# Double-critical graph conjecture for claw-free graphs 

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#### Abstract

A connected graph $G$ with chromatic number $t$ is double-critical if $G \backslash\{x, y\}$ is $(t-2)$-colorable for each edge $x y \in E(G)$. The complete graphs are the only known examples of double-critical graphs. A long-standing conjecture of Erdős and Lovász from 1966, which is referred to as the Double-Critical Graph Conjecture, states that there are no other double-critical graphs. That is, if a graph $G$ with chromatic number $t$ is double-critical, then $G$ is the complete graph on $t$ vertices. This has been verified for $t \leq 5$, but remains open for $t \geq 6$. In this paper, we first prove that if $G$ is a non-complete, double-critical graph with chromatic number $t \geq 6$, then no vertex of degree $t+1$ is adjacent to a vertex of degree $t+1, t+2$, or $t+3$ in $G$. We then use this result to show that the Double-Critical Graph Conjecture is true for double-critical graphs $G$ with chromatic number $t \leq 8$ if $G$ is claw-free.


Keywords: vertex coloring, double-critical graphs, claw-free graphs

## 1 Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph $G$, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $e(G)$ the number of edges, $\alpha(G)$ the independence number, $\omega(G)$ the clique number, $\chi(G)$ the chromatic number, and $\bar{G}$ the complement of $G$, respectively. For a vertex $x \in V(G)$, we will use $N_{G}(x)$ to denote the set of vertices in $G$ which are adjacent to $x$. We define $N_{G}[x]=N_{G}(x) \cup\{x\}$ and $d_{G}(x)=\left|N_{G}(x)\right|$. Given vertex sets $A, B \subseteq V(G)$, we say that $A$ is complete to (resp. anti-complete to) $B$ if for every $a \in A$ and every $b \in B, a b \in E(G)$ (resp. $a b \notin E(G)$ ). The subgraph of $G$ induced by $A$, denoted $G[A]$, is the graph with vertex set $A$ and edge set $\{x y \in E(G): x, y \in A\}$. We denote by $B \backslash A$ the set $B-A, e_{G}(A, B)$ the number of edges between $A$ and $B$ in $G$, and $G \backslash A$ the subgraph of $G$ induced on $V(G) \backslash A$, respectively. If

[^0]$A=\{a\}$, we simply write $B \backslash a, e_{G}(a, B)$, and $G \backslash a$, respectively. A graph $H$ is an induced subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $H=G[V(H)]$. A graph $G$ is claw-free if $G$ does not contain $K_{1,3}$ as an induced subgraph. Given two graphs $G$ and $H$, the union of $G$ and $H$, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Given two isomorphic graphs $G$ and $H$, we may (with a slight but common abuse of notation) write $G=H$. A cycle with $t \geq 3$ vertices is denoted by $C_{t}$. Throughout this paper, a proper vertex coloring of a graph $G$ with $k$ colors is called a $k$-coloring of $G$.

In 1966, the following conjecture of Lovász was published by Erdős [6] and is known as the Erdős-Lovász Tihany Conjecture.

Conjecture 1.1 For any integers $s, t \geq 2$ and any graph $G$ with $\omega(G)<\chi(G)=s+t-1$, there exist disjoint subgraphs $G_{1}$ and $G_{2}$ of $G$ such that $\chi\left(G_{1}\right) \geq s$ and $\chi\left(G_{2}\right) \geq t$.

To date, Conjecture 1.1 has been shown to be true only for values of $(s, t) \in\{(2,2),(2,3)$, $(2,4),(3,3),(3,4),(3,5)\}$. The case $(2,2)$ is trivial. The case $(2,3)$ was shown by Brown and Jung in 1969 [3]. Mozhan [10] and Stiebitz [13] each independently showed the case $(2,4)$ in 1987. The cases $(3,3),(3,4)$, and $(3,5)$ were also settled by Stiebitz in 1987 [14]. Recent work on the Erdős-Lovász Tihany Conjecture has focused on proving the conjecture for certain classes of graphs. Kostochka and Stiebitz [9] showed the conjecture holds for line graphs. Balogh, Kostochka, Prince, and Stiebitz [2] then showed that the conjecture holds for all quasi-line graphs and all graphs $G$ with $\alpha(G)=2$. More recently, Chudnovsky, Fradkin, and Plumettaz [5] proved the following slight weaking of Conjecture 1.1 for claw-free graphs, the proof of which is long and relies heavily on the structure theorem for claw-free graphs developed by Chudnovsky and Seymour [4].

Theorem 1.2 Let $G$ be a claw-free graph with $\chi(G)>\omega(G)$. Then there exists a clique $K$ with $|V(K)| \leq 5$ such that $\chi(G \backslash V(K))>\chi(G)-|V(K)|$.

The most recent result related to the Erdős-Lovász Tihany Conjecture is due to Stiebitz [15], who showed that for integers $s, t \geq 2$, any graph $G$ with $\omega(G)<\chi(G)=s+t-1$ contains disjoint subgraphs $G_{1}$ and $G_{2}$ of $G$ with either $\chi\left(G_{1}\right) \geq s$ and $\operatorname{col}\left(G_{2}\right) \geq t$, or $\operatorname{col}\left(G_{1}\right) \geq s$ and $\chi\left(G_{2}\right) \geq t$, where $\operatorname{col}(H)$ denotes the coloring number of a graph $H$.

If we restrict $s=2$ in Conjecture 1.1, then the Erdős-Lovász Tihany Conjecture states that for any graph $G$ with $\chi(G)>\omega(G) \geq 2$, there exists an edge $x y \in E(G)$ such that $\chi(G \backslash\{x, y\}) \geq \chi(G)-1$. To prove this special case of Conjecture 1.1, suppose for a contradiction that no such edge exists. Then $\chi(G \backslash\{x, y\})=\chi(G)-2$ for every edge $x y \in E(G)$. This motivates the definition of double-critical graphs. A connected graph $G$ is doublecritical if for every edge $x y \in E(G), \chi(G \backslash\{x, y\})=\chi(G)-2$. A graph $G$ is $t$-chromatic if $\chi(G)=t$. We are now ready to state the following conjecture, which is referred to as the Double-Critical Graph Conjecture, due to Erdős and Lovász [6].

Conjecture 1.3 Let $G$ be a double-critical, $t$-chromatic graph. Then $G=K_{t}$.
Since Conjecture 1.3 is a special case of Conjecture 1.1, it has been settled in the affirmative for $t \leq 5$ [10, 13], for line graphs [9], and for quasi-line graphs and graphs with independence number two [2]. Representing a weakening of Conjecture 1.3, Kawarabayashi, Pedersen, and Toft [8] have shown that any double-critical, $t$-chromatic graph contains $K_{t}$ as a minor for $t \in\{6,7\}$. As a further weakening, Pedersen [11] showed that any double-critical, 8 -chromatic graph contains $K_{8}^{-}$as a minor. Albar and Gonçalves [1] later proved that any double-critical, 8-chromatic graph contains $K_{8}$ as a minor. Their proof is computer-assisted. The present authors [12] gave a computer-free proof of the same result and further showed that any double-critical, $t$-chromatic graph contains $K_{9}$ as a minor for all $t \geq 9$. We note here that Theorem 1.2 does not completely settle Conjecture 1.3 for all claw-free graphs. Recently, Huang and Yu [7] proved that the only double-critical, 6-chromatic, claw-free graph is $K_{6}$. We prove the following main results in this paper. Theorem 1.4 is a generalization of a result obtained in [8] that no two vertices of degree $t+1$ are adjacent in any non-complete, double-critical, $t$-chromatic graph.

Theorem 1.4 If $G$ is a non-complete, double-critical, $t$-chromatic graph with $t \geq 6$, then for any vertex $x \in V(G)$ with $d_{G}(x)=t+1$, the following hold:
(a) $e\left(\overline{G\left[N_{G}(x)\right]}\right) \geq 8$; and
(b) for any vertex $y \in N_{G}(x), d_{G}(y) \geq t+4$. Furthermore, if $d_{G}(y)=t+4$, then $\mid N_{G}(x) \cap$ $N_{G}(y) \mid=t-2$ and $\overline{G\left[N_{G}(x)\right]}$ contains either only one cycle, which is isomorphic to $C_{8}$, or exactly two cycles, each of which is isomorphic to $C_{5}$.

Corollary 1.5 below follows immediately from Theorem 1.4.
Corollary 1.5 If $G$ is a non-complete, double-critical, $t$-chromatic graph with $t \geq 6$, then no vertex of degree $t+1$ is adjacent to a vertex of degree $t+1, t+2$, or $t+3$ in $G$.

We then use Corollary 1.5 to prove the following main result.
Theorem 1.6 Let $G$ be a double-critical, $t$-chromatic graph with $t \in\{6,7,8\}$. If $G$ is claw-free, then $G=K_{t}$.

The rest of this paper is organized as follows. In Section 2, we first list some known properties of non-complete, double-critical graphs obtained in [8] and then establish a few new ones. In particular, Lemma 2.4 turns out to be very useful. Our new lemmas lead to a very short proof of Theorem 1.6 for $t=6,7$, which we place at the end of Section 2. We prove the remainder of our main results in Section 3.

## 2 Preliminaries

The following is a summary of the basic properties of non-complete, double-critical graphs shown by Kawarabayashi, Pedersen, and Toft in [8].

Proposition 2.1 If $G$ is a non-complete, double-critical, $t$-chromatic graph, then all of the following are true.
(a) $G$ does not contain $K_{t-1}$ as a subgraph.
(b) For all edges $x y$, every $(t-2)$-coloring $c: V(G) \backslash\{x, y\} \rightarrow\{1,2, \ldots, t-2\}$ of $G \backslash\{x, y\}$, and any non-empty sequence $j_{1}, j_{2}, \ldots, j_{i}$ of $i$ different colors from $\{1,2, \ldots, t-2\}$, there is a path of order $i+2$ with vertices $x, v_{1}, v_{2}, \ldots, v_{i}, y$ in order such that $c\left(v_{k}\right)=j_{k}$ for all $k \in\{1,2, \ldots, i\}$.
(c) For any edge $x y \in E(G), x$ and $y$ have at least one common neighbor in every color class of any $(t-2)$-coloring of $G \backslash\{x, y\}$. In particular, every edge $x y \in E(G)$ belongs to at least $t-2$ triangles.
(d) There exists at least one edge $x y \in E(G)$ such that $x$ and $y$ share a common non-neighbor in $G$.
(e) For any edge $x y \in E(G)$, the subgraph of $G$ induced by $N_{G}(x) \backslash N_{G}[y]$ contains no isolated vertices. In particular, no vertex of $N_{G}(x)$ can have degree one in $\overline{G\left[N_{G}(x)\right]}$.
(f) $\delta(G) \geq t+1$.
(g) For any vertex $x \in V(G), \alpha\left(G\left[N_{G}(x)\right]\right) \leq d_{G}(x)-t+1$.
(h) For any vertex $x$ with at least one non-neighbor in $G$, $\chi\left(G\left[N_{G}(x)\right]\right) \leq t-3$.
(i) For any $x \in V(G)$ with $d_{G}(x)=t+1, \overline{G\left[N_{G}(x)\right]}$ is the union of isolated vertices and cycles of length at least five. Furthermore, there must be at least one such cycle in $\overline{G\left[N_{G}(x)\right]}$.
(j) No two vertices of degree $t+1$ are adjacent in $G$.

We next establish some new properties of non-complete, double-critical graphs.
Lemma 2.2 Let $G$ be a double-critical, $t$-chromatic graph and let $x \in V(G)$. If $d_{G}(x)=$ $|V(G)|-1$, then $G \backslash x$ is a double-critical, $(t-1)$-chromatic graph.

Proof. Let $u v$ be any edge of $G \backslash x$. Clearly, $\chi(G \backslash x)=t-1$. Since $G$ is double-critical, $\chi(G \backslash\{u, v\})=t-2$ and so $\chi(G \backslash\{u, v, x\})=t-3$ because $x$ is adjacent to all the other vertices in $G \backslash\{u, v\}$. Hence $G \backslash x$ is double-critical and $(t-1)$-chromatic.

Lemma 2.3 If $G$ is a non-complete, double-critical, $t$-chromatic graph, then for any $x \in$ $V(G)$ with at least one non-neighbor in $G, \chi\left(G \backslash N_{G}[x]\right) \geq 3$. In particular, $G \backslash N_{G}[x]$ must contain an odd cycle, and so $d_{G}(x) \leq|V(G)|-4$.

Proof. Let $x$ be any vertex in $G$ with $d_{G}(x)<|V(G)|-1$ and let $H=G \backslash N_{G}[x]$. Suppose that $\chi(H) \leq 2$. Since $d_{G}(x)<|V(G)|-1, H$ contains at least one vertex. Let $y \in V(H)$ be adjacent to a vertex $z \in N_{G}(x)$. This is possible because $G$ is connected. If $H$ has no edge, then $G \backslash(V(H) \cup\{z\})$ has a $(t-2)$-coloring $c$, which can be extended to a $(t-1)$-coloring of $G$ by assigning all vertices in $V(H)$ the color $c(x)$ and assigning a new color to the vertex $z$, a contradiction. Thus $H$ must contain at least one edge, and so $\chi(H)=2$. Let $(A, B)$ be a bipartition of $H$. Now $G \backslash H$ has a $(t-2)$-coloring $c^{*}$, which again can be extended to a $(t-1)$-coloring of $G$ by assigning all vertices in $A$ the color $c^{*}(x)$ and all vertices in $B$ the same new color, a contradiction. This proves that $\chi(H) \geq 3$, and so $H$ must contain an odd cycle. Therefore $d_{G}(x) \leq|V(G)|-4$.

Lemma 2.4 Let $G$ be a double-critical, $t$-chromatic graph. For any edge $x y \in E(G)$, let $c$ be any $(t-2)$-coloring of $G \backslash\{x, y\}$ with color classes $V_{1}, V_{2}, \ldots, V_{t-2}$. Then the following two statements are true.
(a) For any $i, j \in\{1,2, \ldots, t-2\}$ with $i \neq j$, if $N_{G}(x) \cap N_{G}(y) \cap V_{i}$ is anti-complete to $N_{G}(x) \cap V_{j}$, then there exists at least one edge between $\left(N_{G}(y) \backslash N_{G}(x)\right) \cap V_{i}$ and $N_{G}(x) \cap V_{j}$ in $G$. In particular, $\left(N_{G}(y) \backslash N_{G}(x)\right) \cap V_{i} \neq \emptyset$.
(b) Assume that $d_{G}(x)=t+1$ and $y$ belongs to a cycle of length $k \geq 5$ in $\overline{G\left[N_{G}(x)\right]}$. $\left(b_{1}\right)$ If $k \geq 7$, then $d_{G}(y) \geq t+e(\overline{G[N(x)]})-4$;
$\left(b_{2}\right)$ If $k=6$, then $d_{G}(y) \geq \max \left\{t+2, t+e\left(\overline{G\left[N_{G}(x)\right]}\right)-5\right\}$; and
$\left(b_{3}\right)$ If $k=5$, then $d_{G}(y) \geq \max \left\{t+2, t+e\left(\overline{G\left[N_{G}(x)\right]}\right)-6\right\}$.

Proof. Let $G, x, y, c$ be as given in the statement. For any $i, j \in\{1,2, \ldots, t-2\}$ with $i \neq j$, if $N_{G}(x) \cap N_{G}(y) \cap V_{i}$ is anti-complete to $N_{G}(x) \cap V_{j}$, then $G$ is non-complete. By Proposition 2.1(b), there must exist a path $x, u_{j}, u_{i}, y$ in $G$ such that $c\left(u_{j}\right)=j$ and $c\left(u_{i}\right)=i$. Clearly, $u_{j} u_{i} \in E(G)$ and $u_{j} \in N_{G}(x) \cap V_{j}$. Since $N_{G}(x) \cap N_{G}(y) \cap V_{i}$ is anti-complete to $N_{G}(x) \cap V_{j}$, we see that $u_{i} \in\left(N_{G}(y) \backslash N_{G}(x)\right) \cap V_{i}$. This proves Lemma 2.4(a).

To prove Lemma [2.4(b), let $H=\overline{G\left[N_{G}(x)\right]}$. Assume that $d_{G}(x)=t+1$ and that $y$ belongs to a cycle, say $C_{k}$, of $H$, where $k \geq 5$. By Proposition 2.1(j), $d_{G}(y) \geq t+2$, and by Proposition 2.1(i), $H$ is the union of isolated vertices and cycles of length at least five. Clearly, $\left|N_{G}(x) \cap N_{G}(y)\right|=t-2$. By Proposition 2.1(c), we may assume that $V_{i} \cap$ $\left(N_{G}(x) \cap N_{G}(y)\right)=\left\{v_{i}\right\}$ for all $i \in\{1, \ldots, t-2\}$. Then $N_{G}(x) \cap N_{G}(y)=\left\{v_{1}, \ldots, v_{t-2}\right\}$. Let $\{a, b\}=N_{G}(x) \backslash N_{G}[y]$. Since $a$ and $b$ are neighbors of $y$ in a cycle of length at least 5 in $H$, $a b \in E(G)$. We may further assume that $a \in V_{1}$ and $b \in V_{2}$. Then $v_{1} a y b v_{2}$ forms a path on five vertices of $C_{k}$, since $v_{1}, a \in V_{1}$ and $v_{2}, b \in V_{2}$. If $k \geq 6$, then $v_{1} v_{2} \in E(G)$ and both $v_{1}$ and $v_{2}$ have precisely one non-neighbor in $\left\{v_{3}, v_{4}, \ldots, v_{t-2}\right\}$. We may assume that $v_{1} v_{3} \notin E(G)$ and $v_{2} v_{\ell} \notin E(G)$, where $\ell=3$ if $k=6$, and $\ell=4$ if $k \geq 7$. For any $i, j \in\{3,4, \ldots, t-2\}$ with
$i \neq j$, if $v_{i} v_{j} \notin E(G)$, then by Lemma 2.4(a), there exists $v_{j}^{\prime} \in V_{j} \backslash v_{j}$ such that $v_{j}^{\prime} y \in E(G)$. By symmetry, there exists $v_{i}^{\prime} \in V_{i} \backslash v_{i}$ such that $v_{i}^{\prime} y \in E(G)$. Therefore, if $C$ is any cycle in $H \backslash V\left(C_{k}\right)$ and $V_{m} \cap V(C) \neq \emptyset$ for some $m \in\{3,4, \ldots, t-2\}$, then $y$ is adjacent to a vertex in $V_{m} \backslash v_{m}$.

Assume that $k=5$. Then $v_{1} v_{2} \notin E(G)$ and so $d_{G}(y) \geq\left|N_{G}(x) \cap N_{G}(y)\right|+|\{x\}|+$ $e\left(H \backslash V\left(C_{k}\right)\right)=(t-2)+1+(e(H)-5)=t+e(H)-6$. Next assume that $k=6$. Then $v_{\ell}=v_{3}$. Since both $N_{G}(x) \cap N_{G}(y) \cap V_{1}$ and $N_{G}(x) \cap N_{G}(y) \cap V_{2}$ are anti-complete to $N_{G}(x) \cap V_{3}$, by Lemma 2.4(a), $N_{G}(y) \cap\left(V_{1} \backslash\left\{a, v_{1}\right\}\right) \neq \emptyset$ and $N_{G}(y) \cap\left(V_{2} \backslash\left\{b, v_{2}\right\}\right) \neq \emptyset$. Then $d_{G}(y) \geq\left|N_{G}(x) \cap N_{G}(y)\right|+|\{x\}|+\left|N_{G}(y) \cap\left(V_{1} \backslash\left\{a, v_{1}\right\}\right)\right|+\left|N_{G}(y) \cap\left(V_{2} \backslash\left\{b, v_{2}\right\}\right)\right|+$ $e\left(H \backslash V\left(C_{k}\right)\right) \geq(t-2)+1+1+1+(e(H)-6)=t+e(H)-5$. Finally assume that $k \geq 7$. Then $v_{\ell}=v_{4}$. Since $N_{G}(x) \cap N_{G}(y) \cap V_{1}$ is anti-complete to $N_{G}(x) \cap V_{3}$ and $N_{G}(x) \cap N_{G}(y) \cap V_{2}$ is anti-complete to $N_{G}(x) \cap V_{4}$, by Lemma2.4(a), $N_{G}(y) \cap\left(V_{1} \backslash\left\{a, v_{1}\right\}\right) \neq \emptyset$ and $N_{G}(y) \cap\left(V_{2} \backslash\left\{b, v_{2}\right\}\right) \neq \emptyset$. As observed earlier, for any $i, j \in\{3,4, \ldots, t-2\}$ with $i \neq j$ and $v_{i} v_{j} \notin E(G), y$ has at least one neighbor in each of $V_{i} \backslash v_{i}$ and $V_{j} \backslash v_{j}$ in $G$. Hence $d_{G}(y) \geq\left|N_{G}(x) \cap N_{G}(y)\right|+|\{x\}|+\left|N_{G}(y) \cap\left(V_{1} \backslash\left\{a, v_{1}\right\}\right)\right|+\left|N_{G}(y) \cap\left(V_{2} \backslash\left\{b, v_{2}\right\}\right)\right|+$ $\left|V\left(C_{k}\right) \backslash\left\{a, b, v_{1}, v_{2}, y\right\}\right|+e\left(H \backslash V\left(C_{k}\right)\right) \geq(t-2)+1+1+1+(k-5)+(e(H)-k)=t+e(H)-4$. Note that since $k \geq 7$, we see that $e(H) \geq 7$, and so $d(y) \geq t+e(H)-4>t+2$. This completes the proof of Lemma 2.4(b).

Lemma 2.5 Let $G$ be a double-critical, $t$-chromatic graph with $t \geq 6$. If $G$ is claw-free, then for any $x \in V(G), d_{G}(x) \leq 2 t-4$. Furthermore, if $d_{G}(x)<|V(G)|-1$, then $d_{G}(x) \leq 2 t-6$.

Proof. Let $x \in V(G)$ be a vertex of maximum degree in $G$, and let $u v$ be any edge of $G \backslash x$. Let $c$ be any $(t-2)$-coloring of $G \backslash\{u, v\}$ with color classes $V_{1}, V_{2}, \ldots, V_{t-2}$. We may assume that $x \in V_{t-2}$. Since $G$ is claw-free, $x$ can have at most two neighbors in each of $V_{1}, \ldots, V_{t-3}$. Additionally, $x$ may be adjacent to $u$ and $v$ in $G$. Therefore $d_{G}(x) \leq 2 t-4$. If $d_{G}(x)<|V(G)|-1$, then $\chi\left(G\left[N_{G}(x)\right]\right) \leq t-3$ by Proposition 2.1(h). Since $G$ is claw-free, each color class in any $(t-3)$-coloring of $G\left[N_{G}(x)\right]$ can contain at most two vertices, and so $d_{G}(x) \leq 2 t-6$.

It is now an easy consequence of Proposition 2.1 and Lemma 2.5 that Theorem 1.6 is true for $t=6,7$.

Proof of Theorem 1.6 for $t=6,7$. Let $G$ and $t \in\{6,7\}$ be as given in the statement. Suppose that $G \neq K_{t}$. By Proposition[2.1(d), there exists an edge $x y \in E(G)$ such that $x$ and $y$ have a common non-neighbor. By Proposition2.1(f) and Lemma 2.5, $t+1 \leq d_{G}(x) \leq 2 t-6$ and $t+1 \leq d_{G}(y) \leq 2 t-6$. Thus $t=7$ and $d_{G}(x)=d_{G}(y)=8$, which contradicts Proposition 2.1(j).

## 3 Proofs of Main Results

In this section, we prove our main results, namely, Theorem 1.4 and Theorem 1.6 for the case $t=8$. We first prove Theorem 1.4.

Proof of Theorem 1.4. Let $G$ and $x$ be as given in the statement. Let $H=\overline{G\left[N_{G}(x)\right]}$. Then $|V(H)|=t+1$. Note that if $d_{G}(x)=|V(G)|-1$, then it follows from Proposition 2.1(f) that $G=K_{t+1}$, a contradiction. Thus $d_{G}(x)<|V(G)|-1$. Now by Proposition [2.1(g) and Proposition 2.1(h) applied to the vertex $x, \alpha(\bar{H}) \leq 2$ and $\chi(\bar{H}) \leq t-3$. Let $c^{*}$ be any $(t-3)$-coloring of $\bar{H}$. Then each color class of $c^{*}$ contains at most two vertices. Since $|V(H)|=t+1$, we see that at least four color classes of $c^{*}$ must each contain two vertices. By Proposition [2.1(e), $H$ has at least eight vertices of degree two and so $e(H) \geq 8$. This proves Theorem 1.4(a).

To prove Theorem 1.4(b), let $y \in N_{G}(x)$. Since $d_{G}(x)=t+1$, by Proposition 2.1(i), either $\left|N_{G}(x) \cap N_{G}(y)\right|=t$ or $\left|N_{G}(x) \cap N_{G}(y)\right|=t-2$. Assume that $\left|N_{G}(x) \cap N_{G}(y)\right|=t-2$. Then $y$ belongs to a cycle of length $k \geq 5$ in $H$ because $H$ is a disjoint union of isolated vertices and cycles. By Proposition 2.1(i), $y$ belongs to a cycle of length at least 5 in H. By Theorem 1.4(a), $e(H) \geq 8$. Note that if $5 \leq k \leq 7$, then by Proposition 2.1(i), $H$ has at least two cycles of length at least 5 , and so $e(H) \geq k+5 \geq 10$. Thus by Lemma 2.4(b), $d_{G}(y) \geq t+4$. If $d_{G}(y)=t+4$, then it follows from Lemma 2.4(b) that either $k=8$ and $H$ is isomorphic to $C_{8} \cup \bar{K}_{t-7}$ or $k=5$ and $H$ is isomorphic to $C_{5} \cup$ $C_{5} \cup \bar{K}_{t-9}$. So we may assume that $\left|N_{G}(x) \cap N_{G}(y)\right|=t$. Let $c$ be any $(t-2)$-coloring of $G \backslash\{x, y\}$ with color classes $V_{1}, V_{2}, \ldots, V_{t-2}$. Since $\alpha(\bar{H}) \leq 2$, we may further assume that $N_{G}(x) \cap V_{1}=\left\{v_{1}, v_{1}^{\prime}\right\}, N_{G}(x) \cap V_{2}=\left\{v_{2}, v_{2}^{\prime}\right\}$, and $N_{G}(x) \cap V_{i}=\left\{v_{i}\right\}$ for all $i \in$ $\{3,4, \ldots, t-2\}$. Then $v_{1} v_{1}^{\prime}, v_{2} v_{2}^{\prime} \in E(H)$. By Proposition 2.1(i) applied to the vertex $x, e_{H}\left(\left\{v_{1}, v_{1}^{\prime}, v_{2}, v_{2}^{\prime}\right\},\left\{v_{3}, v_{4}, \ldots, v_{t-2}\right\}\right) \leq 4$. By Theorem 1.4(a), $e(H) \geq 8$. Thus there must exist at least four vertices in $\left\{v_{3}, v_{4}, \ldots, v_{t-2}\right\}$, say $v_{3}, v_{4}, v_{5}, v_{6}$, such that $d_{H}\left(v_{j}\right)=2$ and $y$ is adjacent to at least one vertex of $V_{j} \backslash v_{j}$ in $G$ for all $j \in\{3,4,5,6\}$. Therefore $\left|N_{G}(y) \backslash N_{G}[x]\right| \geq 4$ and so $d_{G}(y)=\left|N_{G}[x] \cap N_{G}(y)\right|+\left|N_{G}(y) \backslash N_{G}[x]\right| \geq(t+1)+4=t+5$.

This completes the proof of Theorem 1.4.
We are now ready to complete the proof of Theorem 1.6.
Proof of Theorem 1.6 for $t=8$. Let $G$ and $t=8$ be as given in the statement. Suppose that $G \neq K_{8}$. We claim that

Claim 1. $G$ is 10 -regular.
Proof. By Lemma 2.2 and Theorem 1.6 for $t=7, \Delta(G) \leq|V(G)|-2$. By Proposition 2.1)(f) and Lemma 2.5, we see that $9 \leq d_{G}(x) \leq 10$ for all vertices $x \in V(G)$. By Corollary 1.5, $G$ is 10-regular.

Claim 2. For any $x \in V(G), 2 \leq \delta\left(\overline{G\left[N_{G}(x)\right]}\right) \leq \Delta\left(\overline{G\left[N_{G}(x)\right]}\right) \leq 3$.
Proof. Let $x \in V(G)$. Then $x$ has at least one non-neighbor in $G$, otherwise $G=K_{11}$ by Claim 1, a contradiction. By Proposition [2.1(h), $\chi\left(G\left[N_{G}(x)\right]\right) \leq 5$. Since $G$ is claw-free, we see that $\alpha\left(G\left[N_{G}(x)\right]\right)=2$, and so $\chi\left(G\left[N_{G}(x)\right]\right)=5$ since every color class can contain at most two vertices. Thus every vertex of $N_{G}(x)$ has at least one non-neighbor in $G\left[N_{G}(x)\right]$. By Proposition 2.1(e) and Proposition 2.1(c), $2 \leq \delta\left(\overline{G\left[N_{G}(x)\right]}\right) \leq \Delta\left(\overline{G\left[N_{G}(x)\right]}\right) \leq 3$.

Claim 3. For any $x \in V(G), \Delta\left(\overline{G\left[N_{G}(x)\right]}\right)=3$. That is, $\overline{G\left[N_{G}(x)\right]}$ is not 2-regular.
Proof. Suppose that there exists a vertex $x \in V(G)$ such that $\overline{G\left[N_{G}(x)\right]}$ is 2-regular. Let $y \in N_{G}(x)$ and let $c$ be any 6 -coloring of $G \backslash\{x, y\}$ with color classes $V_{1}, V_{2}, \ldots, V_{6}$. Let $W=N_{G}(x) \cap N_{G}(y)$. Then $|W|=7$ because $\overline{G\left[N_{G}(x)\right]}$ is 2-regular. By Proposition 2.1(c), we may assume that $\left|V_{1} \cap W\right|=2$ and $\left|V_{i} \cap W\right|=1$ for every $i \in\{2,3,4,5,6\}$. Let $V_{1} \cap W=\left\{v_{1}, u_{1}\right\}$ and $V_{i} \cap W=\left\{v_{i}\right\}$ for each $i \in\{2,3,4,5,6\}$. Since $G$ is claw-free, we may further assume that $N_{G}(x) \cap V_{2}=\left\{v_{2}, u_{2}\right\}$ and $N_{G}(x) \cap V_{3}=\left\{v_{3}, u_{3}\right\}$. Clearly, $y u_{2}, y u_{3} \notin E(G)$ and thus $u_{2} u_{3} \in E(G)$ because $G$ is claw-free. Since $\overline{G\left[N_{G}(x)\right]}$ is 2regular, we see that $G\left[\left\{v_{4}, v_{5}, v_{6}\right\}\right]$ is not a clique. We may assume that $v_{4} v_{5} \notin E(G)$. By Lemmar 2.4(a), $N_{G}(y) \cap\left(V_{j} \backslash\left\{v_{j}\right\}\right) \neq \emptyset$ for all $j \in\{4,5\}$. Let $w_{4} \in V_{4} \backslash v_{4}$ and $w_{5} \in V_{5} \backslash v_{5}$ be two other neighbors of $y$ in $G$. Then $N_{G}(y) \backslash N_{G}[x]=\left\{w_{4}, w_{5}\right\}$ since $G$ is 10-regular by Claim 11. By Lemma 2.4(a), $v_{6}$ must be complete to $\left\{v_{2}, v_{3}, v_{4}, v_{5}\right\}$ in $G$. Notice that $v_{6}$ is complete to $\left\{u_{2}, u_{3}\right\}$ in $G$ since $\overline{G\left[N_{G}(x)\right]}$ is 2-regular. Thus $v_{6}$ must be anti-complete to $\left\{v_{1}, u_{1}\right\}$ in $G$ and so $G\left[\left\{x, v_{1}, u_{1}, v_{6}\right\}\right]$ is a claw, a contradiction.

From now on, we fix an arbitrary vertex $x \in V(G)$. Let $H=\overline{G\left[N_{G}(x)\right]}$. By Claim 3, let $y \in N_{G}(x)$ with $\left|N_{G}(x) \cap N_{G}(y)\right|=6$. We choose such a vertex $y \in N_{G}(x)$ so that $N_{G}(x) \backslash N_{G}[y]$ contains as many vertices of degree two in $H$ as possible. Let $c$ be any 6-coloring of $G \backslash\{x, y\}$ with color classes $V_{1}, V_{2}, \ldots, V_{6}$. We may assume that $V_{i} \cap N_{G}(x) \cap N_{G}(y)=\left\{v_{i}\right\}$ for all $i \in\{1,2,3,4,5,6\}$. Since $G$ is claw-free, we may further assume that $N_{G}(x) \cap V_{j}=$ $\left\{v_{j}, u_{j}\right\}$ for all $j \in\{1,2,3\}$. Notice that $y$ is anti-complete to $\left\{u_{1}, u_{2}, u_{3}\right\}$ in $G$, and since $G$ is claw-free, $G\left[\left\{u_{1}, u_{2}, u_{3}\right\}\right]=K_{3}$. Let $A=\left\{u_{1}, u_{2}, u_{3}\right\}, B=\left\{v_{1}, v_{2}, v_{3}\right\}$, and $C=\left\{v_{4}, v_{5}, v_{6}\right\}$.

Claim 4. $B$ is not complete to $C$ in $G$.
Proof. Suppose that $B$ is complete to $C$ in $G$. Then $e_{H}(C, A)=\sum_{v \in C} d_{H}(v)-2 e(H[C]) \geq$ $6-2 e(H[C])$. For each $i \in\{1,2,3\}, u_{i} v_{i}, u_{i} y \notin E(G)$ and $d_{H}\left(u_{i}\right) \leq 3$. Thus $e_{H}(A, C) \leq 3$ and so $e(H[C]) \geq 2$. Since $G$ is claw-free, we have $e(H[C])=2$. We may assume that $v_{4} v_{6} \notin E(H)$. Then $v_{4} v_{6} \in E(G)$ and $v_{4} v_{5}, v_{5} v_{6} \notin E(G)$. Since $d_{H}\left(v_{4}\right) \geq 2, d_{H}\left(v_{6}\right) \geq 2$, and $B$ is complete to $C$ in $G$, we may assume that $u_{2} v_{4}, u_{3} v_{6} \notin E(G)$. Note that $H$ is not 3 -regular since $e_{H}(A, C) \leq 3$ and $e_{H}(B, C)=0$. By the choice of $y, d_{H}\left(u_{1}\right)=2$ and
$d_{H}\left(v_{j}\right)=2$ for all $j \in\{4,5,6\}$. Since $d_{H}\left(u_{2}\right)=d_{H}\left(u_{3}\right)=3$, by the choice of $y$ again, $d_{H}\left(v_{2}\right)=d_{H}\left(v_{3}\right)=3$. Thus $G[B]=\overline{K_{3}}$ and so $G[\{x\} \cup B]$ is a claw, a contradiction.

Claim 5. $G[C]=K_{3}$.
Proof. Suppose that $G[C]$ contains a missing edge, say $v_{4} v_{5} \notin E(G)$. By Lemma 2.4(a), there exist $w_{4} \in V_{4} \backslash v_{4}$ and $w_{5} \in V_{5} \backslash v_{5}$ such that $y w_{4}, y w_{5} \in E(G)$. By Claim 4, we may assume that $v_{3} v_{j} \notin E(G)$ for some $j \in\{4,5,6\}$. By Lemma 2.4(a), $y$ has another neighbor, say $w_{3}$, in $V_{3} \backslash v_{3}$. Since $G$ is 10 -regular, $\left\{w_{3}, w_{4}, w_{5}\right\}=N(y) \backslash N[x]$, so by Lemma 2.4(a), $v_{4} v_{5}$ is the only missing edge in $G[C]$ and $\left\{v_{1}, v_{2}\right\}$ is complete to $C$ in $G$. If $e_{H}(A, C)=3$, then $d_{H}\left(u_{i}\right)=3$ for all $i \in\{1,2,3\}$. By the choice of $y, d_{H}\left(v_{3}\right)=3$, or else we could replace $y$ with $u_{3}$. Notice that for all $i \in\{4,5,6\}, e_{H}\left(v_{i}, A \cup\left\{v_{3}\right\}\right) \geq 1$, and so by the choice of $y$, $d_{H}\left(v_{i}\right)=3$, or else we could replace $y$ with $v_{3}$. Thus $e_{H}(A, C) \geq 5$, which is impossible. Hence $e_{H}(A, C) \leq 2$. Notice that $e_{H}(A, C)=\left(d_{H}\left(v_{4}\right)-1\right)+\left(d_{H}\left(v_{5}\right)-1\right)+d_{H}\left(v_{6}\right)-e_{H}\left(v_{3}, C\right) \geq 2$. it follows that $e_{H}(A, C)=2, e_{H}\left(v_{3}, C\right)=2$ and $d_{H}\left(v_{i}\right)=2$ for all $i \in\{4,5,6\}$. Then $N_{G}(x) \backslash N_{G}[y]$ has at most one vertex of degree two in $H$, but $N_{G}(x) \backslash N_{G}\left[v_{3}\right]$ has two vertices of degree two in $H$, contradicting the choice of $y$.

Claim 6. $v_{1} u_{1}, v_{2} u_{2}$, and $v_{3} u_{3}$ are the only edges in $H[A \cup B]$.
Proof. Suppose that $H[A \cup B]$ has at least four edges. By Claim 5 and Claim 2, $e_{H}(A \cup$ $B, C) \geq 6$. On the other hand, $e_{H}(A \cup B, C)=\sum_{v \in A \cup B} d_{H}(v)-2 e(H[A \cup B])-3 \leq$ $15-2 e(H[A \cup B])$. It follows that $e(H[A \cup B])=4$ and $A \cup B$ contains at most one vertex of degree two in $H$. Thus $e_{H}(A \cup B, C) \leq 7$ and so at least two vertices of $C$, say $v_{4}$ and $v_{5}$, are of degree two in $H$. Since $e_{H}(A, C) \leq 3$ and $G[C]=K_{3}$ by Claim 5, we may assume that $v_{4} v_{3} \notin E(G)$. If $d_{H}\left(v_{3}\right)=3$, then since $d_{H}\left(v_{4}\right)=2$ and at most one vertex of $A \cup B$ has degree two in $H$, by the choice of $y$, exactly one of $u_{1}, u_{2}, u_{3}$ has degree two in $H$. Then $e_{H}(A \cup B, C)=6$. Thus $d_{H}\left(v_{j}\right)=2$ for all $j \in\{4,5,6\}$ and by the choice of $y$, each vertex of $B$ is adjacent to at most one vertex of $C$ in $H$. Thus $e_{H}(A \cup B, C) \leq 5$, a contradiction. Hence $d_{H}\left(v_{3}\right)=2$. Now $d_{H}\left(u_{i}\right)=3$ for all $i \in\{1,2,3\}$ because at most one vertex of $A \cup B$ has degree two in $H$. We see that $N(x) \backslash N[y]$ has no vertex of degree two in $H$ but $N(x) \backslash N\left[u_{3}\right]$ has at least one vertex of degree two in $H$, contrary to the choice of $y$.

By Claim 6, we see that for any $i \in\{1,2,3\}, v_{i} v_{j} \notin E(G)$ for some $j \in\{4,5,6\}$. By Lemma 2.4(a), let $w_{i} \in V_{i} \backslash v_{i}$ be such that $y w_{i} \in E(G)$ for all $i \in\{1,2,3\}$. Let $D=$ $\left\{w_{1}, w_{2}, w_{3}\right\}$. Then $N_{G}(y) \backslash N_{G}[x]=D$ and $G[D]=K_{3}$ because $G$ is claw-free. Clearly, $D$ is not complete to $C$ in $G$, otherwise $G[\{y\} \cup D \cup C]=K_{7}$, contrary to Proposition 2.1(a). We may assume that $w_{3} v_{4} \notin E(G)$. For each $i \in\{1,2\}, v_{i} v_{3}, v_{i} u_{3} \in E(G)$ by Claim 6. Thus $v_{1} w_{3}, v_{2} w_{3} \notin E(G)$ because $G$ is claw-free. Notice that $w_{3}, x, v_{1}, v_{2}, v_{4} \in N_{G}(y)$ and $w_{3}$ is anti-complete to $\left\{x, v_{1}, v_{2}, v_{4}\right\}$ in $G$. Thus $\Delta\left(\overline{G\left[N_{G}(y)\right]}\right) \geq 4$, contrary to Claim 2.

This completes the proof of Theorem 1.6.

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