Nonempty Intersection of Longest Paths in Series-Parallel Graphs

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In 1966 Gallai asked whether all longest paths in a connected graph have nonempty intersection. This is not true in general and various counterexamples have been found. However, the answer to Gallai's question is positive for several well-known classes of graphs, as for instance connected outerplanar graphs, connected split graphs, and 2-trees. A graph is series-parallel if it does not contain K_4 as a minor. Series-parallel graphs are also known as partial 2-trees, which are arbitrary subgraphs of 2-trees. We present a proof that every connected series-parallel graph has a vertex that is common to all of its longest paths. Since 2-trees are maximal series-parallel graphs, and outerplanar graphs are also series-parallel, our result captures these two classes in one proof and strengthens them to a larger class of graphs. We also describe how this vertex can be found in linear time.

1 Introduction

A path in a graph is a *longest path* if there exists no other path in the same graph that is strictly longer. The study of intersections of longest paths has a long history and, in particular, the question of whether every connected graph has a vertex that is common to all of its longest paths was raised by Gallai [12] in 1966. For some years it was not clear whether the answer is positive or negative until, finally, Walther [25] found a graph on 25 vertices that answers Gallai's question negatively.

Today, the smallest known graph answering Gallai's question negatively is a graph on 12 vertices, found by Walther and Voss [26], and independently by Zamfirescu [28] (see Figure 1). To see that the depicted graph does not have a vertex common to all longest paths, one can identify the three leaves to obtain the Petersen graph, which is hypohamiltonian, meaning that it does not have a Hamiltonian cycle but every vertex-deleted subgraph is Hamiltonian. Note that the length of a longest path in the depicted graph can be at most 10 since at most two of its three leaves can be contained in a longest path. But any path of length 10 in the depicted graph would correspond to a Hamiltonian cycle in the Petersen graph. It follows first that the length of a longest path is at most 9 (and it is exactly 9) and second that the intersection of all longest paths is empty.

Figure 1: The counterexample of Walther, Voss, and Zamfirescu.

These are by far not the only counterexamples. In fact, there are infinitely many (even planar) ones since every hypotraceable graph, meaning a graph having no Hamiltonian path whose all vertex-deleted subgraphs have a Hamiltonian path, is obviously a counterexample. Thomassen proved in [22] that there are infinitely many such graphs.

Since the answer to Gallai's question is negative in general, it seems natural to restrict the problem to subsets of a fixed size of all longest paths. It is well-known [19] that any two longest paths of a connected graph share a common vertex. However, considering the intersection of more than two longest paths gets more intriguing. It is still unknown whether any three longest paths of every connected graph share a common vertex. Zamfirescu asked this question several times [24, 29] and it was mentioned at the 15th British Combinatorial Conference [8]. It is presented as a conjecture in [14] and as an open problem in the list collected by West [27]. Progress in this direction was made by de Rezende, Martin, Wakabayashi, and the second author [11], who proved that, if all non-trivial blocks of a connected graph are Hamiltonian, then any three longest paths of the graph share a vertex. Skupień [21] showed that for every $p \geq 7$, there exists a connected graph such that p longest paths have no common vertex and every $p - 1$ longest paths have a common vertex.

Even though it seems as if the property of having a vertex common to all longest paths is too strong, there are some classes of connected graphs for which this property holds. A simple example is the class of trees since in a tree all longest paths contain its center(s). Moreover, Klavžar and Petkovšek [17] proved that the intersection of all longest paths of a (connected) split graph is nonempty. Furthermore, they showed in [17] that, if every block of a connected graph G is Hamilton-connected, almost Hamilton-connected, or a cycle, then there exists a vertex common to all its longest paths. The latter result implies immediately that the answer to Gallai's question is positive for the class of (connected) cacti, where a graph is a cactus if and only if every block is either a simple cycle or a vertex or a single edge.

In 2013, de Rezende et al. proved the following two theorems.

Theorem 1 ([11])
For every connected outerplanar graph, there exists a vertex common to all its longest paths.

Theorem 2 ([11])
For every 2-tree, there exists a vertex common to all its longest paths.

Theorem 1 is a strengthening of a theorem by Axenovich [2], which states that any three longest paths in a connected outerplanar graph share a vertex.

In this paper we treat the general case of nonempty intersection of all longest paths and prove that the answer to Gallai's question is positive for the class of connected series-parallel graphs settling a question raised in [11]. Note that a joint paper presenting two different proofs for this statement is to appear in [9]. Since not only all trees and cacti but also outerplanar graphs and 2-trees are series-parallel, our result gives a unified proof for Theorems 1 and 2 and generalizes them to a larger class of graphs.

The rest of the paper is organized as follows. In Section 2 we give essential definitions and prove some statements that will be useful in what follows. In Section 3 we prove the main theorem by proceeding in three steps. First we fix a 2-tree that has the given series-parallel graph as a spanning subgraph. The 2-tree captures the structure of the given graph and guides us in the proof. The main obstacle is that we are not sure of which edges of this 2-tree truly exist in the given graph, so the techniques used for 2-trees in [11] fail. Therefore, we are somehow obliged to work with edges that exist in the 2-tree and may or may not exist in the given series-parallel graph. Roughly, the only things that we can rely on are the facts that our graph is connected, that two longest paths intersect, and on the special structure of the so-called components of the series-parallel graph, inherited from the 2-tree. In Section 4 we show that finding a vertex contained in all longest paths can be done in quadratic time for series-parallel graphs. Finally, in Section 5 we state several open problems concerning the intersection of longest paths in specific classes of graphs.

2 Preliminaries and definitions

We start with a few basic definitions which we use in the subsequent part of our paper. All graphs in this paper are undirected and finite. We write $H \subseteq G$ if the graph H is a subgraph of the graph G. Also, we denote by $V(G)$ the set of vertices of G.

Let G be a graph and s and t be two of its vertices. We say G is *series-parallel with terminals s and t* if it can be turned into K_2 by a sequence of the following operations: replacement of a pair of parallel edges with a single edge that connects their common endpoints, or replacement of a pair of edges incident to a vertex of degree 2 other than s or t with a single edge. A graph G is *2-terminal series-parallel* if there are vertices s and t in G such that G is series-parallel with terminals s and t. A graph G is *series-parallel* if each of its 2-connected components is a 2-terminal series-parallel graph. (See [7, Sec. 11.2].)

A *2-tree* can be defined in the following way. A single edge is a 2-tree. If T is not a single edge, then T is a 2-tree if and only if there exists a vertex v of degree 2 such that its neighbors are adjacent and $T - v$ is also a 2-tree. A graph is a *partial 2-tree* if it is a subgraph of a 2-tree. (See [7, Sec. 11.1].) We say a partial 2-tree is *trivial* if it consists of a single vertex or a single edge. Note that every edge in a non-trivial 2-tree is contained in a triangle.

It is well-known that a graph is a partial 2-tree if and only if it is K_4-minor free. Partial 2-trees are exactly the series-parallel graphs, and are also known for being the graphs with tree width at most 2. (See [7, Sec. 11.1].)

Next we present some notation we use in our proofs.

The *length* of a path P, denoted by $|P|$, is the number of edges in P. Let $L(G)$ denote the length of a longest path in the graph G. Let $\mathscr{L}(G)$ denote the set of all longest paths in G, that is, $\mathscr{L}(G) = \{P \mid P \text{ is a path in } G \text{ and } |P| = L(G)\}$. If the graph G is clear from the context, we simply write L for $L(G)$ and $\mathscr{L}$ for $\mathscr{L}(G)$.

By the intersection $P \cap P'$ of two paths P and P', we mean the intersection of their vertex sets. If v is a vertex of the path P, we write $v \in P$. If P and P' have a common endpoint x but no other common vertex, then the *union* $P \cup P'$ is simply defined as the path obtained by concatenating the path P and the path P' at the vertex x.

A subpath of a path P is called a *tail* of P if it contains an endpoint of P. Given a vertex x in P, the path P can be split into two subpaths P' and P'' such that $P' \cap P'' = \{x\}$; we call them *tails of P starting at x*. If $|P'| \geq |P''|$, then P' is called a *longer tail* of P starting at x.

Given a second path Q such that $Q \cap P \neq \varnothing$ and such that at least one endpoint x of P is not contained in Q, we define the *bridge path* $P \times_x Q$ as the path starting at the endpoint x going along P until the first intersection with Q.

For some subgraph H, we define $P[H]$ to be the induced subgraph of P in H, that is, the collection of (maximal) subpaths of P that lie in H. Note that this might be more than one path.

In the next, we borrow some definitions and results presented by Tutte [23]. Let G be a graph and H be a subgraph of G. A *vertex of attachment* of H in G is a vertex of H that is incident to some edge of G that is not an edge of H. Let T be a 2-tree and $\{x, y\} \in E(T)$. An $\{x, y\}$-*bridge* in T is a minimal subgraph B of T containing a vertex other than x and y and whose vertices of attachment are contained in $\{x, y\}$. For each common neighbor z of x and y, let $B_{\{x,y\},z}(T)$ be the $\{x, y\}$-bridge in T containing z. There is a unique such bridge because, by Theorem I.51 [23], the intersection of two distinct $\{x, y\}$-bridges in T lies in $\{x, y\}$. The *interior* of the $B_{\{x,y\},z}(T)$ is the graph $B^{\circ}_{\{x,y\},z}(T) = B_{\{x,y\},z}(T) - x - y$.

In what follows, for a series-parallel graph $G = (V, E)$, we let $T(G) = (V, F)$ denote an arbitrary but fixed 2-tree that contains G as a spanning subgraph. We say a *virtual edge/triangle* of G is an edge/triangle in $T(G)$ independent of its existence in G.

We denote by $C_{\{x,y\},z}(G)$ the maximal subgraph of G contained in $B_{\{x,y\},z}(T(G))$. Note that such a subgraph of G may be disconnected. We call this subgraph the *component* of G generated by the virtual edge $\{x, y\}$ in direction z. Similarly, $C^{\circ}_{\{x,y\},z}(G) = C_{\{x,y\},z}(G) - x - y$ is called

the *interior* of the component $C_{\{x,y\},z}(G)$. Again, if the graph is clear from the context, we write $C_{\{x,y\},z} = C_{\{x,y\},z}(G)$.

Also, let $\mathscr{C}_{\{x,y\}}(G) = \{C_{\{x,y\},z}(G) : z \text{ is adjacent to } x \text{ and } y \text{ in } T(G)\}$ be the set of all components generated by the virtual edge $\{x,y\}$. Further, define $\mathscr{C}_{\{x,y\}|z}(G) = \mathscr{C}_{\{x,y\}}(G)\backslash\{C_{\{x,y\},z}(G)\}$ for a virtual triangle $\{x,y,z\}$.

A set of vertices W is called a *Gallai set* (for G) if $W \cap P \neq \emptyset$ for all $P \in \mathscr{L}(G)$. If W is a Gallai set and if its vertices are pairwise connected by virtual edges, we call this set a *virtual Gallai edge* and a *virtual Gallai triangle*, when W has size two or three, respectively.

We say a vertex $v \in V$ is a *Gallai vertex* if $\{v\}$ is a Gallai set. Note that the intersection of all longest paths of a graph G is nonempty if and only if G has a Gallai vertex.

For a given virtual edge $\{u,v\}$, let $\mathscr{L}_{uv} = \{P \in \mathscr{L} \,|\, u,v \in P\}$ and $\mathscr{L}_{u\bar{v}} = \{P \in \mathscr{L} \,|\, u \in P,\ v \notin P\}$. Similarly, for a given virtual triangle $\Delta = \{u,v,w\}$, we define $\mathscr{L}_{uvw} = \{P \in \mathscr{L} \,|\, u,v,w \in P\}$, $\mathscr{L}_{uv\bar{w}} = \{P \in \mathscr{L} \,|\, u,v \in P,\ w \notin P\}$, and $\mathscr{L}_{u\bar{vw}} = \{P \in \mathscr{L} \,|\, u \in P,\ v,w \notin P\}$. Moreover, let $\mathscr{L}_{(uvw)} = \{P \in \mathscr{L}_{uvw} \,|\, v \text{ is between } u \text{ and } w \text{ in } P\}$.

We end this section with the following auxiliary results that will be useful in Section 3. Note that the first four results hold for general graphs, not only for series-parallel graphs.

Proposition 3 ([19])
Any two longest paths in a connected graph share a common vertex.

Lemma 4
In a graph, let P_1 and P_2 be two paths with tails R_1 and R_2, respectively (that is, subpaths containing an endpoint of P_1 or P_2) such that $R_1 \cap P_2 = \emptyset$ and $R_2 \cap P_1 = \emptyset$. If there exists a *connecting path* P such that $\emptyset \neq P \cap P_1 \subseteq R_1$ and $\emptyset \neq P \cap P_2 \subseteq R_2$, then P_1 and P_2 cannot both be longest paths.

Proof. Assume for a contradiction that both P_1 and P_2 are longest paths. For $i \in \{1,2\}$, let R_i' denote the other tail of P_i so that $P_i = R_i \cup R_i'$ and R_i and R_i' intersect at only one vertex. By assumption, both R_1 and R_2 intersect P. Hence, there exist vertices x and y such that $x \in R_1 \cap P$, $y \in R_2 \cap P$, and the interior of the subpath of P starting at x and ending in y does not contain vertices in P_1 or P_2. Let Q_1 denote the path obtained from going along R_1', along R_1 until x, along P until y, and then along R_2 until the endpoint that is not in R_2'. Let Q_2 denote the path obtained from going along R_2', along R_2 until y, along P until x, and then along R_1 until the endpoint that is not in R_1'. Now, as $|Q_1| + |Q_2| > |P_1| + |P_2|$, we have that $|Q_1| > |P_1|$ or $|Q_2| > |P_2|$, a contradiction. $\square$

Lemma 5
In a graph, let P_1 and P_2 be two paths that share a common vertex z and let R_1 and R_2 be two subpaths of P_1 and P_2, respectively, both having z as an endpoint, such that $R_1 \cap P_2 = \{z\}$ and $R_2 \cap P_1 = \{z\}$. If there exists a *connecting path* P such that $z \notin P$, $\emptyset \neq P \cap P_1 \subseteq R_1$, and $\emptyset \neq P \cap P_2 \subseteq R_2$, then P_1 and P_2 cannot both be longest paths.

Figure 2: Situation for Lemmas 4 and 5.

Proof. Assume for a contradiction that both P_1 and P_2 are longest paths. By assumption, both R_1 and R_2 intersect P in a vertex other than z. Hence, there exist vertices x and y distinct from z such that $x \in R_1 \cap P$, $y \in R_2 \cap P$, and the interior of the subpath of P starting at x and ending in y does not contain vertices in P_1 or P_2. Let $\tilde{R}_1$ denote the path starting at z, going along R_1, and ending in x, and let $\tilde{R}_2$ denote the path starting at z, going along R_2, and ending in y. Let R'_1 and R'_2 denote the tails of P_1 and P_2 starting at z not containing R_1 and R_2, respectively. If $|\tilde{R}_1| \geq |\tilde{R}_2|$, then by combining R'_2, $\tilde{R}_1$, the subpath of P starting at x and ending in y, and the tail of P_2 starting at y and not containing z, we get a path strictly longer than P_2, a contradiction. If, on the other hand, $|\tilde{R}_1| < |\tilde{R}_2|$, then by combining R'_1, $\tilde{R}_2$, the subpath of P starting at y and ending in x, and the tail of P_1 starting at x and not containing z, we get a path strictly longer than P_1, a contradiction. □

Observe that Lemma 4 is not a consequence of Lemma 5. Indeed, in the situation of Lemma 4, starting from x and going along P_1, the first common vertex with P_2 might not be the same as the first common vertex of P_2 with P_1, starting from y. Thus, the vertex z as required in Lemma 5 might not exist.

At some points of the proofs in the next section, we are in a situation where one of the two lemmas above apply. The next corollary describes this situation.

Figure 3: Situation for Corollary 6.

Corollary 6
In a graph, let P_1 and P_2 be two paths that share a common vertex z and let R_1 be a tail of P_1 starting at z. Let R_2 be a union of pairwise internally vertex disjoint subpaths of P_2 (that is, they may have common endpoints) such that all paths in R_2 have as one endpoint z or an endpoint of P_2. Suppose $R_1 \cap P_2 = \{z\}$ and $R_2 \cap P_1 \subseteq \{z\}$. If there exists a *connecting path P* such that $z \notin P$, $\varnothing \neq P \cap P_1 \subseteq R_1$, and $\varnothing \neq P \cap P_2 \subseteq R_2$, then P_1 and P_2 cannot both be longest paths.

Proof. There exist vertices $x \in P \cap R_1$ and $y \in P \cap R_2$ such that the interior of the subpath $P^{x,y}$ of P starting at x and ending in y does not contain any other vertices in P_1 or P_2. Let R'_2 be

the path in R_2 that contains y. If R_2' contains z then the statement follows from Lemma 5 for longest paths P_1 and P_2 with their subpaths R_1 and R_2', respectively, and connecting path $P^{x,y}$. Otherwise, the statement follows from Lemma 4 again for longest paths P_1 and P_2, tails R_1 and R_2', and connecting path $P^{x,y}$. □

The next two results are specific for series-parallel graphs.

Lemma 7
Let $\Delta = \{v_1, v_2, v_3\}$ be a virtual triangle in a connected series-parallel graph G. If R_i is a path in G with v_i as an endpoint and $R_i \cap \Delta = \{v_i\}$ for each $i \in \{1, 2, 3\}$, then only one of the sets $R_1 \cap R_2$, $R_1 \cap R_3$, and $R_2 \cap R_3$ can be nonempty. Furthermore, if $R_i \cap R_j \neq \varnothing$, then $R_i \cup R_j \subseteq C$ for some component $C \in \mathscr{C}_{\{v_i, v_j\}|v_k}$.

Proof. For the first statement, assume without loss of generality that $R_1 \cap R_2 \neq \varnothing$. Then we have a path S from v_1 to v_2 consisting of $R_1 \bowtie_{v_1} R_2$ and the tail of R_2 containing v_2. Note that this path does not use v_3 or the virtual edge $\{v_1, v_2\}$ because R_i contains only v_i in Δ for $i = 1, 2$.

If additionally $R_1 \cap R_3 \neq \varnothing$ or $R_2 \cap R_3 \neq \varnothing$, then v_3 is connected to S by a path $S' \cup S''$, where S' is the shortest tail of R_3 from v_3 to a vertex u in $R_1 \cup R_2$, and S'' is a shortest path from u to S in the connected graph $R_1 \cup R_2$. Let x be the endpoint of S'' in S. Observe that x is an internal vertex of S. So $\{x, v_1, v_2, v_3\}$ determines a K_4 minor in $T(G)$, a contradiction.

For the second statement, suppose that R_i and R_j intersect. Obviously, $H = (R_i \cup R_j) - v_i - v_j$ must lie in the interior of a $\{v_i, v_j\}$-bridge B of $T(G)$. Also, the edge $\{v_i, v_j\}$ is a cut set in $T(G)$, separating v_k from H. Otherwise we would have three paths as above, namely R_i, R_j, and the path from v_k to H avoiding v_i and v_j, and at least two of the pairs within these three paths would intersect. Therefore $B \in \mathscr{C}_{\{v_i, v_j\}|v_k}(T(G))$ and thus $R_i \cup R_j \subseteq C = G[V(B)] \in \mathscr{C}_{\{v_i, v_j\}|v_k}(G)$. □

Lemma 8
Let $\Delta = \{v_1, v_2, v_3\}$ be a virtual triangle in a connected series-parallel graph G, and R be a path in G with v_i as endpoint and $R \cap \Delta = \{v_i\}$, for some i in $\{1, 2, 3\}$. Let j and k be such that $\{i, j, k\} = \{1, 2, 3\}$. If S_1 is a path with endpoints v_i and v_j such that $S_1 \cap \Delta = \{v_i, v_j\}$, and S_2 is a path with endpoints v_j and v_k such that $S_2 \cap \Delta = \{v_j, v_k\}$, then $R \cap S_2 = \varnothing$ and $S_1 \cap S_2 = \{v_j\}$.

Proof. Assume for a contradiction that there is some vertex $x \in R \cap S_2$. Split the path S_2 at x and look at the two tails S_2^j and S_2^k starting at x and ending at v_j and v_k, respectively. Now R, S_2^j, and S_2^k are three paths as in Lemma 7 and they all intersect at x, a contradiction.

Similarly, if $y \in (S_1 \cap S_2) \backslash \{v_j\}$, split S_2 analogously at y obtaining S_2^j and S_2^k. Then $S_1 - v_j, S_2^j$, and S_2^k are three paths as in Lemma 7 and they all intersect at y, again a contradiction. □

3 Intersection of longest paths in series-parallel graphs

As we have already mentioned in Section 1, de Rezende at el. [11] proved that the intersection of all longest paths of a 2-tree is nonempty. In this section, we extend this result proving that all connected subgraphs of 2-trees, that is, all series-parallel graphs, have also this property.

We proceed in four steps. First, we prove in Lemma 9 that there exists a virtual Gallai triangle. Then, we show in Lemma 10 that actually one virtual edge of this triangle is a virtual Gallai edge and there exists a component generated by this virtual edge that satisfies certain properties. In Lemma 12 we prove that either one of the endpoints of this virtual edge is a Gallai vertex or we can find an adjacent virtual Gallai edge and a strictly smaller component satisfying the same properties. By iterating, we end up with a Gallai vertex since we only consider finite graphs.

Lemma 9
In every non-trivial connected series-parallel graph G, there exists a virtual Gallai triangle.

Proof. Take any virtual triangle Δ_0 of a non-trivial connected series-parallel graph G. Note that, in every connected series-parallel graph, every virtual edge is either a cut set of G or is contained in exactly one virtual triangle. Assume that there exists a longest path P_0 in G containing no vertex of Δ_0. Then there exists a virtual edge $e_0 \subseteq \Delta_0$, which is a cut set, and a vertex $z_0 \notin \Delta_0$ such that z_0 is adjacent to both endpoints of e_0 in $T(G)$ and P_0 lies in the component generated by e_0 in direction z_0, that is, $P_0 \subseteq C^\circ_{e_0,z_0}$. By Proposition 3, all longest paths must intersect P_0 and so they have at least one vertex in $C^\circ_{e_0,z_0}$. Note that $\Delta_1 = e_0 \cup \{z_0\}$ is a triangle in $T(G)$ and thus a virtual triangle in C_{e_0,z_0}. Now either all longest paths contain a vertex of Δ_1 and we are done, or there exist a longest path P_1, a virtual edge $e_1 \subseteq \Delta_1$, where $e_1 \neq e_0$ and e_1 is a cut set, and a vertex $z_1 \notin \Delta_1$ such that z_1 is adjacent to both endpoints of e_1 in $T(G)$ and $P_1 \subseteq C_{e_1,z_1}$. Note that $C_{e_1,z_1} \subsetneq C_{e_0,z_0}$, as P_1 must intersect P_0 in $C^\circ_{e_0,z_0}$ again by Proposition 3. Iteratively, obtain Δ_2 and C_{e_2,z_2} and eventually a strictly decreasing sequence of components $C_{e_0,z_0} \supsetneq C_{e_1,z_1} \supsetneq C_{e_2,z_2} \supsetneq \cdots \supsetneq C_{e_k,z_k}$. Since G is finite, this process ends with some triangle $\Delta = \Delta_k$ such that all longest paths contain one vertex of Δ. $\qquad\square$

Next we prove that one of the virtual edges of a Gallai triangle is a virtual Gallai edge and there exists a component generated by this virtual edge that satisfies certain properties.

Lemma 10
For every connected series-parallel graph $G = (V, E)$, there exists a Gallai vertex, or a virtual Gallai edge $\{u, v\}$ and a component $C \in \mathscr{C}_{\{u,v\}}$ such that, for every pair

$$(P, P') \in (\mathscr{L}_{u\bar{v}} \times \mathscr{L}_{v\bar{u}}) \cup (\mathscr{L}_{u\bar{v}} \times \mathscr{L}_{uv}) \cup (\mathscr{L}_{v\bar{u}} \times \mathscr{L}_{uv}),$$

there exists a vertex in $C^\circ \cap P \cap P'$.

Before presenting the proof of Lemma 10, we prove an intermediate result stated in the next lemma. Throughout the next proofs, we keep Lemmas 7 and 8 in mind and use them implicitly *whenever* we claim that certain constructions are indeed paths and whenever we claim that a path lies in a certain component.

For the proof of Lemmas 11 and 12, we use the following notation. Every path $P \in \mathscr{L}_{uv\overline{w}}$ can be split at u and v, resulting in three subpaths. Let $P^{(u)}$ and $P^{(v)}$ be the tails of P starting at vertex u and vertex v, respectively. The remaining subpath, joining u and v, is denoted by $P^{(u,v)}$. Analogously, a path $P \in \mathscr{L}_{(uvw)}$ is split into $P^{(u)}$, $P^{(u,v)}$, $P^{(v,w)}$, and $P^{(w)}$ (see Figure 4).

Figure 4: Splitting $P \in \mathscr{L}_{uv\overline{w}}$ at vertices u and v, and $P \in \mathscr{L}_{(uvw)}$ at vertices u, v, and w.

Lemma 11

Let Δ be a virtual Gallai triangle in a non-trivial connected series-parallel graph G. If $\mathscr{L}_{xy\overline{z}} \neq \varnothing$ for every x, y, z such that $\{x, y, z\} = \Delta$, then, for some u, v, w such that $\{u, v, w\} = \Delta$, there is a component $C \in \mathscr{C}_{\{u,w\}|v}$ such that, for every pair

$$(P, P') \in \left(\bigcup_{\{x,y,z\}=\Delta} \mathscr{L}_{xy\overline{z}} \times \mathscr{L}_{xz\overline{y}} \right) \cup (\mathscr{L}_{uv\overline{w}} \times \mathscr{L}_{uvw}) \cup (\mathscr{L}_{vw\overline{u}} \times \mathscr{L}_{uvw}),$$

there exists a vertex in $C^\circ \cap P \cap P'$.

Proof (Lemma 11) Let $P \in \mathscr{L}_{uv\overline{w}} \cup \mathscr{L}_{uw\overline{v}} \cup \mathscr{L}_{vw\overline{u}}$, where $\{u, v, w\} = \Delta$, and $x \in \Delta$ be such that x is in P and $P^{(x)}$ is as long as possible. Without loss of generality, we may assume $P \in \mathscr{L}_{uv\overline{w}}$ and $x = u$. In what follows, we use P_z to refer to an arbitrary path in $\mathscr{L}_{xy\overline{z}}$ where $\{x, y, z\} = \Delta$.

First note that $P^{(u)}$ intersects every $P_u^{(w)}$, otherwise $P^{(u)} \cup P^{(u,v)} \cup P_u^{(v,w)} \cup P_u^{(w)}$ is a path of length strictly greater than L by the choice of P. Thus, $P^{(u)}$ must lie in a component $C \in \mathscr{C}_{\{u,w\}|v}$. We will prove that C has the property stated in the lemma.

We start by proving that each path in $\mathscr{L}_{uv\overline{w}}$ intersects in C° every path in $\mathscr{L}_{vw\overline{u}}$, that is, we show that each $P_w^{(u)}$ intersects every $P_u^{(w)}$. Observe that $|P_w^{(u)}| = |P^{(u)}|$, otherwise $P^{(u)} \cup P_w^{(u,v)} \cup P_w^{(v)}$ would be a path of length strictly greater than L. So the argument previously applied to P, now with P_w instead, implies that each $P_w^{(u)}$ intersects every $P_u^{(w)}$.

Now we prove that each path in $\mathscr{L}_{uv\overline{w}} \cup \mathscr{L}_{vw\overline{u}}$ intersects in C° every path in $\mathscr{L}_{uw\overline{v}} \cup \mathscr{L}_{uvw}$. First note that each $P_w^{(u)}$ intersects $Q^{(u,w)}$ in C° for every $Q \in \mathscr{L}_{uw\overline{v}} \cup \mathscr{L}_{(uwv)} \cup \mathscr{L}_{(vuw)}$, otherwise $P_w^{(u)} \cup Q^{(u,w)} \cup P_u^{(w,v)} \cup P_u^{(v)}$ is a path of length strictly greater than L by the choice of P. As both $\mathscr{L}_{uw\overline{v}}$ and $\mathscr{L}_{vw\overline{u}}$ are nonempty, there exist at least one such Q and one such P_u. So Q and $P_u^{(w)}$ must intersect in C°, otherwise we derive a contradiction from Lemma 5 for subpaths $P_u^{(w)}$ and $Q^{(u,w)}$, $z = w$, and a connecting path contained in $P_w^{(u)}$. Second, we prove that each $P_w^{(u)}$ and $P_u^{(w)}$ intersect every $Q \in \mathscr{L}_{(uvw)}$ in C°. Observe that $|Q^{(u)}| \leq |P^{(u)}|$, otherwise either $Q^{(u)} \cup P^{(u,v)} \cup P^{(v)}$ or $Q^{(u)} \cup P^{(u,w)} \cup P_v^{(w)}$ is a path of length strictly greater than L by the choice of P. If $|Q^{(u)}| < |P^{(u)}| = |P_w^{(u)}|$, then $P_w^{(u)}$ intersects $Q^{(w)}$, otherwise $P_w^{(u)} \cup Q^{(u,v)} \cup Q^{(v,w)} \cup Q^{(w)}$ is a path of length strictly greater than L. So $P_u^{(w)}$ and Q must intersect in C°, otherwise we derive a contradiction from Corollary 6 for longest paths P_u and Q with $z = w$, tail $P_u^{(w)}$, subpaths $Q[C]$, and connecting path $P_u^{(u)}$. If $|Q^{(u)}| = |P^{(u)}|$, then $Q^{(u)}$ intersects $P_u^{(w)}$, otherwise $Q^{(u)} \cup Q^{(u,v)} \cup P_u^{(v,w)} \cup P_u^{(w)}$ is a path of length strictly greater than L by the choice of P. So $P_w^{(u)}$

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and Q must intersect in C°, otherwise we derive a contradiction from Corollary 6 for longest paths P_w and Q with $z = u$, tail $P_w^{(u)}$, subpaths $Q[C]$, and connecting path $P_u^{(w)}$. $\qquad\square$

Proof (Lemma 10) If G is trivial, there exists a Gallai vertex. Otherwise, let $\Delta = \{u, v, w\}$ be a virtual Gallai triangle, which exists by Lemma 9.

First, we show that at least one of the edges of Δ is a virtual Gallai edge. Assume for a contradiction that no edge of Δ is a virtual Gallai edge, which means that there are three longest paths $P_u \in \mathscr{L}_{u\overline{vw}}$, $P_v \in \mathscr{L}_{v\overline{uw}}$, and $P_w \in \mathscr{L}_{w\overline{uv}}$. Thus, there exist three distinct components C_{uv}, C_{uw}, and C_{vw} generated by the virtual edges of Δ such that, for every x, y in Δ, all intersection points of P_x and P_y lie in the component C_{xy}. Without loss of generality, let $|P_u[C_{uv}]| \geq L/2$. Then, by combining the paths $P_u[C_{uv}]$, $P_u[C_{uw}] \mathbin{\rotatebox[origin=c]{90}{$\ltimes$}}_u P_w$, and a longer tail of P_w, we obtain a path of length strictly greater than L, a contradiction. So, there exists a virtual Gallai edge in Δ.

If all edges of Δ are virtual Gallai edges, then $\mathscr{L}_{u\overline{vw}} = \mathscr{L}_{v\overline{uw}} = \mathscr{L}_{w\overline{uv}} = \varnothing$. If moreover at least one among $\mathscr{L}_{uv\overline{w}}$, $\mathscr{L}_{uw\overline{v}}$, $\mathscr{L}_{vw\overline{u}}$ is empty, then one of the vertices in Δ is a Gallai vertex. Otherwise, we are in the situation of Lemma 11 and the statement of the lemma follows immediately.

Without loss of generality, we may assume $\{u, v\}$ is a virtual Gallai edge. Hence, $\mathscr{L}_{w\overline{uv}} = \varnothing$. Let $P_u \in \mathscr{L}_{u\overline{vw}} \cup \mathscr{L}_{v\overline{uw}}$ and $x \in \{u, v\}$ be such that P_u has a tail starting at x that is as long as possible. Without loss of generality, we may assume $x = u$ and thus $P_u \in \mathscr{L}_{u\overline{vw}}$. Let P_u' be such a longer tail. (If both tails of P_u starting at u have same length, choose P_u' to be any one of them.) Let P_u'' be the other tail of P_u. As $\mathscr{L}_{u\overline{vw}}$ is nonempty, $\{v, w\}$ is not a virtual Gallai edge. If all longest paths contain u, then we are done since u is a Gallai vertex. Otherwise, $\mathscr{L}_{v\overline{u}}$ is nonempty.

Each $P_v \in \mathscr{L}_{v\overline{u}}$ intersects P_u' because otherwise, by combining P_u', $P_u'' \mathbin{\rotatebox[origin=c]{90}{$\ltimes$}}_u P_v$, and a longer tail of P_v, we get a path of length strictly greater than L by the choice of P_u. If $\{u, v\}$ is the only virtual Gallai edge in Δ, then $\mathscr{L}_{v\overline{uw}} \neq \varnothing$ and P_u' has to lie in a component $C \in \mathscr{C}_{\{u,v\}|w}$, so that it intersects every $P_v \in \mathscr{L}_{v\overline{uw}}$. Otherwise, $\{u, w\}$ is also a virtual Gallai edge and P_u' lies in a component C either in $\mathscr{C}_{\{u,v\}|w}$ or in $\mathscr{C}_{\{u,w\}|v}$. (Note that in this case $\mathscr{L}_{vw\overline{u}} \neq \varnothing$.) Without loss of generality, we assume $C \in \mathscr{C}_{\{u,v\}|w}$. We claim that C has the desired properties.

First, we prove that each path in $\mathscr{L}_{v\overline{u}}$ intersects in C° every path in $\mathscr{L}_{u\overline{v}} \cup \mathscr{L}_{uv}$. Let $P_v \in \mathscr{L}_{v\overline{u}}$. Suppose that there exists a path $Q \in \mathscr{L}_{u\overline{v}} \cup \mathscr{L}_{uv}$ such that P_v does not intersect Q in C°. Either Q intersects P_u' in C°, or P_u' and both tails of Q starting at u have length $L/2$ (see Figure 5). In the former case, since P_u' intersects both P_v and Q in C°, we can apply Lemma 4 if $Q \in \mathscr{L}_{u\overline{v}}$ (with paths P_v and Q and connecting path P_u') or Corollary 6 if $Q \in \mathscr{L}_{uv}$ (with paths P_v and Q, $z = v$, a suitable tail of P_v starting at v, subpaths $Q[C]$, and a connecting path contained in P_u') deriving a contradiction. So, suppose now that P_u' and both tails of Q starting at u have equal length. The path P_v intersects both tails of Q starting at u because otherwise such a tail of Q, $P_u' \mathbin{\rotatebox[origin=c]{90}{$\ltimes$}}_u P_v$, and a longer tail of P_v would be a path of length strictly greater than L. Therefore and since the combination of P_u' with one tail of Q starting at u and the combination of P_u' with the other tail of Q starting at u are both longest paths, we can apply Lemma 5 with a connecting path contained in P_v and $z = u$, deriving again a contradiction. Hence, each path in $\mathscr{L}_{v\overline{u}}$ intersects in C° every path in $\mathscr{L}_{u\overline{v}} \cup \mathscr{L}_{uv}$.

Next, we prove that each path in $\mathscr{L}_{uv}$ intersects in C° every path in $\mathscr{L}_{u\overline{v}}$. If $\mathscr{L}_{uv} \neq \varnothing$, let P be a path in $\mathscr{L}_{uv}$ and Q_u in $\mathscr{L}_{u\overline{v}}$. Assume that P does not intersect Q_u in C°. Let $P_v \in \mathscr{L}_{v\overline{u}}$ and note that such a longest path must exist. Since P_v intersects Q_u in C° and P in C°, we

Figure 5: Left: P_v and Q do not intersect in C (apply either Lemma 4 or Corollary 6); Right: P_v intersects both tails of Q outside of C (apply Lemma 5).

derive a contradiction longest paths Q_u and P with $z = u$, a suitable tail of Q_u starting at u, subpaths $P[C]$, and a connecting path contained in $P_v[C]$. □

Lemma 12

In a non-trivial connected series-parallel graph G, let $e = \{u,v\}$ be a virtual Gallai edge and $C \in \mathscr{C}_e$ be a component such that all pairs of longest paths mutually intersect in at least one vertex of C and all pairs of longest paths in $\mathscr{L}_{u\bar{v}} \times \mathscr{L}_{v\bar{u}}$, $\mathscr{L}_{u\bar{v}} \times \mathscr{L}_{uv}$, and $\mathscr{L}_{v\bar{u}} \times \mathscr{L}_{uv}$ mutually intersect in C° (as in Lemma 10). Let w be the unique vertex in C adjacent in $T(G)$ to both u and v. Then u, v, or w is a Gallai vertex, or there is a virtual edge f incident to u or v and a component $C_1 \in \mathscr{C}_f$, $C_1 \subsetneq C$, with the properties of Lemma 10.

Proof. Let $\Delta = \{u,v,w\}$. Assume neither u nor v are Gallai vertices (otherwise there is nothing more to prove). Thus, both $\mathscr{L}_{u\bar{v}}$ and $\mathscr{L}_{v\bar{u}}$ are nonempty. All pairs of paths in $\mathscr{L}_{u\bar{v}} \times \mathscr{L}_{v\bar{u}}$ intersect in C° by the assumptions of the lemma. By Lemma 7, all paths in $\mathscr{L}_{u\bar{v}}$ or all paths in $\mathscr{L}_{v\bar{u}}$ must contain w since $C \notin \mathscr{C}_{\{u,v\}|w}$. Without loss of generality we may assume that all paths in $\mathscr{L}_{u\bar{v}}$ contain w. Therefore, $\mathscr{L}_{u\bar{v}\bar{w}} = \varnothing$ and $\{v,w\}$ is a virtual Gallai edge.

We distinguish two cases. First, we consider the case in which there exists a path in $\mathscr{L}_{v\bar{u}}$ that does not contain w (that is, a path in $\mathscr{L}_{v\bar{u}\bar{w}}$) and then the case in which all paths in $\mathscr{L}_{v\bar{u}}$ contain w (that is, $\mathscr{L}_{v\bar{u}\bar{w}} = \varnothing$). In both cases we show that either there exists a Gallai vertex, or a virtual Gallai edge and a component strictly smaller than C that fulfill the requirements of Lemma 12.

Case 1. The set $\mathscr{L}_{v\bar{u}\bar{w}}$ is nonemtpy.

For every path $P_u \in \mathscr{L}_{u\bar{v}} = \mathscr{L}_{uw\bar{v}}$, the tail $P_u^{(w)}$ must intersect every path in $\mathscr{L}_{v\bar{u}\bar{w}}$ by assumption. Let $C_1 \subsetneq C$, $C_1 \in \mathscr{C}_{\{v,w\}|u}$ be the unique component where they mutually intersect.

We claim that the virtual edge $\{v,w\}$ together with the component C_1 fulfills the requirements of Lemma 12.

If $\mathscr{L}_{uv\bar{w}} \neq \varnothing$, let $P \in \mathscr{L}_{uv\bar{w}}$ and $P_v \in \mathscr{L}_{v\bar{u}\bar{w}}$. Assume for a contradiction that there exists a path $P_u \in \mathscr{L}_{u\bar{v}}$ such that P does not intersect P_u in C_1°. Note that P_v and P must intersect in C° by

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Figure 6: The scenario of the proof of Lemma 12: Case 1 (left) and Case 2 (right).

assumption and hence P_u and $P^{(v)}$ are disjoint. Since P_v intersects P_u in C_1° and P in C_u (at least in vertex v), we can apply Lemma 4 (with paths P_u and P, tails $P_u^{(w)}$ and $P^{(v)}$, and a connecting path contained in $P_v[C_1]$) to derive a contradiction.

Next, we prove that every path in $\mathscr{L}_{vw\overline{u}} \cup \mathscr{L}_{uvw}$ intersects every path in $\mathscr{L}_{v\overline{uw}} \cup \mathscr{L}_{u\overline{v}} \cup \mathscr{L}_{uv\overline{w}}$ in C_1°.

Let $P \in \mathscr{L}_{vw\overline{u}} \cup \mathscr{L}_{uvw}$ and $P_v \in \mathscr{L}_{v\overline{uw}}$. Note that P_v intersects P in C° by assumption if $P \in \mathscr{L}_{uvw}$. Otherwise, $P \in \mathscr{L}_{vw\overline{u}}$, and they also intersect in C°, or not both P_v and P could be longest paths by Lemma 5 for $z = v$, a suitable tail of $P_v[C_1]$, tail $P^{(v,w)} \cup P^{(w)}$, and connecting path $P_u^{(w)}$ for some $P_u \in \mathscr{L}_{u\overline{v}}$. Furthermore, P must intersect P_v in C_1°. Otherwise, P_v would have a tail starting at v completely in a component $C' \in \mathscr{C}_{\{v,w\}|u}$, $C' \neq C_1$. Then $|P_v[C']| < L/2$ since $P_v[C']$ is disjoint from P_u and so $P_v[C'] \cup P_v[C_1] \boxtimes_v P_u$, and a longer tail of P_u would be a path of length strictly greater than L. But now, by combining $P_v[C_1]$ and a longer tail of P starting at v, we get a path of length strictly greater than L, a contradiction. For every $P_u \in \mathscr{L}_{u\overline{v}}$, by applying Corollary 6 (with paths P_u and P, $z = w$, tail $P_u^{(w)}$, subpaths $P[C_1]$, and connecting path $P_v[C_1]$), we can deduce that P intersects P_u in C_1°. Every $P_w \in \mathscr{L}_{uv\overline{w}}$ must intersect P in C_1°, otherwise we get a contradiction by applying Corollary 6 (with paths P and P_w, $z = v$, tail $P_w[C_1]$, subpaths $P[C_1]$, and connecting path $P_u^{(w)}$).

Case 2. The set $\mathscr{L}_{v\overline{uw}}$ is empty.

If the set $\mathscr{L}_{uv\overline{w}}$ is empty, then all longest paths contain w, therefore w is a Gallai vertex and the requirements of Lemma 12 are fulfilled. So, from now on, we assume that the set $\mathscr{L}_{uv\overline{w}}$ is nonempty.

First, we prove that, for each $P \in \mathscr{L}_{uv\overline{w}}$, either $P^{(v)}$ intersects $P_u^{(w)}$ for every $P_u \in \mathscr{L}_{u\overline{v}} = \mathscr{L}_{uw\overline{v}}$, or $P^{(u)}$ intersects $P_v^{(w)}$ for every $P_v \in \mathscr{L}_{v\overline{u}} = \mathscr{L}_{vw\overline{u}}$. Assume for a contradiction that there exist $P \in \mathscr{L}_{uv\overline{w}}$, $P_u \in \mathscr{L}_{u\overline{v}}$, and $P_v \in \mathscr{L}_{v\overline{u}}$ such that $P^{(u)}$ does not intersect $P_v^{(w)}$ and $P^{(v)}$ does not intersect $P_u^{(w)}$. By the assumptions of the lemma, P has to intersect both P_u and P_v in C°. Therefore, P intersects P_u in the interior of some component of $\mathscr{C}_{\{u,w\}|v}$ and P_v in the interior of some component of $\mathscr{C}_{\{v,w\}|u}$. (See Figure 7.) First, suppose that $P_u^{(u)}$ and $P_v^{(v)}$ do not intersect. By combining $P^{(v)}$, $P^{(v,u)} \boxtimes_v P_u$, and a longer tail of P_u, we get a path that cannot be of length strictly greater than L, hence $|P^{(v)}| < L/2$. Combining $P^{(u)} \cup P^{(u,v)} \cup P_v^{(v,w)} \cup P_v^{(w)}$ would be a path of length strictly greater than L if $|P_v^{(v)}| \leq L/2$. However, by combining $P_v^{(v)}$, $P_v^{(v,w)} \boxtimes_v P_u$, and a

longer tail of P_u, we get a path of length strictly greater than L, a contradiction. If, on the other hand, $P_u^{(u)}$ and $P_v^{(v)}$ intersect, then $P^{(u)}$ and $P_u^{(u)}$ are disjoint except for the vertex u. Note that $|P^{(u)}| = |P_u^{(u,w)} \cup P_u^{(w)}|$ since otherwise P and P_u would not be longest paths. The combination of $P_u^{(u)}$ and $P^{(u)}$ is therefore a longest path in $\mathscr{L}_{uvw}$, which is a contradiction. Therefore, $P^{(u)}$ intersects $P_v^{(w)}$, or $P^{(v)}$ intersects $P_u^{(w)}$.

Figure 7: Left: $P_u^{(u)}$ and $P_v^{(v)}$ do not intersect; Right: $P_u^{(u)}$ and $P_v^{(v)}$ intersect.

We claim that if $R^{(u)}$, for some longest path R in $\mathscr{L}_{uv\overline{w}}$, does not intersect $P_v^{(w)}$ for some path $P_v \in \mathscr{L}_{v\overline{u}}$, then for every longest path $P \in \mathscr{L}_{uv\overline{w}}$, the tail $P^{(v)}$ intersects $P_u^{(w)}$ for every path $P_u \in \mathscr{L}_{u\overline{v}}$. Indeed, let P be a path in $\mathscr{L}_{uv\overline{w}}$. Assume for a contradiction that there exists a path $P_u \in \mathscr{L}_{u\overline{v}}$ such that $P^{(v)}$ does not intersect $P_u^{(w)}$. Note that by the latter paragraph $P_u^{(w)}$ must intersect $R^{(v)}$ and $P_v^{(w)}$ must intersect $P^{(u)}$. If $P^{(v)}$ lies in some component of $\mathscr{C}_{\{v,w\}|u}$, then by Lemma 7 $P^{(v)}$ cannot intersect $R^{(u)}$ and $R^{(u,v)} - v$, and $R^{(v)}$ cannot intersect $P^{(u)}$ and $P^{(u,v)} - v$ since $R^{(v)}$ also lies in a component of $\mathscr{C}_{\{v,w\}|u}$. Hence, we have $|P^{(v)}| = |R^{(v)}|$ and $Q = R^{(u)} \cup R^{(u,v)} \cup P^{(v)}$ is a path in $\mathscr{L}_{uv\overline{w}}$ such that $Q^{(u)}$ does not intersect $P_v^{(w)}$, and $Q^{(v)}$ does not intersect $P_u^{(w)}$, a contradiction. Analogously, if $R^{(u)}$ lies in a component of $\mathscr{C}_{\{u,w\}|v}$, the path $R^{(u)} \cup P^{(u,v)} \cup P^{(v)}$ yields a contradiction. Thus, we may assume that both $R^{(u)}$ and $P^{(u)}$ lie in a component of $C_{\{u,v\}|w}$ and are therefore disjoint from $P^{(u)}$ except for u, and from $R^{(v)}$ except for v, respectively. Now, we get a contradiction from Lemma 4 for longest paths R and P, tails $R^{(v)}$ and $P^{(u)}$, and a connecting path contained in $(P_u^{(w)} \diagdown_w R) \cup (P_v^{(w)} \diagdown_w P)$.

Without loss of generality, we may assume that, for each $P \in \mathscr{L}_{uv\overline{w}}$, the tail $P^{(v)}$ intersects $P_u^{(w)}$ for every $P_u \in \mathscr{L}_{u\overline{v}}$. Let $C_1 \in \mathscr{C}_{\{v,w\}|u}$ be the unique component where they mutually intersect. Note that the virtual edge $\{v, w\}$ is indeed a virtual Gallai edge since the set $\mathscr{L}_{u\overline{vw}}$ is empty. Let $P \in \mathscr{L}_{uv\overline{w}}$ and $P_u \in \mathscr{L}_{u\overline{v}}$ be arbitrary but fixed. Note that P_u intersects P in C_1°. Assume for a contradiction that there exists a longest path $P_v \in \mathscr{L}_{vw\overline{u}} \cup \mathscr{L}_{uvw}$ such that P and P_v do not intersect each other in C_1°. Then not both P and P_v can be longest paths by Corollary 6 for $z = v$, tail $P^{(v)}$, subpaths $P_v[C_1]$, and connecting path $P_u^{(w)}$, a contradiction. Therefore, the virtual edge $\{v, w\}$ together with the component C_1 fulfills the requirements of Lemma 12. $\square$

Theorem 13

For every connected series-parallel graph G, there exists a vertex v such that all longest paths in G contain v.

Proof. This follows from Lemma 10 and by iteratively applying Lemma 12 since G is finite. $\square$

4 Algorithmic remarks

For any hereditary class of graphs for which there is a polynomial-time algorithm that computes (the length of) a longest path, it is easy to derive a polynomial-time algorithm that finds all Gallai vertices. Indeed, one just has to compute the length L of a longest path in the given (connected) graph G, and then to check, for each vertex v, whether the length of a longest path in $G - v$ remains the same. If not, v is a Gallai vertex.

It is a well-known result that one can use dynamic programming to solve many combinatorial problems on graphs of bounded treewidth in polynomial or even linear time [1, 4]. In particular, Bodlaender [5, Thm. 2.2] claims a linear-time algorithm following these lines to find a longest path in a graph with bounded treewidth. (See also [6] on how to obtain in linear time a tree decomposition for graphs with bounded treewidth.) Therefore, using the idea described in the previous paragraph, one can find all Gallai vertices in time quadratic on the number of vertices of the given connected series-parallel graph.

In fact, one can do better by applying the same strategy used to compute the length of a longest path in a partial k-tree, but carrying more information during the process. Given a connected graph G of treewidth k, compute in linear time a (nice) tree decomposition for G (as done in [10] for instance). Then run a dynamic programming algorithm on top of this tree decomposition, to compute the length L of a longest path in G. Roughly speaking, this algorithm computes the length of longest parts of paths within the subgraph induced by the vertices in clusters already traversed of the tree decomposition, and puts together this information while going through the tree decomposition. Specifically, when visiting a node u of the tree decomposition, if H_u is the subgraph of G induced by the vertices in the cluster X_u or clusters of nodes below u, for each different way that a path can behave in the cluster X_u and in H_u, we have a configuration as the ones described in Figure 8 for the case in which X_u has three vertices.

Figure 8: A sample of the configurations for a cluster with three vertices. The whole set of configurations has to consider the labels of the vertices in the cluster.

The number of such configurations depends only on the treewidth. For each such configuration, the algorithm computes the length of a longest part of a path in H_u that "agrees" with that configuration. It does this using dynamic programming, that is, it computes such length for a node u and one of the configurations using the information that it already computed for the children of u in the tree. Some of the configurations of the children, together with new edges within X_u, combine into each configuration for u. The combinations that give raise to the longest parts are the ones of interest, and give the length of a longest part for that configuration for u.

In a first traversal of the tree decomposition, the value of L is computed. Now, as it is usual in dynamic programming, in a reverse traversal of the tree, retracing backwards what was done to find out L, one can mark, for each node and each configuration, if that configuration at that

node gives raise to a path of length L in G. Once this is done for a node, the algorithm checks whether the configurations for that node that give raise to a longest path all contain one of the vertices in the cluster of that node. If so, this is a Gallai vertex. Otherwise the algorithm proceeds to the next node in the reverse traversal.

This process finishes with a Gallai vertex as long as the graph has one such vertex. In particular, for partial 2-trees, this process will find a Gallai vertex in the reverse traversal as soon as it reaches the first cluster of the tree that contains a Gallai vertex. By proceeding with the reverse traversal in this way, one can find all Gallai vertices. For bounded k, the running time of this algorithm is linear on the number of vertices of the graph. (Note that the number of edges in a partial k-tree is at most kn, where n is the number of vertices in the partial k-tree.) Indeed, first computing a (nice) tree decomposition can be done in linear time. Second, the number of configurations depends only on k, and the processing of each node of the tree decomposition depends only on the number of configurations (and on the size of the cluster, which is bounded by $k + 1$ and thus also by the number of configurations). Therefore, for series-parallel graphs, this algorithm finds a Gallai vertex in time that is linear on the number of vertices of the graph.

5 Related results and open questions

There are several questions related to Gallai's original question that remain open. For instance, it was asked [16, 29] whether there is a vertex common to all longest paths in all 4-connected graphs. This problem is open so far, and even the more general question for k-connected graphs with larger k has not been answered. There are 3-connected examples known for which Gallai's question has a negative answer [13].

In [11], where a proof that all 2-trees have nonempty intersection of all longest paths was presented, it was asked whether the same holds for k-trees with larger values of k. As far as we know, this also has not yet been answered. In the present paper, we have proven that all connected subgraphs of 2-trees have nonempty intersection of all longest paths. We observe that the same does not hold for all subgraphs of 3-trees. Indeed, the counterexample by Walther, Voss, and Zamfirescu in [26, 28] is a connected spanning subgraph of a 3-tree (see Figure 9).

Figure 9: The counterexample of Walther, Voss, and Zamfirescu as a subgraph of a 3-tree. Missing edges are dotted. The number next to each vertex indicates the sequence in which they are added to the 3-tree.

In other words, Gallai's question has a positive answer for connected graphs with treewidth at most 2 (series-parallel graphs), but a negative answer for connected graphs with treewidth

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at most 3. As series-parallel graphs are the class of K_4 minor free graphs, one might also ask whether the answer is positive for all (connected) K_5 minor free graphs, but there are planar counterexamples known [22].

As split graphs and 2-trees are chordal, a natural question raised by Balister et al. [3] is whether all longest paths share a vertex in all chordal graphs. Recently, Michel Habib (personal communication) suggested that the answer to Gallai's question might be positive in co-comparability graphs. For this class of graphs, as well as for series-parallel graphs, there is a polynomial-time algorithm to compute a longest path [15]. (For chordal graphs, computing a longest path is NP-hard [18].)

As already stated in Section 1, instead of looking at the intersection of all longest paths, Zamfirescu asked whether any p longest paths in an arbitrary connected graph contain a common vertex. This is certainly true for $p = 2$, proven to be false [20, 21] for $p \geq 7$, but still open for p in $\{3, 4, 5, 6\}$.

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