Erdös-Gallai-type problems for distance-edge-monitoring numbers *

Zhen Ji, Ralf Klasing, Wen Li, Yaping Mao, Xiaoyan Zhang

Erdös-Gallai-type problems for distance-edge-monitoring numbers *

Zhen Ji †, Ralf Klasing ‡, Wen Li †, Yaping Mao §, Xiaoyan Zhang¶

Abstract

Foucaud et al. recently introduced and initiated the study of a new graph-theoretic concept in the area of network monitoring. For every edge e of G and a set $M \subseteq V(G)$, M is a *distance-edge-monitoring (DEM for short) set* if there are a vertex x of M and a vertex y of G such that e belongs to all shortest paths between x and y. The *DEM number* $\operatorname{dem}(G)$ is the smallest size of such a set in G. The vertices of M represent distance probes in a network modeled by G; when the edge e fails, the distance from x to y increases, and thus we are able to detect the failure. In this paper, we study Erdös-Gallai-type problems for DEM numbers of general graphs. The exact values or bounds of $\operatorname{dem}(G)$ for radix n-triangular mesh networks and hexagonal networks are also given.

Keywords: Distance; Distance-edge-monitoring number; Hexagonal network; Radix n-triangular mesh network

AMS subject classification 2020: 05C12; 05C35; 05C82.

1 Introduction

In 2022, Foucaud *et al.* [11] introduced a new graph-theoretic concept called *distance-edge-monitoring set*, which means network monitoring using distance probes. Networks are naturally modeled by finite undirected simple connected graphs, whose vertices represent computers and

*Supported by the National Science Foundation of China (Nos. 12061059, 12271259) and the Qinghai Key Laboratory of Internet of Things Project (2017-ZJ-Y21).

†School of Computer, Qinghai Normal University, Xining, Qinghai 810008, China. `jizhen219@163.com; liwzjn@163.com`

‡Corresponding author: Université de Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR 5800, Talence, France. `ralf.klasing@labri.fr`

§Faculty of Environment and Information Sciences, Yokohama National University, 79-2 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan. `maoyaping@ymail.com`

¶School of Mathematical Science, Nanjing Normal University, Nanjing 210023, China. `zhangxiaoyan@njnu.edu.cn`

whose edges represent connections between them. When a connection (an edge) fails in the network, we can detect this failure, and thus achieve the purpose of monitoring the network. Probes are made up of vertices we choose in the network. At any given moment, a probe of the network can measure its graph distance to any other vertex of the network. Whenever an edge of the network fails, one of the measured distances changes, so the probes are able to detect the failure of any edge. Probes that measure distances in graphs are present in real-life networks. They are e.g. useful in the fundamental task of routing [12, 10] and are also frequently used for problems concerning network verification [2, 3, 5].

1.1 Distance-edge-monitoring numbers

We now proceed with the formal definition of our main concept.

All graphs considered in this paper are undirected, finite and simple. We refer to the book [7] for graph theoretical notation and terminology not described here. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, respectively. And we use $e(G)$ to express the number of edges in G, that is $e(G) = |E(G)|$. Let K_n be the complete graph of order n. In this paper, for a graph G and $x, y \in V(G)$, we denote by $d_G(x, y)$ the shortest distance between two vertices x and y in a graph G. If there is no path between the vertices u and v in G, then let $d_G(u, v) = \infty$. For an edge set Y of G, we denote by $G - Y$ the graph obtained by deleting all edges in Y from G. If $Y = \{e\}$, we simply write $G - e$ for $G - Y$. We use $X \setminus S$ to denote the vertex subset of X obtained by removing all the vertices of S from X and $Y - W$ to denote the edge subset of Y obtained by removing all the edges of W from Y. If $S = \{v\}$, we simply write $X \setminus v$ for $X \setminus S$.

Definition 1. *For a set M of vertices and an edge e of a graph G, let $P(M, e)$ be the set of pairs (x, y) with x a vertex of M and y a vertex of $V(G)$ such that $d_G(x, y) \neq d_{G-e}(x, y)$. In other words, e belongs to all shortest paths between x and y in G.*

Definition 2. *For a vertex x, let $EM(x)$ be the set of edges e such that there exists a vertex v in G with $(x, v) \in P(\{x\}, e)$. If $e \in EM(x)$, we say that e* is monitored by *x.*

Definition 3. *A set M of vertices of a graph G is* distance-edge-monitoring (DEM for short) set *if every edge e of G is monitored by some vertex of M, that is, the set $P(M, e)$ is nonempty. Equivalently, $\bigcup_{x \in M} EM(x) = E(G)$.*

Definition 4. *The* DEM number $\operatorname{dem}(G)$ *of a graph G is defined as the smallest size of DEM sets of G.*

For the convenience of readers' understanding, we give the following example.

Example 1. *Let the vertex set $M = \{v_1, v_3\}$ and $e = v_4v_5$ be an edge of G, where the graph G is shown in Figure 1. Then $P(M, e) = \{(v_3, v_5), (v_5, v_3), (v_3, v_6), (v_6, v_3)\}$. For a vertex v_4, we have $EM(v_4) = \{v_1v_4, v_2v_4, v_3v_4, v_4v_5, v_5v_6\}$. Let $M_1 = \{v_i \mid 1 \leq i \leq 4\}$, $M_2 = \{v_1, v_2, v_4\}$ and $M_3 = \{v_2, v_5, v_6\}$. Then M_1 and M_2 are DEM sets of the graph G, but M_3 is not.*

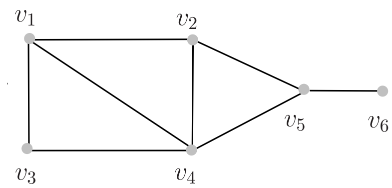

Figure 1: The graph G in Example 1.

For a graph G, the vertex set $V(G)$ is always a DEM set of G, and hence $\mathrm{dem}(G)$ is well-defined. However, the vertex set $V(G)$ is bad as DEM set in G, and hence people are always looking for k such that $\mathrm{dem}(G) \leq k$ ($k > 0$), normally, we build the smallest possible DEM set of G.

In the recent years, Bampas et al. [2] and Beerliova et al. [3] studied a weaker model as a network discovery problem, that is, where we seek a set U of vertices such that for each edge e, there exist a vertex x of U and a vertex y of G such that e belongs to some shortest path from x to y. In [4], Bejeranoa et al. studied a different and weaker model as the link monitoring problem. One seeks to monitor the edges of a graph network by selecting vertices to act as probes. To each probe is assigned a routing tree (a DFS tree spanning the whole graph), and it is essentially required that each edge of the graph belongs to one of the trees. For more results on the DEM set, we can refer to the papers [13, 15, 16, 20, 23].

1.2 Recent progress and our results

A vertex set U is a *vertex cover* of G if every edge of G has one of its endpoints in U, and the smallest size of a vertex cover of G is denoted by $vc(G)$.

Foucaud et al. [11] derived the following result for complete graphs.

Theorem 1.1. [11] *In any graph G of order n, any vertex cover $vc(G)$ is a DEM set, and thus* $\mathrm{dem}(G) \leq vc(G) \leq n-1$. *Moreover,* $\mathrm{dem}(G) = n-1$ *if and only if G is the complete graph of order n.*

Given a vertex x of a graph G and an integer i, we let $r_i(x)$ denote the set of vertices at distance i of x in G.

Lemma 1.1. [11] *Let x be a vertex of a connected graph G. Then, an edge uv belongs to $EM(x)$ if and only if $u \in r_i(x)$ and v is the only neighbor of u in $r_{i-1}(x)$.*

Lemma 1.2. [11] *Let G be a graph and x a vertex of G. Then, for any edge e incident with x, $e \in EM(x)$.*

Foucaud et al. [11] gave the DEM number of a complete bipartite graph, the grid and the hypercube.

Theorem 1.2. [11] For a complete bipartite graph $K_{a,b}$ with parts of sizes a and b, $\mathrm{dem}(K_{a,b}) = \min\{a, b\}$.

Theorem 1.3. [11] Let $G_{a,b}$ denote the grid of dimension $a \times b$ for $a, b \geq 2$. Then $\mathrm{dem}(G_{a,b}) = \max\{a, b\}$.

Theorem 1.4. [11] For the hypercube Q_n of dimension n, $\mathrm{dem}(Q_n) = 2^{n-1}$.

Let $t(G)$ be a graph parameter of G. The Erdös-Gallai-type problems are stated as follows.

Problem 1. *Given two positive integers n and k, compute the minimum integer $f(n, k)$ such that for every connected graph G of order n, if $e(G) \geq f(n, k)$ then $t(G) \geq k$.*

Problem 2. *Given two positive integers n and k, compute the maximum integer $g(n, k)$ such that for every connected graph G of order n, if $e(G) \leq g(n, k)$ then $t(G) \leq k$.*

In recent years, Wang et al. [21] investigated some extremal problems on matching preclusion number $mp(G)$. In 2019, Jiang et al. [14] obtained Erdös-Gallai-type results for total monochromatic connection $tmc(G)$ of graphs. In 2022, Li and Li [17] solved the Erdös-Gallai-type problems for the monochromatic disconnection $md(G)$. For more results on Erdös-Gallai-type problems, we refer to [1, 8].

In this paper, we consider Erdös-Gallai-type problems for the DEM numbers, where $t(G) = \mathrm{dem}(G)$ in the problems. In Section 2, we derive the following results for Problems 1 and 2.

Theorem 1.5. *Let n, k be two positive integers with $n \geq 6$, $4 \leq k \leq n - 2$. Then*

$$n + 2 \leq f(n, k) \leq \binom{n}{2} - \binom{n-k}{2}.$$

In addition, $f(n, 1) = n - 1$; $f(n, 2) = n$; $n + 1 \leq f(n, 3) \leq 2n - 2$ for $n \geq 6$; $f(n, n - 1) = \binom{n}{2}$. Moreover, the bounds are sharp.

Theorem 1.6. *Let n, k be two positive integers with $n \geq 9$. Then*

$$n + 2 \leq g(n, k) \leq \begin{cases} (k + 1)(n - 1) - 1, & \text{if } 4 \leq k \leq \lfloor (n - 1)/2 \rfloor; \\ \binom{n}{2} - \binom{n-k}{2}, & \text{if } \lceil n/2 \rceil \leq k \leq n - 2. \end{cases}$$

In addition, $g(n, 1) = n - 1$; $n \leq g(n, 2) \leq 2n - 4$ for $n \geq 5$; $n + 1 \leq g(n, 3) \leq 3n - 6$ for $n \geq 6$; $g(n, n - 1) = \binom{n}{2}$. Moreover, the bounds are sharp.

A *radix n-triangular mesh network*, denoted by T_n, is the graph with $V(T_n) = \{(x, y) \,|\, 0 \leq x + y \leq n - 1\}$ in which any two vertices (x_1, y_1) and (x_2, y_2) are connected by an edge if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$, or $x_2 = x_1 + 1$ and $y_2 = y_1 - 1$, or $x_2 = x_1 - 1$ and $y_2 = y_1 + 1$, and we write an edge as $((x_1, y_1), (x_2, y_2))^*$; see [22] for more details. The number of vertices and edges in T_n is equal to $n(n + 1)/2$ and $3n(n - 1)/2$, respectively; see Figure 2.

In Section 3, we get the DEM numbers of radix n-triangular mesh networks.

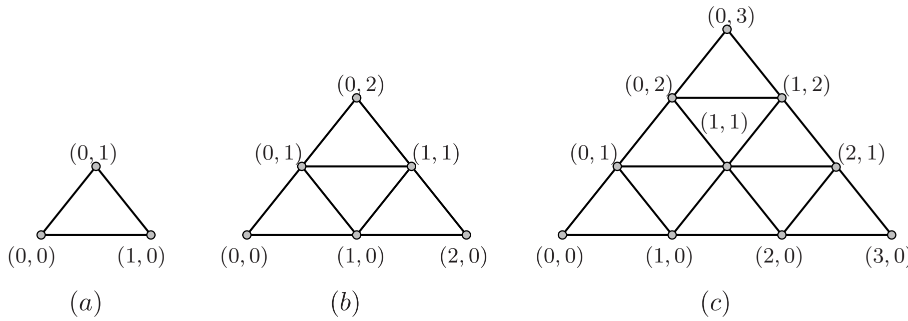

Figure 2: (a) T_2; (b) T_3; (c) T_4

Theorem 1.7. *For a radix n-triangular mesh network T_n with $n \geq 2$, we have*

$$\mathrm{dem}(T_n) = \begin{cases} 2, & n = 2; \\ 3, & n = 3; \\ (3n-6)/2, & n > 2 \text{ and } n \text{ is even}; \\ (3n-5)/2, & n > 3 \text{ and } n \text{ is odd}. \end{cases}$$

The following corollary shows the relation between the size and DEM numbers of a radix n-triangular mesh network.

Corollary 1.8. *For a radix n-triangular mesh network T_n, if $n \geq 4$, then*

$$\mathrm{dem}(T_n) = \begin{cases} (\sqrt{9+24e(T_n)}-9)/4, & n \text{ is even}; \\ (\sqrt{9+24e(T_n)}-7)/4, & n \text{ is odd}. \end{cases}$$

It is known that there exist three regular plane tessellations, composed of the same kind of regular polygons: triangular, square, and hexagonal. The triangular tessellation is used to define Hexagonal networks [9].

A *hexagonal network* $HX(n)$ of dimension n has $3n^2-3n+1$ vertices and $9n^2-15n+6$ edges, where n ($n \geq 2$) is the number of vertices on one side of the hexagon [9, 18]. There are six vertices of degree three which we call as corner vertices a, b, c, d, f, g; see Figure 3. There is exactly one vertex v at distance $n-1$ from each of the corner vertices. This vertex is called the *centre* of $HX(n)$ and is represented by o.

In Section 4, we give the results about DEM numbers of hexagonal networks.

Theorem 1.9. *For a hexagonal network $HX(n)$ ($n \geq 2$), we have $2n-1 \leq \mathrm{dem}(HX(n)) \leq 3n-3$.*

The following corollary shows the relation between the size and DEM numbers of a hexagonal network.

Corollary 1.10. *For a hexagonal network $HX(n)$ ($n \geq 2$), let $t = e(HX(n))$. Then we have $(\sqrt{1+4t}+2)/3 \leq \mathrm{dem}(HX(n)) \leq (\sqrt{1+4t}-1)/2$.*

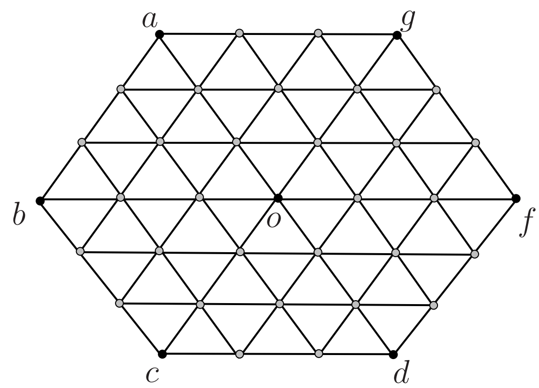

Figure 3: A hexagonal network of dimension 4.

2 Erdös-Gallai-type problems for general graphs

Foucaud et al. [11] obtained the following results.

Theorem 2.1. [11] *For any graph G of order $n \geq 4$ and size m,* $\mathrm{dem}(G) \geq \frac{m}{n-1}$.

Theorem 2.2. [11] *Let G be a connected graph with at least one edge. We have* $\mathrm{dem}(G) = 1$ *if and only if G is a tree.*

The following corollary is immediate.

Corollary 2.3. *Let G be a connected graph with $|V(G)| = n$ and $e(G) \geq n$. Then we have* $\mathrm{dem}(G) \geq 2$.

Proposition 2.1. *Let K_n be a complete graph and $e \in E(K_n)$. Then* $\mathrm{dem}(K_n - e) = n - 2$.

Proof. Let the graph $G = K_n - e$ and the edge $e = uv$. From Theorem 1.1, we have $\mathrm{dem}(K_n - e) \leq n - 2$. To show $\mathrm{dem}(K_n - e) \geq n - 2$, let $V(K_n) = \{v_i \mid 1 \leq i \leq n\}$. Suppose that the vertex set $U \subseteq V(G)$ with $|U| = n - 3$ is a DEM set. Without loss of generality, let $U = \{v_i \mid 1 \leq i \leq n - 3\}$. Since e is incident to at most two vertices in $\{v_{n-2}, v_{n-1}, v_n\}$, say $v_{n-2} \notin \{u, v\} \cup U$, it follows that $d_G(v_i, v_{n-2}) = d_G(v_i, v_{n-1})$ for each $1 \leq i \leq n - 3$, and hence the edge $v_{n-2}v_{n-1} \notin \cup_{x \in U} EM(x)$, and so $\mathrm{dem}(K_n - e) \geq n - 2$. $\square$

Lemma 2.1. *For a connected graph G, if G contains a subgraph K_r ($r \geq 2$), then* $\mathrm{dem}(G) \geq r - 1$.

Proof. Let $G' = K_r$ be a complete graph with vertex set $V(G') = \{v_i \mid 1 \leq i \leq r\}$. Suppose that the vertex set Q with $|Q| = r - 2$ is a DEM set of the graph G. If $|Q \cap V(G')| = r - 2$, then $Q \subseteq V(G')$. From Theorem 1.1, there exists an edge $e \in E(G')$ such that $e \notin \cup_{x \in Q} EM(x)$, a contradiction. If $|Q \cap V(G')| < r - 2$, then let the vertex set $U = Q \setminus V(G')$. For each vertex $u \in U$, there exists a vertex $v_i \in V(G')$, where $1 \leq i \leq r$, such that $d_{G-E(G')}(u, v_j) \geq d_{G-E(G')}(u, v_i) = k \geq 1$ for any $1 \leq j \leq r$ with $j \neq i$. If $d_{G-E(G')}(u, v_j) \geq k + 2$, then the edge set $\{v_iv_t \mid 1 \leq t \neq i \leq r\} \subseteq EM(u)$, which implies that $EM(u) \cap E(G') = EM(v_i) \cap E(G')$. If $d_{G-E(G')}(u, v_j) = k$ for $1 \leq j \neq i \leq r$, then it follows from Definition 1 and Lemma 1.1 that $E(G') \nsubseteq EM(u)$. If $d_{G-E(G')}(u, v_j) = k+1$ for $1 \leq j \neq i \leq r$, then it follows from Definition 1 and Lemma 1.1 that the edge set $\{v_iv_t \mid 1 \leq t \neq i \leq r\} - \{v_iv_j\} \subseteq EM(u)$, which implies that $EM(u) \cap E(G') \subset EM(v_i) \cap E(G')$. Therefore, the vertex set Q can monitor at most $\binom{r}{2} - 1$ edges of $E(G')$, which contradicts the fact that $e(G') = \binom{r}{2}$. □

Lemma 2.2. *Let n, k be two positive integers with $n \geq 2$. Then*

(1) $f(n, 1) = n - 1$;

(2) $f(n, 2) = n$;

(3) $n + 1 \leq f(n, 3) \leq 2n - 2$ *for* $n \geq 6$;

(4) $f(n, n-1) = \binom{n}{2}$ *for* $n \geq 4$.

Proof. (1) Let G_1 be a connected graph with order n. Then $f(n, 1) \geq n - 1$. If G_1 is a tree, then it follows from Theorem 2.2 that $\mathrm{dem}(G_1) = 1$, and hence $f(n, 1) \leq n - 1$, and so $f(n, 1) = n - 1$.

(2) Let G_2 be a connected graph with n vertices such that $e(G_2) \geq n$. It follows from Corollary 2.3 that $\mathrm{dem}(G_2) \geq 2$, and so $f(n, 2) \leq n$. To show $f(n, 2) \geq n$, we let G be a connected graph of order n and size $n - 1$. From Theorem 2.2, we have $\mathrm{dem}(G) = 1 < 2$, and hence $f(n, 2) \geq n$, and so $f(n, 2) = n$.

(3) Let G_3 be a connected graph with order n and $e(G_3) \leq n$. Clearly, $\mathrm{dem}(G_3) \leq 2$. Let F_1 be a connected graph of order $n \geq 6$. We construct a graph F_2 as follows: F_2 is the base graph grid $G_{2,3}$ of F_1. Note that the base graph of a graph F_1 is the graph obtained from F_1 by iteratively removing vertices of degree 1. One can easily check that $e(F_1) = n+1$ and $\mathrm{dem}(F_1) = 3$, and hence $f(n, 3) \geq n + 1$, which shows that the lower bound is sharp. To show the upper bound, we let F_3 be the graph obtained from t ($t \geq 2$) triangles with unique common edge e, by adding the edge w_1w_2, where $e = uv$ and $w_1, w_2, \ldots, w_t$ are the vertices except u and v in t triangles. Let F_4 be the graph obtained from F_3 by adding all possible edges between the vertices in $\{w_i \mid 1 \leq i \leq t\}$, besides the edge w_1w_2. Then, there exists a clique K_4 induced by the vertices in $\{u, v, w_1, w_2\}$, then it follows from Lemma 2.1 that $\mathrm{dem}(F_4) \geq 3$, and hence $f(n, 3) \leq e(F_3) = 2n - 2$. Moreover, F_3 can reach a graph whose upper bound is sharp.

(4) Let G_4 be a connected graph with order n. Since $\mathrm{dem}(G_4) = n-1$, it follows from Theorem 1.1 that G_4 is a complete graphK_n, and hence $f(n, n-1) \leq \binom{n}{2}$. For a connected graph G' with order n and $e(G') = \binom{n}{2} - 1$, by Proposition 2.1, we have $\mathrm{dem}(G') = n - 2 < n - 1$, and hence $f(n, n-1) \geq \binom{n}{2}$, and so $f(n, n-1) = \binom{n}{2}$. □

A *feedback edge set* of a graph G is a set of edges such that removing them from G leaves a forest. The smallest size of a feedback edge set of G is denoted by $\mathrm{fes}(G)$.

In Figure 4, let two edge sets $E_1 = \{v_1v_2, v_2v_4, v_4v_5, v_3v_5, v_5v_6\}$ and $E_2 = \{v_2v_4, v_2v_5, v_4v_5, v_3v_5\}$ in H. Then, the graphs H_1 and H_2 obtained by removing E_1 and E_2 from H are two forests, respectively. Therefore, E_1 and E_2 are two feedback edge sets of H.

Theorem 2.4. [11] *If* $\mathrm{fes}(G) \leq 2$, *then* $\mathrm{dem}(G) \leq \mathrm{fes}(G) + 1$. *Moreover, if* $\mathrm{fes}(G) \leq 1$, *then the equality holds.*

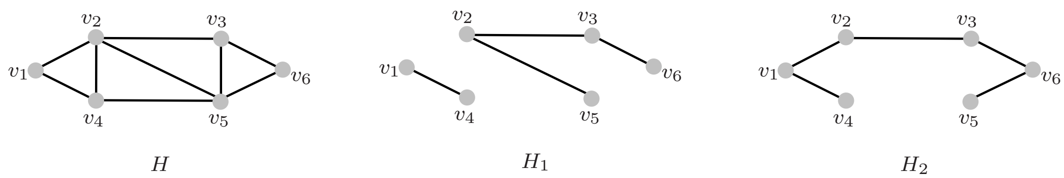

Figure 4: The graphs as an example of feedback edge set.

The following corollary is immediate.

Corollary 2.5. *For a connected graph G, if $e(G) \leq n+k-2$ ($k=2,3$), then* $\mathrm{dem}(G) \leq k$.

The DEM number of complete multipartite graph is given below.

Proposition 2.2. *Let r be an integer with $r \geq 3$. For a complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$, $n_1 \leq n_2 \leq \cdots \leq n_r$, we have* $\mathrm{dem}(K_{n_1,n_2,\ldots,n_r}) = \sum\limits_{i=1}^{r-1} n_i$.

Proof. Let $G = K_{n_1,n_2,\ldots,n_r}$, and A_i be the vertex set of the part i in G with $|A_i| = n_i$, $1 \leq i \leq r$. Note that $V(G) = A_1 \cup A_2 \cup \cdots \cup A_r$. Let the vertex set $U = \cup_{i=1}^{r-1} A_i$. For each vertex $v \in U$, it follows from Lemma 1.2 that v can monitor all the edges incident with v, and so $EM(U) = E(G)$. Since $EM(U)$ represents the union of edge sets monitored by each $v \in U$, it follows that $\mathrm{dem}(G) \leq \sum\limits_{i=1}^{r-1} n_i$.

To show the lower bound, we arbitrarily choose a vertex set $M \subseteq V(G)$ as a DEM set with $|M| = \sum\limits_{i=1}^{r-1} n_i - 1$. If $|M \cap (A_1 \cup A_2 \cup \cdots \cup A_{r-1})| = \sum\limits_{i=1}^{r-1} n_i - 1$, then there exists a vertex v such that $v \in A_i$ but M, $1 \leq i \leq r-1$. Then for a vertex $u \in A_r$ and any vertex w of M, we have $d_G(w,v) = d_G(w,u) = 1$ if $w \in A_j, j \neq i$; there exist two shortest paths from w to u if $w \in A_i$, and hence the edge uv cannot be monitored by M. If $|M \cap (A_1 \cup A_2 \cup \cdots \cup A_{r-1})| < \sum\limits_{i=1}^{r-1} n_i - 1$, then we take $|M \cap A_r| = t \geq 1$. Let $M \cap A_r = \{u_1, u_2, \ldots, u_t\}$ and $(A_1 \cup A_2 \cup \cdots \cup A_{r-1}) \setminus M = \{v_1, v_2, \ldots, v_{t+1}\}$. Suppose that the vertices $v_1, v_2, \ldots, v_{t+1}$ are not in the same part of G. Without loss of generality, let $v_1 \in A_1$ and $v_2 \in A_2$. Then $d_G(w_1, v_1) = d_G(w_1, v_2)$ for any vertex $w_1 \in M \cap (\cup_{i=3}^{r} A_i)$. For any vertex $w_2 \in M \cap (A_1 \cup A_2)$, says $w_2 \in A_1$, we can obtain the two shortest paths $w_2v_2v_1$ and $w_2w_3v_1$ from w_2 to v_1, and hence the edge v_1v_2 cannot be monitored by M, where $w_3 \in \cup_{i=3}^{r} A_i$. Otherwise, the vertices $v_1, v_2, \ldots, v_{t+1}$ are all in one part of G, says V_{r-1}. Obviously, since $n_1 \leq n_2 \leq \cdots \leq n_r$, then $|A_r| \geq t+1$, note that $u_{t+1} \notin M$. From Definition 1 and Lemma 1.1, the edge v_1u_{t+1} cannot be monitored by M, and hence $\mathrm{dem}(G) \geq \sum\limits_{i=1}^{r-1} n_i$, and so $\mathrm{dem}(G) = \sum\limits_{i=1}^{r-1} n_i$. □

We are now in a position to give the proof of the upper and lower bounds for $f(n,k)$.

Proof of Theorem 1.5: For any connected graph G with order $n \geq 6$ and $e(G) \leq n+1$, it follows from Corollary 2.5 that $\mathrm{dem}(G) \leq 3$, and hence $f(n,4) \geq n+2$, and so $f(n,k) \geq f(n,4) \geq n+2$

for $4 \leq k \leq n-2$. Now we construct a connected graph F whose the base graph is grid $G_{2,4}$, then $e(F) = n+2$. By Theorem 1.3, $\mathrm{dem}(F) = 4$, and hence the lower bound is sharp.

Let H be the connected graph of order n obtained from $K_{n_1,n_2,\ldots,n_r}$ by adding the all edges formed by every two pairs of vertices in V_i for each $1 \leq i \leq r-1$, where V_i is the vertex set of part i in $K_{n_1,n_2,\ldots,n_r}$, and $|V_i| = n_i$ for $1 \leq i \leq r$ and $n_1 \leq n_2 \leq \cdots \leq n_r$. Let $\sum\limits_{i=1}^{r-1} n_i = k$, where $4 \leq k \leq n-2$, which implies that $n_r \geq 2$. Let the vertex set $Q = V_1 \cup V_2 \cup \cdots \cup V_{r-1}$, and hence $EM(Q) = E(H)$. Since $EM(Q)$ represents the union of edge sets monitored by each vertex of Q, it follows from Proposition 2.2 that $\mathrm{dem}(H) \leq \sum\limits_{i=1}^{r-1} n_i = k$. But adding all possible edges formed by every two pairs of vertices in V_r, we can obtain a new graph H' and the following claim holds.

Claim 1. $\mathrm{dem}(H') \geq k+1$.

Proof. We arbitrarily choose a vertex set M in $V(H')$ as a DEM set with $|M| = k$. Let $|M \cap V_r| = t \geq 0$ and $|M \cap (\cup_{i=1}^{r-1} V_i)| = k-t$. If $t = 0$, then we take $u_1, u_2 \in V_r$ and $u_1u_2 \in E(H')$, and hence $d_{H'}(w, u_1) = d_{H'}(w, u_2)$ for any vertex $w \in M$, and so the edge u_1u_2 cannot monitored by M. If $t = n_r$, then there exist two vertices v_1 and v_2 in H' such that $v_1, v_2 \in \cup_{i=1}^{r-1} V_i$ but M, and $d_{H'}(w, v_1) = d_{H'}(w, v_2)$ for any vertex $w \in M$, and so the edge v_1v_2 cannot monitored by M. If $1 < t < n_r$, then assume that $u_3 \in V_r, v_3 \in \cup_{i=1}^{r-1} V_i$ but $u_3, v_3 \notin M$. Therefore, $d_{H'}(w_1, v_3) = d_{H'}(w_1, u_3)$ for any vertex $w_1 \in M \cap (\cup_{i=1}^{r-1} V_i)$. For any vertex $w_2 \in M \cap V_r$, let $v_4 \in \cup_{i=1}^{r-1} V_i$ ($v_4 \neq v_3$). If w_1 and w_2 are not adjacent, then there are two shortest paths $w_2v_3u_3$ and $w_2v_4u_3$ from w_2 to u_3. Otherwise, $d_{H'}(w_2, v_3) = d_{H'}(w_2, u_3)$, and hence the edge u_3v_3 cannot be monitored by M, a contradiction. $\square$

Therefore, $g(n,k) \leq e(H) = \binom{n}{2} - \binom{n_r}{2} = \binom{n}{2} - \binom{n-k}{2}$ for $4 \leq k \leq n-2$ and $n \geq 6$. In addition, by Lemma 2.2, we have $f(n,1) = n-1$ and $f(n,2) = n$ for $n \geq 2$, $n+1 \leq f(n,3) \leq 2n-2$ for $n \geq 6$ and $f(n,n-1) = \binom{n}{2}$ for $n \geq 4$. $\square$

Lemma 2.3. *Let n, k be two positive integers with $n \geq 2$. Then*

(1) $g(n,1) = n-1$;

(2) $n \leq g(n,2) \leq 2n-4$ *for* $n \geq 5$;

(3) $n+1 \leq g(n,3) \leq 3n-6$ *for* $n \geq 6$;

(4) $g(n,n-1) = \binom{n}{2}$.

Proof. By Theorems 1.1 and 2.2, we have $g(n,1) = n-1$ and $g(n,n-1) = \binom{n}{2}$, and so (1) and (4) hold. By Corollary 2.5, $\mathrm{dem}(G) \leq 2$, for a connected graph G with $e(G) \leq n$, and hence $g(n,2) \geq n$. Moreover, let R be a connected graph with order n such that the base graph of R is a cycle. Then $g(n,2) = n$ for the graph R, and hence the lower bound is sharp. To show $g(n,2) \leq 2n-4$, where $n \geq 5$, let G_1 be the graph obtained from t ($t \geq 1$) triangles with unique common edge uv, suspending a new triangle on an endpoint v of uv, where $w_1, w_2, \ldots, w_t$ are the vertices except u, v in the t triangles and x, y are the vertices except v in the new triangle.

Obviously, we have $\text{dem}(G_1) = 2$. Let G_1' be the graph obtained from G_1 by adding an edge e which is not in $E(G_1)$. Now we give the following claim.

Claim 2. $\text{dem}(G_1') \geq 3$.

Proof. Let the vertex set X be a DEM set of G_1' with $|X| = 2$. If the edge $e = w_i w_j$, where $1 \leq i, j \leq t$ and $i \neq j$, then the graph induced by the vertex set $\{u, v, w_i, w_j\}$ is a complete graph K_4, and hence from Lemma 2.1, $\text{dem}(G_1') \geq 3$, a contradiction. If the edge $e = uy$ *or* yw_s, $1 \leq s \leq t$, then the edges in $\{uw_i \mid 1 \leq i \leq t\} \cup \{xy\}$ only can be monitored by its endpoints. If $X \subseteq \{w_i \mid 1 \leq i \leq t\}$, then the edge uv cannot be monitored by X, and hence $u \in X$. Similarly, if $X = \{x, y\}$, then the edges in $\{uw_i \mid 1 \leq i \leq t\}$ cannot be monitored, and hence x *or* $y \in X$, and so vy cannot be monitored if $x \in X$; vw_s cannot be monitored if $y \in X$, a contradiction. □

Therefore, we have $g(n, 2) \leq 2n - 4$ for the graph G_1 with $|V(G_1)| = n$ and $e(G_1) = 2t + 4$, where $n \geq 5$, and hence (2) holds. Moreover, $g(n, 2) = 2n - 4$ for graph G_1, and hence the upper bound is sharp.

For a connected graph G with $e(G) \leq n + 1$, it follows from Corollary 2.5 that $\text{dem}(G) \leq 3$, and hence $g(n, 3) \geq n + 1$. Moreover, for a connected graph H of order n, where the base graph of H is a grid $G_{2,3}$, we have $g(n, 3) = n + 1$, and hence the lower bound is sharp.

We now show the upper bound of (3). Let G_2 be the graph obtained from a complete bipartite graph $K_{3,n-3}$, where $n \geq 6$, by adding all edges in $\{v_i v_j \mid 1 \leq i \neq j \leq 3\}$. Note that $V(K_{3,n-3}) = A \cup B$, $A = \{v_i \mid 1 \leq i \leq 3\}$ and $B = \{u_i \mid 1 \leq i \leq n-3\}$, where A and B are the vertex sets of the two parts in $K_{3,n-3}$. By Theorem 1.2, we have $\text{dem}(G_2) = 3$. Let G_2' be the graph obtained from G_2 by adding one edge between the vertices in $\{u_i \mid 1 \leq i \leq n - 3\}$. Now we give the following claim.

Claim 3. $\text{dem}(G_2') \geq 4$.

Proof. Assume, to the contrary, that $\text{dem}(G_2') \leq 3$. Arbitrarily choose a vertex set U in G_2' as a DEM set with $|U| = 3$. If $|U \cap A| = 3$, then for any edge $u_j u_k$ of the added edges, $1 \leq j \neq k \leq n-3$, we can get that $d_{G_2'}(v_i, u_j) = d_{G_2'}(v_i, u_k)$, and hence the edge $u_j u_k$ cannot be monitored by U. If $|U \cap A| = 2$, then let $U \cap A = \{v_1, v_2\}$ and $U \cap B = \{u_1\}$, without loss of generality. From Definition 1 and Lemma 1.1, the edge $v_3 u_2$ cannot be monitored by U. If $|U \cap A| \leq 1$, then there exist two vertices in A but U, says v_1 and v_2. Obviously, the edge $v_1 v_2$ cannot be monitored by U, a contradiction. □

Therefore, we have $g(n, 3) \leq e(G_2') \leq 3n - 6$, and hence (3) holds. □

Theorem 2.6. [11] *For a graph G, if* $\text{fes}(G) = t$ $(t \geq 3)$*, then* $\text{dem}(G) \leq 2t - 2$.

We first give the corollary of Theorem 2.6 as follows.

Corollary 2.7. *For a connected graph G with order n, if $e(G) \leq n + \lfloor k/2 \rfloor$, $k \geq 4$, then* $\text{dem}(G) \leq k$.

Proof. For a connected graph G with order n, if $\mathrm{fes}(G) = t$, then $e(G) = n + t - 1$. Let $t = \lfloor (k+2)/2 \rfloor$. Then $e(G) = n + \lfloor k/2 \rfloor$, where $k \geq 4$. From Theorem 2.6, we have $\mathrm{dem}(G) \leq k$. Let G' be the connected graph obtained from G by deleting some edges. Therefore, $e(G') = n + \lfloor k'/2 \rfloor \leq n + \lfloor k/2 \rfloor = e(G)$, where $k' \leq k$, and hence $\mathrm{dem}(G') \leq k$. □

Proof of Theorem 1.6: We now give the lower bound. Let G be a connected graph with order $n \geq 9$. For $4 \leq k \leq n-2$, it follows from Corollary 2.7 that $g(n,k) \geq n + \lfloor k/2 \rfloor \geq n+2$. Moreover, let H be a connected graph with order n, where the base graph of H is a grid $G_{2,4}$. Then, $g(n,k) = n+2$ for the graph H, and hence the lower bound is sharp.

To show the upper bound, for a connected graph H_1 with order n and $e(H_1) = (k+1)(n-1)-1$, it follows from Theorem 2.1 that $\mathrm{dem}(H_1) \geq \frac{(k+1)(n-1)-1}{n-1} \geq k$. However, for any connected graph H_2 with order n, if $e(H_1) \geq (k+1)(n-1)$, then $\mathrm{dem}(H_2) \geq k+1$, and hence $g(n,k) \leq (k+1)(n-1)-1$ for $4 \leq k \leq \lfloor (n-1)/2 \rfloor$. Now we give the upper bound for $\lceil n/2 \rceil \leq k \leq n-2$. Let H_3 be the connected graph obtained from $K_{n_1,n_2,\ldots,n_r}$ by adding the all edges formed by every pair of vertices in V_i for each $1 \leq i \leq r-1$. Note that V_i is the vertex set of part i in a complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$, and $|V_i| = n_i$, $1 \leq i \leq r$. Let $\sum\limits_{i=1}^{r-1} n_i = k$, where $\lceil n/2 \rceil \leq k \leq n-2$, and the vertex set $U = V_1 \cup V_2 \cup \cdots \cup V_{r-1}$. Then, we have $EM(U) = E(H_3)$. Since $EM(U)$ represents the union of edge sets monitored by each vertex of U, it follows that $\mathrm{dem}(H_3) \leq \sum\limits_{i=1}^{r-1} n_i = k$. But adding one edge formed by every pair of vertices in V_r, we can obtain a new graph H_3' such that $\mathrm{dem}(H_3') \geq k+1$, and hence $g(n,k) \leq e(H_3) = \binom{n}{2} - \binom{n_r}{2} = \binom{n}{2} - \binom{n-k}{2}$, which implies that the upper bound is sharp.

In addition, by Lemma 2.3, we have $g(n,1) = n-1$ and $g(n,n-1) = \binom{n}{2}$ for $n \geq 2$, $n \leq g(n,2) \leq 2n-4$ for $n \geq 5$ and $n+1 \leq g(n,3) \leq 3n-6$ for $n \geq 6$. □

3 Results for radix n-triangular mesh networks

For an integer t ($1 \leq t \leq n-1$) and any two edges $((x_1,y_1),(x_2,y_2))^*$ and $((x_3,y_3),(x_4,y_4))^*$ of the radix n-triangular mesh networks T_n, we call $((x_1,y_1),(x_2,y_2))^*$ and $((x_3,y_3),(x_4,y_4))^*$ **the linear edges** if the two edges satisfied one of the following cases

(1) $x_i = x_j = t-1$ ($i,j = 1,2,3,4$);

(2) $y_i = y_j = t-1$ ($i,j = 1,2,3,4$);

(3) $x_i + y_i = x_j + y_j = t$ ($i,j = 1,2,3,4$).

Otherwise, **the nonlinear edges**. Let M_t^i be the edge set satisfying the case (i) for each $1 \leq t \leq n-1$ and V_t^i be the endpoint set of all edges in the edge set M_t^i, where $1 \leq i \leq 3$. Note that $|M_t^1| = n-t$, $|M_t^2| = n-t$, $|M_t^3| = t$, $|V_t^1| = n-t+1$, $|V_t^2| = n-t+1$ and $|V_t^3| = t+1$.

Theorem 3.1. *For any vertex $v = (x,y) \in V(T_n)$, $EM(v) = M_{x+1}^1 \cup M_{y+1}^2 \cup M_{x+y}^3$.*

Proof. For any $uw \in M_{x+1}^1$ with $d_{T_n}(v,u) > d_{T_n}(v,w)$, since there exists only one shortest path P_{vu} from v to u in the graph T_n, where $E(P_{vu}) \subseteq M_{x+1}^1$, it follows that $d_{T_n}(v,u) \neq d_{T_n-uw}(v,u)$, and hence $uw \in EM(v)$, and so all edges in M_{x+1}^1 can be monitored by v. Similarly, the all edges in M_{y+1}^2 and M_{x+y}^3 can be monitored by v. For any edge $uw \in E(T_n) - M_{x+1}^1 \cup M_{y+1}^2 \cup M_{x+y}^3$, it follows from Definition 1 and Lemma 1.1 that the vertex v cannot monitor the edge uw, and hence v cannot monitor all edges in $E(T_n) - M_{x+1}^1 \cup M_{y+1}^2 \cup M_{x+y}^3$. Therefore, $EM(v) = M_{x+1}^1 \cup M_{y+1}^2 \cup M_{x+y}^3$. $\square$

Since $|M_{x+1}^1| = n-(x+1)$, $|M_{y+1}^2| = n-(y+1)$ and $|M_{x+y}^3| = x+y$ for any vertex $v=(x,y) \in V(T_n)$, it follows that $|EM(v)| = 2(n-1)$, and hence the following corollary holds.

Corollary 3.2. *Let T_n be a radix n-triangular mesh network. Then we have $|EM(v)| = 2(n-1)$ for any vertex $v \in V(T_n)$.*

Proposition 3.1. [23] *Let G be a connected graph and $M_1, M_2 \subseteq V(G)$. For any $e \in E(G)$, if $M_1 \subseteq M_2$, then $P(M_1,e) \subseteq P(M_2,e)$.*

Proposition 3.2. *For $v \in M \subseteq V(T_n)$ and $e \in E(T_n)$, we have $|P(M \setminus v, e)| \leq |P(M,e)|$. Moreover, if $v \in M \subseteq V_t^i$ and $e \in M_t^i$, then $P(M \setminus v, e) \subset P(M,e)$; if $e \in M_t^i$, then $P(M,e) = P(M \cap V_t^i, e)$, where $1 \leq i \leq 3$.*

Proof. By Proposition 3.1, we have $P(M \setminus v, e) \subseteq P(M,e)$, and hence $|P(M \setminus v,e)| \leq |P(M,e)|$. For $1 \leq i \leq 3$, let $v \in V_t^i$ and $e = uw \in M_t^i$. Without loss of generality, we assume $d_{T_n}(v,w) < d_{T_n}(v,u)$, then there exists the unique shortest path P_{vu} from v to u such that $uw \in E(P_{vu})$, and hence $d_{T_n}(v,w) \neq d_{T_n-e}(v,w)$ and so the vertex pair $(v,w) \in P(M,e)$ and $(v,w) \notin P(M \setminus v, e)$. Therefore, $P(M \setminus v, e) \subset P(M,e)$.

For any $v \in V(T_n) \setminus V_t^i$ and $e \in M_t^i$, then there exists a shortest path P_{vy} from v to y such that $E(P_{vy}) \cap M_t^i = \emptyset$ for any $y \in V(T_n)$, and hence $d_{T_n}(v,y) = d_{T_n-e}(v,y)$, and so $P(\{v\},e) = \emptyset$. Therefore, $P(M,e) = P(M \setminus v, e)$, and so $P(M,e) = P(M \setminus (V(T_n) \setminus V_t^i), e) = P(M \cap V_t^i, e)$. $\square$

Theorem 3.3. *For a radix n-triangular mesh network T_n ($n \geq 2$), let $M \subseteq V(T_n)$ and $e \in E(T_n)$. Then we have $0 \leq |P(M,e)| \leq 2\lfloor n/2 \rfloor \lceil n/2 \rceil$.*

Proof. By Definition 1, we have $|P(M,e)| \geq 0$. For any edge $e = uv \in E(T_n)$, there exists a M_t^i such that $e \in M_t^i$. Let $M \subseteq V(T_n) \setminus V_t^i$. Since there exists a shortest path P_{xy} from x to y such that $E(P_{xy}) \cap M_t^i = \emptyset$ for any $x \in M$ and $y \in V(T_n)$, it follows that $d_{T_n}(x,y) = d_{T_n-e}(x,y)$, and hence $|P(M,e)| = 0$, and so the lower bound is sharp.

For any edge $e = uv \in E(T_n)$, there exists a M_t^i such that $e \in M_t^i$. For any $M \subseteq V(T_n)$, from Proposition 3.2, we have $P(M,e) = P(M \cap V_t^i, e)$, and hence $|P(M,e)| = |P(M \cap V_t^i, e)| \leq |P(V_t^i, e)|$. Let $X \subseteq V_t^i$ be the vertex set such that $d_{T_n}(u,x) < d_{T_n}(v,x)$ for any $x \in X$, and $Y \subseteq V_t^i$ be the vertex set such that $d_{T_n}(u,x) > d_{T_n}(v,x)$ for any $y \in Y$. Then, $X \cup Y = V_t^i$. Since $d_{T_n}(x,y) \neq d_{T_n-e}(x,y)$, it follows that $(x,y),(y,x) \in P(V_t^i, e)$, and hence $|P(V_t^i,e)| = 2|X| \cdot |Y|$. Since $|V_t^1| = n-t+1$, $|V_t^2| = n-t+1$ and $|V_t^3| = t+1$, it follows that $|V_t^i| \leq n$, where $1 \leq i \leq 3$, and hence $|P(V_t^i,e)| = 2|X||Y| \leq 2\lfloor n/2 \rfloor \lceil n/2 \rceil$. $\square$

The following example shows the upper bound in Theorem 3.3 is sharp.

Example 2. *For the odd $n \geq 3$, let $e_1 = ((0, \frac{n-1}{2}), (0, \frac{n+1}{2}))^*$ and $M = \{(0, i) \mid 0 \leq i \leq n-1\}$. By Theorem 3.1, we have $P(M, e_1) = \{((0, i), (0, j)), ((0, j), (0, i)) \mid 0 \leq i \leq \frac{n-1}{2}, \frac{n+1}{2} \leq j \leq n-1\}$, and hence $|P(M, e_1)| = \frac{n^2-1}{2}$. For the even $n \geq 2$, let $e_2 = ((0, \frac{n-2}{2}), (0, \frac{n}{2}))^*$. By Theorem 3.1, we have $P(M, e_2) = \{((0, i), (0, j)), ((0, j), (0, i)) \mid 0 \leq i \leq \frac{n-2}{2}, \frac{n}{2} \leq j \leq n-1\}$, and hence $|P(M, e_2)| = \frac{n^2}{2}$, which implies that the upper bound is sharp.*

We are now in a position to give the proof of Theorem 1.7 by the following two propositions.

Proposition 3.3. *Let T_n be a radix n-triangular mesh network, where n is even. Then, we have*

$$\mathrm{dem}(T_n) = \begin{cases} 2 & n = 2, \\ (3n-6)/2 & n > 2. \end{cases}$$

Proof. If $n = 2$, then T_2 is a complete graph K_3 of order 3. From Theorem 1.1, we have $\mathrm{dem}(T_2) = 2$. For $n > 2$, we let $M_1 = \{(0, v) \mid 1 \leq v \leq \frac{n-2}{2}\}$, $M_2 = \{(u, 0) \mid \frac{n}{2} \leq u \leq n-2\}$ and $M_3 = \{(u, v) \mid u + v = n-1, 1 \leq u \leq \frac{n-2}{2}, \frac{n}{2} \leq v \leq n-2\}$.

Let $M = M_1 \cup M_2 \cup M_3$. Then, $|M| = (3n-6)/2$. For each vertex $(0, v) \in M_1$, by Theorem 3.1, we have $EM((0, v)) = \{((0, i)(0, i+1))^* \mid 0 \leq i \leq n-2\} \cup \{((j, v)(j+1, v))^* \mid 0 \leq j \leq n-2-v\} \cup \{((j, i)(j+1, i-1))^* \mid 0 \leq j \leq v-1, j+i = v\}$.

Similarly, we have $EM((u, 0)) = \{((i, 0)(i+1, 0))^* \mid 0 \leq i \leq n-2\} \cup \{((u, j)(u, j+1))^* \mid 0 \leq j \leq n-2-u\} \cup \{((j, i)(j+1, i-1))^* \mid 0 \leq j \leq u-1, j+i = u\}$ for each $(u, 0)$ of M_2, and $EM((u, v)) = \{((i, v)(i+1, v))^* \mid 0 \leq i \leq n-2-v\} \cup \{((u, j)(u, j+1))^* \mid 0 \leq j \leq n-2-u\} \cup \{((i, j)(i+1, j-1))^* \mid 0 \leq i \leq u+v-1, i+j = u+v\}$ for each (u, v) of M_3. Since $\left(\cup_{(0,v) \in M_1} EM((0, v))\right) \cup \left(\cup_{(u,0) \in M_2} EM((u, 0))\right) \cup \left(\cup_{(u,v) \in M_3} EM((u, v))\right) = E(T_n)$, it follows that $\mathrm{dem}(T_n) \leq (3n-6)/2$ for $n > 2$.

To show $\mathrm{dem}(T_n) \geq (3n-6)/2$ for $n > 2$, let the vertex set $Q \subseteq V(T_n)$ with $|Q| = (3n-6)/2 - 1$ be a DEM set of T_n. Choose the edge set $I = (\cup_{i=n/2}^{n-2} M_{i+1}^1) \cup (\cup_{j=n/2}^{n-2} M_{j+1}^2) \cup (\cup_{k=1}^{(n-2)/2} M_k^3)$ and the vertex set $R = (\cup_{i=n/2}^{n-2} V_{i+1}^1) \cup (\cup_{j=n/2}^{n-2} V_{j+1}^2) \cup (\cup_{k=1}^{(n-2)/2} V_k^3)$. For any edge $e \in M_i^j \subseteq I$, where $1 \leq i \leq n-1$ and $1 \leq j \leq 3$, from Proposition 3.2, we have $P(M, e) = P(M \cap V_i^j, e)$ for any $M \subseteq V(T_n)$, and hence $P(\{u\}, e) = \emptyset$ for any $u \in V(T_n) \setminus V_i^j$, and so e can only be monitored by some vertex v in $V_i^j \subseteq R$. Thus, $Q \cap V_i^j \neq \emptyset$, for any $V_i^j \subseteq R$, and so $|Q| \geq (3n-6)/2$, which contradicts the fact that $|Q| = (3n-6)/2 - 1$. Therefore, we have $\mathrm{dem}(T_n) \geq (3n-6)/2$, and hence $\mathrm{dem}(T_n) = (3n-6)/2$. □

Proposition 3.4. *For a radix n-triangular mesh network T_n with n odd, we have*

$$\mathrm{dem}(T_n) = \begin{cases} 3, & n = 3; \\ (3n-5)/2, & n > 3. \end{cases}$$

Proof. For $n = 3$, we choose the vertex set $M = \{(0,1),(1,0),(1,1)\}$ in T_3. By Theorem 3.1, we have

$$\begin{aligned} EM((0,1)) &= \{((0,0)(0,1))^*, ((0,2)(0,1))^*, ((1,0)(0,1))^*\}, \\ EM((1,0)) &= \{((0,0)(1,0))^*, ((2,0)(1,0))^*, ((1,0)(1,1))^*\}, \\ EM((1,1)) &= \{((0,1)(1,1))^*, ((2,0)(1,1))^*, ((0,2)(1,1))^*\}. \end{aligned}$$

Since $EM((0,1)) \cup EM((1,0)) \cup EM((1,1)) = E(T_3)$, it follows that $\mathrm{dem}(T_3) \le 3$. To show $\mathrm{dem}(T_3) \ge 3$, let the vertex set $Q \subseteq V(T_3)$ with $|Q| = 2$ be a DEM set of T_3. For any vertex $v \in V(T_3)$, from Theorem 3.2, $|EM(v)| = 2(n-1) = 4$, and hence $|\cup_{x \in Q} EM(x)| \le 8 < e(T_3) = 9$, and so Q is not a DEM set of T_3. Therefore, $\mathrm{dem}(T_3) \ge 3$, and so $\mathrm{dem}(T_3) = 3$.

For $n > 3$, let $M_1 = \{(0,v) \mid 1 \le v \le \frac{n-1}{2}\}$, $M_2 = \{(u,0) \mid \frac{n-1}{2} \le u \le n-2\}$ and $M_3 = \{(u,v) \mid 1 \le u \le \frac{n-3}{2}, \frac{n+1}{2} \le v \le n-2, u+v = n-1\}$. Choose the vertex set $M = M_1 \cup M_2 \cup M_3$ with $|M| = (3n-5)/2$ in T_n. For each vertex $(0,v) \in M_1$, by Lemma 3.1, we have $EM((0,v)) = \{((0,i)(0,i+1))^* \mid 0 \le i \le n-2\} \cup \{((j,v)(j+1,v))^* \mid 0 \le j \le n-2-v\} \cup \{((j,i)(j+1,i-1))^* \mid 0 \le j \le v-1, j+i = v\}$.

Similarly, we have $EM((u,0)) = \{((i,0)(i+1,0))^* \mid 0 \le i \le n-2\} \cup \{((u,j)(u,j+1))^* \mid 0 \le j \le n-2-u\} \cup \{((j,i)(j+1,i-1))^* \mid 0 \le j \le u-1, j+i = u\}$ for each $(u,0)$ of M_2, and $EM((u,v)) = \{((i,v)(i+1,v))^* \mid 0 \le i \le n-2-v\} \cup \{((u,j)(u,j+1))^* \mid 0 \le j \le n-2-u\} \cup \{((i,j)(i+1,j-1))^* \mid 0 \le i \le u+v-1, k+j = u+v\}$ for each (u,v) of M_3. Since $(\cup_{(0,v)\in M_1} EM((0,v))) \cup (\cup_{(u,0)\in M_2} EM((u,0))) \cup (\cup_{(u,v)\in M_3} EM((u,v)))$, it follows that $\mathrm{dem}(T_n) \le |M| = (3n-5)/2$ for $n > 3$.

To show $\mathrm{dem}(T_n) \ge (3n-5)/2$ for $n > 3$, let the vertex set $Q \subseteq V(T_n)$ with $|Q| = (3n-5)/2 - 1$ be a DEM set of T_n. Choose the edge set $I = (\cup_{i=(n+1)/2}^{n-2} M_{i+1}^1) \cup (\cup_{j=(n+1)/2}^{n-2} M_{j+1}^2) \cup (\cup_{k=1}^{(n-1)/2} M_k^3)$ and the vertex set $R = (\cup_{i=(n+1)/2}^{n-2} V_{i+1}^1) \cup (\cup_{j=(n+1)/2}^{n-2} V_{j+1}^2) \cup (\cup_{k=1}^{(n-1)/2} V_k^3)$. For any edge $e \in M_i^j \subseteq I$, where $1 \le i \le n-1$ and $1 \le j \le 3$, from Proposition 3.2, we have $P(M,e) = P(M \cap V_i^j, e)$ for any $M \subseteq V(T_n)$, and hence $P(u,e) = \emptyset$ for any $u \in V(T_n) \setminus V_i^j$, and so e can only be monitored by some vertex v in $V_i^j \subseteq R$. Thus, $Q \cap V_i^j \ne \emptyset$, for any $V_i^j \subseteq R$, and so $|Q \cap R| \ge (3n-5)/2 - 2$. In fact, there exist three edge sets $M_{(n+1)/2}^1$, $M_{(n+1)/2}^2$ and $M_{(n-1)/2}^3$ such that $(M_{(n+1)/2}^1 \cup M_{(n+1)/2}^2 \cup M_{(n-1)/2}^3) \cap I = \emptyset$. Similarly, from Proposition 3.2, the edge $e \in M_{(n+1)/2}^j$ can only be monitored by some vertex $v \in V_{(n+1)/2}^j$, where $1 \le j \le 2$, and the edge $e \in M_{(n-1)/2}^3$ can only be monitored by some vertex $v \in V_{(n-1)/2}^3$. Since $V_{(n+1)/2}^1 \cap V_{(n+1)/2}^2 \ne \emptyset$, $V_{(n+1)/2}^1 \cap V_{(n-1)/2}^3 \ne \emptyset$ and $V_{(n+1)/2}^2 \cap V_{(n-1)/2}^3 \ne \emptyset$, it follows that $|Q \cap (V(T_n) \setminus R)| \ge 2$, and hence $|Q| \ge (3n-5)/2$, which contradicts the fact that $|Q| = (3n-5)/2 - 1$. Therefore, $\mathrm{dem}(T_n) = (3n-5)/2$. □

4 Results for hexagonal networks

Now, we construct a coordinate system for $HX(n)$. Let a, b, c, d, f, g be the corner vertices of $HX(n)$; see Figure 3. In this scheme, the three axes, X, Y and Z parallel to three edge directions and at mutual angle of 120 degrees between any two of them are introduced, where the directions from a to d, b to f and c to g are the directions of X, Y and Z, respectively. We call lines parallel to the coordinate axes as X-lines, Y-lines and Z-lines. Further, we use X_i-line to denote a line of X-lines with the distance of i from the X-axis for $1-n \leq i \leq n-1$. Note that X_0-line is the X-axis, X_k-line lies in upper side of X-axis, and X_{-k}-line lies in under side of X-axis, where $1 \leq k \leq n-1$. Let $X_i, \widehat{X}_i$ be the edge set and the vertex set of X_i-line, respectively; similarly, we define $Y_i, \widehat{Y}_i$, Z_i and $\widehat{Z}_i$, where $1-n \leq i \leq n-1$.

For each vertex v of $HX(n)$, we can always use $x_iy_jz_k$ to express v, where $\widehat{X}_i \cap \widehat{Y}_j \cap \widehat{Z}_k = \{x_iy_jz_k\}$, where $1-n \leq i, j, k \leq n-1$. Note that $k = j-i$ for any vertex $x_iy_jz_k$. For $u, v \in V(HX(n))$, if uv is an edge of $HX(n)$, then we use $(u, v)^*$ to represent it. For example, the corner vertex d can be represented as $x_0y_{1-n}z_{1-n}$, and the edges associated with d can be written as $(x_0y_{1-n}z_{1-n}, x_0y_{2-n}z_{2-n})^*$, $(x_0y_{1-n}z_{1-n}, x_{-1}y_{1-n}z_{2-n})^*$ and $(x_0y_{1-n}z_{1-n}, x_1y_{2-n}z_{1-n})^*$. These definitions will help us to prove the following results.

Lemma 4.1. *For a vertex $v = x_iy_jz_k$ of $HX(n)$, we have $EM(v) = X_i \cup Y_j \cup Z_k$, where $1-n \leq i, j, k \leq n-1$.*

Proof. For any $uw \in X_i \cup Y_j \cup Z_k$ with $d_{HX(n)}(v, u) > d_{HX(n)}(v, w)$, since there exists only one shortest path P_{vu} from v to u in the graph $HX(n)$, where $E(P_{vu}) \subseteq X_i \cup Y_j \cup Z_k$, it follows that $d_{HX(n)}(v, u) \neq d_{HX(n)-uw}(v, u)$, and hence $uw \in EM(v)$, and so $X_i \cup Y_j \cup Z_k \subseteq EM(v)$. For any edge $uw \in E(HX(n)) - X_i \cup Y_j \cup Z_k$, it follows from Definition 1 and Lemma 1.1 that the edge $uw \notin EM(v)$, and hence $EM(v) \cap (E(HX(n)) - X_i \cup Y_j \cup Z_k) = \emptyset$. Therefore, we have $EM(v) = X_i \cup Y_j \cup Z_k$, where $v = x_iy_jz_k$ and $1-n \leq i, j, k \leq n-1$. $\square$

Proposition 4.1. *Let $M \subseteq V(HX(n))$. For $v \in M$ and $e \in E(HX(n))$, we have $|P(M \setminus v, e)| \leq |P(M, e)|$. Moreover, if $v \in M \subseteq \widehat{X}_t$ and $e \in X_t$, then $P(M \setminus v, e) \subset P(M, e)$; if $e \in X_t$, then $P(M, e) = P(M \cap \widehat{X}_t, e)$, where $1-n \leq t \leq n-1$. (The cases of Y_t and Z_t are symmetric.)*

Proof. By Proposition 3.1, we have $P(M \setminus v, e) \subseteq P(M, e)$, and hence $|P(M \setminus v, e)| \leq |P(M, e)|$. Without loss of generality, let $d_{HX(n)}(v, w) < d_{HX(n)}(v, u)$. Since $v \in M \subseteq \widehat{X}_t$ and $e = uw \in X_t$, where $1-n \leq t \leq n-1$, it follows that there exists the unique shortest path P_{vu} from v to u such that $uw \in E(P_{vu})$, and hence $d_{HX(n)}(v, w) \neq d_{HX(n)-e}(v, w)$, and so the vertex pair $(v, w) \in P(M, e)$ and $(v, w) \notin P(M \setminus v, e)$. Therefore, $P(M \setminus v, e) \subset P(M, e)$.

For $e \in X_t$ and $v \in M \setminus \widehat{X}_t$, there exists a shortest path P_{vy} from v to y such that $E(P_{vy}) \cap X_t = \emptyset$ for any $y \in V(HX(n))$, and hence $d_{HX(n)}(v, y) = d_{HX(n)-e}(v, y)$, and so $P(\{v\}, e) = \emptyset$. Therefore, $P(M, e) = P(M \setminus v, e)$, and so $P(M, e) = P(M \setminus (V(HX(n)) \setminus \widehat{X}_t), e) = P(M \cap \widehat{X}_t, e)$. $\square$

Theorem 4.1. *For a hexagonal network $HX(n)$, let $M \subseteq V(HX(n))$ and $e \in E(HX(n))$. Then we have $0 \leq |P(M, e)| \leq 2n(n-1)$.*

Proof. By Definition 1, we have $|P(M, e)| \geq 0$. For any edge $e \in E(HX(n))$, there exists a X_t such that $e \in X_t$, where $1-n \leq t \leq n-1$. Let $M \subseteq V(HX(n)) \setminus \widehat{X}_t$. Since there exists a shortest path P_{xy} from x to y such that $E(P_{xy}) \cap X_t = \emptyset$ for any $x \in M$ and $y \in V(HX(n))$, it follows that $d_{HX(n)}(x, y) = d_{HX(n)-e}(x, y)$, and hence $|P(M, e)| = 0$, and so the lower bound is sharp.

For any edge $e = uv \in E(HX(n))$, there exists a X_t such that $e \in X_t$, where $1-n \leq t \leq n-1$. For any $M \subseteq V(HX(n))$, from Proposition 4.1, we have $P(M, e) = P(M \cap \widehat{X}_t, e)$, and hence $|P(M, e)| = |P(M \cap \widehat{X}_t, e)| \leq |P(\widehat{X}_t, e)|$. Let $A \subseteq \widehat{X}_t$ be the vertex set such that $d_{HX(n)}(u, x) < d_{HX(n)}(v, x)$ for any $x \in A$, and $B \subseteq \widehat{X}_t$ be the vertex set such that $d_{T_n}(u, x) > d_{T_n}(v, x)$ for any $y \in B$. Then, $A \cup B = \widehat{X}_t$. Since $d_{HX(n)}(x, y) \neq d_{HX(n)-e}(x, y)$, it follows that $(x, y), (y, x) \in P(\widehat{X}_t, e)$, and hence $|P(\widehat{X}_t, e)| = 2|A| \cdot |B|$. Since $|\widehat{X}_t| = 2n - 1 - |t|$, it follows that $|\widehat{X}_t| \leq 2n - 1$, and hence $|P(\widehat{X}_t, e)| = 2|A| \cdot |B| \leq 2n(n-1)$. $\square$

Example 3. *Choose the edge* $e = (x_0y_1z_1, x_0y_0z_0)^*$ *and the vertex set* $M = \{x_0y_iz_i \mid 1-n \leq i \leq n-1\}$*. By Proposition 4.1, we have* $P(M, e) = \{(x_0y_iz_i, x_0y_jz_j) \mid 1-n \leq i \leq 0, 1 \leq j \leq n-1\} \cup \{(x_0y_iz_i, x_0y_jz_j) \mid 1 \leq i \leq n-1, 1-n \leq j \leq 0\}$*, then* $|P(M, e)| = 2n(n-1)$*, and hence the upper bound is sharp.*

Theorem 4.2. *For a hexagonal network* $HX(n)$*, we have* $4(n-1) \leq |EM(v)| \leq 6(n-1)$ *for any vertex* $v \in V(HX(n))$*.*

Proof. Let $v = x_iy_jz_k$, where $1-n \leq i, j, k \leq n-1$. By Lemma 4.1, we have $EM(v) = X_i \cup Y_j \cup Z_k$. Since $X_i \cap Y_j = \emptyset$, $X_i \cap Z_k = \emptyset$ and $Y_j \cap Z_k = \emptyset$, it follows that $|EM(v)| = |X_i| + |Y_j| + |Z_k|$. Clearly, $|X_i|, |Y_j|, |Z_k| \leq 2(n-1)$, and hence we have $|EM(v)| \leq 6(n-1)$. Now we proof the lower bound. Since $k = j - i$ for any vertex $v = x_iy_jz_k$, it follows from Lemma 4.1 that $EM(v) = X_i \cup Y_j \cup Z_{j-i}$. Then $|EM(v)| = |X_i| + |Y_j| + |Z_{j-i}| = (2(n-1) - |i|) + (2(n-1) - |j|) + (2(n-1) - |j-i|)$, and hence $|EM(v)| = 6(n-1) - (|i| + |j| + |j-i|)$, where $1-n \leq i, j \leq n-1$. Since $|i| + |j| + |j-i| \leq 2(n-1)$ it follows that $|EM(v)| \geq 4(n-1)$. $\square$

To show the sharpness of the bounds of Theorem 4.2, we give the following example.

Example 4. *For the vertex* $u = x_0y_{n-1}z_{n-1}$*, from Lemma 4.1,* $EM(u) = X_0 \cup Y_{n-1} \cup Z_{n-1}$*, then* $|EM(u)| = 2(n-1) + (n-1) + (n-1) = 4(n-1)$*. For the vertex* o *of* $HX(n)$*, it follows from Lemma 4.1 that* $EM(o) = EM(x_0y_0z_0) = X_0 \cup Y_0 \cup Z_0$*. Clearly,* $|X_0| = |Y_0| = |Z_0| = 2(n-1)$*, then* $|EM(o)| = 6(n-1)$*. Therefore, the bounds are sharp.*

Proof of Theorem 1.9: To show the upper bound, let $M_1 = \{x_0y_iz_i \mid 1 \leq i \leq n-1\}$, $M_2 = \{x_iy_0z_{-i} \mid 1 \leq i \leq n-1\}$, $M_3 = \{x_{-i}y_{-i}z_0 \mid 1 \leq i \leq n-1\}$. Choose the vertex set $M = M_1 \cup M_2 \cup M_3$ with $|M| = 3n - 3$ in $HX(n)$. From Lemma 4.1, we let

$$\begin{aligned}
\mathscr{E}_1 &= \cup_{v \in M_1} EM(v) = X_0 \cup (\cup_{i=1}^{n-1} Y_i) \cup (\cup_{i=1}^{n-1} Z_i), \\
\mathscr{E}_2 &= \cup_{v \in M_2} EM(v) = (\cup_{i=1}^{n-1} X_i) \cup Y_0 \cup (\cup_{i=1}^{n-1} Z_{-i}), \\
\mathscr{E}_3 &= \cup_{v \in M_3} EM(v) = (\cup_{i=1}^{n-1} X_{-i}) \cup (\cup_{i=1}^{n-1} Y_{-i}) \cup Z_0.
\end{aligned}$$

Since $\mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3 = E(HX(n))$, it follows that $\mathrm{dem}(HX(n)) \leq |M| = 3n - 3$.

We now prove the lower bound. Let $Q \subseteq V(HX(n))$ be a DEM set of $HX(n)$ with $|Q| = 2n-2$. By Proposition 4.1, we have $P(M, e) = P(M \cap \widehat{X}_t, e)$ for any $M \subseteq V(HX(n))$ and $e \in X_t$, and hence $P(\{u\}, e) = \emptyset$ for any $u \in V(HX(n)) \setminus \widehat{X}_t$, and so X_t can only be monitored by the vertices in $\widehat{X}_t$ for each t $(1-n \leq t \leq n-1)$. Therefore, $|Q| \geq 2n-1$, which contradicts the fact that $|Q| = 2n-2$, $\square$

5 Concluding remark

In this paper, we studied some extremal problems for DEM numbers. For Problems 1 and 2, it is natural to improve and get some better bounds for $3 \leq k \leq n-2$.

For further future work, it would be interesting to study DEM sets in further standard graph classes, including pyramids, Sierpiński-type graphs, circulant graphs, graph products, or line graphs. In addition, it would be of interest to characterize the graphs with $\mathrm{dem}(G) = n-2$, as well as clarifying further the relation of the parameter $\mathrm{dem}(G)$ to other standard graph parameters, such as arboricity, vertex cover number and feedback edge set number.

Acknowledgement. We would like to thank the anonymous referees for a number of helpful comments and suggestions.

References

[1] J. Balogh, A. Dudek, L. Li, An analogue of the Erdős-Gallai theorem for random graphs, *European J. Combin.* 91 (2021), 103200.

[2] E. Bampas, D. Bilò, G. Drovandi, L. Gualà, R. Klasing, G. Proietti, Network verification via routing table queries, *J. Comput. System Sci.* 81(1) (2015), 234–248.

[3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalák, L.S. Ram, Network discovery and verification, *IEEE J. Sel. Areas Commun.* 24(12) (2006), 2168–2181.

[4] Y. Bejerano, R. Rastogi, Robust monitoring of link delays and faults in IP networks, *IEEE/ACM Trans. Netw.* 14(5) (2006), 1092–1103.

[5] D. Bilò, T. Erlebach, M. Mihalák, P. Widmayer, Discovery of network properties with all-shortest-paths queries, *Theoret. Comput. Sci.* 411(14-15) (2010), 1626–1637.

[6] B. Bollobás, *Extremal graph theory*, Courier Corporation, 2004.

[7] J. A. Bondy, U. S. R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[8] Q. Cai, X. Li, D. Wu, Erdős-Gallai-type results for colorful monochromatic connectivity of a graph, *J. Comb. Optim.* 33(1) (2017), 123–131.

[9] M. Chen, K. Shin, D. Kandlur, Addressing, routing, and broadcasting in hexagonal mesh multiprocessors, *IEEE Trans. Comput.* 39(1) (1990), 10–18.

[10] L. Dall'Asta, J. Alvarez-Hamelin, A. Barrat, A. Vázquez, A. Vespignani, Exploring networks with traceroute-like probes: Theory and simulations, *Theoret. Comput. Sci.* 355(1) (2006), 6–24.

[11] F. Foucaud, S.S. Kao, R. Klasing, M. Miller, J. Ryan, Monitoring the edges of a graph using distances, *Discrete Appl. Math.* 319 (2022), 424–438.

[12] R. Govindan, H. Tangmunarunkit, Heuristics for Internet map discovery, in: Proceedings of the 19th IEEE International Conference on Computer Communications, *INFOCOM'00*, 2000, pp. 1371–1380.

[13] F. Harary, R. Melter, On the metric dimension of a graph, *Ars Combin.* 2 (1976), 191–195.

[14] H. Jiang, X. Li, Y. Zhang, Erdös-Gallai-type results for total monochromatic connection of graphs, *Discuss. Math. Graph Theory* 39(4) (2019), 775–785.

[15] A. Kelenc, D. Kuziak, A. Taranenko, I.G. Yero, Mixed metric dimension of graphs, *Appl. Math. Comput.* 314 (2017), 429–438.

[16] A. Kelenc, N. Tratnik, I.G. Yero, Uniquely identifying the edges of a graph: The edge metric dimension, *Discrete Appl. Math.* 251 (2018), 204–220.

[17] P. Li, X. Li, Monochromatic disconnection: Erdös-Gallai-type problems and product graphs, *J Comb Optim.* 44 (2022), 136–153.

[18] P. Manuel, B. Rajan, I. Rajasingh, C. Monica, On minimum metric dimension of honeycomb networks, *J. Discrete Algor.* 6 (2008), 20–27.

[19] N. Salia, C. Tompkins, O. Zamora, An Erdős-Gallai type theorem for vertex colored graphs, *Graphs Comb.* 35 (2019), 689–694.

[20] P.J. Slater, Leaves of trees, *Congr. Numer.* 14 (1975), 549–559.

[21] Z. Wang, Y. Mao, E. Cheng, J. Zou, Matching preclusion number of graphs, *Theor. Comput. Sci.* 759 (2018), 61–71.

[22] D.A. Xavier and C. J. Deeni, Matching preclusion number of radix triangular mesh, *J. Theor. Comput. Math.* 1(1) (2015), 28–34.

[23] C. Yang, R. Klasing, Y. Mao, X. Deng, On the distance-edge-monitoring numbers of graphs, accepted by *Discrete Appl. Math.*