

On the genus of the complete tripartite graph $K_{n,n,1}$

Valentas Kurauskas^{*†}

2016-09-09

Abstract

For even n we prove that the genus of the complete tripartite graph $K_{n,n,1}$ is $\lceil (n-1)(n-2)/4 \rceil$. This is the least number of bridges needed to build a complete n -way road interchange where changing lanes is not allowed. Both the theoretical result, and the surprising link to modelling road intersections are new.

1 Introduction

We start with some basics of topological graph theory, for details and definitions that are not given here, see [9, 14]. Let $V(G)$ denote the set of vertices and $E(G)$ the set of edges of a graph G . The *genus* of G , denoted $g(G)$, is the minimum number g such that G can be embedded on the orientable surface \mathbb{S}_g , the sphere with g handles. If G has genus g , there is a *2-cell (cellular) embedding* of G into \mathbb{S}_g , that is, an embedding where the interior of each region (face) is homeomorphic to an open disk. A *rotation system* of G is a set $\{\pi_v : v \in V(G)\}$ where π_v is a cyclic permutation of edges incident to v , called a *rotation* at v . There is a well-known correspondence between orientable 2-cell embeddings and rotation systems: the rotation π_v corresponds to the clockwise ordering of edges emanating from v in the embedding.

Ringel [15] showed that for any positive integers n and m the genus of the complete bipartite graph $K_{n,m}$ is $L(n, m)$, where $L(n, m) = \left\lceil \frac{(n-2)(m-2)}{4} \right\rceil$.

^{*}Vilnius University, Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania.

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An alternative proof was given by Bouchet [1]. The inequality $g(K_{n,m}) \geq L(n, m)$ follows easily by Euler's formula $2 - 2g = v + f - e$, which holds for any 2-cell embedding of a graph with v vertices, e edges and f faces into \mathbb{S}_g [9].

White [17] conjectured in 1965 that the genus of a complete tripartite graph $K_{n,r,s}$ with $n \geq r \geq s$, satisfies

$$g(K_{n,r,s}) = L(n, r + s) = \left\lceil \frac{(n-2)(r+s-2)}{4} \right\rceil. \quad (1)$$

Since $K_{n,r+s}$ is a subgraph of $K_{n,r,s}$, we know that $g(K_{n,r,s})$ must be at least $L(n, r + s)$, see, for example, [16]. The challenge is to construct embeddings of such genus.

White's conjecture has been confirmed for complete tripartite graphs with even part sizes [10], the graphs $K_{n,r,r}$, $n \geq r \geq 2$ and several other classes, see [2, 12, 16]. However, it remains open in general. A corresponding conjecture for non-orientable embeddings has been settled for all complete tripartite graphs by Ellingham, Stephens and Zha [7].

We prove

Theorem 1.1 *For any even positive integer n there is a 2-cell embedding of $K_{n,n}$ into $\mathbb{S}_{\lceil (n-1)(n-2)/4 \rceil}$ which has a face bounded by a Hamiltonian cycle.*

This confirms (1) for even n and r with $n = r$ and $s = 1$. Let $L(n) = \lceil (n-1)(n-2)/4 \rceil$. For $n \geq r$ and r even we can use the *diamond sum* operation [1, 12] to see that $g(K_{n,r,1}) \leq L(n, r + 1) + 1$ and (1) holds if either $r \bmod 4 = 2$ or both $r \bmod 4 = 0$ and $n \bmod 4 \in \{0, 1\}$. The same technique easily yields $g(K_{n,n,1}) \leq L(n) + 1$ for any odd n .

To our knowledge, prior to this work, $g(K_{n,n,1}) = L(n)$ and, implicitly, Theorem 1.1, has been proved only for an infinite sequence of n of the form $3^q(2^p + \frac{1}{2}) + \frac{1}{2}$ where $q \geq 0$ and $p \geq 3$ are integers. This follows from the embeddings of K_{n+1} where all faces are bounded by Hamiltonian cycles, constructed by Ellingham and his co-authors [5, 6].

The authors in [7] claimed a proof of (1) for a very general family of graphs $K_{n,r,s}$, including the cases studied in this paper. However, no proofs have appeared since the publication of [7] in 2006. Our idea is simple, original and does not involve the traditional voltage graph [5, 9] or transition graph constructions [6, 7].

We conjecture that Theorem 1.1 also holds for any odd integer $n \geq 3$. The main result and the application described below leads to a more general question: when, among all minimum genus embeddings of a graph G there is one with a Hamiltonian cycle bounding a face?

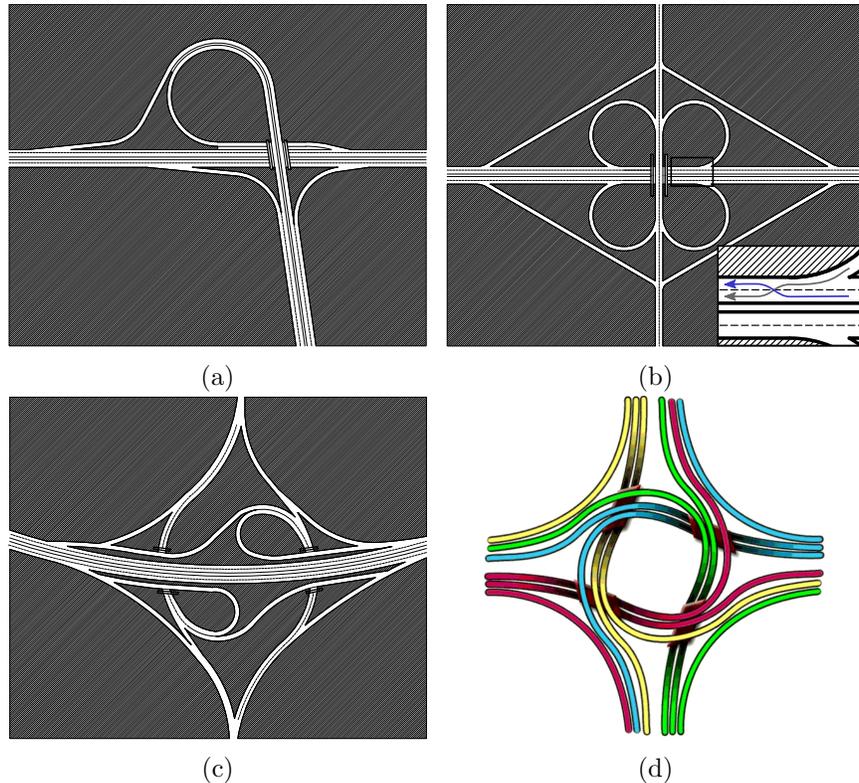


Figure 1: Some junction types: (a) trumpet, (b) cloverleaf (an example of traffic weaving in the magnified rectangle), (c) double trumpet, (d) Pinavia [11]. Images (a)-(c) from Davies and Jokiniemi [3] used with authors' permission; image (d) from www.pinavia.com used with authors' permission.

2 Road interchanges

Our work is motivated by a beautiful road junction optimisation problem, which, to our knowledge, is being described here for the first time.

Many types of road interchanges are known by engineers and built in practice, see, for example, Chapter 7 of [8]. A popular design is the 4-way *cloverleaf interchange*. A 3-way example is the *trumpet interchange*, see Figure 1. To drive through certain junctions, some vehicles must cross each other's path and change their lanes. For example, drivers approaching a cloverleaf from the south and going west, need a maneuver similar to the one depicted in a corner of Figure 1b (assuming right-hand traffic). In

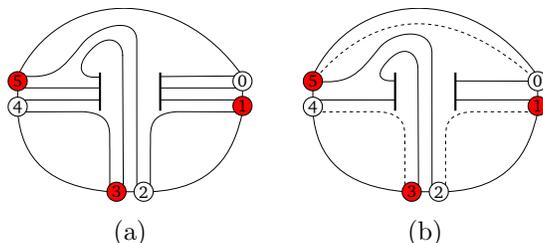


Figure 2: (a) Representation of the trumpet interchange by a multigraph embedded into a genus 1 surface; $H = (0, 1, 2, 3, 4, 5)$. (b) A different complete interchange, an embedding of $K_{3,3}$, shown in solid lines. Lanes connecting every pair of motorways are obtained by duplicating appropriate edges of H (dashed lines).

engineering, a situation when a “vehicle first merges into a stream of traffic, obliquely crosses that stream, and then merges into a second stream moving in the same direction” is called *traffic weaving* [8].

Traffic weaving, which is generally undesirable, can be avoided in the trumpet, the *all-directional four leg* [8], also called *four-level stack*, and, for example, recently invented *Pinavia* interchanges (Figure 1d). In these interchanges the lanes can be completely separated so that the exit motorway and lane of a vehicle are determined by the lane it enters the junction (lane changes inside the junction are not necessary or not allowed). We call such interchanges *weaving-free*.

Let $n \geq 2$ be an integer. Based on an idea by Rimvydas Krasauskas (personal communication) and Mikhail Skopenkov, see also [13], we propose to model a *weaving-free n -way interchange* by a quadruple $(G, H, \mathcal{M}, \mathbb{S})$. Here G is a bipartite multigraph with n white and n black vertices as its parts, H is a directed Hamiltonian cycle on which the vertex colours alternate, \mathbb{S} is a closed connected orientable surface and \mathcal{M} is an embedding of G into \mathbb{S} such that H bounds a *face* (a region homeomorphic to an open disc). The i -th motorway is represented by one white vertex a_i (the incoming direction) and one black vertex b_i (the outgoing direction). The cycle H corresponds to the order the motorways enter and leave the junction. In particular, if traffic is right-hand and the clockwise order in which the motorways join the junction is $(1, \dots, n)$, then $H = (a_1, b_1, \dots, a_n, b_n)$. Finally, the connections between the ingoing and outgoing lanes are represented by the remaining edges $uv \in E(G)$ such that $uv \notin E(H)$. For example, the trumpet interchange corresponds to the embedding shown in Figure 2a.

The *number of bridges* in an interchange $I = (G, H, \mathcal{M}, \mathbb{S})$ is defined to be the genus of \mathbb{S} . We call I *complete* if for each white vertex u and each black vertex v there is an edge $uv \in E(G)$. If I is complete we can iteratively remove repeated edges from G which do not lie on H to obtain a complete interchange $I' = (G', H, \mathcal{M}, \mathbb{S})$ on the same surface with G' a complete bipartite graph $K_{n,n}$, see Figure 2b. Similarly we can insert new lanes connecting arbitrary pairs of motorways, for example, by duplicating edges of G without changing \mathbb{S} . Therefore we focus on complete interchanges with G isomorphic to $K_{n,n}$ below.

Krasauskas was interested in the minimum number of bridges of a complete n -way weaving-free interchange, for a given $n \geq 2$. We call the interchanges that achieve this minimum *optimal* for n . By the next lemma, Theorem 1.1 provides a solution for all even n .

Lemma 2.1 *Let G be a complete bipartite graph $K_{n,n}$. An interchange $(G, H, \mathcal{M}, \mathbb{S})$ is optimal for n if and only if \mathcal{M} minimizes the genus over 2-cell embeddings of G which have a face bounded by a Hamiltonian cycle.*

Proof Suppose I is an optimal interchange for n . Let $I = (G, H, \mathcal{M}, \mathbb{S})$. It suffices to prove that \mathcal{M} is 2-cell. Suppose the contrary.

Youngs, see paragraph 3.5 of [18], showed, that in this case the surface \mathbb{S} can be transformed into a surface \mathbb{S}' , such that \mathcal{M} is a 2-cell embedding, when viewed as an embedding from G to \mathbb{S}' , all 2-cell faces of \mathcal{M} are preserved (in particular, the face bounded by H) and the genus of \mathbb{S}' is smaller than the genus of \mathbb{S} . This is a contradiction to the optimality of I . \square

3 Solutions for small n

Given a graph G with a rotation system $\{\pi_v : v \in V(G)\}$, the boundary circuit of each face in the corresponding 2-cell embedding can be obtained by a simple *face-tracing algorithm* [9]. This algorithm repeats the following procedure until all edges are traversed (once in both directions). Start with an arbitrary unvisited edge $e_0 = (v_0, v_1)$. Then for $i \geq 1$ if $e_i = (v_{i-1}, v_i)$ define $e_{i+1} = \pi_{v_i}(e_i) = (v_i, v_{i+1})$, that is, at each vertex make a clockwise turn. Repeat this until $e_{t+1} = e_0$ for some $t > 0$, which closes a directed walk (v_0, v_1, \dots, v_t) .

Once we know the number of faces, Euler's formula, yields the genus of the surface. Thus, we can find optimal interchanges for small n by running the above algorithm for all rotation systems of $K_{n,n}$ with a face bounded by a fixed Hamiltonian cycle. The number of such rotation systems, $(n-2)!^{2n}$,

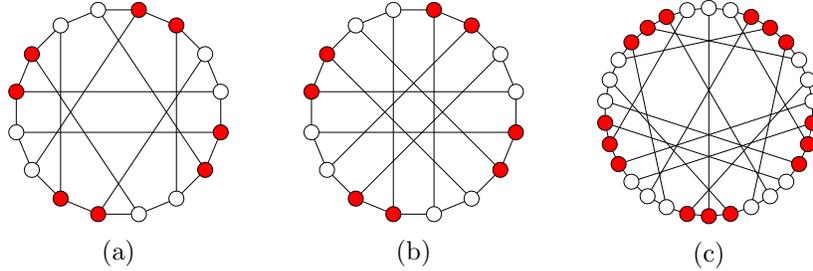


Figure 3: The unique (up to isomorphism) optimal interchanges for $n = 4$ and $n = 5$ expanded to Hamiltonian cubic graphs. To get the rotation system for $K_{n,n}$, contract same colour paths on the outer circle.

grows with n fast, the embeddings with minimal genus are very rare and exhaustive search is feasible only for $n \leq 5$. Using two different methods we verified Theorem 1.1 for all $n \in \{2, \dots, 11\}$. The first method was a randomized version of [1], see also [12, 14]. The second one was restricting the exhaustive search to certain symmetry patterns; this yielded more symmetric solutions, especially for even n . One of the two solutions for $n = 4$, see Figure 3a, is known to engineers as the *double trumpet* interchange. It has been built in practice and has $L(4) = 2$ bridges (Figure 1c). The other solution, see Figure 3b, has the same rotation system as the four-level stack interchange. However, the construction used in practice see, for example, [8], corresponds to a non-cellular embedding \mathcal{M} and a surface \mathbb{S} of genus larger than two.

4 Proofs

In the proof we work only with 2-cell embeddings of simple graphs. Therefore we use terms *embedding* and *rotation system* synonymously and we use *face* to refer to *boundary circuit*. The genus of an embedding is the genus of the corresponding surface. Also, we represent a rotation π_v as a cyclic permutation of vertices (rather than edges) incident to v and use the notation $v : \pi_v$. For example, the rotation $v : xyz$ or $\pi_v = (x, y, z)$ represents 3 directed edges (arcs) emanating from v in this clockwise order: $((v, x), (v, y), (v, z))$.

We will need the next simple result.

Lemma 4.1 (Ringel [15]) *Let n and m be even. The following rotation*

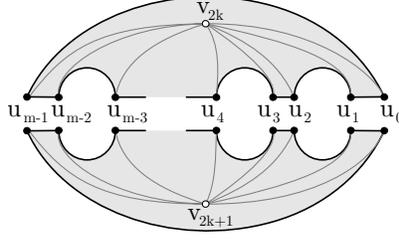


Figure 4: The surface of Lemma 4.1 is a “book” with arched leaves.

system gives a minimum genus embedding of $K_{n,m}$ with parts $\{u_s : s \in \mathbb{Z}_m\}$ and $\{v_t : t \in \mathbb{Z}_n\}$:

$$\begin{aligned} u_{2j} &: v_{n-1}v_{n-2} \dots v_0; & u_{2j+1} &: v_0v_1 \dots v_{n-1}; & j &= 0, \dots, m/2 - 1; \\ v_{2k} &: u_0u_1 \dots u_{m-1}; & v_{2k+1} &: u_{m-1}u_{m-2} \dots u_0; & k &= 0, \dots, n/2 - 1. \end{aligned}$$

Proof The faces of the proposed embedding are

$$\{v_t u_{s+1} v_{t+1} u_s : s \text{ even}, t \text{ even}\} \cup \{u_s v_{t+1} u_{s+1} v_t : s \text{ odd}, t \text{ odd}\}.$$

This is a minimum genus embedding of $K_{n,m}$ because all faces are of length 4, which is minimum possible for a bipartite graph. Indeed, there are $v = 2n$ vertices, $e = nm$ edges and $f = \frac{nm}{2}$ faces in total. By Euler’s formula the genus g satisfies $2 - 2g = v + f - e$, or $g = (n - 2)(m - 2)/4 = L(n, m)$. \square

The corresponding surface can be visualised as a book with $\frac{n}{2}$ thick leaves, where each leaf has $\frac{m}{2} - 1$ arcs, see Figure 4. For $k \in \{0, \dots, \frac{n}{2} - 1\}$, glue faces $2k$ and $2k + 1$ (bounded by the black edges in the figure) along the alternating edges $u_{m-1}u_{m-2}, u_{m-3}u_{m-4}, \dots, u_1u_0$. Also glue faces $2k$ and $(2k - 1) \bmod (2n)$ along the remaining edges $u_{m-2}u_{m-3}, u_{m-4}u_{m-5}, \dots, u_2u_1$, forming a “leaf”. Each of the n faces is bounded by a Hamiltonian cycle. Now, as in Bouchet [1], the embedding of Lemma 4.1 results by placing a vertex v_j inside each face j and connecting it to each vertex u_i (grey edges in the picture).

Note that for $m = n$ the embedding of Lemma 4.1 contains a family

$$\mathcal{F}(n) = \{v_s u_{s+1} v_{s+1} u_s : s \text{ even}\} \quad (2)$$

of $n/2$ disjoint faces that covers all of the vertices.

$$\mathcal{F}'(n) = \{u_s v_{s+1} u_{s+1} v_s : s \text{ odd}\} \quad (3)$$

is another such family. The union of $\mathcal{F}(n)$ and $\mathcal{F}'(n)$ makes up a cylindrical band of rectangles glued along their opposite sides, see Figure 5a. Below, given an embedding of $K_{n,n}$ as in Lemma 4.1, we call faces in $\mathcal{F}(n)$ *special*.

We prove Theorem 1.1 by combining 4 minimum genus embeddings of complete bipartite graphs with equal part sizes. Our key insight comes after a careful analysis of symmetric optimal n -way interchanges for $n = 4$ and $n = 8$ generated by computer.

Proof of Theorem 1.1. We aim to construct an embedding of a complete bipartite graph with parts $\{0, 2, \dots, 2n - 2\}$ and $\{1, 3, \dots, 2n - 1\}$, such that one of the faces is the Hamiltonian cycle $(0, 1, \dots, 2n - 1)$. We view the vertex set of the resulting graph as \mathbb{Z}_{2n} and perform arithmetics in the proof modulo $2n$.

The case $n \equiv 0 \pmod{4}$. We start by partitioning the edges of H into four parts P_i , $i \in \{0, 1, 2, 3\}$, where

$$P_i = \{(t - 1, t) : t \equiv i \pmod{4}, t \in \mathbb{Z}_{2n}\}$$

For each $i \in \{0, 1, 2, 3\}$ the pairs in P_i cover $n/2$ even and $n/2$ odd vertices. We denote these sets by $U_i = \{u : uv \in P_i\}$ and $V_i = \{v : uv \in P_i\}$.

For each i we apply Lemma 4.1 to get an embedding \mathcal{R}_i of G_i , where G_i is $K_{n/2, n/2}$ with parts U_i and V_i . To apply the lemma, it suffices to choose a permutation $\bar{u}_i = u_{i,0}u_{i,1} \dots u_{i, \frac{n}{2}-1}$ of U_i as $u_0u_1 \dots u_{\frac{n}{2}-1}$ and a permutation $\bar{v}_i = v_{i,0}v_{i,1} \dots v_{i, \frac{n}{2}-1}$ of V_i as $v_0v_1 \dots v_{\frac{n}{2}-1}$. We define these permutations by setting for $i \in \{0, 1, 2, 3\}$ and $k \in \{0, \dots, \frac{n}{4} - 1\}$

$$(v_{i,2k}, u_{i,2k+1}, v_{i,2k+1}, u_{i,2k}) = C_{i,k}. \quad (4)$$

Here for $k \neq 0$

$$C_{i,k} = \begin{cases} (4k - 1, 4k, 2n - 4k - 1, 2n - 4k) & i = 0; \\ (4k, 4k + 1, 2n - 4k, 2n - 4k + 1) & i = 1; \\ (4k + 1, 4k + 2, 2n - 4k + 1, 2n - 4k + 2) & i = 2; \\ (4k + 2, 4k + 3, 2n - 4k - 2, 2n - 4k - 1) & i = 3; \end{cases}$$

and for $k = 0$

$$C_{i,k} = \begin{cases} (i - 1, i, n + i - 1, n + i) & i \in \{0, 1, 2\}; \\ (2, 3, 2n - 2, 2n - 1) & i = 3. \end{cases}$$

Let $\mathcal{F}_i = \{C_{i,k} : k \in \{0, \dots, \frac{n}{4} - 1\}\}$. Note that \mathcal{F}_i is the family $\mathcal{F}(n/2)$ of special faces of \mathcal{R}_i defined in (2). Importantly, each special face of \mathcal{F}_i is

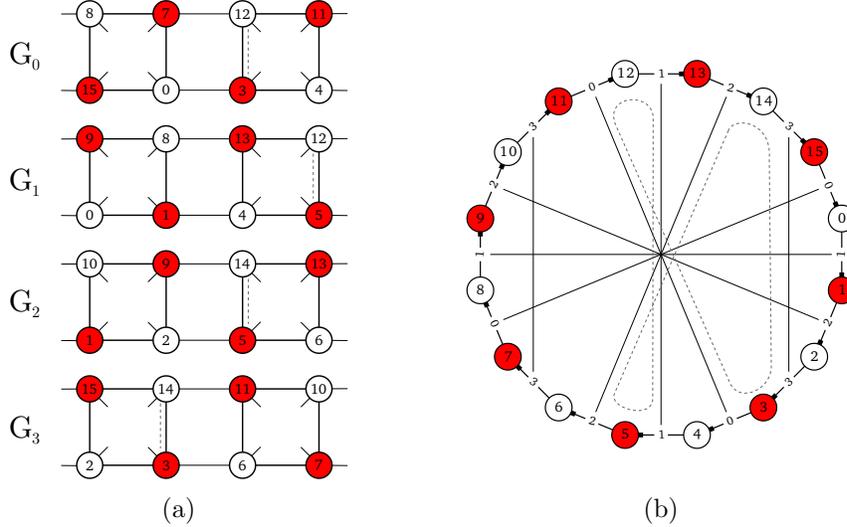


Figure 5: The case $n = 8$. (a) In \mathcal{R}_i , the arcs of P_i are paired to make up the $\frac{n}{4}$ disjoint special faces. The marks in their corners show where the rotations are cut in the concatenation step. (b) The matchings of P_i , $i \in \{0, 1, 2, 3\}$. The arc label i indicates the subgraph G_i . The dashed cycle shows one of the new faces, $F_{1,1} = (3, 14, 5, 12)$.

made of exactly two arcs in P_i . The family \mathcal{F}_i can alternatively be seen as a matching of the elements of P_i .

Let us now describe how $\mathcal{R}_0, \dots, \mathcal{R}_3$ are combined into an embedding of $K_{n,n}$. By the definition of G_i , any vertex $v \in \mathbb{Z}_{2n}$ belongs to exactly two of the four graphs, $G_{v \bmod 4}$ and $G_{(v+1) \bmod 4}$. Let $y_v = (y_{v,0} y_{v,1} \dots y_{v, \frac{n}{2}-1})$ and $z_v = (z_{v,0} z_{v,1} \dots z_{v, \frac{n}{2}-1})$ be the rotations at v in their respective embeddings. Without loss of generality we can assume that $y_{v, \frac{n}{2}-1} = v-1$ and $z_{v,0} = v+1$. Note that the sets of elements of y_v and z_v are disjoint and partition the odd (even) vertices of \mathbb{Z}_{2n} if v is even (odd).

The rotation system $\mathcal{R} = \{\pi_v : v \in \mathbb{Z}_{2n}\}$ where

$$\pi_v = (y_{v,0}, \dots, y_{v, \frac{n}{2}-1}, z_{v,0}, \dots, z_{v, \frac{n}{2}-1}) = (\dots, v-1, v+1, \dots) \quad (5)$$

defines an embedding of $K_{n,n}$. Notice the special role of vertices $y_{v,0}$, $y_{v, \frac{n}{2}-1} = v-1$, $z_{v,0} = v+1$ and $z_{v, \frac{n}{2}-1}$: the embedding $\mathcal{R}_{v \bmod 4}$ contains a special face $(v-1, v, y_{v,0}, y_{v,0}+1)$. Similarly $\mathcal{R}_{(v+1) \bmod 4}$ contains a special face $(v, v+1, z_{v, \frac{n}{2}-1}-1, z_{v, \frac{n}{2}-1})$.

What are the faces of \mathcal{R} ? Note that if \mathcal{R}_i has a rotation $a : (\dots, b, c, \dots)$ where (b, a) and (a, c) are not both directed edges of a special face, then \mathcal{R}

also has a rotation $a : (\dots, b, c, \dots)$. Thus, if (a, b, c, d) is a non-special face of \mathcal{R}_i then it is also a face of \mathcal{R} . It follows by (5) that the new faces of \mathcal{R} are formed only from the directed edges of the cycles in $\cup_i \mathcal{F}_i$, see Figure 5a.

The right side of (5) implies that H is one of the new faces. Also, by (4) and (5) we get that for each $k \in \{1, \dots, \frac{n}{4} - 1\}$ the embedding \mathcal{R} contains a face

$$F_{1,k} = (4k - 1, 2n - 4k + 2, 4k + 1, 2n - 4k)$$

and a face

$$F_{2,k} = (4k, 2n - 4k - 1, 4k + 2, 2n - 4k + 1).$$

Finally, the arcs of $C_{i,0}$ for $i \in \{0, 1, 2\}$ not lying on H , together with the unused edges $(2n - 1, 2)$ and $(n - 1, n + 2)$ of $C_{3,0}$ and $C_{3, \frac{n}{4} - 1}$ respectively yield a cycle $(0, n - 1, n + 2, 1, n, 2n - 1, 2, n + 1)$, which we denote C_8 .

Write $\mathcal{F} = \{F_{1,k} : k \in \{1, \dots, \frac{n}{4} - 1\}\} \cup \{F_{2,k} : k \in \{1, \dots, \frac{n}{4} - 1\}\}$ and note that the arcs of the cycles in $\{H, C_8\} \cup \mathcal{F}$ cover all edges of $\cup_i \mathcal{F}_i$. Thus, the lengths of face boundaries of \mathcal{R} are $2n, 8, 4, 4, \dots, 4$, the number of faces is $2 + \frac{2n^2 - 2n - 8}{4} = \frac{n(n-1)}{2}$ and by Euler's formula, the genus is $\frac{n^2 - 3n + 4}{4} = L(n)$.

Interestingly, all the faces in $\{H, C_8\} \cup \mathcal{F}$ can be obtained by a face-tracing algorithm on a graph formed by placing H on a circle and connecting midpoints of the the elements of P_i that are matched (i.e. lying on the same special face in \mathcal{F}_i) by a chord, see Figure 5b.

The case $n \equiv 2 \pmod{4}$. For $i \in \{0, 1, 2, 3\}$ set

$$\begin{aligned} \tilde{P}_i = & \{(t - 1, t) : t \equiv i \pmod{4}, t \in \{0, \dots, n - 1\}\} \\ & \cup \{(t - 1, t) : (t - n) \equiv i \pmod{4}, t \in \{n, \dots, 2n - 1\}\}. \end{aligned}$$

Let $P_i = \tilde{P}_i$ for $i \in \{1, 2\}$ but $P_0 = \tilde{P}_0 \setminus \{(2n - 1, 0), (n - 1, n)\}$ and $P_3 = \tilde{P}_3 \cup \{(2n - 1, 0), (n - 1, n)\}$. Define $U_i = \{u : uv \in P_i\}$ and $V_i = \{v : uv \in P_i\}$ for $i \in \{0, 1, 2\}$, but $U_3 = \{u : uv \in \tilde{P}_3\} \cup \{0, n\}$ and $V_3 = \{v : uv \in \tilde{P}_3\} \cup \{n - 1, 2n - 1\}$. Note that $|U_i| = |V_i| = n_i$ is even: $n_i = \frac{n-2}{2}$ for $i \in \{0, 2\}$ and $n_i = \frac{n+2}{2}$ for $i \in \{1, 3\}$.

We again use Lemma 4.1 for each $i \in \{0, 1, 2, 3\}$ to construct an embedding \mathcal{R}_i of a complete bipartite graph G_i with parts (U_i, V_i) , with the property that the arcs in P_i lie on the special faces of \mathcal{R}_i . Specifically, we define the permutations $\bar{u}_i = u_{i,0}u_{i,1} \dots u_{i,n_i-1}$ and $\bar{v}_i = v_{i,0}v_{i,1} \dots v_{i,n_i-1}$ (and, implicitly, the matchings of P_i) by setting the following cycles as special faces of the embedding \mathcal{R}_i , $i \in \{0, 1, 2, 3\}$:

$$(v_{i,2k}, u_{i,2k+1}, v_{i,2k+1}, u_{i,2k}) = \begin{cases} C_{i,k}, & k \in S_i; \\ (n, 2n - 1, 0, n - 1), & i = 3, k = \frac{n-2}{4}. \end{cases}$$

Here $u_{i,n_i} = u_{i,0}$, $v_{i,n_i} = v_{i,0}$ and $C_{i,k}$, S_i are as follows (see also Figure 6)

i	$C_{i,k}$	S_i
0	$(4k+3, 4k+4, 2n-4k-3, 2n-4k-2)$	$0, \dots, \frac{n-2}{4} - 1$
1	$(4k, 4k+1, 2n-4k-2, 2n-4k-1)$	$0, \dots, \frac{n-2}{4}$
2	$(4k+1, 4k+2, 2n-4k-5, 2n-4k-4)$	$0, \dots, \frac{n-2}{4} - 1$
3	$(4k+2, 4k+3, 2n-4k-4, 2n-4k-3)$	$0, \dots, \frac{n-2}{4} - 1$

We will now combine $\mathcal{R}_0, \dots, \mathcal{R}_3$ similarly as in the case $n \equiv 0 \pmod{4}$, but with one important difference: before concatenating rotations, we glue \mathcal{R}_1 and \mathcal{R}_3 along the face on 4 shared vertices.

The gluing operation is carried out as follows. Let $\mathcal{F}_i, \mathcal{F}'_i$ be the sets of faces $\mathcal{F}(n_i), \mathcal{F}'(n_i)$ defined in (2) and (3) respectively for \mathcal{R}_i . Note that \mathcal{F}'_1 contains a face

$$F' = (u_{1, \frac{n}{2}}, v_{1,0}, u_{1,0}, v_{1, \frac{n}{2}}) = (n-1, 0, 2n-1, n),$$

and \mathcal{F}_3 contains a face

$$F = (v_{3, \frac{n-2}{2}}, u_{3, \frac{n}{2}}, v_{3, \frac{n}{2}}, u_{3, \frac{n-2}{2}}) = (n, 2n-1, 0, n-1).$$

Let $S = \{0, n-1, n, 2n-1\}$. We have $V(G_1) \cap V(G_3) = S$.

For $i \in \{0, 1, 2, 3\}$ and $v \in V(G_i)$ let $p_v^i = (p_{v,0}^i, \dots, p_{v, n_i-1}^i)$ be the rotation at v in \mathcal{R}_i . Choose the indices so that $p_{v,0}^i$ and p_{v, n_i-1}^i are the neighbours of v on the special face F_i containing v . Also for $v \in S$ define permutations $q_v^1 = (q_{v,0}^1, \dots, q_{v, n_1-1}^1)$, such that q_v^1 is p_v^1 with $q_{v,0}^1$ and q_{v, n_1-1}^1 the neighbours of v on the face F' .

Let G_{13} be a graph with vertex set $V(G_1) \cup V(G_3)$ and edge set $E(G_1) \cup E(G_3)$. This graph is bipartite, with parts U_{13} and V_{13} , where

$$U_{13} = \{0, 2, \dots, 2n-2\}, \quad V_{13} = \{1, 3, \dots, 2n-1\}.$$

Since F and F' have opposite directions, $q_{v,0}^1 = p_{v, n_3-1}^3$ and $q_{v, n_1-1}^1 = p_{v,0}^3$ for $v \in S$. Let the embedding of G_{13} be $\mathcal{R}_{13} = \{p_v^{13} : v \in V(G_{13})\}$ where

$$p_v^{13} = \begin{cases} (q_{v,0}^1, q_{v,1}^1, \dots, q_{v, n_1-1}^1, p_{v,1}^3, p_{v,2}^3, \dots, p_{v, n_3-2}^3), & v \in S; \\ p_v^1, & v \in V(G_1) \setminus S; \\ p_v^3, & v \in V(G_3) \setminus S. \end{cases}$$

The faces of \mathcal{R}_{13} are the union of faces of \mathcal{R}_1 and \mathcal{R}_3 except $\{F, F'\}$; in particular they are all quadrangular. Call the faces in the set $\mathcal{F}_{13} = \mathcal{F}_1 \cup$

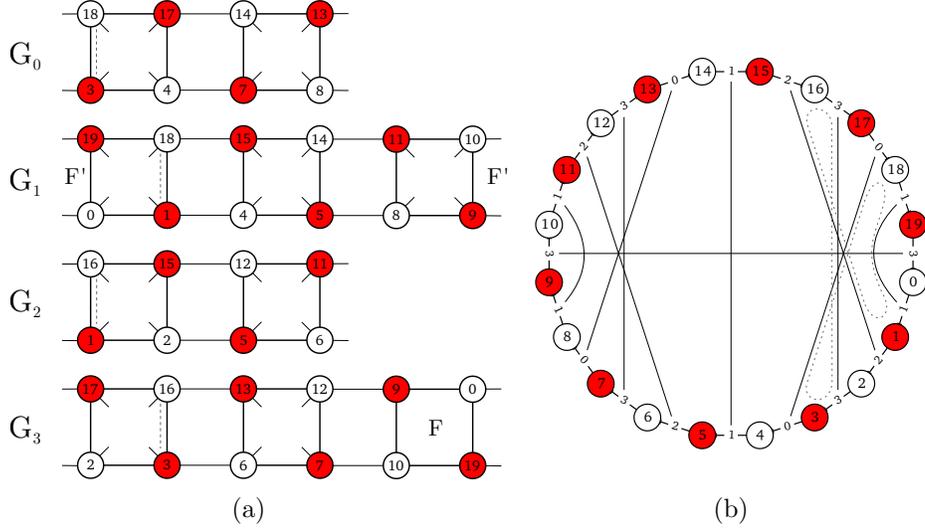


Figure 6: The case $n = 10$. (a) The embeddings of G_i (only faces in $\mathcal{F}_i \cup \mathcal{F}'_i$ shown). (b) The matchings of arcs in P_i , $i = 0, 1, 2, 3$. The dashed edges show the face $F_{1,0} = (1, 18, 3, 16)$.

($\mathcal{F}_3 \setminus \{F\}$), the *special faces* of \mathcal{R}_{13} . \mathcal{F}_{13} consists of $\frac{n-2}{2} + 1$ vertex-disjoint faces and covers $V(G_{13}) = \mathbb{Z}_{2n}$. Note that for $v \in S$, p_v^{13} is a permutation of V_{13} (U_{13}) if v is even (odd).

Now the sets $V(G_0)$ and $V(G_2)$ partition $\mathbb{Z}_{2n} \setminus S$ into two subsets, each containing $\frac{n-2}{2}$ even and $\frac{n-2}{2}$ odd vertices. For $v \in \mathbb{Z}_{2n} \setminus S$ we let $t(v) \in \{0, 2\}$ denote the index of the graph $G_{t(v)}$ it belongs to. It is easy to see that the neighbours of v in G_{13} and the neighbours of v in $G_{t(v)}$ partition the set of odd (even) vertices in \mathbb{Z}_{2n} if v is even (odd).

Thus the rotation system $\mathcal{R} = \{\pi_v : v \in \mathbb{Z}_{2n}\}$ defined by

$$\pi_v = \begin{cases} p_v^{13}, & v \in S; \\ (p_{v,0}^0; p_{v,1}^0, \dots, p_{v, \frac{n-2}{2}}^0, p_{v,0}^{13}, p_{v,1}^{13}, \dots, p_{v, \frac{n+2}{2}}^{13}), & v \in G_0; \\ (p_{v,0}^2; p_{v,1}^2, \dots, p_{v, \frac{n-2}{2}}^2, p_{v,0}^{13}, p_{v,1}^{13}, \dots, p_{v, \frac{n+2}{2}}^{13}), & v \in G_2. \end{cases}$$

is an embedding of $K_{n,n}$ with parts $\{0, 2, \dots, 2n-2\}$ and $\{1, 3, \dots, 2n-1\}$.

Suppose $v \in \mathbb{Z}_{2n} \setminus S$. By the definition of \mathcal{R}_i and p_v^i we have that $p_{v,0}^i = v+1$ if v is even and $i \in \{1, 3\}$ or v is odd and $i \in \{0, 2\}$. Similarly, $p_{v, n_i-1}^i = v-1$ if v is odd and $i \in \{1, 3\}$ or v is even and $i \in \{0, 2\}$. Thus $\pi_v = (\dots, v-1, v+1, \dots)$ if v is even and $\pi_v = (v+1, \dots, v-1)$ if v is odd. Now \mathcal{R}_1 and hence also \mathcal{R}_{13} contains the face $C_{1,0} = (0, 1, 2n-2, 2n-1)$ and the

face $C_{1, \frac{n-2}{4}} = (n-2, n-1, n, n+1)$. This implies $p_v^{13} = (\dots, v-1, v+1, \dots)$ for $v \in S$. Thus $\pi_v = (\dots, v-1, v+1, \dots)$ for all $v \in \mathbb{Z}_{2n}$ and H is a face of \mathcal{R} .

To complete the proof, we show that all faces of \mathcal{R} , apart from H , are of length 4. Then by Euler's formula the genus g of \mathcal{R} satisfies $2 - 2g = 2n - n^2 + (2n^2 - 2n)/4 + 1$, or $g = (n^2 - 3n + 2)/4 = L(n)$, as stated.

As before, we only need to check the lengths of new faces in \mathcal{R} , that is, the faces formed from the arcs of the special faces of \mathcal{R}_0 , \mathcal{R}_2 and \mathcal{R}_{13} . Since these embeddings have $\frac{n-2}{4}$, $\frac{n-2}{4}$ and $\frac{n-2}{2} + 1$ special faces respectively, there are in total $4(n-1)$ such arcs.

Now \mathcal{R} contains for each $k \in \{0, \dots, \frac{n-2}{4} - 1\}$ a pair of faces $F_{1,k}$ and $F_{2,k}$, where

$$\begin{aligned} F_{1,k} &= (4k + 1, 2n - 4k - 2, 4k + 3, 2n - 4k - 4), \\ F_{2,k} &= (4k + 2, 2n - 4k - 5, 4k + 4, 2n - 4k - 3). \end{aligned}$$

The faces in $\mathcal{F} = \{F_{1,k} : \{0, \dots, \frac{n-2}{4} - 1\}\} \cup \{F_{2,k} : \{0, \dots, \frac{n-2}{4} - 1\}\}$ are pairwise arc-disjoint and there are $2(n-2)$ of them. Thus $\{H\} \cup \mathcal{F}$ covers all $2(n-2) + 2n = 4(n-1)$ arcs from the special faces. This completes the proof.

The faces in $\{H\} \cup \mathcal{F}$ can again be traced in a simple arc matching graph, see Figure 6. The illustration indicates one extra face $F = (0, n-1, n, 2n-1)$ which is lost from \mathcal{R} when gluing. \square

Acknowledgement. *I would like to sincerely thank Rimvydas Krauskas who introduced me to the road interchange problem and suggested to model it via graph embeddings.*

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