₁) Consider the case $\mathbf{m} = \{1, 1, ..., 1\}$. Note that $A_{\mathbf{m}}(x, y)$ is bivariate γ -positive with the center of symmetry $\frac{m+1}{2}$. By (7) and Lemma 6, we have

$$\begin{aligned} A_{\overline{\mathbf{m}}}(x,y) &= T\left(A_{\mathbf{m}}(x,y)\right) \\ &= T\left(\sum_{k=1}^{\lfloor (m+1)/2 \rfloor} a(m,k)(xy)^k (x+y)^{m+1-2k}\right) \\ &= \sum_k a(m,k)[k(xy)^k (x+y)^{m+2-2k} + 2(m+1-2k)(xy)^{k+1} (x+y)^{m-2k}], \end{aligned}$$

Setting $\widetilde{a}(m,k) = ka(m,k) + 2(m+3-2k)a(m,k-1)$, we get

$$A_{\overline{\mathbf{m}}}(x,y) = \sum_{k=1}^{\lfloor (m+2)/2 \rfloor} \widetilde{a}(m,k)(xy)^k (x+y)^{m+2-2k}.$$
(10)

Thus the γ -expansion of $A_{\overline{\mathbf{m}}}(x, y)$ belongs to the type (7).

Consider the action of G on the basis element $(xy)^k(x+y)^{m+1-2k}$. We get

$$G\left((xy)^{k}(x+y)^{m+1-2k}\right) = G_1\left((xy)^{k}(x+y)^{m+1-2k}\right) + (G_2+G_3)\left((xy)^{k}(x+y)^{m+1-2k}\right),$$

where G_1, G_2 and G_3 are defined by (5). After some calculations, this gives the following:

$$G_{1}\left((xy)^{k}(x+y)^{m+1-2k}\right) = y\left[k(xy)^{k}(x+y)^{m+2-2k} + 2(m+1-2k)(xy)^{k+1}(x+y)^{m-2k}\right],$$

$$(G_{2}+G_{3})\left((xy)^{k}(x+y)^{m+1-2k}\right) = \binom{k}{2}(xy)^{k}(x+y)^{m+3-2k} + k(xy)^{k+1}(x+y)^{m+3-2(k+1)} + 2k(m+1-2k)(xy)^{k+1}(x+y)^{m+3-2(k+1)} + 2(m+1-2k)(m-2k)(xy)^{k+2}(x+y)^{m+3-2(k+2)},$$

Thus $G_1((xy)^k(x+y)^{m+1-2k})$ and $(G_2+G_3)((xy)^k(x+y)^{m+1-2k})$ are both bivariate γ -positive polynomials with the center of symmetry $\frac{m+2}{2}$ and $\frac{m+3}{2}$, respectively. Therefore, the γ -expansion of $G(A_{\mathbf{m}}(x,y))$ belongs to the type (9). More precisely, there exist nonnegative real numbers $\tilde{c}(m,k)$ and $\tilde{d}(m,k)$ such that

$$G(A_{\mathbf{m}}(x,y)) = \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} \widetilde{c}(m,k)(xy)^{k}(x+y)^{m+3-2k} + y^{\lfloor m/2 \rfloor} y \sum_{k=1}^{\lfloor m/2 \rfloor} \widetilde{d}(m,k)(xy)^{k}(x+y)^{m+2-2k}.$$
(11)

 (a_2) Consider the case $\mathbf{m} = \{2, 2, \dots, 2\}$. By (8) and Lemma 6, we have

$$\begin{split} T\left(A_{\mathbf{m}}(x,y)\right) &= T\left(y\sum_{k=1}^{\lfloor m/2 \rfloor} b(m,k)(xy)^{k}(x+y)^{m-2k}\right) \\ &= \sum_{k=1}^{\lfloor m/2 \rfloor} b(m,k)(xy)^{k+1}(x+y)^{m-2k} + yT\left(\sum_{k=1}^{\lfloor m/2 \rfloor} b(m,k)(xy)^{k}(x+y)^{m-2k}\right) \\ &= \sum_{i=2}^{\lfloor (m+2)/2 \rfloor} b(m,i-1)(xy)^{i}(x+y)^{m+2-2k} + \\ &y\sum_{k} b(m,k) \left[k(xy)^{k}(x+y)^{m+1-2k} + 2(m-2k)(xy)^{k+1}(x+y)^{m-2k-1}\right], \\ &= \sum_{i=2}^{\lfloor (m+2)/2 \rfloor} b(m,i-1)(xy)^{i}(x+y)^{m+2-2k} + y\sum_{k=1}^{\lfloor (m+1)/2 \rfloor} \tilde{b}(m,k)(xy)^{k}(x+y)^{m+1-2k} + 2(m-2k)(xy)^{k+1}(x+y)^{m-2k-1} \right], \end{split}$$

where b(m,k) = kb(m,k) + 2(m-2k+2)b(m,k-1). Thus the γ -expansion of $T(A_{\mathbf{m}}(x,y))$ belongs to the type (9).

Consider the action of the operator G on the basis element $y(xy)^p(x+y)^q$. After some simplifications, we obtain that $G(x^py^{p+1}(x+y)^q)$ has the following expansion:

$$y(xy)^{p}(x+y)^{q-2}\left[\binom{p}{2}(x+y)^{4} + (1+p)(1+2q)(xy)(x+y)^{2} + 4\binom{q}{2}(xy)^{2}\right],$$

which yields that the γ -expansion of $G(A_{\mathbf{m}}(x, y))$ belongs to the type (8). More precisely, there exist nonnegative real numbers $\tilde{b}(m, k)$ such that

$$G\left(y\sum_{k=1}^{\lfloor m/2 \rfloor} b(m,k)(xy)^k (x+y)^{m-2k}\right) = y\sum_{k=1}^{\lfloor (m+2)/2 \rfloor} \widetilde{b}(m,k)(xy)^k (x+y)^{m+2-2k}.$$
 (12)

(a₃) Consider $\mathbf{m} = \{m_1, m_2, \dots, m_n\}$, where $\#\{m_i \in \mathbf{m} : m_i = 1\} = r$ and $\#\{m_i \in \mathbf{m} : m_i = 2\} = s$. Without loss of generality, assume that $1 \leq r, s < n$ and r + s = n. Combining Lemma 6 and Theorem 7, we have $A_{\mathbf{m}}(x, y) = G^s(T^r(x))$. Using (10), we see that there exist nonnegative real numbers a(r, k) such that

$$G^{s}(T^{r}(x)) = G^{s}\left(\sum_{k=1}^{\lfloor (r+1)/2 \rfloor} a(r,k)(xy)^{k}(x+y)^{r+1-2k}\right)$$

Repeatedly using (11) and (12), we deduce that

$$A_{\mathbf{m}}(x,y) = \sum_{k \ge 1} c(r+2s,k)(xy)^k (x+y)^{r+2s+1-2k} + y \sum_{k \ge 1} d(r+2s,k)(xy)^k (x+y)^{r+2s-2k}.$$

When y = 1, we arrive at

$$A_{\mathbf{m}}(x) = \sum_{k \ge 1} c(r+2s,k) x^k (1+x)^{r+2s+1-2k} + \sum_{k \ge 1} d(r+2s,k) x^k (1+x)^{r+2s-2k},$$

as desired. This completes the proof.

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