

The Coloring Game on Planar Graphs with Large Girth, by a result on Sparse Cactuses

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Abstract

We denote by $\chi_g(G)$ the *game chromatic number* of a graph G , which is the smallest number of colors Alice needs to win the *coloring game* on G . We know from Montassier et al. [M. Montassier, P. Ossona de Mendez, A. Raspaud and X. Zhu, *Decomposing a graph into forests*, J. Graph Theory Ser. B, 102(1):38-52, 2012] and, independantly, from Wang and Zhang, [Y. Wang and Q. Zhang. *Decomposing a planar graph with girth at least 8 into a forest and a matching*, Discrete Maths, 311:844-849, 2011] that planar graphs with girth at least 8 have game chromatic number at most 5.

One can ask if this bound of 5 can be improved for a sufficiently large girth. In this paper, we prove that it cannot. More than that, we prove that there are *cactuses* CT (i.e. graphs whose edges only belong to at most one cycle each) having $\chi_g(CT) = 5$ despite having arbitrary large girth, and even arbitrary large distance between its cycles.

1 Introduction

We only consider in this paper simple, finite, and undirected graphs. The *length* of a path or cycle is the cardinal of its edge-set. The *girth* $g(G)$ of a graph G is the length of its smallest cycle. A *cactus* is a graph G in which any edge belongs to at most one cycle. The *cycle-distance* of a cactus is the length of its smallest path between two vertices belonging to different cycles. For a vertex v , we call *v-leaf* a vertex of degree 1 (or *leaf*) whose neighbor is v .

The *coloring game* on a graph G is a two-player non-cooperative game on the vertices of G , introduced by Brams [?] and rediscovered ten years after by Bodlaender [?]. Given a set of k colors, Alice and Bob take turns coloring properly an uncolored vertex, with aim for Alice to color entirely G , and for Bob to prevent Alice from winning. The

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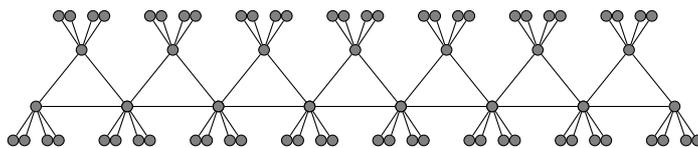


Figure 1: A cactus with game chromatic number 5 [?]

game chromatic number $\chi_g(G)$ of G is the smallest number of colors insuring Alice's victory. This graph invariant has been extensively studied these past twenty years, see for example [?, ?, ?, ?].

In [?], Bodlaender proved that every forest F has $\chi_g(F) \leq 5$, and exhibited trees T with $\chi_g(T) \geq 4$. In [?], Faigle et al. showed that every forest F has $\chi_g(F) \leq 4$. Conditions for trees to have game chromatic number 3 were recently studied by Dunn et al. [?].

A graph is said $(1, k)$ -decomposable if its edge set can be partitioned into two sets, one inducing a forest and the other inducing a graph with maximum degree at most k . Using the notion of *marking game* introduced by Zhu in [?], He et al. observed in [?] that every $(1, k)$ -decomposable graph has $\chi_g(G) \leq k + 4$, then deduced upper bounds for the game chromatic number of planar graphs with given girth. Among other results, they proved that planar graphs with girth at least 11 are $(1, 1)$ -decomposable, and therefore their game chromatic number is at most 5. Later, were proved successively the $(1, 1)$ -decomposability of planar graphs with girth 10 by Bassa et al. [?], girth 9 by Borodin et al. [?], and girth 8 by Montassier et al. [?] and Wang and Zhang [?] independantly. There exist planar graphs with girth 7 that are not $(1, 1)$ -decomposable.

Borodin et al. [?] gave conditions for planar graphs with no small cycles except triangles to be $(1, 1)$ -decomposable, in terms of distance between the triangles and of minimal length of a non-triangle cycle. In [?], Sidorowicz, arguing that cactuses are $(1, 1)$ -decomposable, showed that every cactus CT has $\chi_g(CT) \leq 5$. Moreover, she exhibited a cactus with game chromatic number 5, depicted in Figure 1. As one can see, this cactus has intersecting triangles.

The work we present here started as we tried to answer the following question:

Question 1. *Is there an integer g such that every planar graph G with girth at least g has $\chi_g(G) \leq 4$?*

We answer negatively, with a result going way beyond the question we initially asked.

Theorem 1. *For any integers d, k , there are cactuses CT with girth at least k , cycle-distance at least d and $\chi_g(G) = 5$.*

This proves that the upper bound of 5 for the game chromatic number of the classes of $(1, 1)$ -decomposable graphs considered in [?, ?, ?, ?, ?, ?] are best possible.

As our construction needs odd cycles, this asks whether or not this result generalizes to bipartite graphs, seeming more difficult to handle. This question is still open. As a

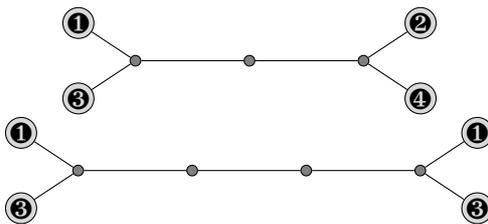


Figure 2: Two winning paths (Lemma 2)

partial result, we can find in [?] a proof by Andres and Hochstättler that every *forest with thin 4-cycles*, which is a cactus constructed from a forest by replacing some edges uv by a pair of 2-vertices both adjacent to u and v (and which is bipartite), has game chromatic number at most 4.

2 Proof of Theorem 1

We consider Alice and Bob playing the coloring game on a graph G with a set of four colors $\mathcal{C} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$. At each time of the game, we denote by $\phi(v)$ the color of a vertex v (if v is colored), and by $\Phi(v)$ the set of colors in the neighborhood of v : if v is uncolored, then this is the set of colors forbidden for v . We call *surrounded* an uncolored vertex v with $\Phi(v) = \mathcal{C}$. Bob wins if he can surround a vertex.

We give further the construction of G . For now just assume that *every cycle of G is odd* and that *every non-leaf vertex of G is adjacent to a large number of leaves*, say at least 8 leaves.

We also give further Bob's strategy in details, but assume that *Bob only plays on the leaves of G* . Also, for any vertex v , when Bob colors a v -leaf, he uses a color that is not already in $\Phi(v)$. If Alice colors a v -leaf during the game, then Bob always colors another v -leaf if possible. So we can assume that *for any uncolored vertex v , there is always at least one uncolored leaf of v* . Moreover, coloring a v -leaf for Alice does not suppress any possibility for Bob (this forbid Bob to color v with the color Alice used, but Bob had no intention to color v anyway) and only increase $\Phi(v)$, coloring a leaf is always unoptimal for Alice. So we assume *Alice never colors a leaf during the game*.

We describe some *winning positions* for Bob (i.e. partial colorings from where Bob has a winning strategy) in the following lemmas. In the description of every winning position, we assume that it is Alice's turn to play.

Lemma 2. *Suppose that G contains a path of length d , $P = v_0 \dots v_d$, of uncolored non-leaf vertices. Also suppose $|\Phi(v_0)| \geq 2$, say $\{\mathbf{1}, \mathbf{3}\} \subseteq \Phi(v_0)$. The game is in a winning position for Bob if (see Figure 2):*

- d is odd and $\{\mathbf{1}, \mathbf{3}\} \subseteq \Phi(v_d)$.
- d is even and $\{\mathbf{2}, \mathbf{4}\} \subseteq \Phi(v_d)$.

We say that P is a winning path.

Proof. Recall that, by assumption, every vertex of P has at least one uncolored leaf.

In the case $d = 0$, P is reduced to a single surrounded vertex and Bob wins.

The case $d = 1$ corresponds to an edge v_0v_1 with v_0 and v_1 uncolored and $|\Phi(v_0) \cap \Phi(v_1)| \geq 2$, say $\{\mathbf{1}, \mathbf{3}\} \subseteq \Phi(v_0) \cap \Phi(v_1)$. If Alice colors v_0 , she has to use an even color, Bob colors a v_1 -leaf with the other even color, surrounds v_1 , and wins. If Alice colors v_1 , then Bob can surround v_0 as well. So Bob's winning strategy consists in coloring leaves of v_0 with different colors until v_0 is surrounded or Alice colors v_0 or v_1 .

We prove the other cases by induction. Assume that our lemma is true for every value of d strictly smaller than an integer q and consider the case $d = q$. Bob's strategy is to color a v_0 -leaf until v_0 is surrounded or Alice colors a vertex of P . When Alice colors a vertex v_i of P , we consider two cases.

- If $i = 0$ or $i = q$, say w.l.o.g. $i = 0$, then she uses an even color. Bob colors a v_1 -leaf with the other even color. By induction, Bob wins since $P - v_0$ is a winning path of length $q - 1$.
- If $i \neq 0$ and $i \neq q$, then let $P_1 = v_0 \dots v_{i-1}$ and $P_2 = v_{i+1} \dots v_q$. We denote by d_1 and d_2 the length of P_1 and P_2 respectively. We have $d_1 + d_2 = q - 2$. We consider two subcases:
 - Either q is even, and d_1 and d_2 are either both even or both odd. Say we have $\{\mathbf{1}, \mathbf{3}\} \subseteq \Phi(v_0)$ and $\{\mathbf{2}, \mathbf{4}\} \subseteq \Phi(v_q)$. Without loss of generality, assume Alice colored v_i with $\mathbf{1}$. If d_1 and d_2 are both odd, then Bob colors a v_{i-1} -leaf with $\mathbf{3}$. If they are both even, then Bob colors a v_{i+1} -leaf with $\mathbf{2}$. Path P_1 or P_2 respectively is a winning path and Bob wins by induction hypothesis.
 - Either q is odd, and say $\{\mathbf{1}, \mathbf{3}\} \subseteq \Phi(v_0) \cap \Phi(v_1)$. Among d_1 and d_2 , one is even and one is odd, say d_1 is even and d_2 is odd. If Alice colored v_i with an odd color, then Bob colors a v_{i+1} -leaf with the other odd color. If Alice colored v_i with an even color, then Bob colors a v_{i-1} -leaf with the other even color. Path P_1 or P_2 respectively is a winning path and Bob wins by induction hypothesis.

This concludes our proof. □

Lemma 3. *Suppose that C is an (odd) cycle of G of uncolored vertices. If there are two neighbors u and v with $|\Phi(u)| \geq 2$ and $|\Phi(u) \cap \Phi(v)| = 1$ (see Figure 3), then G is in a winning position for Bob. We say that C is a winning cycle.*

Proof. Let $C = v_0v_1 \dots v_k$ be an odd cycle of uncolored vertices with, say, $\{\mathbf{1}, \mathbf{2}\} \subseteq \Phi(v_0)$ and $\mathbf{1} \in \Phi(v_1)$. If the next Alice's move is not to color v_0 or v_1 , then Bob colors a leaf of v_1 with $\mathbf{2}$ and Bob wins by Lemma 2 (v_0v_1 is a winning path). If Alice colors v_0 , then she has to use a color different to $\mathbf{1}$ and, at his turn, Bob colors a v_k -leaf with $\mathbf{1}$. Since C is an odd cycle, Bob wins by Lemma 2 (path $v_1 \dots v_k$ is a winning path). Similarly, if

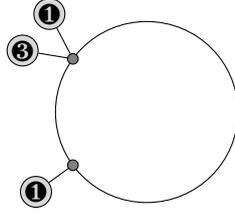


Figure 3: A winning cycle (Lemma 3)

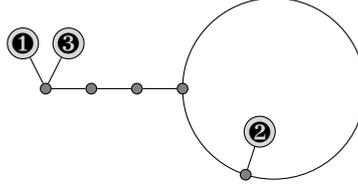


Figure 4: A winning cycle-path (Lemma 4)

Alice colors v_1 , Bob can color a v_2 -leaf with a color among $\mathbf{1}$ and $\mathbf{2}$ different from $\phi(v_1)$ and $v_2 \dots v_k v_0$ is a winning path. \square

Lemma 4. *Suppose that G contains a path of length d , $P = v_0 \dots v_d$, of uncolored non-leaf vertices. Also suppose $|\Phi(v_0)| \geq 2$, say $\{\mathbf{1}, \mathbf{3}\} \subseteq \Phi(v_0)$, and suppose v_d belongs to an odd cycle C of uncolored vertices, with w being a neighbor of v_d in this cycle. Graph G is in a winning position for Bob if (see Figure 4):*

- d is even and $\mathbf{1}$ or $\mathbf{3}$ is in $\Phi(w)$.
- d is odd and $\mathbf{2}$ or $\mathbf{4}$ is in $\Phi(w)$.

We say that $P \cup C$ is a winning cycle-path.

Proof. The case $d = 0$ is true by Lemma 3, since C is then a winning cycle.

We prove the other cases by induction on d . Assume our lemma is true for every value of d strictly smaller than q . We consider the case $d = q$. We have different cases depending of Alice's move:

- If Alice colors w , then let w' be the neighbor of w different from v_q . The uncolored vertices of $P \cup C$ form a path $P' = v_0, \dots, v_q, \dots, w'$. Since C has odd length, one path among P and P' is odd and the other is even. If $\phi(w) \in \Phi(v_0)$, then Bob can color a leaf of v_q or w' such that the odd path among P and P' become a winning path. If $\phi(w) \notin \Phi(v_0)$, then Bob colors a leaf of v_q or w' such that the even path among P and P' is a winning path. By Lemma 2, Bob is winning the game.
- If Alice colors v_0 , then Alice has to use an even color, Bob colors a v_1 -leaf with the other even color and $(P - v_0) \cup C$ is a winning cycle-path, Bob wins by induction hypothesis.

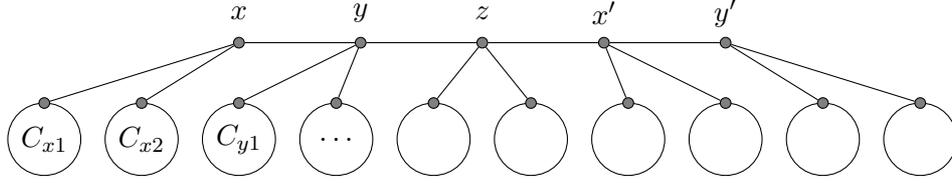


Figure 5: A sketch of graph G

- If Alice colors v_q , then let w'' be the other neighbor of v_d in C than w . If there is in $\Phi(w)$ a color c different to $\phi(v_q)$, then Bob colors w'' with c and $C - v_d$ is winning path since C has odd length. By Lemma 2, Bob wins. If $\Phi(w) = \phi(v_d)$, then we have two cases:
 - If d is even, then $\phi(v_d) = \mathbf{1}$ or $\mathbf{3}$, say $\mathbf{1}$. Bob colors a v_{d-1} -leaf with $\mathbf{3}$ and $P - v_d$ is a winning path.
 - If d is odd, then $\phi(v_d) = \mathbf{2}$ or $\mathbf{4}$, say $\mathbf{2}$. Bob colors a v_{d-1} -leaf with $\mathbf{4}$ and $P - v_d$ is a winning path.

Bob wins by Lemma 2.

- If Alice colors v_i , $i \neq 0$ and $i \neq q$, then let $P_1 = v_0 \dots v_{i-1}$ and $P_2 = v_{i+1} \dots v_q$. We denote by d_1 and d_2 the length of P_1 and P_2 respectively. We have $d_1 + d_2 = q - 2$. We consider two subcases:
 - If d_1 is odd and Alice used an odd color on v_i , or if d_1 is even and Alice used an even color on v_i , then Bob colors v_{i-1} with the other odd color (resp. the other even color), and Bob wins by Lemma 2 (P_1 is a winning path).
 - If d_1 is odd and Alice used an even color on v_i , or if d_1 is even and Alice used an odd color on v_i , then Bob colors v_{i+1} with the other even color (resp. the other odd color). Bob wins by induction hypothesis, since $P_2 \cup C$ is a winning cycle-path of smaller path-length.
- If Alice colors another vertex, then one can observe that Bob can color a w -leaf in such a way that $P + w$ is a winning path.

Our lemma is true for every d by induction. This concludes our proof. \square

Corollary 5. *If, at Bob's turn, there is in G an uncolored path $P = v_0, \dots, v_d$ such that $|\Phi(v_0)| \geq 2$ and v_d belongs to an uncolored odd cycle, then Bob has a winning strategy (he can obtain a winning cycle-path in one move).*

We describe now explicitly the graph G on which Alice and Bob are playing, depicted in Figure 5. Here paths and cycles have arbitrary large length and cycles are odd. For a vertex v , we say we *attach a cycle C to v with a path P* if we add C and P such that the endvertices of P are v and a vertex of C . We start from a path $xyz'y'x'$, then for

every vertex v in this path, we attach two odd cycles C_{v1} and C_{v2} to v with two paths P_{v1} and P_{v2} respectively. We denote $G_v = C_{v1} + C_{v2} + P_{v1} + P_{v2}$. Finally, we copy our graph (i.e. we add a similar second connected component to the graph). Then we add a large number of leaves (at least 8) to every vertex to obtain G .

Now we give the winning strategy for Bob. As we assumed, Bob only plays on leaves. After Alice's first move, Bob considers the connected component of the graph where Alice did not play on and will only play on it.

Step 1. *Bob plays in G_z . He aims to end Step 1 with his victory or with Alice coloring z . Moreover, Bob wants to be sure Alice colors in Step 1 at most one vertex not in G_z .*

At his first move, Bob colors a z -leaf with an arbitrary color. If Alice colors z at her second move, then Bob goes to Step 2. Otherwise, Bob colors another z -leaf with another color. If Alice colors z , then Alice played at most one move out of G_z and Bob goes to Step 2. Otherwise,

- If Alice has played her two moves in G_z , then Bob colors another z -leaf and Alice has to color z for Bob not to surround it next turn. Bob goes to Step 2.
- Otherwise, then we can assume without loss of generality that $P_{z1} \cup C_{z1}$ is uncolored, and Bob wins by Corollary 5.

Step 2. *Since Alice only played once out of G_z , we assume that $G_x \cup G_y$ is uncolored. Now Bob plays in G_x . He aims to end Step 2 with his victory or with Alice coloring x with a color different from $\phi(z)$. Moreover, Bob wants to be sure Alice colors in Step 2 at most one vertex not in G_x , and this vertex, if it exists, is not y .*

Step 2 is quite similar to Step 1. Bob begins by coloring a x -leaf with $\phi(z)$. If Alice colors y , she has to use a color different from $\phi(z)$ and Bob wins by Corollary 5. If she colors x , then Bob goes to Step 3. If she colors any other vertex, then Bob colors another x -leaf, and, from this point, this is similar to Step 1.

Step 3. *We can assume without loss of generality that $P_{y1} \cup C_{y1}$ is uncolored. Since $|\Phi(y)| \geq 2$, Bob wins by Corollary 5.*

This ends the proof of Theorem 1.