# COMMUTING EULERIAN OPERATORS 

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#### Abstract

Motivated by the work of Visontai and Dey-Sivasubramanian on the gammapositivity of some polynomials, we find the commutative property of a pair of Eulerian operators. As an application, we show the bi-gamma-positivity of the descent polynomials on permutations of the multiset $\left\{1^{a_{1}}, 2^{a_{2}}, \ldots, n^{a_{n}}\right\}$, where $0 \leqslant a_{i} \leqslant 2$. Therefore, these descent polynomials are all alternatingly increasing, and so they are unimodal with modes in the middle.


Keywords: Eulerian operators; Eulerian polynomials; Unimodality; Gamma-positivity

## 1. Introduction

Let $f(x)=\sum_{i=0}^{n} f_{i} x^{i}$ be a polynomial with nonnegative coefficients. We say that $f(x)$ is unimodal if $f_{0} \leqslant f_{1} \leqslant \cdots \leqslant f_{k} \geqslant f_{k+1} \geqslant \cdots \geqslant f_{n}$ for some $k$, where the index $k$ is called the mode of $f(x)$. It is well known that if $f(x)$ with only nonpositive real zeros, then $f(x)$ is unimodal (see [6, p. 419] for instance). If $f(x)$ is symmetric with the center of symmetry $\lfloor n / 2\rfloor$, i.e., $f_{i}=f_{n-i}$ for all indices $0 \leqslant i \leqslant n$, then it can be expanded as

$$
f(x)=\sum_{k=0}^{\lfloor n / 2\rfloor} \gamma_{k} x^{k}(1+x)^{n-2 k} .
$$

The polynomial $f(x)$ is $\gamma$-positive if $\gamma_{k} \geqslant 0$ for all $0 \leqslant k \leqslant\lfloor n / 2\rfloor$. Clearly, $\gamma$-positivity implies symmetry and unimodality. Let $f(x, y)=\sum_{i=0}^{n} f_{i} x^{i} y^{n-i}$ be a homogeneous bivariate polynomial. We say that $f(x, y)$ is bivariate $\gamma$-positive with the center of symmetry $\frac{n}{2}$ if $f(x, y)$ can be written as follows:

$$
f(x, y)=\sum_{k=0}^{\lfloor n / 2\rfloor} \gamma_{k}(x y)^{k}(x+y)^{n-2 k} .
$$

There has been considerable recent interest in the study of the $\gamma$-positivity of polynomials, see [1, 4] for details. In particular, Brändén [4, Remark 7.3.1] noted that if $f(x)$ is symmetric and has only real zeros, then it is $\gamma$-positive.

Let $f(x)=\sum_{i=0}^{n} f_{i} x^{i}$, where $f_{n} \neq 0$. Following [2, 5], there is a unique symmetric decomposition $f(x)=a(x)+x b(x)$, where

$$
a(x)=\frac{f(x)-x^{n+1} f(1 / x)}{1-x}, b(x)=\frac{x^{n} f(1 / x)-f(x)}{1-x} .
$$

According to [15, Definition 8], the polynomial $f(x)$ is said to be $b i-\gamma$-positive if both $a(x)$ and $b(x)$ are $\gamma$-positive. Thus $\gamma$-positivity is a special case of bi- $\gamma$-positivity. Following [17, Definition 2.9], the polynomial $f(x)$ is alternatingly increasing if

$$
f_{0} \leqslant f_{n} \leqslant f_{1} \leqslant f_{n-1} \leqslant \cdots \leqslant f_{\lfloor(n+1) / 2\rfloor} .
$$

Brändén and Solus [5] pointed out that $f(x)$ is alternatingly increasing if and only if the pair of polynomials in its symmetric decomposition are both unimodal and have only nonnegative coefficients. Therefore, bi- $\gamma$-positivity is stronger than alternatingly increasing property. The alternatingly increasing property first appeared in the work of Beck and Stapledon [2]. Recently, Beck-Jochemko-McCullough [3], Brändén-Solus [5] and Solus [19] studied the alternatingly increasing property of several $h^{*}$-polynomials as well as some refined Eulerian polynomials.

A multipermutation of a multiset is a sequence of its elements. Throughout this paper, we always let $\mathbf{m}=\left(m_{1}, m_{2}, \ldots, m_{n}\right) \in \mathbb{P}^{n}$. Denote by $\mathfrak{S}_{\mathbf{m}}$ the set of all multipermutations of the multiset $\left\{1^{m_{1}}, 2^{m_{2}}, \ldots, n^{m_{n}}\right\}$, where $i$ appears $m_{i}$ times. Set $m=\sum_{i=1}^{n} m_{i}$. For $\pi=\pi_{1} \pi_{2} \ldots \pi_{m} \in \mathfrak{S}_{\mathbf{m}}$, we always assume that $\pi_{0}=\pi_{m+1}=0$ (except where explicitly stated). If $i \in\{0,1,2, \ldots, m\}$, then $\pi_{i}$ is called an ascent (resp. descent, plateau) if $\pi_{i}<\pi_{i+1}$ (resp. $\pi_{i}>\pi_{i+1}, \pi_{i}=\pi_{i+1}$ ). Let asc $(\pi)$ (resp. des $(\pi)$, plat $\left.(\pi)\right)$ be the number of ascents (resp. descents, plateaux) of $\pi$. The multiset Eulerian polynomials $A_{\mathbf{m}}(x)$ are defined by

$$
A_{\mathbf{m}}(x)=\sum_{\pi \in \mathfrak{G}_{\mathbf{m}}} x^{\operatorname{asc}(\pi)}=\sum_{\pi \in \mathfrak{G}_{\mathbf{m}}} x^{\operatorname{des}(\pi)}
$$

A classical result of MacMahon [16, Vol 2, Chapter IV, p. 211] says that

$$
\begin{equation*}
\frac{A_{\mathbf{m}}(x)}{(1-x)^{1+m}}=\sum_{k \geqslant 0}\binom{k+m_{1}}{m_{1}}\binom{k+m_{2}}{m_{2}} \cdots\binom{k+m_{n}}{m_{n}} x^{k+1} . \tag{1}
\end{equation*}
$$

Let $\mathfrak{S}_{n}$ be the set of all permutations of $\{1,2, \ldots, n\}$. As usual, we write $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in \mathfrak{S}_{n}$. Denote by $A_{\pi(\mathbf{m})}(x)$ the descent polynomial on multipermutations of $\left\{\pi_{1}^{m_{1}}, \pi_{2}^{m_{2}}, \ldots, \pi_{n}^{m_{n}}\right\}$. It follows from (1) that

$$
\begin{equation*}
A_{\mathbf{m}}(x)=A_{\pi(\mathbf{m})}(x) \tag{2}
\end{equation*}
$$

When $\mathbf{m}=(1,1, \ldots, 1)$, the polynomial $A_{\mathbf{m}}(x)$ is reduced to the classical Eulerian polynomial $A_{n}(x)$. In other words,

$$
A_{n}(x)=\sum_{\pi \in \mathfrak{S}_{n}} x^{\operatorname{asc}(\pi)}=\sum_{\pi \in \mathfrak{S}_{n}} x^{\operatorname{des}(\pi)} .
$$

Simion [18, Section 2] found that $A_{\mathbf{m}}(x)$ is real-rootedness for any $\mathbf{m}$. When $\mathbf{m}=(p, p, \ldots, p)$, Carlitz-Hoggatt [7] showed that $A_{\mathbf{m}}(x)$ is symmetric, where $p$ is a given positive integer. By [4, Remark 7.3.1], an immediate consequence is the following well known result.

Proposition 1. For any $\mathbf{m}$, the multiset Eulerian polynomials $A_{\mathbf{m}}(x)$ are all unimodal. When $\mathbf{m}=(p, p, \ldots, p)$, the polynomial $A_{\mathbf{m}}(x)$ is $\gamma$-positive, and so its mode is in the middle.

Recently, there has been much work on the descent polynomials of permutations over multisets, see [11, 12, 13, 14, 21] for instance. In particular, Lin-Xu-Zhao [13] found a combinatorial interpretation for the $\gamma$-coefficients of $A_{\mathrm{m}}(x)$ via the model of weakly increasing trees, where $\mathbf{m}=(p, p, \ldots, p)$. Motivated by Proposition [it is natural to consider the following problem.

Problem 2. For any $\mathbf{m}$, could we characterize the location of the mode of $A_{\mathbf{m}}(x)$ ?
A bivariate version of the Eulerian polynomial over the symmetric group is given as follows:

$$
A_{n}(x, y)=\sum_{\pi \in \mathfrak{S}_{n}} x^{\operatorname{asc}(\pi)} y^{\operatorname{des}(\pi)}
$$

In particular, $A_{n}(x, 1)=A_{n}(1, x)=A_{n}(x)$. Carlitz and Scoville [8] found that

$$
A_{n+1}(x, y)=x y\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) A_{n}(x, y), A_{1}(x, y)=x y
$$

Using the following Eulerian operator

$$
\begin{equation*}
T=x y\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) \tag{3}
\end{equation*}
$$

Foata and Schützenberger [10] discovered that

$$
A_{n}(x, y)=\sum_{k=1}^{\lfloor(n+1) / 2\rfloor} \gamma(n, k)(x y)^{k}(x+y)^{n+1-2 k}
$$

where $\gamma(n, k)$ are all nonnegative integers. Applying the same idea, Visontai [20] investigated the joint generating polynomial of descents and inverse descents, Dey-Sivasubramanian [9 studied the descent polynomials on permutations in the alternating group. As an illustration, we now recall a result on the Eulerian operator $T$, which is a slightly variant of [9, Lemma 5].

Lemma 3. Let $f(x, y)$ be a bivariate $\gamma$-positive polynomial with the center of symmetry $\frac{n}{2}$. Then $T(f(x, y))$ is a bivariate $\gamma$-positive polynomial with the center of symmetry $\frac{n+1}{2}$.

Motivated by the work of Visontai [20] and Dey-Sivasubramanian [9, in this paper we introduce the following Eulerian operator

$$
\begin{equation*}
G=x y^{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+\frac{x^{2} y^{2}}{2}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)+x^{2} y^{2} \frac{\partial^{2}}{\partial x \partial y} . \tag{4}
\end{equation*}
$$

In the next section, we prove the commutative property of the Eulerian operators $T$ and $G$. In Section 3, we prove following result, which gives a partial answer to Problem 2,

Theorem 4. Let $\mathbf{m}=\left\{m_{1}, m_{2}, \ldots, m_{n}\right\}$, where $0 \leqslant m_{i} \leqslant 2$. The Eulerian polynomials $A_{\mathbf{m}}(x)$ are all bi- $\gamma$-positive, and so $A_{\mathbf{m}}(x)$ are all alternating increasing. More precisely, when $\mathbf{m}=$ $\{1,1, \ldots, 1\}$ or $\mathbf{m}=\{2,2, \ldots, 2\}$, the polynomial $A_{\mathbf{m}}(x)$ is $\gamma$-positive; for the other cases, the polynomial $A_{\mathbf{m}}(x)$ can be written as a sum of two $\gamma$-positive polynomials.

In the following discussion, we always set $\mathbf{m}=\left\{m_{1}, m_{2}, \ldots, m_{n}\right\}$, where $0 \leqslant m_{i} \leqslant 2$. Let

$$
A_{\mathbf{m}}(x, y)=\sum_{\pi \in \mathfrak{S}_{\mathbf{m}}} x^{\operatorname{des}(\pi)} y^{m+1-\operatorname{des}(\pi)}
$$

where $m=\sum_{i=1}^{n} m_{i}$. Clearly, $A_{\mathbf{m}}(x, 1)=A_{\mathbf{m}}(x)$. For convenience, set $A_{\emptyset}(x, y)=x$.
Example 5. We have

$$
\begin{gathered}
A_{\{1\}}(x, y)=x y, A_{\{2\}}(x, y)=x y^{2}, A_{\{1,1\}}(x, y)=x y(x+y), \\
A_{\{1,1,1\}}(x, y)=x y\left(x^{2}+4 x y+y^{2}\right), A_{\{1,2\}}(x, y)=A_{\{2,1\}}(x, y)=x y^{2}(y+2 x), \\
A_{\{2,2\}}(x, y)=x y^{2}\left(y^{2}+4 x y+x^{2}\right), A_{\{2,1,2\}}(x, y)=x y^{2}\left(y^{3}+12 x y^{2}+15 x^{2} y+2 x^{3}\right) .
\end{gathered}
$$

## 2. The commutative property of Eulerian operators

Lemma 6. Let $\mathbf{m}=\left\{m_{1}, m_{2}, \ldots, m_{n}\right\}$, where $0 \leqslant m_{i} \leqslant 2$. Set $\overline{\mathbf{m}}=\mathbf{m} \cup\{n+1\}$ and $\underline{\mathbf{m}}=\mathbf{m} \cup\{n+1, n+1\}$. Let $T$ and $G$ be the Eulerian operators defined by (3) and (4), respectively. Then we have $A_{\overline{\mathbf{m}}}(x, y)=T\left(A_{\mathbf{m}}(x, y)\right)$ and $A_{\underline{\mathbf{m}}}(x, y)=G\left(A_{\mathbf{m}}(x, y)\right)$.

Proof. Let $\pi \in \mathfrak{S}_{\mathbf{m}}$. We introduce a labeling of $\pi$ as follows:
$\left(L_{1}\right)$ if $\pi_{i}$ is a descent, then put a superscript label $x$ right after it;
$\left(L_{2}\right)$ if $\pi_{i}$ is an ascent or a plateau, then put a superscript label $y$ right after it.
For example, for $\pi=12125433$, the labeling of $\pi$ is given by ${ }^{y} 1^{y} 2^{x} 1^{y} 2^{y} 5^{x} 4^{x} 3^{y} 3^{y}$.
When we insert the letter $n+1$ into $\pi$, we always get a label $x$ just before $n+1$ as well as a label $y$ right after $n+1$. This corresponds to the substitution rule of labels: $x \rightarrow x y$ or $y \rightarrow x y$. Thus the term $T\left(A_{\mathbf{m}}(x, y)\right)$ gives the contribution of all $\pi^{\prime} \in \mathfrak{S}_{\overline{\mathbf{m}}}$ in which the element $n+1$ appears in positions $j$, where $0 \leqslant j \leqslant m$. Therefore, one has $A_{\overline{\mathbf{m}}}(x, y)=T\left(A_{\mathbf{m}}(x, y)\right)$.

When we insert two elements $n+1$ into $\pi$, we distinguish among three distinct cases:
$\left(c_{1}\right)$ If the pair $(n+1)(n+1)$ is inserted in a position of $\pi$, then the changes of labeling are illustrated as follows:

$$
\begin{aligned}
& \cdots \pi_{i}^{x} \pi_{i+1} \cdots \rightarrow \cdots \pi_{i}^{y}(n+1)^{y}(n+1)^{x} \pi_{i+1} \cdots \\
& \cdots \pi_{i}^{y} \pi_{i+1} \cdots \rightarrow \cdots \pi_{i}^{y}(n+1)^{y}(n+1)^{x} \pi_{i+1} \cdots
\end{aligned}
$$

This explains the term $x y^{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$;
$\left(c_{2}\right)$ If the two $n+1$ are inserted into two different positions with the same label, then the changes of labeling are illustrated as follows:

$$
\begin{aligned}
& \cdots \pi_{i}^{x} \pi_{i+1} \cdots \pi_{j}^{x} \pi_{j+1} \cdots \rightarrow \cdots \pi_{i}^{y}(n+1)^{x} \pi_{i+1} \cdots \pi_{j}^{y}(n+1)^{x} \pi_{j+1} \cdots, \\
& \cdots \pi_{i}^{y} \pi_{i+1} \cdots \pi_{j}^{y} \pi_{j+1} \cdots \rightarrow \cdots \pi_{i}^{y}(n+1)^{x} \pi_{i+1} \cdots \pi_{j}^{y}(n+1)^{x} \pi_{j+1} \cdots .
\end{aligned}
$$

This explains the term $\frac{x^{2} y^{2}}{2}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)$;
$\left(c_{3}\right)$ If the two $n+1$ are inserted into two different positions with different labels, then the changes of labeling are illustrated as follows:

$$
\begin{aligned}
& \cdots \pi_{i}^{x} \pi_{i+1} \cdots \pi_{j}^{y} \pi_{j+1} \cdots \rightarrow \cdots \pi_{i}^{y}(n+1)^{x} \pi_{i+1} \cdots \pi_{j}^{y}(n+1)^{x} \pi_{j+1} \cdots, \\
& \cdots \pi_{i}^{y} \pi_{i+1} \cdots \pi_{j}^{x} \pi_{j+1} \cdots \rightarrow \cdots \pi_{i}^{y}(n+1)^{x} \pi_{i+1} \cdots \pi_{j}^{y}(n+1)^{x} \pi_{j+1} \cdots
\end{aligned}
$$

This explains the term $x^{2} y^{2} \frac{\partial^{2}}{\partial x \partial y}$.
Therefore, the action of $G$ on the set of labeled multipermutations in $\mathfrak{S}_{\mathbf{m}}$ gives the set of labeled multipermutations in $\mathfrak{S}_{\underline{\mathbf{m}}}$. This yields $A_{\underline{\mathbf{m}}}(x, y)=G\left(A_{\mathbf{m}}(x, y)\right)$.

We can now present the following result.
Theorem 7. The Eulerian operators $T$ and $G$ are commutative, i.e., $T G=G T$.

Proof. Let $G=G_{1}+G_{2}+G_{3}$, where

$$
\begin{equation*}
G_{1}=x y^{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right), G_{2}=\frac{x^{2} y^{2}}{2}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right), G_{3}=x^{2} y^{2} \frac{\partial^{2}}{\partial x \partial y} \tag{5}
\end{equation*}
$$

It is easily checked that

$$
\begin{aligned}
& G_{1} T=x y^{2}\left[(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+x y\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+2 \frac{\partial^{2}}{\partial x \partial y}\right)\right], \\
& G_{2} T=\frac{x^{2} y^{2}}{2}\left[2 y \frac{\partial^{2}}{\partial x^{2}}+2 x \frac{\partial^{2}}{\partial y^{2}}+2(x+y) \frac{\partial^{2}}{\partial x \partial y}+x y\left(\frac{\partial^{3}}{\partial x^{3}}+\frac{\partial^{3}}{\partial y^{3}}+\frac{\partial^{3}}{\partial x^{2} \partial y}+\frac{\partial^{3}}{\partial y^{2} \partial x}\right)\right], \\
& G_{3} T=x^{2} y^{2}\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+(x+y) \frac{\partial^{2}}{\partial x \partial y}+x \frac{\partial^{2}}{\partial x^{2}}+y \frac{\partial^{2}}{\partial y^{2}}+x y\left(\frac{\partial^{3}}{\partial x^{2} \partial y}+\frac{\partial^{3}}{\partial y^{2} \partial x}\right)\right], \\
& T G_{1}=x y\left[\left(2 x y+y^{2}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+x y^{2}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+2 \frac{\partial^{2}}{\partial x \partial y}\right)\right], \\
& T G_{2}=x y\left[\left(x y^{2}+x^{2} y\right)\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)+\frac{x^{2} y^{2}}{2}\left(\frac{\partial^{3}}{\partial x^{3}}+\frac{\partial^{3}}{\partial y^{3}}+\frac{\partial^{3}}{\partial x^{2} \partial y}+\frac{\partial^{3}}{\partial y^{2} \partial x}\right)\right], \\
& T G_{3}=x y\left[2\left(x^{2} y+x y^{2}\right) \frac{\partial^{2}}{\partial x \partial y}+x^{2} y^{2}\left(\frac{\partial^{3}}{\partial x^{2} \partial y}+\frac{\partial^{3}}{\partial y^{2} \partial x}\right)\right] .
\end{aligned}
$$

Thus we obtain

$$
\begin{align*}
& G T=T G=\left(x y^{3}+2 x^{2} y^{2}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+\left(2 x^{2} y^{3}+x^{3} y^{2}\right)\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)+ \\
& \left(4 x^{2} y^{3}+2 x^{3} y^{2}\right) \frac{\partial^{2}}{\partial x \partial y}+\frac{x^{3} y^{3}}{2}\left(\frac{\partial^{3}}{\partial x^{3}}+\frac{\partial^{3}}{\partial y^{3}}\right)+\frac{3 x^{3} y^{3}}{2}\left(\frac{\partial^{3}}{\partial x^{2} \partial y}+\frac{\partial^{3}}{\partial y^{2} \partial x}\right) . \tag{6}
\end{align*}
$$

This completes the proof.
Example 8. Note that $A_{\{2\}}(x, y)=x y^{2}$. Using (6), one has

$$
G T\left(x y^{2}\right)=T G\left(x y^{2}\right)=x y^{2}\left(y^{3}+12 x y^{2}+15 x^{2} y+2 x^{3}\right)=A_{\{2,1,2\}}(x, y)=A_{\{2,2,1\}}(x, y) .
$$

## 3. The proof of Theorem 4

We claim that the bivariate $\gamma$-expansions of $A_{\mathbf{m}}(x, y)$ has three types:

$$
\begin{gather*}
A_{\mathbf{m}}(x, y)=\sum_{k=1}^{\lfloor(m+1) / 2\rfloor} a(m, k)(x y)^{k}(x+y)^{m+1-2 k},  \tag{7}\\
A_{\mathbf{m}}(x, y)=y \sum_{k=1}^{\lfloor m / 2\rfloor} b(m, k)(x y)^{k}(x+y)^{m-2 k},  \tag{8}\\
A_{\mathbf{m}}(x, y)=\sum_{k=1}^{\lfloor(m+1) / 2\rfloor} c(m, k)(x y)^{k}(x+y)^{m+1-2 k}+y \sum_{k=1}^{\lfloor m / 2\rfloor} d(m, k)(x y)^{k}(x+y)^{m-2 k}, \tag{9}
\end{gather*}
$$

where the first expansion corresponds to $\mathbf{m}=\{1,1, \ldots, 1\}$, the second expansion corresponds to $\mathbf{m}=\{2,2, \ldots, 2\}$, and the last expansion corresponds to the other cases.

As illustrated by Example 5, the claim holds for any $m \leqslant 4$. We proceed by induction. It suffices to distinguish among three distinct cases:
$\left(a_{1}\right)$ Consider the case $\mathbf{m}=\{1,1, \ldots, 1\}$. Note that $A_{\mathbf{m}}(x, y)$ is bivariate $\gamma$-positive with the center of symmetry $\frac{m+1}{2}$. By (7) and Lemma 6, we have

$$
\begin{aligned}
A_{\overline{\mathbf{m}}}(x, y) & =T\left(A_{\mathbf{m}}(x, y)\right) \\
& =T\left(\sum_{k=1}^{\lfloor(m+1) / 2\rfloor} a(m, k)(x y)^{k}(x+y)^{m+1-2 k}\right) \\
& =\sum_{k} a(m, k)\left[k(x y)^{k}(x+y)^{m+2-2 k}+2(m+1-2 k)(x y)^{k+1}(x+y)^{m-2 k}\right]
\end{aligned}
$$

Setting $\widetilde{a}(m, k)=k a(m, k)+2(m+3-2 k) a(m, k-1)$, we get

$$
\begin{equation*}
A_{\overline{\mathbf{m}}}(x, y)=\sum_{k=1}^{\lfloor(m+2) / 2\rfloor} \widetilde{a}(m, k)(x y)^{k}(x+y)^{m+2-2 k} \tag{10}
\end{equation*}
$$

Thus the $\gamma$-expansion of $A_{\overline{\mathbf{m}}}(x, y)$ belongs to the type (7).
Consider the action of $G$ on the basis element $(x y)^{k}(x+y)^{m+1-2 k}$. We get

$$
G\left((x y)^{k}(x+y)^{m+1-2 k}\right)=G_{1}\left((x y)^{k}(x+y)^{m+1-2 k}\right)+\left(G_{2}+G_{3}\right)\left((x y)^{k}(x+y)^{m+1-2 k}\right)
$$

where $G_{1}, G_{2}$ and $G_{3}$ are defined by (5). After some calculations, this gives the following:

$$
\begin{aligned}
& G_{1}\left((x y)^{k}(x+y)^{m+1-2 k}\right)=y\left[k(x y)^{k}(x+y)^{m+2-2 k}+2(m+1-2 k)(x y)^{k+1}(x+y)^{m-2 k}\right] \\
& \left(G_{2}+G_{3}\right)\left((x y)^{k}(x+y)^{m+1-2 k}\right)=\binom{k}{2}(x y)^{k}(x+y)^{m+3-2 k}+k(x y)^{k+1}(x+y)^{m+3-2(k+1)}+ \\
& 2 k(m+1-2 k)(x y)^{k+1}(x+y)^{m+3-2(k+1)}+2(m+1-2 k)(m-2 k)(x y)^{k+2}(x+y)^{m+3-2(k+2)}
\end{aligned}
$$

Thus $G_{1}\left((x y)^{k}(x+y)^{m+1-2 k}\right)$ and $\left(G_{2}+G_{3}\right)\left((x y)^{k}(x+y)^{m+1-2 k}\right)$ are both bivariate $\gamma$-positive polynomials with the center of symmetry $\frac{m+2}{2}$ and $\frac{m+3}{2}$, respectively. Therefore, the $\gamma$-expansion of $G\left(A_{\mathbf{m}}(x, y)\right)$ belongs to the type (9). More precisely, there exist nonnegative real numbers $\widetilde{c}(m, k)$ and $\widetilde{d}(m, k)$ such that

$$
\begin{array}{r}
G\left(A_{\mathbf{m}}(x, y)\right)=\sum_{k=1}^{\lfloor(m+1) / 2\rfloor} \widetilde{c}(m, k)(x y)^{k}(x+y)^{m+3-2 k}+  \tag{11}\\
y \sum_{k=1}^{\lfloor m / 2\rfloor} \widetilde{d}(m, k)(x y)^{k}(x+y)^{m+2-2 k}
\end{array}
$$

$\left(a_{2}\right)$ Consider the case $\mathbf{m}=\{2,2, \ldots, 2\}$. By (8) and Lemma 6, we have

$$
\begin{aligned}
& T\left(A_{\mathbf{m}}(x, y)\right) \\
& =T\left(y \sum_{k=1}^{\lfloor m / 2\rfloor} b(m, k)(x y)^{k}(x+y)^{m-2 k}\right) \\
& =\sum_{k=1}^{\lfloor m / 2\rfloor} b(m, k)(x y)^{k+1}(x+y)^{m-2 k}+y T\left(\sum_{k=1}^{\lfloor m / 2\rfloor} b(m, k)(x y)^{k}(x+y)^{m-2 k}\right) \\
& =\sum_{i=2}^{\lfloor(m+2) / 2\rfloor} b(m, i-1)(x y)^{i}(x+y)^{m+2-2 k}+ \\
& y \sum_{k} b(m, k)\left[k(x y)^{k}(x+y)^{m+1-2 k}+2(m-2 k)(x y)^{k+1}(x+y)^{m-2 k-1}\right], \\
& =\sum_{i=2}^{\lfloor(m+2) / 2\rfloor} b(m, i-1)(x y)^{i}(x+y)^{m+2-2 k}+y \sum_{k=1}^{\lfloor(m+1) / 2\rfloor} \widetilde{b}(m, k)(x y)^{k}(x+y)^{m+1-2 k},
\end{aligned}
$$

where $\widetilde{b}(m, k)=k b(m, k)+2(m-2 k+2) b(m, k-1)$. Thus the $\gamma$-expansion of $T\left(A_{\mathbf{m}}(x, y)\right)$ belongs to the type (9).

Consider the action of the operator $G$ on the basis element $y(x y)^{p}(x+y)^{q}$. After some simplifications, we obtain that $G\left(x^{p} y^{p+1}(x+y)^{q}\right)$ has the following expansion:

$$
y(x y)^{p}(x+y)^{q-2}\left[\binom{p}{2}(x+y)^{4}+(1+p)(1+2 q)(x y)(x+y)^{2}+4\binom{q}{2}(x y)^{2}\right],
$$

which yields that the $\gamma$-expansion of $G\left(A_{\mathbf{m}}(x, y)\right)$ belongs to the type (8). More precisely, there exist nonnegative real numbers $\widetilde{b}(m, k)$ such that

$$
\begin{equation*}
G\left(y \sum_{k=1}^{\lfloor m / 2\rfloor} b(m, k)(x y)^{k}(x+y)^{m-2 k}\right)=y \sum_{k=1}^{\lfloor(m+2) / 2\rfloor} \widetilde{b}(m, k)(x y)^{k}(x+y)^{m+2-2 k} . \tag{12}
\end{equation*}
$$

$\left(a_{3}\right)$ Consider $\mathbf{m}=\left\{m_{1}, m_{2}, \ldots, m_{n}\right\}$, where $\#\left\{m_{i} \in \mathbf{m}: m_{i}=1\right\}=r$ and $\#\left\{m_{i} \in \mathbf{m}: m_{i}=\right.$ $2\}=s$. Without loss of generality, assume that $1 \leqslant r, s<n$ and $r+s=n$. Combining Lemma 6 and Theorem 7, we have $A_{\mathbf{m}}(x, y)=G^{s}\left(T^{r}(x)\right)$. Using (10), we see that there exist nonnegative real numbers $a(r, k)$ such that

$$
G^{s}\left(T^{r}(x)\right)=G^{s}\left(\sum_{k=1}^{\lfloor(r+1) / 2\rfloor} a(r, k)(x y)^{k}(x+y)^{r+1-2 k}\right) .
$$

Repeatedly using (11) and (12), we deduce that

$$
A_{\mathbf{m}}(x, y)=\sum_{k \geqslant 1} c(r+2 s, k)(x y)^{k}(x+y)^{r+2 s+1-2 k}+y \sum_{k \geqslant 1} d(r+2 s, k)(x y)^{k}(x+y)^{r+2 s-2 k} .
$$

When $y=1$, we arrive at

$$
A_{\mathbf{m}}(x)=\sum_{k \geqslant 1} c(r+2 s, k) x^{k}(1+x)^{r+2 s+1-2 k}+\sum_{k \geqslant 1} d(r+2 s, k) x^{k}(1+x)^{r+2 s-2 k},
$$

as desired. This completes the proof.

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