A note on median eigenvalues of subcubic graphs *

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Abstract

Let G be an simple graph of order n whose adjacency eigenvalues are $\lambda_1 \geq \cdots \geq \lambda_n$. The HL–index of G is defined to be $R(G) = \max\{|\lambda_h|, |\lambda_l|\}$ with $h = \left\lfloor \frac{n+1}{2} \right\rfloor$ and $l = \left\lceil \frac{n+1}{2} \right\rceil$. Mohar conjectured that $R(G) \leq 1$ for every planar subcubic graph G. In this note, we prove that Mohar's Conjecture holds for every K_4-minor-free subcubic graph. In addition, $R(G) \leq 1$ for every subcubic graph G which contains a subgraph $K_{2,3}$.

Key words: Median eigenvalue; HL-index; subcubic graphs; K_4-minor-free graphs; series–parallel graphs.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $|G|$ the order of graph G and $N_G(v)$ the neighborhood of a vertex v of G, and write $N(v)$ when G is clear. For a subset $A \subseteq V(G)$, denote by $G[A]$ the subgraph of G induced by A, and write $G - A$ for $G[V(G) - A]$ sometimes.

A graph is called *subcubic* if its maximum degree is at most 3. In mathematical chemistry, every subcubic graph is regarded as a chemical graph. It is known that the HOMO–LUMO separation, which is the gap between the Highest Occupied Molecular Orbital(HOMO) and Lowest Unoccupied Molecular Orbital (LUMO), is related linearly to median eigenvalues of a graph [6, 7]. Therefore, it is worthwhile to estimate the median eigenvalues. Fowler and Pisanski [6, 7] introduced the notion of HL–index of a graph (see also Jaklič et al. [8]). For a simple graph G of order n, let $\lambda_i(G)$ be the i-th largest eigenvalue of the adjacency matrix of G (counting multiplicities).The HL-index of G is defined as

$$R(G) = \max\{|\lambda_h(G)|, |\lambda_l(G)|\},$$

where $h = \left\lfloor \frac{n+1}{2} \right\rfloor$ and $l = \left\lceil \frac{n+1}{2} \right\rceil$.

In 2015, Mohar [11] proved that $R(G) \leq \sqrt{2}$ for every subcubic graph G. Further, he proposed the following conjecture.

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Conjecture 1.1. *[11] For every subcubic planar graph G, $R(G) \leq 1$.*

Mohar [10] confirmed that Conjecture 1.1 holds for bipartite subcubic planar graphs. Later, Mohar [12] proved that $R(G) \leq 1$ holds for every bipartite subcubic graph G except the Heawood graph, whose median eigenvalues are $\pm\sqrt{2}$. In addition, Benediktovich [2] confirmed Conjecture 1.1 for subcubic outerplanar graphs. For more results on median eigenvalues, see [9, 13–15].

A graph H is a *minor* (or *H-minor*) of a graph G, if a copy of H can be obtained from G by deleting and/or contracting edges of G. A graph is *$\mathcal{H}$-minor-free* if H is not a minor of it for every $H \in \mathcal{H}$. When $\mathcal{H} = \{H\}$, we simply write *H-minor-free.* In this note, we prove the following result.

Theorem 1.2. *For every subcubic graph G that contains a subgraph $K_{2,3}$, we have $R(G) \leq 1$.*

In electronic engineering and computer science, K_4-minor-free graphs, which are also called series–parallel graphs, are of great interest, because these graphs can be applied to model series and parallel electric circuits. More results on series–parallel graphs may be referred to [4, 5] and references therein. Another main result of this note is as follows.

Theorem 1.3. *For every K_4-minor-free subcubic graph G, we have $R(G) \leq 1$.*

Remark 1. *On one hand, Theorem 1.3 extends the result for subcubic outplanar graphs in [2], since G is outerplanar if and only if neither K_4 nor $K_{2,3}$ is a minor of G. On the other hand, Theorem 1.3 confirms that Conjecture 1.1 holds for K_4-minor-free subcubic graphs, as every K_4-minor-free graph is planar, equivalently, $\{K_5, K_{3,3}\}$-minor-free.*

The rest of this note is organized as follows. In Section 2, some notations and lemmas are presented. In Section 3, we prove Theorems 1.2 and 1.3.

2 Preliminaries

First, we need the following two well-known theorems in spectral graph theory (for example, see [3, pp. 17–20]).

Theorem 2.1 (Eigenvalue interlacing theorem)**.** *Let G be a simple graph G of order n. Let $A \subseteq V(G)$ be a vertex set of size k. Then for $i \in \{1, 2, \cdots, n-k\}$,*

$$\lambda_i(G) \geq \lambda_i(G - A) \geq \lambda_{i+k}(G).$$

Theorem 2.2. *Let G be a simple graph of order n. Suppose $\{E_1, E_2\}$ is a partition of $E(G)$. For $i \in \{1, 2\}$, let $G_i = (V(G), E_i)$ be the spanning graph of G. Then*

$$\lambda_i(G) \leq \lambda_j(G_1) + \lambda_{i-j+1}(G_2) \ (n \geq i \geq j \geq 1),$$

$$\lambda_i(G) \geq \lambda_j(G_1) + \lambda_{i-j+n}(G_2) \ (1 \leq i \leq j \leq n).$$

We also need a simple fact.

Lemma 2.3. *Let G be a simple graph with two distinct vertices u and v such that $N(u) = N(v)$. Then zero is an adjacency eigenvalue of G. Furthermore, if G is bipartite, then $R(G) = 0$.*

Proof. The determinant of the adjacency matrix A of G is equal to 0, so zero is an eigenvalue of A. Furthermore, if G is bipartite, then the adjacency eigenvalues of A is symmetric with respect to origin 0; so $R(G) = 0$. $\square$

Let G be a simple graph of order n. A partition $\{A, B\}$ of vertex set of G is called *unfriendly* if every vertex in A has at least as many neighbors in B as in A, and every vertex in B has at least as many neighbors in A as in B. A partition $\{A, B\}$ of vertex set of G is called *unbalanced* if $|A| \neq |B|$; *balanced*, otherwise. We include Lemma 2.4 which was initially presented in [1], and later applied to median eigenvalues by Mohar in [10, 11].

Lemma 2.4. *[1] Every graph has an unfriendly partition.*

By applying Theorem 2.1, Mohar [10, Lemma 2.4] proved the following lemma.

Lemma 2.5. *[10] If G is a subcubic graph with an unbalanced unfriendly partition, then $R(G) \leq 1$.*

Finally, we need the following lemma, which was verified by Mohar in the proof of [10, Lemma 2.1].

Lemma 2.6. *[10] For every subcubic graph G of odd order, $R(G) \leq 1$.*

3 Proof of Theorems 1.2 and 1.3.

In this section, we give the proof of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let G be a subcubic graph such that $K_{2,3}$ is a subgraph of G, where $V(K_{2,3}) = \{x_1, x_2, y_1, y_2, y_3\} \subseteq V(G)$ with bipartite partition $\{x_1, x_2\} \cup \{y_1, y_2, y_3\}$. Take an unfriendly partition $\{A, B\}$ of $V(G)$, as there exists one by Lemma 2.4. If $x_1 \in A$ and $x_2 \in B$, then any way of putting y_1, y_2 and y_3 into either A or B can not build the unfriendly partition $\{A, B\}$. Thus we may assume that x_1 and x_2 belong to the same part, say A. Then $\{y_1, y_2, y_3\} \subseteq B$.

Let $G_1 = (V(G), E_1)$ and $G_2 = (V(G), E(G) \setminus E_1)$ be two spanning subgraphs of G, where $E_1 = E(A, B)$ consists of all edges with one end in A and the other in B. Then G_1 is a bipartite graph with $N_{G_1}(x_1) = N_{G_1}(x_2) = \{y_1, y_2, y_3\}$. Hence, $\lambda_h(G_1) = 0$ with where $h = \lfloor \frac{n+1}{2} \rfloor$ by Lemma 2.3. In addition, since $\{A, B\}$ is an unfriendly partition of G, the graph G_2 consists of independent edges and isolated vertices, which implies that $\lambda_1(G_2) \leq 1$. Therefore, by Theorem 2.2,

$$\lambda_h(G) \leq \lambda_h(G_1) + \lambda_1(G_2) \leq 1,$$

where $h = \lfloor \frac{n+1}{2} \rfloor$. By a similar argument, we can also prove that $\lambda_l(G) \geq -1$ with $l = \lceil \frac{n+1}{2} \rceil$. Hence $R(G) \leq 1$. $\square$

Proof of Theorem 1.3. If the assertion of Theorem 1.3 holds for connected graphs, then it also holds for disconnected graphs by considering all the components. Thus we may assume that G is a K_4-minor-free subcubic connected graph . And by Lemma 2.6, we may assume that $|G| = n$ with even n. We use the induction method on even n.

If $n = 2$, then $G = K_2$ and the assertion holds. In addition, if $n = 4$, then G must be one of the five graphs: star $K_{1,3}$, 4-cycle C_4, kite $K_4 - e$ and paw which is a graph obtained by connecting a vertex v to one vertex of K_3. It is easy to see the second largest eigenvalue of the five graphs at least 1, i.e., $R(G) \leq 1$. So the assertion holds.

Suppose the assertion holds for all the graphs of order less than n, where $n \geq 6$. Now we consider the following two cases.

Case 1: There is a cut vertex v in G. Then $G - v$ has a component G_1 of odd order. And $G_2 := G - v - V(G_1)$ has even number of vertices. Denote by $n_1(H)$ the number of eigenvalues of a graph H that are larger than 1. Since $|G_1|$ is odd, $R(G_1) \leq 1$ by Lemma 2.6, and $n_1(G_1) \leq \frac{|G_1|-1}{2}$. Since G is subcubic, G_2 has at most two component. If G_2 is connected or has two components of even order, then apply the inductive hypothesis to components of G_2; otherwise, apply Lemma 2.6 to two components of odd order of G_2. In all the cases above for G_2, we can obtain $n_1(G_2) \leq \frac{|G_2|-2}{2}$. Thus,

$$n_1(G - v) = n_1(G_1) + n_1(G_2) \leq \frac{|G_1| + |G_2| - 3}{2} = \frac{n-4}{2},$$

which implies that $\lambda_{\frac{n}{2}-1}(G - v) \leq 1$. Hence, by Theorem 2.1, $\lambda_{\frac{n}{2}}(G) \leq \lambda_{\frac{n}{2}-1}(G - v) \leq 1$. By a similar argument, we can also prove that $\lambda_{\frac{n}{2}+1}(G) \geq -1$. Therefore, $R(G) \leq 1$.

Case 2: G is a 2-connected graph. If G is a cycle, then the assertion holds. Thus we may assume that G is not a cycle. Let C be a longest cycle in G. Throughout this proof, we say a path lies in C (outside of C, resp.) if all the edges of the path are in C (are not in C, resp.). Then there is a path P with two end vertices $u, v \in V(C)$ such that P contains no edges in $E(C)$. Let $N(u) \cap V(C) = \{u_1, u_2\}$, and $N(v) \cap V(C) = \{v_1, v_2\}$. Moreover, there is a u_1, v_1–path in C that contains neither u nor v. If $u_1 = v_1$ and $u_2 = v_2$, then $G = K_{2,3}$, since G is K_4-minor-free and C is a longest cycle of G. It is a contradiction. Hence we may assume that $u_2 \neq v_2$ and have the following Claim 1.

Claim 1. $G - \{u_2, v\}$ is not connected.

Proof. We use the method of contradiction. Suppose that $G - \{u_2, v\}$ is connected. Then there exists a path Q from u to v_2 in $G - \{u_2, v\}$. Let y denote the vertex of Q that is adjacent to u, which is either the vertex of P (only happens when $|V(P)| > 2$) or u_1. Thus there is a K_4-minor of G on $\{u, v, v_2, y\}$, a contradiction that G is K_4-minor-free. □

Let W be the set of vertices of the component of $G - \{u_2, v\}$ that contains u (for example, the black vertices in Figure 1 represent all the vertices in W.).

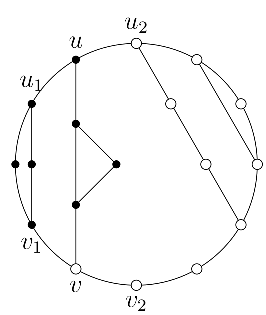

Figure 1: An example for G

Claim 2. If u is adjacent to v, then $G[W - u]$ is connected and a component of $G - \{u, v\}$; if u is not adjacent to v, then there are two connected components in $G[W - u]$ which are also two components of $G - \{u, v\}$.

Proof. If u is adjacent to v, then u has degree 1 in $G[W]$. Thus $G[W-u]$ is still connected and a component of $G-\{u,u_2,v\}$.

If u is not adjacent to v, then $|V(P)|>2$. Then $G[W-u]$ has two components. In fact, if $G[W-u]$ is connected, then there is a path from u_1 to y in $G[W-u]$ with $y \in N(u)\cap V(P)$. Hence there is a K_4-minor in G on $\{u,v,u_1,y\}$, which contradicts to that G is K_4-minor-free. Therefore $G[W-u]$ is disconnected. Moreover, since the degree of u in $G[W]$ is 2, there are exactly two components in $G[W-u]$.

Let H_1 be the component of $G[W-u]$ containing u_1 and H_2 be the rest component of $G[W-u]$. Then by the definition of W, H_1 and H_2 are also two components of $G-\{u,u_2,v\}$. Since G is K_4-minor-free, there is no paths in $G-\{u,v\}$ from u_2 to vertices of H_1 or H_2. Hence H_1 and H_2 are two components of $G-\{u,v\}$. □

If $|W|$ is even, let $G_1 := G[W]$ and $x := u_2$; if $|W|$ is odd, let $G_1 := G[W-u]$ and $x := u$. Then $|G_1|$ is even. Furthermore, by the definition of W and Claim 2, the components of G_1 are always the components of $G-\{x,v\}$. Hence by the inductive hypothesis on G_1, we have $n_1(G_1) \leq \frac{|G_1|-2}{2}$. In addition, let $G_2 := G-\{x,v\}-V(G_1)$. Then G_2 also has even number of vertices. Since $u_2 \neq v_2$, we have $v_2 \in V(G_2)$; so $|G_2| \geq 2$. Since G is 2-connected, there are at most two components in G_2. By Lemma 2.6 and the inductive hypothesis, we can obtain $n_1(G_2) \leq \frac{|G_2|-2}{2}$. Therefore, we have

$$n_1(G-\{u_2,x\}) \leq n_1(G_1)+n_1(G_2) \leq \frac{|G_1|+|G_2|-4}{2} = \frac{n-6}{2},$$

which implies that $\lambda_{\frac{n}{2}-2}(G-\{u_2,x\}) \leq 1$. By Theorem 2.1, $\lambda_{\frac{n}{2}}(G) \leq \lambda_{\frac{n}{2}-2}(G-\{u_2,x\}) \leq 1$.

By a similar argument, we can also prove that $\lambda_{\frac{n}{2}+1}(G) \geq -1$. Therefore, $R(G) \leq 1$, and we complete the proof. □

Data Availability

No data was used for the research described in the article.

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