# On the number of mutually disjoint pairs of S-permutation matrices 

Krasimir Yordzhev<br>Faculty of Mathematics and Natural Sciences<br>South-West University, Blagoevgrad, Bulgaria<br>E-mail: yordzhev@swu.bg


#### Abstract

This work examines the concept of S-permutation matrices, namely $n^{2} \times n^{2}$ permutation matrices containing a single 1 in each canonical $n \times n$ subsquare (block). The article suggests a formula for counting mutually disjoint pairs of $n^{2} \times n^{2}$ S-permutation matrices in the general case by restricting this task to the problem of finding some numerical characteristics of the elements of specially defined for this purpose factorset of the set of $n \times n$ binary matrices. The paper describe an algorithm that solves the main problem. To do that, every $n \times n$ binary matrix is represented uniquely as a n-tuple of integers.


Keyword: binary matrix; S-permutation matrix; disjoint matrices; Sudoku; factor-set; n-tuple of integers

2010 Mathematics Subject Classification: 05B20

## 1 Introduction and notation

Let $n$ be a positive integer. By $[n]$ we denote the set $[n]=\{1,2, \ldots, n\}$.
A binary (or boolean, or ( 0,1 )-matrix) is a matrix all of whose elements belong to the set $\mathfrak{B}=\{0,1\}$. In this paper we will consider only square binary matrices. With $\mathfrak{B}_{n}$ we will denote the set of all $n \times n$ binary matrices. With $\mathfrak{B}_{n, k}$ we will denote the set of all $n \times n$ binary matrices containing exactly $k$ elements equal to 1 .

Two $n \times n$ binary matrices $A=\left(a_{i j}\right) \in \mathfrak{B}_{n}$ and $B=\left(b_{i j}\right) \in \mathfrak{B}_{n}$ will be called disjoint if there are not integers $i, j \in[n]$ such that $a_{i j}=b_{i j}=1$, i.e. if $a_{i j}=1$ then $b_{i j}=0$ and if $b_{i j}=1$ then $a_{i j}=0$.

Let $n$ be a positive integer and let $A \in \mathfrak{B}_{n^{2}}$ be a $n^{2} \times n^{2}$ binary matrix. With the help of $n-1$ horizontal lines and $n-1$ vertical lines $A$ has been separated into $n^{2}$ of number non-intersecting $n \times n$ square sub-matrices $A_{k l}, 1 \leq k, l \leq n$, e.i.

$$
A=\left[\begin{array}{cccc}
A_{11} & A_{12} & \cdots & A_{1 n}  \tag{1}\\
A_{21} & A_{22} & \cdots & A_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n 1} & A_{n 2} & \cdots & A_{n n}
\end{array}\right]
$$

The sub-matrices $A_{k l}, 1 \leq k, l \leq n$ will be called blocks.

A matrix $A \in \mathfrak{B}_{n^{2}}$ is called an $S$-permutation if in each row, in each column, and in each block of $A$ there is exactly one 1 . Let the set of all $n^{2} \times n^{2}$ Spermutation matrices be denoted by $\Sigma_{n^{2}}$.

The concept of S-permutation matrix was introduced by Geir Dahl [2] in relation to the popular Sudoku puzzle. Sudoku is a very popular game. On the other hand, it is well known that Sudoku matrices are special cases of Latin squares in the class of gerechte designs [1].

Obviously a square $n^{2} \times n^{2}$ matrix $M$ with elements of $\left[n^{2}\right]=\left\{1,2, \ldots, n^{2}\right\}$ is a Sudoku matrix if and only if there are matrices $A_{1}, A_{2}, \ldots, A_{n^{2}} \in \Sigma_{n^{2}}$, each two of them are disjoint and such that $P$ can be given in the following way:

$$
\begin{equation*}
M=1 \cdot A_{1}+2 \cdot A_{2}+\cdots+n^{2} \cdot A_{n^{2}} \tag{2}
\end{equation*}
$$

Some algorithms for obtaining random Sudoku matrices and their valuation are described in detail in (5) and (4).

In 44 Roberto Fontana offers an algorithm which randomly gets a family of $n^{2} \times n^{2}$ mutually disjoint S-permutation matrices, where $n=2,3$. In $n=3$ he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then using (2) he obtained $9!\cdot 105=38102400$ Sudoku matrices.

But it is known [3] that the total number of $9 \times 9$ Sudoku matrices is

$$
9!\cdot 72^{2} \cdot 2^{7} \cdot 27704267971=6670903752021072936960
$$

Thus, in relation with Fontana's algorithm, it looks useful to calculate the probability of two randomly generated S-permutation matrices to be disjoint. So the question of enumerating all disjoint pairs of S-permutation matrices naturally arises. This work is devoted to this task.

As we have shown in [6], with hand calculations of the so assigned task with small values of $n(n=2,3)$, it is convenient to use the apparatus of graph theory. Unfortunately, when $n \geq 4$ this approach is inefficient. In this article, we will use only the operations of matrix analysis, which are not difficult to process with computers.

## 2 A representation of S-permutation matrices

Let $n$ be a positive integer. If $z_{1} z_{2} \ldots z_{n}$ is a permutation of the elements of the set $[n]=\{1,2, \ldots, n\}$ and let us shortly denote $\sigma$ this permutation. Then in this case we will denote by $\sigma(i)$ the $i$-th element of this permutation, i.e. $\sigma(i)=z_{i}, i=1,2, \ldots, n$.

Definition 1 Let $\Pi_{n}$ denotes the set of all $n \times n$ matrices, constructed such that $\pi \in \Pi_{n}$ if and only if the following three conditions are true:
i) the elements of $\pi$ are ordered pairs of numbers $\langle i, j\rangle$, where $1 \leq i, j \leq n$;
ii) if

$$
\left[\begin{array}{llll}
\left\langle a_{1}, b_{1}\right\rangle & \left\langle a_{2}, b_{2}\right\rangle & \cdots & \left.\left\langle a_{n}, b_{n}\right\rangle\right]
\end{array}\right.
$$

is the $i$-th row of $\pi$ for any $i \in[n]=\{1,2, \ldots, n\}$, then $a_{1} a_{2} \ldots a_{n}$ in this order is a permutation of the elements of the set $[n]$;
iii) if

$$
\left[\begin{array}{c}
\left\langle a_{1}, b_{1}\right\rangle \\
\left\langle a_{2}, b_{2}\right\rangle \\
\vdots \\
\left\langle a_{n}, b_{n}\right\rangle
\end{array}\right]
$$

is the $j$-th column of $\pi$ for any $j \in[n]$, then $b_{1}, b_{2}, \ldots, b_{n}$ in this order is a permutation of the elements of the set $[n]$.

From Definition 1, it follows that we can represent each row and each column of a matrix $M \in \Pi_{n}$ with the help of a permutation of elements of the set $[n]$.

Conversely for every ( $2 n$ )-tuple

$$
\left\langle\left\langle\rho_{1}, \rho_{2}, \ldots, \rho_{n}\right\rangle,\left\langle\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n}\right\rangle\right\rangle,
$$

where

$$
\begin{gathered}
\rho_{i}=\rho_{i}(1) \rho_{i}(2) \ldots \rho_{i}(n), \quad 1 \leq i \leq n \\
\sigma_{j}=\sigma_{j}(1) \sigma_{j}(2) \ldots \sigma_{j}(n), \quad 1 \leq j \leq n
\end{gathered}
$$

are $2 n$ permutations of elements of $[n]$ (not necessarily different), then the matrix

$$
\pi=\left[\begin{array}{cccc}
\left\langle\rho_{1}(1), \sigma_{1}(1)\right\rangle & \left\langle\rho_{1}(2), \sigma_{2}(1)\right\rangle & \cdots & \left\langle\rho_{1}(n), \sigma_{n}(1)\right\rangle \\
\left\langle\rho_{2}(1), \sigma_{1}(2)\right\rangle & \left\langle\rho_{2}(2), \sigma_{2}(2)\right\rangle & \cdots & \left\langle\rho_{2}(n), \sigma_{n}(2)\right\rangle \\
\vdots & \vdots & \ddots & \vdots \\
\left\langle\rho_{n}(1), \sigma_{1}(n)\right\rangle & \left\langle\rho_{n}(2), \sigma_{2}(n)\right\rangle & \cdots & \left\langle\rho_{n}(n), \sigma_{n}(n)\right\rangle
\end{array}\right]
$$

is matrix of $\Pi_{n}$. Hence

$$
\begin{equation*}
\left|\Pi_{n}\right|=(n!)^{2 n} \tag{3}
\end{equation*}
$$

Definition 2 We say that matrices $\pi^{\prime}=\left[p^{\prime}{ }_{i j}\right]_{n \times n} \in \Pi_{n}$ and $\pi^{\prime \prime}=\left[p^{\prime \prime}{ }_{i j}\right]_{n \times n} \in$ $\Pi_{n}$ are disjoint, if $p^{\prime}{ }_{i j} \neq p^{\prime \prime}{ }_{i j}$ for every $i, j \in[n]$.

Definition 3 Let $\pi^{\prime}, \pi^{\prime \prime} \in \Pi_{n}, \pi^{\prime}=\left[p^{\prime}{ }_{i j}\right]_{n \times n}, \pi^{\prime \prime}=\left[p^{\prime \prime}{ }_{i j}\right]_{n \times n}$ and let the integers $i, j \in[n]$ are such that $p^{\prime}{ }_{i j}=p^{\prime \prime}{ }_{i j}$. In this case we will say that $p^{\prime}{ }_{i j}$ and $p^{\prime \prime}{ }_{i j}$ are component-wise equal elements.

Obviously two $\Pi_{n}$-matrices are disjoint if and only if they do not have component-wise equal elements.

Example 1 We consider the following $\Pi_{3}$-matrices:

$$
\begin{gathered}
\pi^{\prime}=\left[p_{i j}^{\prime}\right]=\left[\begin{array}{lll}
\langle 3,1\rangle & \langle 2,1\rangle & \langle 1,2\rangle \\
\langle 2,3\rangle & \langle 3,2\rangle & \langle 1,1\rangle \\
\langle 3,2\rangle & \langle 1,3\rangle & \langle 2,3\rangle
\end{array}\right] \\
\pi^{\prime \prime}=\left[p_{i j}^{\prime \prime}\right]=\left[\begin{array}{lll}
\langle 3,2\rangle & \langle 1,3\rangle & \langle 2,1\rangle \\
\langle 3,3\rangle & \langle 1,1\rangle & \langle 2,2\rangle \\
\langle 2,1\rangle & \langle 1,2\rangle & \langle 3,3\rangle
\end{array}\right] \\
\pi^{\prime \prime \prime}=\left[p_{i j}^{\prime \prime \prime}\right]=\left[\begin{array}{lll}
\langle 3,1\rangle & \langle 1,3\rangle & \langle 2,2\rangle \\
\langle 2,2\rangle & \langle 3,1\rangle & \langle 1,1\rangle \\
\langle 2,3\rangle & \langle 1,2\rangle & \langle 3,3\rangle
\end{array}\right]
\end{gathered}
$$

Matrices $\pi^{\prime}$ and $\pi^{\prime \prime}$ are disjoint, because they do not have component-wise equal elements.

Matrices $\pi^{\prime}$ and $\pi^{\prime \prime \prime}$ are not disjoint, because they have two component-wise equal elements: $p_{11}^{\prime}=p_{11}^{\prime \prime \prime}=\langle 3,1\rangle$ and $p_{23}^{\prime}=p_{23}^{\prime \prime \prime}=\langle 1,1\rangle$.

Matrices $\pi^{\prime \prime}$ and $\pi^{\prime \prime \prime}$ are not disjoint, because they have three componentwise equal elements: $p_{12}^{\prime \prime}=p_{12}^{\prime \prime \prime}=\langle 1,3\rangle, p_{32}^{\prime \prime}=p_{32}^{\prime \prime \prime}=\langle 1,2\rangle$, and $p_{33}^{\prime}=p_{33}^{\prime \prime \prime}=$ $\langle 3,3\rangle$.

The relationship between S-permutation matrices and the matrices from the set $\Pi_{n}$ are given by the following theorem:

Theorem 1 Let $n$ be an integer, $n \geq 2$. Then there is one to one correspondence between the sets $\Sigma_{n^{2}}$ and $\Pi_{n}$.

Proof. Let $A \in \Sigma_{n^{2}}$. Then $A$ is constructed with the help of formula (1) and for every $i, j \in[n]$ in the block $A_{i j}$ there is only one 1 and let this 1 has coordinates $\left(a_{i}, b_{j}\right)$. For every $i, j \in[n]$ we obtain ordered pairs of numbers $\left\langle a_{i}, b_{j}\right\rangle$ corresponding to these coordinates. As in every row and every column of $A$ there is only one 1 , then the matrix $\left[\alpha_{i j}\right]_{n \times n}$, where $\alpha_{i j}=\left\langle a_{i}, b_{j}\right\rangle, 1 \leq i, j \leq n$, which is obtained by the ordered pairs of numbers is matrix of $\Pi_{n}$, i.e. matrix for which the conditions i), ii) and iii) are true.

Conversely, let $\left[\alpha_{i j}\right]_{n \times n} \in \Pi_{n}$, where $\alpha_{i j}=\left\langle a_{i}, b_{j}\right\rangle, i, j \in[n], a_{i}, b_{j} \in[n]$. Then for every $i, j \in[n]$ we construct a binary $n \times n$ matrices $A_{i j}$ with only one 1 with coordinates $\left(a_{i}, b_{j}\right)$. Then we obtain the matrix of type (11). According to the properties i), ii) and iii), it is obvious that the obtained matrix is S permutation matrix.

Corollary 1 The number of all pairs of disjoint matrices from $\Sigma_{n^{2}}$ is equal to the number of all pairs of disjoint matrices from $\Pi_{n}$.

Proof. It is easy to see that with respect of the described in Theorem 1 one to one correspondence, every pair of disjoint matrices of $\Sigma_{n^{2}}$ will correspond to a pair of disjoint matrices of $\Pi_{n}$ and conversely every pair of disjoint matrices of $\Pi_{n}$ will correspond to a pair of disjoint matrices of $\Sigma_{n^{2}}$.

Corollary 2 [2] The number of all $n^{2} \times n^{2} S$-permutation matrices is equal to

$$
\begin{equation*}
\left|\Sigma_{n^{2}}\right|=(n!)^{2 n} \tag{4}
\end{equation*}
$$

Proof. It follows immediately from Theorem 1 and formula (3).

## 3 A formula for counting all disjoint pairs of $n^{2} \times$ $n^{2}$ S-permutation matrices

Let $A=\left[a_{i j}\right]_{n \times n} \in \mathfrak{B}_{n}$. We define the following numerical characteristics of the binary matrix A:
$r_{k}(A)$ - the number of rows in $A$ having exactly $k$ units, $k=0,1,2, \ldots, n$;
$c_{k}(A)$ - the number of columns in $A$ having exactly $k$ units, $k=0,1,2, \ldots, n$;
$\psi_{k}(A)=r_{k}(A)+c_{k}(A), k=0,1,2, \ldots, n ;$
$\varepsilon(A)$ - the number of units in $A$.
Let $A, B \in \mathfrak{B}_{n}$. We will say that $A \sim B$, if $B$ is obtained from $A$ after dislocation of some of the rows of $A$. Obviously, the relation defined like that is an equivalence relation. The factor-set $\mathfrak{B}_{n / \sim}$, i.e. the set of equivalence classes on the above defined relation we denote with $\overline{\mathfrak{B}}_{n}$. If $A \in \mathfrak{B}_{n}$, then with $\bar{A}$ we will denote the set $\bar{A}=\left\{\underline{B} \in \mathfrak{B}_{n} \mid B \sim A\right\}$. Thus $|\bar{A}|=\left|\left\{B \in \mathfrak{B}_{n} \mid B \sim A\right\}\right|$ is the cardinality of the set $\bar{A}$. By definition $\overline{\mathfrak{B}}_{n, k}=\mathfrak{B}_{n, k / \sim}$

Obviously if $A, B \in \mathfrak{B}_{n}$ and $A \sim B$, then $r_{k}(A)=r_{k}(B), c_{k}(A)=c_{k}(B)$, $\psi_{k}(A)=\psi_{k}(B), \varepsilon_{k}(A)=\varepsilon_{k}(B), k=0,1,2, \ldots, n$. So in a natural way we can define the functions $r_{k}, c_{k}, \psi_{k}$ and $\varepsilon$ in the factor-set $\overline{\mathfrak{B}}_{n}=\mathfrak{B}_{n / \sim}$ as $r_{k}(\bar{A})$, $c_{k}(\bar{A}), \psi_{k}(\bar{A})$ and $\varepsilon(\bar{A})$ will mean respectively $r_{k}(A), c_{k}(A), \psi_{k}(A) \varepsilon(A)$, where $A$ is an arbitrary representative of the set $\bar{A}=\left\{B \in \mathfrak{B}_{n} \mid B \sim A\right\}$.

Lemma 2 Let $\pi \in \Pi_{n}$. Then the number $q(n, k)$ of all matrices $\pi^{\prime} \in \Pi_{n}$ (including $\pi$ ), having at least $k, k=0,1, \ldots, n^{2}$ component-wise equal elements to the matrix $\pi$ is equal to

$$
\begin{equation*}
q(n, k)=\sum_{\bar{A} \in \overline{\mathfrak{B}}_{n, k}}|\bar{A}| \prod_{i=0}^{n-2}[(n-i)!]^{\psi_{i}(\bar{A})} \tag{5}
\end{equation*}
$$

Proof. Let $\pi=\left[p_{i j}\right]_{n \times n}, \pi^{\prime}=\left[p_{i j}^{\prime}\right]_{n \times n} \in \Pi_{n}$ and let $\pi$ and $\pi^{\prime}$ have exactly $k$ component-wise equal elements. Then we uniquely obtain the binary $n \times n$ matrix $A=\left[a_{i j}\right]_{n \times n}$, such that $a_{i j}=1$ if and only if $p_{i j}=p_{i j}^{\prime}, i, j \in[n]$.

Inversely, let $A=\left[a_{i j}\right]_{n \times n} \in \mathfrak{B}_{n}$ and let $\pi=\left[p_{i j}\right]_{n \times n}$ be an arbitrary matrix from $\Pi_{n}$. We search for the number $h(\pi, A)$ of all matrices $\pi^{\prime}=\left[p_{i j}^{\prime}\right]_{n \times n} \in \Pi_{n}$, such that $p_{i j}^{\prime}=p_{i j}$, if $a_{i j}=1$. (It is assumed that there exist $s, t \in[n]$ such that $a_{s t}=0$ and $p_{s t}^{\prime}=p_{s t}$.)

Let us denote with $\gamma_{s}$ the number of 1 in $s$-th row of $A$ and let the $s$-th row of $\pi$ correspond to the permutation $\rho_{s}$ of the elements of $[n], s=1,2, \ldots, n$. Then there exist $\left(n-\gamma_{s}\right)$ ! permutations $\rho^{\prime}$ of the elements of $[n]$, such that if $a_{s t}=1$, then $\rho_{s}(t)=\rho^{\prime}(t), t \in[n]$. Likewise we also prove the respective statement for the columns of $\pi$. Therefore

$$
h(\pi, A)=\prod_{i=0}^{n}[(n-i)!]^{r_{i}(A)} \prod_{i=0}^{n}[(n-i)!]^{c_{i}(A)}=\prod_{i=0}^{n}[(n-i)!]^{\psi_{i}(A)}
$$

From everything said so far it follows that for each $\pi \in \Pi_{n}$ there exist

$$
q(n, k)=\sum_{A \in \mathfrak{B}_{n, k}} \prod_{i=0}^{n}[(n-i)!]^{\psi_{i}(A)}=\sum_{\bar{A} \in \overline{\mathfrak{B}}_{n, k}}|\bar{A}| \prod_{i=0}^{n}[(n-i)!]^{\psi_{i}(\bar{A})}
$$

matrices from $\Pi_{n}$, which have at least $k$ elements that are component-wise equal to the respective elements of $\pi$.

And since $(n-n)!=0!=1$ and $[n-(n-1)]!=1!=1$, then we finally obtain formula (5).

Lemma 3 For every integer $n \geq 2$

$$
q(n, 0)=q(n, 1)=(n!)^{2 n}=\left|\Pi_{n}\right|=\left|\Sigma_{n^{2}}\right| .
$$

Proof. Let $k=0$. Then $\mathfrak{B}_{n, 0}$ contains only the matrix, all elements of which are equal to 0 . So $\left|\mathfrak{B}_{n, 0}\right|=1$ and if $A \in \mathfrak{B}_{n, 0}$ then $|\bar{A}|=1, \psi_{0}(A)=2 n$ and $\psi_{i}(A)=0$ when $i \geq 1$. Therefore $q(n, 0)=\sum_{\bar{A} \in \overline{\mathfrak{B}}_{n, 0}}|\bar{A}| \prod_{i=0}^{n-2}[(n-i)!]^{\psi_{i}(\bar{A})}=$ $1 \cdot[(n-0)!]^{2 n} \prod_{i=1}^{n-2}[(n-i)!]^{0}=(n!)^{2 n}$.

When $k=1$, there are $n^{2}$ matrices $A \in \mathfrak{B}_{n, 1}$. It is easy to see that $|\overline{\mathfrak{B}}|=n$ and for every $\bar{A} \in \overline{\mathfrak{B}}_{n, 1},|\bar{A}|=n, \psi_{0}(\bar{A})=2(n-1), \psi_{1}(\bar{A})=2$ and $\psi_{i}(\bar{A})=0$ for $i>1$. Therefore $q(n, 1)=n^{2}[(n-0)!]^{2 n-2}[(n-1)!]^{2}=(n!)^{2 n-2}(n!)^{2}=(n!)^{2 n}$.

Theorem 4 Let $A \in \Sigma_{n^{2}}$. Then the number $\xi_{n}$ of all matrices $B \in \Sigma_{n^{2}}$ which are disjoint with $A$ does not depend on $A$ and is equal to

$$
\begin{equation*}
\xi_{n}=\sum_{\bar{A} \in \overline{\mathfrak{B}}_{n}, \varepsilon(\bar{A}) \geq 2}(-1)^{\varepsilon(\bar{A})}|\bar{A}| \prod_{i=0}^{n-2}[(n-i)!]^{\psi_{i}(\bar{A})} \tag{6}
\end{equation*}
$$

Proof. Let $n \geq 2$ be an integer. Then applying Theorem11, Lemma2, Lemma 3 and the principle of inclusion and exclusion we obtain that the number $\xi_{n}$ of all matrices $B \in \Sigma_{n^{2}}$ which are disjoint with $A$ is equal to

$$
\begin{array}{rlr}
\xi_{n} & = & \left|\Pi_{n}\right|+\sum_{k=1}^{n^{2}}(-1)^{k} q(n, k) \\
& =\quad(n!)^{2 n}-(n!)^{2 n}+\sum_{k=2}^{n^{2}}(-1)^{k} q(n, k) \\
& = & \sum_{k=2}^{n^{2}}(-1)^{k} q(n, k)
\end{array}
$$

where the function $q(n, k)$ is calculated with the help of formula (5). Thus we obtain the proof to formula (6).

Corollary 3 The cardinality $\eta_{n}$ of the set of all disjoint non-ordered pairs of $n^{2} \times n^{2} S$-permutation matrices is equal to

$$
\begin{equation*}
\eta_{n}=\frac{(n!)^{2 n}}{2} \xi_{n} \tag{7}
\end{equation*}
$$

where $\xi_{n}$ is described using formula 6.
Proof. It follows directly from formula (4) and having in mind that the "disjoint" relation is symmetric and antireflexive.

Corollary 4 The probability $p_{n}$ of two randomly generated $n^{2} \times n^{2} S$-permutation matrices to be disjoint is equal to

$$
\begin{equation*}
p_{n}=\frac{\xi_{n}}{(n!)^{2 n}-1} \tag{8}
\end{equation*}
$$

where $\xi_{n}$ is described using formula (6).

Proof. Applying Corollary 3 and formula (4), we obtain:

$$
p_{n}=\frac{\eta_{n}}{\binom{\left|\Sigma_{n^{2}}\right|}{2}}=\frac{\frac{(n!)^{2 n}}{2} \xi_{n}}{\frac{(n!)^{2 n}\left((n!)^{2 n}-1\right)}{2}}=\frac{\xi_{n}}{(n!)^{2 n}-1}
$$

## 4 An algorithm for counting

There is one to one correspondence between the representation of the integers in decimal and in binary notations. So a square binary $n \times n$ matrix can be represented using ordered $n$-tuple of nonnegative integers, which belong to the closed interval $\left[0,2^{n}-1\right]$. Let the integer $a \in\left[0,2^{n}-1\right]$. Then $a$ is represented uniquely in the form:

$$
a=\sum_{u=0}^{n-1} b_{u}(a) 2^{u}
$$

where $b_{u}(a) \in \mathfrak{B}=\{0,1\}, u=0,1, \ldots, n-1$. We assume that we have implemented an algorithm for calculating the functions $b_{u}(a)$ for every $u=$ $0,1, \ldots, n-1$ and for every $a \in\left[0,2^{n}-1\right]$. For example, in the programming languages $\mathrm{C}++$ and Java, $b_{u}(a)$ can be calculated using the expression

$$
\text { bu }=(\mathrm{a} \&(1 \ll \mathrm{u}))==0 \text { ? } 0: 1
$$

Let $A \in \mathfrak{B}_{n}$. With $\rho(A)$ we will denote the ordered $n$-tuple

$$
\rho(A)=\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle
$$

where $0 \leq x_{i} \leq 2^{n}-1, i=1,2, \ldots n$ and $x_{i}$ is the integer written in binary notation with the help of the $i$-th row of $A$.

We consider the set:

$$
\begin{aligned}
\mathfrak{R}_{n} & =\left\{\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle \mid 0 \leq x_{i} \leq 2^{n}-1, i=1,2, \ldots n\right\} \\
& =\left\{\rho(A) \mid A \in \mathfrak{B}_{n}\right\}
\end{aligned}
$$

Thus we define the mapping $\rho: \mathfrak{B}_{n} \rightarrow \mathfrak{R}_{n}$, which is bijective and therefore $\mathfrak{B}_{n} \cong \mathfrak{R}_{n}$.

If $A \in \mathfrak{B}_{n}$ and $\rho(A)=\alpha \in \mathfrak{R}_{n}$, then by analogy we define the numerical characteristics of the element $\alpha \in \mathfrak{R}_{n}: r_{k}(\alpha)=r_{k}(A), c_{k}(\alpha)=c_{k}(A), \psi_{k}(\alpha)=$ $r_{k}(\alpha)+c_{k}(\alpha)=\psi_{k}(A), k=0,1,2, \ldots, n$ and $\varepsilon(\alpha)=\varepsilon(A)$. We assume $|\alpha|=|\bar{A}|$, where $\bar{A}=\left\{B \in \mathfrak{B}_{n} \mid B \sim A\right\}$.

Let $\alpha=\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle \in \mathfrak{R}_{n}$ and let $s$ be the number of different elements in $\alpha=\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle$. Then the set $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ can be divide into parts

$$
X=X_{1} \cup X_{2} \cup \cdots \cup X_{s}
$$

such that for every $k \in[s]$ and every $i, j \in[n], i \neq j$ the condition $x_{i}, x_{j} \in X_{k}$ is satisfied if and only if $x_{i}=x_{j}$. We assume

$$
z_{i}=\left|X_{i}\right|, \quad i=1,2, \ldots s
$$

It is easily seen that

$$
|\alpha|=\frac{n!}{\prod_{i=1}^{s} z_{i}!}
$$

Let

$$
\bar{\Re}_{n}=\left\{\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle \mid 0 \leq x_{1} \leq x_{2} \leq \cdots \leq x_{n} \leq 2^{n}-1\right\} \subset \Re_{n}
$$

It is easily seen that $\overline{\mathfrak{B}}_{n} \cong \overline{\mathfrak{}}_{n}$, which gives the basis to construct the following algorithm for calculating $\xi_{n}$ :

## Algorithm 5 Calculation of $\xi_{n}$.

## begin

$\xi_{n}:=0$;
For every $\alpha=\left\langle x_{1} x_{2}, \ldots, x_{n}\right\rangle \in \overline{\mathfrak{R}}_{n}$ do
\{

$$
s:=1
$$

$$
\varepsilon(\alpha):=0
$$

$$
\text { For } i=1,2, \ldots, n \text { do }
$$

\{
$z_{s}:=z_{s}+1 ;$
$t=0$;
For $u=0,1, \ldots, n-1 d o$
\{

$$
t:=t+b_{u}\left(x_{i}\right) ;
$$

\}
$r_{t}(\alpha):=r_{t}(\alpha)+1 ;$
$\varepsilon(\alpha):=\varepsilon(\alpha)+t ;$
If $i<n$ and $x_{i}<x_{i+1}$ then $s:=s+1$;
\}
If $\varepsilon(\alpha)=0$ or $\varepsilon(\alpha)=1$ then go to next $\alpha$;
For $u=0,1, \ldots, n-1$ do
\{
$t:=0$;
For $i=1,2, \ldots, n$ do
\{

$$
t=t+b_{u}\left(x_{i}\right) ;
$$

        \}
        \(c_{t}(\alpha):=c_{t}(\alpha)+1 ;\)
    \}
    For \(k=0,1, \ldots, n\) do
    \{
        \(\psi_{k}(\alpha):=r_{k}(\alpha)+c_{k}(\alpha) ;\)
        \}
        \(|\alpha|:=\frac{n!}{\prod_{i=1}^{s} z_{i}!} ;\)
        \(T(\alpha):=(-1)^{\varepsilon(\alpha)}|\alpha| \prod_{i=0}^{n-2}[(n-i)!]^{\psi_{i}(\alpha)} ;\)
    ```
    \(\xi_{n}:=\xi_{n}+T(\alpha) ;\)
    \}
end.
```


## 5 Conclusion

On the basis of algorithm 5 with programming language Java, we made a computer program for calculating $\xi_{n}, \eta_{n}$ and $p_{n}$ and we received the following results:

$$
\begin{gathered}
\xi_{2}=7 \\
\xi_{3}=17972 \\
\xi_{4}=41685061617 \\
\xi_{5}=232152032603580176504 \\
\xi_{6}=7236273578711450275537707547657855 \\
\eta_{2}=56 \\
\eta_{3}=419250816 \\
\eta_{4}=2294248126968596791296 \\
\eta_{5}=71871209790288983974921874964480000000000 \\
\eta_{6}=7022228210556132949916635069726824032981704989720182784 \cdot 10^{13}
\end{gathered}
$$

$$
\begin{gathered}
p_{2}=0.4666666666666667 \\
p_{3}=0.38521058836137606 \\
p_{4}=0.3786958223051558 \\
p_{5}=0.37493849344703684 \\
p_{6}=0.3728421644517476
\end{gathered}
$$

For $\mathrm{n}=2$ and $\mathrm{n}=3$, the results that we get here coincide with the calculations made by hand in [6], where we used a graph theory approach.

## References

[1] R.A. Bailey, P.J. Cameron, and R. Connelly. Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and hamming codes. Amer. Math. Monthly, (115):383-404, 2008.
[2] Geir Dahl. Permutation matrices related to sudoku. Linear Algebra and its Applications, 430(8-9):2457-2463, 2009.
[3] Bertram Felgenhauer and Frazer Jarvis. Enumerating possible sudoku grids, 2005. http://www.afjarvis.staff.shef.ac.uk/sudoku/sudoku.pdf.
[4] Roberto Fontana. Fractions of permutations. an application to sudoku. Journal of Statistical Planning and Inference, 141(12):3697-3704, 2011.
[5] Krasimir Yordzhev. Random permutations, random sudoku matrices and randomized algorithms. International J. of Math. Sci. \& Engg. Appls., 6(VI):291-302, 2012.
[6] Krasimir Yordzhev. Calculation of the number of all pairs of disjoint spermutation matrices. Applied Mathematics and Computation, 268:1-11, 2015.

