# Double-critical graph conjecture for claw-free graphs

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#### Abstract

A connected graph G with chromatic number t is double-critical if  $G \setminus \{x, y\}$  is (t-2)-colorable for each edge  $xy \in E(G)$ . The complete graphs are the only known examples of double-critical graphs. A long-standing conjecture of Erdős and Lovász from 1966, which is referred to as the Double-Critical Graph Conjecture, states that there are no other double-critical graphs. That is, if a graph G with chromatic number t is double-critical, then G is the complete graph on t vertices. This has been verified for  $t \leq 5$ , but remains open for  $t \geq 6$ . In this paper, we first prove that if G is a non-complete, double-critical graph with chromatic number  $t \geq 6$ , then no vertex of degree t+1 is adjacent to a vertex of degree t+1, t+2, or t+3 in G. We then use this result to show that the Double-Critical Graph Conjecture is true for double-critical graphs G with chromatic number  $t \leq 8$  if G is claw-free.

Keywords: vertex coloring, double-critical graphs, claw-free graphs

### 1 Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph G, we will use V(G) to denote the vertex set, E(G) the edge set, e(G) the number of edges,  $\alpha(G)$  the independence number,  $\omega(G)$  the clique number,  $\chi(G)$  the chromatic number, and  $\overline{G}$  the complement of G, respectively. For a vertex  $x \in V(G)$ , we will use  $N_G(x)$  to denote the set of vertices in G which are adjacent to x. We define  $N_G[x] = N_G(x) \cup \{x\}$  and  $d_G(x) = |N_G(x)|$ . Given vertex sets  $A, B \subseteq V(G)$ , we say that A is complete to (resp. anti-complete to) B if for every  $a \in A$  and every  $b \in B$ ,  $ab \in E(G)$  (resp.  $ab \notin E(G)$ ). The subgraph of G induced by A, denoted G[A], is the graph with vertex set A and edge set  $\{xy \in E(G) : x, y \in A\}$ . We denote by  $B \setminus A$  the set B - A,  $e_G(A, B)$  the number of edges between A and B in G, and  $G \setminus A$  the subgraph of G induced on  $V(G) \setminus A$ , respectively. If

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 $A = \{a\}$ , we simply write  $B \setminus a$ ,  $e_G(a, B)$ , and  $G \setminus a$ , respectively. A graph H is an induced subgraph of a graph G if  $V(H) \subseteq V(G)$  and H = G[V(H)]. A graph G is claw-free if G does not contain  $K_{1,3}$  as an induced subgraph. Given two graphs G and H, the union of G and H, denoted  $G \cup H$ , is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write G = H. A cycle with  $t \geq 3$  vertices is denoted by  $C_t$ . Throughout this paper, a proper vertex coloring of a graph G with k colors is called a k-coloring of G.

In 1966, the following conjecture of Lovász was published by Erdős [6] and is known as the Erdős-Lovász Tihany Conjecture.

**Conjecture 1.1** For any integers  $s, t \ge 2$  and any graph G with  $\omega(G) < \chi(G) = s + t - 1$ , there exist disjoint subgraphs  $G_1$  and  $G_2$  of G such that  $\chi(G_1) \ge s$  and  $\chi(G_2) \ge t$ .

To date, Conjecture 1.1 has been shown to be true only for values of  $(s,t) \in \{(2,2), (2,3), (2,4), (3,3), (3,4), (3,5)\}$ . The case (2,2) is trivial. The case (2,3) was shown by Brown and Jung in 1969 [3]. Mozhan [10] and Stiebitz [13] each independently showed the case (2,4) in 1987. The cases (3,3), (3,4), and (3,5) were also settled by Stiebitz in 1987 [14]. Recent work on the Erdős-Lovász Tihany Conjecture has focused on proving the conjecture for certain classes of graphs. Kostochka and Stiebitz [9] showed the conjecture holds for line graphs. Balogh, Kostochka, Prince, and Stiebitz [2] then showed that the conjecture holds for all quasi-line graphs and all graphs G with  $\alpha(G) = 2$ . More recently, Chudnovsky, Fradkin, and Plumettaz [5] proved the following slight weaking of Conjecture 1.1 for claw-free graphs, the proof of which is long and relies heavily on the structure theorem for claw-free graphs developed by Chudnovsky and Seymour [4].

**Theorem 1.2** Let G be a claw-free graph with  $\chi(G) > \omega(G)$ . Then there exists a clique K with  $|V(K)| \leq 5$  such that  $\chi(G \setminus V(K)) > \chi(G) - |V(K)|$ .

The most recent result related to the Erdős-Lovász Tihany Conjecture is due to Stiebitz [15], who showed that for integers  $s, t \ge 2$ , any graph G with  $\omega(G) < \chi(G) = s + t - 1$ contains disjoint subgraphs  $G_1$  and  $G_2$  of G with either  $\chi(G_1) \ge s$  and  $\operatorname{col}(G_2) \ge t$ , or  $\operatorname{col}(G_1) \ge s$  and  $\chi(G_2) \ge t$ , where  $\operatorname{col}(H)$  denotes the coloring number of a graph H.

If we restrict s = 2 in Conjecture 1.1, then the Erdős-Lovász Tihany Conjecture states that for any graph G with  $\chi(G) > \omega(G) \ge 2$ , there exists an edge  $xy \in E(G)$  such that  $\chi(G \setminus \{x, y\}) \ge \chi(G) - 1$ . To prove this special case of Conjecture 1.1, suppose for a contradiction that no such edge exists. Then  $\chi(G \setminus \{x, y\}) = \chi(G) - 2$  for every edge  $xy \in E(G)$ . This motivates the definition of double-critical graphs. A connected graph G is doublecritical if for every edge  $xy \in E(G)$ ,  $\chi(G \setminus \{x, y\}) = \chi(G) - 2$ . A graph G is t-chromatic if  $\chi(G) = t$ . We are now ready to state the following conjecture, which is referred to as the Double-Critical Graph Conjecture, due to Erdős and Lovász [6].

#### **Conjecture 1.3** Let G be a double-critical, t-chromatic graph. Then $G = K_t$ .

Since Conjecture 1.3 is a special case of Conjecture 1.1, it has been settled in the affirmative for  $t \leq 5$  [10, 13], for line graphs [9], and for quasi-line graphs and graphs with independence number two [2]. Representing a weakening of Conjecture 1.3, Kawarabayashi, Pedersen, and Toft [8] have shown that any double-critical, *t*-chromatic graph contains  $K_t$  as a minor for  $t \in \{6, 7\}$ . As a further weakening, Pedersen [11] showed that any double-critical, 8-chromatic graph contains  $K_8^-$  as a minor. Albar and Gonçalves [1] later proved that any double-critical, 8-chromatic graph contains  $K_8$  as a minor. Their proof is computer-assisted. The present authors [12] gave a computer-free proof of the same result and further showed that any double-critical, *t*-chromatic graph contains  $K_9$  as a minor for all  $t \geq 9$ . We note here that Theorem 1.2 does not completely settle Conjecture 1.3 for all claw-free graphs. Recently, Huang and Yu [7] proved that the only double-critical, 6-chromatic, claw-free graph is  $K_6$ . We prove the following main results in this paper. Theorem 1.4 is a generalization of a result obtained in [8] that no two vertices of degree t + 1 are adjacent in any non-complete, double-critical, *t*-chromatic graph.

**Theorem 1.4** If G is a non-complete, double-critical, t-chromatic graph with  $t \ge 6$ , then for any vertex  $x \in V(G)$  with  $d_G(x) = t + 1$ , the following hold:

- (a)  $e(\overline{G[N_G(x)]}) \ge 8$ ; and
- (b) for any vertex  $y \in N_G(x)$ ,  $d_G(y) \ge t + 4$ . Furthermore, if  $d_G(y) = t + 4$ , then  $|N_G(x) \cap N_G(y)| = t 2$  and  $\overline{G[N_G(x)]}$  contains either only one cycle, which is isomorphic to  $C_8$ , or exactly two cycles, each of which is isomorphic to  $C_5$ .

Corollary 1.5 below follows immediately from Theorem 1.4.

**Corollary 1.5** If G is a non-complete, double-critical, t-chromatic graph with  $t \ge 6$ , then no vertex of degree t + 1 is adjacent to a vertex of degree t + 1, t + 2, or t + 3 in G.

We then use Corollary 1.5 to prove the following main result.

**Theorem 1.6** Let G be a double-critical, t-chromatic graph with  $t \in \{6, 7, 8\}$ . If G is claw-free, then  $G = K_t$ .

The rest of this paper is organized as follows. In Section 2, we first list some known properties of non-complete, double-critical graphs obtained in [8] and then establish a few new ones. In particular, Lemma 2.4 turns out to be very useful. Our new lemmas lead to a very short proof of Theorem 1.6 for t = 6, 7, which we place at the end of Section 2. We prove the remainder of our main results in Section 3.

### 2 Preliminaries

The following is a summary of the basic properties of non-complete, double-critical graphs shown by Kawarabayashi, Pedersen, and Toft in [8].

**Proposition 2.1** If G is a non-complete, double-critical, t-chromatic graph, then all of the following are true.

- (a) G does not contain  $K_{t-1}$  as a subgraph.
- (b) For all edges xy, every (t-2)-coloring  $c: V(G) \setminus \{x, y\} \to \{1, 2, \dots, t-2\}$  of  $G \setminus \{x, y\}$ , and any non-empty sequence  $j_1, j_2, \dots, j_i$  of i different colors from  $\{1, 2, \dots, t-2\}$ , there is a path of order i + 2 with vertices  $x, v_1, v_2, \dots, v_i, y$  in order such that  $c(v_k) = j_k$  for all  $k \in \{1, 2, \dots, i\}$ .
- (c) For any edge  $xy \in E(G)$ , x and y have at least one common neighbor in every color class of any (t-2)-coloring of  $G \setminus \{x, y\}$ . In particular, every edge  $xy \in E(G)$  belongs to at least t-2 triangles.
- (d) There exists at least one edge  $xy \in E(G)$  such that x and y share a common non-neighbor in G.
- (e) For any edge  $xy \in E(G)$ , the subgraph of G induced by  $N_G(x) \setminus N_G[y]$  contains no isolated vertices. In particular, no vertex of  $N_G(x)$  can have degree one in  $\overline{G[N_G(x)]}$ .

(f) 
$$\delta(G) \ge t+1$$
.

- (g) For any vertex  $x \in V(G)$ ,  $\alpha(G[N_G(x)]) \leq d_G(x) t + 1$ .
- (h) For any vertex x with at least one non-neighbor in G,  $\chi(G[N_G(x)]) \leq t 3$ .
- (i) For any  $x \in V(G)$  with  $d_G(x) = t+1$ ,  $\overline{G[N_G(x)]}$  is the union of isolated vertices and cycles of length at least five. Furthermore, there must be at least one such cycle in  $\overline{G[N_G(x)]}$ .
- (j) No two vertices of degree t + 1 are adjacent in G.

We next establish some new properties of non-complete, double-critical graphs.

**Lemma 2.2** Let G be a double-critical, t-chromatic graph and let  $x \in V(G)$ . If  $d_G(x) = |V(G)| - 1$ , then  $G \setminus x$  is a double-critical, (t - 1)-chromatic graph.

**Proof.** Let uv be any edge of  $G \setminus x$ . Clearly,  $\chi(G \setminus x) = t - 1$ . Since G is double-critical,  $\chi(G \setminus \{u, v\}) = t - 2$  and so  $\chi(G \setminus \{u, v, x\}) = t - 3$  because x is adjacent to all the other vertices in  $G \setminus \{u, v\}$ . Hence  $G \setminus x$  is double-critical and (t - 1)-chromatic.

**Lemma 2.3** If G is a non-complete, double-critical, t-chromatic graph, then for any  $x \in V(G)$  with at least one non-neighbor in G,  $\chi(G \setminus N_G[x]) \geq 3$ . In particular,  $G \setminus N_G[x]$  must contain an odd cycle, and so  $d_G(x) \leq |V(G)| - 4$ .

**Proof.** Let x be any vertex in G with  $d_G(x) < |V(G)| - 1$  and let  $H = G \setminus N_G[x]$ . Suppose that  $\chi(H) \leq 2$ . Since  $d_G(x) < |V(G)| - 1$ , H contains at least one vertex. Let  $y \in V(H)$  be adjacent to a vertex  $z \in N_G(x)$ . This is possible because G is connected. If H has no edge, then  $G \setminus (V(H) \cup \{z\})$  has a (t-2)-coloring c, which can be extended to a (t-1)-coloring of G by assigning all vertices in V(H) the color c(x) and assigning a new color to the vertex z, a contradiction. Thus H must contain at least one edge, and so  $\chi(H) = 2$ . Let (A, B) be a bipartition of H. Now  $G \setminus H$  has a (t-2)-coloring  $c^*$ , which again can be extended to a (t-1)-coloring of G by assigning all vertices in A the color  $c^*(x)$  and all vertices in B the same new color, a contradiction. This proves that  $\chi(H) \geq 3$ , and so H must contain an odd cycle. Therefore  $d_G(x) \leq |V(G)| - 4$ .

**Lemma 2.4** Let G be a double-critical, t-chromatic graph. For any edge  $xy \in E(G)$ , let c be any (t-2)-coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \ldots, V_{t-2}$ . Then the following two statements are true.

- (a) For any  $i, j \in \{1, 2, ..., t 2\}$  with  $i \neq j$ , if  $N_G(x) \cap N_G(y) \cap V_i$  is anti-complete to  $N_G(x) \cap V_j$ , then there exists at least one edge between  $(N_G(y) \setminus N_G(x)) \cap V_i$  and  $N_G(x) \cap V_j$  in G. In particular,  $(N_G(y) \setminus N_G(x)) \cap V_i \neq \emptyset$ .
- (b) Assume that  $d_G(x) = t + 1$  and y belongs to a cycle of length  $k \ge 5$  in  $\overline{G[N_G(x)]}$ . (b<sub>1</sub>) If  $k \ge 7$ , then  $d_G(y) \ge t + e(\overline{G[N(x)]}) - 4$ ; (b<sub>2</sub>) If k = 6, then  $d_G(y) \ge \max\{t + 2, t + e(\overline{G[N_G(x)]}) - 5\}$ ; and
  - (b<sub>3</sub>) If k = 5, then  $d_G(y) \ge \max\{t + 2, t + e(\overline{G[N_G(x)]}) 6\}$ .

**Proof.** Let G, x, y, c be as given in the statement. For any  $i, j \in \{1, 2, ..., t - 2\}$  with  $i \neq j$ , if  $N_G(x) \cap N_G(y) \cap V_i$  is anti-complete to  $N_G(x) \cap V_j$ , then G is non-complete. By Proposition 2.1(b), there must exist a path  $x, u_j, u_i, y$  in G such that  $c(u_j) = j$  and  $c(u_i) = i$ . Clearly,  $u_j u_i \in E(G)$  and  $u_j \in N_G(x) \cap V_j$ . Since  $N_G(x) \cap N_G(y) \cap V_i$  is anti-complete to  $N_G(x) \cap V_j$ , we see that  $u_i \in (N_G(y) \setminus N_G(x)) \cap V_i$ . This proves Lemma 2.4(a).

To prove Lemma 2.4(b), let  $H = \overline{G[N_G(x)]}$ . Assume that  $d_G(x) = t + 1$  and that y belongs to a cycle, say  $C_k$ , of H, where  $k \ge 5$ . By Proposition 2.1(j),  $d_G(y) \ge t + 2$ , and by Proposition 2.1(i), H is the union of isolated vertices and cycles of length at least five. Clearly,  $|N_G(x) \cap N_G(y)| = t - 2$ . By Proposition 2.1(c), we may assume that  $V_i \cap (N_G(x) \cap N_G(y)) = \{v_i\}$  for all  $i \in \{1, \ldots, t-2\}$ . Then  $N_G(x) \cap N_G(y) = \{v_1, \ldots, v_{t-2}\}$ . Let  $\{a, b\} = N_G(x) \setminus N_G[y]$ . Since a and b are neighbors of y in a cycle of length at least 5 in H,  $ab \in E(G)$ . We may further assume that  $a \in V_1$  and  $b \in V_2$ . Then  $v_1 v_2 \in E(G)$  and both  $v_1$  and  $v_2$  have precisely one non-neighbor in  $\{v_3, v_4, \ldots, v_{t-2}\}$ . We may assume that  $v_1v_3 \notin E(G)$  and  $v_2v_\ell \notin E(G)$ , where  $\ell = 3$  if k = 6, and  $\ell = 4$  if  $k \ge 7$ . For any  $i, j \in \{3, 4, \ldots, t-2\}$  with

 $i \neq j$ , if  $v_i v_j \notin E(G)$ , then by Lemma 2.4(a), there exists  $v'_j \in V_j \setminus v_j$  such that  $v'_j y \in E(G)$ . By symmetry, there exists  $v'_i \in V_i \setminus v_i$  such that  $v'_i y \in E(G)$ . Therefore, if C is any cycle in  $H \setminus V(C_k)$  and  $V_m \cap V(C) \neq \emptyset$  for some  $m \in \{3, 4, \ldots, t-2\}$ , then y is adjacent to a vertex in  $V_m \setminus v_m$ .

Assume that k = 5. Then  $v_1v_2 \notin E(G)$  and so  $d_G(y) \ge |N_G(x) \cap N_G(y)| + |\{x\}| + e(H \setminus V(C_k)) = (t-2) + 1 + (e(H) - 5) = t + e(H) - 6$ . Next assume that k = 6. Then  $v_\ell = v_3$ . Since both  $N_G(x) \cap N_G(y) \cap V_1$  and  $N_G(x) \cap N_G(y) \cap V_2$  are anti-complete to  $N_G(x) \cap V_3$ , by Lemma 2.4(a),  $N_G(y) \cap (V_1 \setminus \{a, v_1\}) \neq \emptyset$  and  $N_G(y) \cap (V_2 \setminus \{b, v_2\}) \neq \emptyset$ . Then  $d_G(y) \ge |N_G(x) \cap N_G(y)| + |\{x\}| + |N_G(y) \cap (V_1 \setminus \{a, v_1\})| + |N_G(y) \cap (V_2 \setminus \{b, v_2\})| + e(H \setminus V(C_k)) \ge (t-2) + 1 + 1 + 1 + (e(H) - 6) = t + e(H) - 5$ . Finally assume that  $k \ge 7$ . Then  $v_\ell = v_4$ . Since  $N_G(x) \cap N_G(y) \cap V_1$  is anti-complete to  $N_G(x) \cap V_3$  and  $N_G(x) \cap N_G(y) \cap V_2$  is anti-complete to  $N_G(x) \cap V_4$ , by Lemma 2.4(a),  $N_G(y) \cap (V_1 \setminus \{a, v_1\}) \neq \emptyset$  and  $N_G(y) \cap (V_2 \setminus \{b, v_2\}) \neq \emptyset$ . As observed earlier, for any  $i, j \in \{3, 4, \dots, t-2\}$  with  $i \neq j$  and  $v_i v_j \notin E(G)$ , y has at least one neighbor in each of  $V_i \setminus v_i$  and  $V_j \setminus v_j$  in G. Hence  $d_G(y) \ge |N_G(x) \cap N_G(y)| + |\{x\}| + |N_G(y) \cap (V_1 \setminus \{a, v_1\})| + |N_G(y) \cap (V_2 \setminus \{b, v_2\})| + |V(C_k) \setminus \{a, b, v_1, v_2, y\}| + e(H \setminus V(C_k)) \ge (t-2) + 1 + 1 + 1 + (k-5) + (e(H) - k) = t + e(H) - 4$ . Note that since  $k \ge 7$ , we see that  $e(H) \ge 7$ , and so  $d(y) \ge t + e(H) - 4 > t + 2$ . This completes the proof of Lemma 2.4(b).

**Lemma 2.5** Let G be a double-critical, t-chromatic graph with  $t \ge 6$ . If G is claw-free, then for any  $x \in V(G)$ ,  $d_G(x) \le 2t - 4$ . Furthermore, if  $d_G(x) < |V(G)| - 1$ , then  $d_G(x) \le 2t - 6$ .

**Proof.** Let  $x \in V(G)$  be a vertex of maximum degree in G, and let uv be any edge of  $G \setminus x$ . Let c be any (t-2)-coloring of  $G \setminus \{u, v\}$  with color classes  $V_1, V_2, \ldots, V_{t-2}$ . We may assume that  $x \in V_{t-2}$ . Since G is claw-free, x can have at most two neighbors in each of  $V_1, \ldots, V_{t-3}$ . Additionally, x may be adjacent to u and v in G. Therefore  $d_G(x) \leq 2t - 4$ . If  $d_G(x) < |V(G)| - 1$ , then  $\chi(G[N_G(x)]) \leq t - 3$  by Proposition 2.1(h). Since G is claw-free, each color class in any (t-3)-coloring of  $G[N_G(x)]$  can contain at most two vertices, and so  $d_G(x) \leq 2t - 6$ .

It is now an easy consequence of Proposition 2.1 and Lemma 2.5 that Theorem 1.6 is true for t = 6, 7.

**Proof of Theorem 1.6 for** t = 6, 7. Let G and  $t \in \{6, 7\}$  be as given in the statement. Suppose that  $G \neq K_t$ . By Proposition 2.1(d), there exists an edge  $xy \in E(G)$  such that x and y have a common non-neighbor. By Proposition 2.1(f) and Lemma 2.5,  $t+1 \leq d_G(x) \leq 2t-6$ and  $t + 1 \leq d_G(y) \leq 2t - 6$ . Thus t = 7 and  $d_G(x) = d_G(y) = 8$ , which contradicts Proposition 2.1(j).

#### **3** Proofs of Main Results

In this section, we prove our main results, namely, Theorem 1.4 and Theorem 1.6 for the case t = 8. We first prove Theorem 1.4.

**Proof of Theorem 1.4.** Let G and x be as given in the statement. Let  $H = \overline{G[N_G(x)]}$ . Then |V(H)| = t+1. Note that if  $d_G(x) = |V(G)| - 1$ , then it follows from Proposition 2.1(f) that  $G = K_{t+1}$ , a contradiction. Thus  $d_G(x) < |V(G)| - 1$ . Now by Proposition 2.1(g) and Proposition 2.1(h) applied to the vertex x,  $\alpha(\overline{H}) \leq 2$  and  $\chi(\overline{H}) \leq t - 3$ . Let  $c^*$  be any (t-3)-coloring of  $\overline{H}$ . Then each color class of  $c^*$  contains at most two vertices. Since |V(H)| = t + 1, we see that at least four color classes of  $c^*$  must each contain two vertices. By Proposition 2.1(e), H has at least eight vertices of degree two and so  $e(H) \geq 8$ . This proves Theorem 1.4(a).

To prove Theorem 1.4(b), let  $y \in N_G(x)$ . Since  $d_G(x) = t + 1$ , by Proposition 2.1(i), either  $|N_G(x) \cap N_G(y)| = t$  or  $|N_G(x) \cap N_G(y)| = t-2$ . Assume that  $|N_G(x) \cap N_G(y)| = t-2$ . Then y belongs to a cycle of length  $k \ge 5$  in H because H is a disjoint union of isolated vertices and cycles. By Proposition 2.1(i), y belongs to a cycle of length at least 5 in H. By Theorem 1.4(a),  $e(H) \ge 8$ . Note that if  $5 \le k \le 7$ , then by Proposition 2.1(i), H has at least two cycles of length at least 5, and so  $e(H) \ge k+5 \ge 10$ . Thus by Lemma 2.4(b),  $d_G(y) \ge t + 4$ . If  $d_G(y) = t + 4$ , then it follows from Lemma 2.4(b) that either k = 8 and H is isomorphic to  $C_8 \cup \overline{K}_{t-7}$  or k = 5 and H is isomorphic to  $C_5 \cup$  $C_5 \cup \overline{K}_{t-9}$ . So we may assume that  $|N_G(x) \cap N_G(y)| = t$ . Let c be any (t-2)-coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \ldots, V_{t-2}$ . Since  $\alpha(\overline{H}) \leq 2$ , we may further assume that  $N_G(x) \cap V_1 = \{v_1, v_1'\}, N_G(x) \cap V_2 = \{v_2, v_2'\}, \text{ and } N_G(x) \cap V_i = \{v_i\} \text{ for all } i \in \{v_i\}$  $\{3, 4, \ldots, t-2\}$ . Then  $v_1v_1', v_2v_2' \in E(H)$ . By Proposition 2.1(i) applied to the vertex  $x, e_H(\{v_1, v'_1, v_2, v'_2\}, \{v_3, v_4, \dots, v_{t-2}\}) \leq 4$ . By Theorem 1.4(a),  $e(H) \geq 8$ . Thus there must exist at least four vertices in  $\{v_3, v_4, \ldots, v_{t-2}\}$ , say  $v_3, v_4, v_5, v_6$ , such that  $d_H(v_i) = 2$ and y is adjacent to at least one vertex of  $V_j \setminus v_j$  in G for all  $j \in \{3, 4, 5, 6\}$ . Therefore  $|N_G(y) \setminus N_G[x]| \ge 4$  and so  $d_G(y) = |N_G[x] \cap N_G(y)| + |N_G(y) \setminus N_G[x]| \ge (t+1) + 4 = t + 5.$ 

This completes the proof of Theorem 1.4.

We are now ready to complete the proof of Theorem 1.6.

**Proof of Theorem 1.6 for** t = 8. Let G and t = 8 be as given in the statement. Suppose that  $G \neq K_8$ . We claim that

Claim 1. G is 10-regular.

**Proof.** By Lemma 2.2 and Theorem 1.6 for t = 7,  $\Delta(G) \leq |V(G)| - 2$ . By Proposition 2.1(f) and Lemma 2.5, we see that  $9 \leq d_G(x) \leq 10$  for all vertices  $x \in V(G)$ . By Corollary 1.5, G is 10-regular.

Claim 2. For any  $x \in V(G)$ ,  $2 \le \delta(\overline{G[N_G(x)]}) \le \Delta(\overline{G[N_G(x)]}) \le 3$ .

**Proof.** Let  $x \in V(G)$ . Then x has at least one non-neighbor in G, otherwise  $G = K_{11}$  by Claim 1, a contradiction. By Proposition 2.1(h),  $\chi(G[N_G(x)]) \leq 5$ . Since G is claw-free, we see that  $\alpha(G[N_G(x)]) = 2$ , and so  $\chi(G[N_G(x)]) = 5$  since every color class can contain at most two vertices. Thus every vertex of  $N_G(x)$  has at least one non-neighbor in  $G[N_G(x)]$ . By Proposition 2.1(e) and Proposition 2.1(c),  $2 \leq \delta(\overline{G[N_G(x)]}) \leq \Delta(\overline{G[N_G(x)]}) \leq 3$ .

**Claim 3.** For any  $x \in V(G)$ ,  $\Delta(\overline{G[N_G(x)]}) = 3$ . That is,  $\overline{G[N_G(x)]}$  is not 2-regular.

**Proof.** Suppose that there exists a vertex  $x \in V(G)$  such that  $\overline{G[N_G(x)]}$  is 2-regular. Let  $y \in N_G(x)$  and let c be any 6-coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \ldots, V_6$ . Let  $W = N_G(x) \cap N_G(y)$ . Then |W| = 7 because  $\overline{G[N_G(x)]}$  is 2-regular. By Proposition 2.1(c), we may assume that  $|V_1 \cap W| = 2$  and  $|V_i \cap W| = 1$  for every  $i \in \{2, 3, 4, 5, 6\}$ . Let  $V_1 \cap W = \{v_1, u_1\}$  and  $V_i \cap W = \{v_i\}$  for each  $i \in \{2, 3, 4, 5, 6\}$ . Since G is claw-free, we may further assume that  $N_G(x) \cap V_2 = \{v_2, u_2\}$  and  $N_G(x) \cap V_3 = \{v_3, u_3\}$ . Clearly,  $yu_2, yu_3 \notin E(G)$  and thus  $u_2u_3 \in E(G)$  because G is claw-free. Since  $\overline{G[N_G(x)]}$  is 2regular, we see that  $G[\{v_4, v_5, v_6\}]$  is not a clique. We may assume that  $v_4v_5 \notin E(G)$ . By Lemma 2.4(a),  $N_G(y) \cap (V_j \setminus \{v_j\}) \neq \emptyset$  for all  $j \in \{4, 5\}$ . Let  $w_4 \in V_4 \setminus v_4$  and  $w_5 \in V_5 \setminus v_5$ be two other neighbors of y in G. Then  $N_G(y) \setminus N_G[x] = \{w_4, w_5\}$  since G is 10-regular by Claim 1. By Lemma 2.4(a),  $v_6$  must be complete to  $\{v_2, v_3, v_4, v_5\}$  in G. Notice that  $v_6$  is complete to  $\{u_2, u_3\}$  in G since  $\overline{G[N_G(x)]}$  is 2-regular. Thus  $v_6$  must be anti-complete to  $\{v_1, u_1\}$  in G and so  $G[\{x, v_1, u_1, v_6\}]$  is a claw, a contradiction.

From now on, we fix an arbitrary vertex  $x \in V(G)$ . Let  $H = \overline{G[N_G(x)]}$ . By Claim 3, let  $y \in N_G(x)$  with  $|N_G(x) \cap N_G(y)| = 6$ . We choose such a vertex  $y \in N_G(x)$  so that  $N_G(x) \setminus N_G[y]$  contains as many vertices of degree two in H as possible. Let c be any 6-coloring of  $G \setminus \{x, y\}$  with color classes  $V_1, V_2, \ldots, V_6$ . We may assume that  $V_i \cap N_G(x) \cap N_G(y) = \{v_i\}$ for all  $i \in \{1, 2, 3, 4, 5, 6\}$ . Since G is claw-free, we may further assume that  $N_G(x) \cap V_j =$  $\{v_j, u_j\}$  for all  $j \in \{1, 2, 3\}$ . Notice that y is anti-complete to  $\{u_1, u_2, u_3\}$  in G, and since G is claw-free,  $G[\{u_1, u_2, u_3\}] = K_3$ . Let  $A = \{u_1, u_2, u_3\}, B = \{v_1, v_2, v_3\}$ , and  $C = \{v_4, v_5, v_6\}$ .

Claim 4. B is not complete to C in G.

**Proof.** Suppose that B is complete to C in G. Then  $e_H(C, A) = \sum_{v \in C} d_H(v) - 2e(H[C]) \ge 6 - 2e(H[C])$ . For each  $i \in \{1, 2, 3\}$ ,  $u_i v_i, u_i y \notin E(G)$  and  $d_H(u_i) \le 3$ . Thus  $e_H(A, C) \le 3$  and so  $e(H[C]) \ge 2$ . Since G is claw-free, we have e(H[C]) = 2. We may assume that  $v_4 v_6 \notin E(H)$ . Then  $v_4 v_6 \in E(G)$  and  $v_4 v_5, v_5 v_6 \notin E(G)$ . Since  $d_H(v_4) \ge 2$ ,  $d_H(v_6) \ge 2$ , and B is complete to C in G, we may assume that  $u_2 v_4, u_3 v_6 \notin E(G)$ . Note that H is not 3-regular since  $e_H(A, C) \le 3$  and  $e_H(B, C) = 0$ . By the choice of  $y, d_H(u_1) = 2$  and

 $d_H(v_j) = 2$  for all  $j \in \{4, 5, 6\}$ . Since  $d_H(u_2) = d_H(u_3) = 3$ , by the choice of y again,  $d_H(v_2) = d_H(v_3) = 3$ . Thus  $G[B] = \overline{K_3}$  and so  $G[\{x\} \cup B]$  is a claw, a contradiction.

Claim 5.  $G[C] = K_3$ .

**Proof.** Suppose that G[C] contains a missing edge, say  $v_4v_5 \notin E(G)$ . By Lemma 2.4(a), there exist  $w_4 \in V_4 \setminus v_4$  and  $w_5 \in V_5 \setminus v_5$  such that  $yw_4, yw_5 \in E(G)$ . By Claim 4, we may assume that  $v_3v_j \notin E(G)$  for some  $j \in \{4, 5, 6\}$ . By Lemma 2.4(a), y has another neighbor, say  $w_3$ , in  $V_3 \setminus v_3$ . Since G is 10-regular,  $\{w_3, w_4, w_5\} = N(y) \setminus N[x]$ , so by Lemma 2.4(a),  $v_4v_5$  is the only missing edge in G[C] and  $\{v_1, v_2\}$  is complete to C in G. If  $e_H(A, C) = 3$ , then  $d_H(u_i) = 3$  for all  $i \in \{1, 2, 3\}$ . By the choice of  $y, d_H(v_3) = 3$ , or else we could replace y with  $u_3$ . Notice that for all  $i \in \{4, 5, 6\}$ ,  $e_H(v_i, A \cup \{v_3\}) \ge 1$ , and so by the choice of y,  $d_H(v_i) = 3$ , or else we could replace y with  $v_3$ . Thus  $e_H(A, C) \ge 5$ , which is impossible. Hence  $e_H(A, C) \le 2$ . Notice that  $e_H(A, C) = (d_H(v_4) - 1) + (d_H(v_5) - 1) + d_H(v_6) - e_H(v_3, C) \ge 2$ . it follows that  $e_H(A, C) = 2$ ,  $e_H(v_3, C) = 2$  and  $d_H(v_i) = 2$  for all  $i \in \{4, 5, 6\}$ . Then  $N_G(x) \setminus N_G[y]$  has at most one vertex of degree two in H, but  $N_G(x) \setminus N_G[v_3]$  has two vertices of degree two in H, contradicting the choice of y.

**Claim 6.**  $v_1u_1, v_2u_2$ , and  $v_3u_3$  are the only edges in  $H[A \cup B]$ .

**Proof.** Suppose that  $H[A \cup B]$  has at least four edges. By Claim 5 and Claim 2,  $e_H(A \cup B, C) \geq 6$ . On the other hand,  $e_H(A \cup B, C) = \sum_{v \in A \cup B} d_H(v) - 2e(H[A \cup B]) - 3 \leq 15 - 2e(H[A \cup B])$ . It follows that  $e(H[A \cup B]) = 4$  and  $A \cup B$  contains at most one vertex of degree two in H. Thus  $e_H(A \cup B, C) \leq 7$  and so at least two vertices of C, say  $v_4$  and  $v_5$ , are of degree two in H. Since  $e_H(A, C) \leq 3$  and  $G[C] = K_3$  by Claim 5, we may assume that  $v_4v_3 \notin E(G)$ . If  $d_H(v_3) = 3$ , then since  $d_H(v_4) = 2$  and at most one vertex of  $A \cup B$  has degree two in H, by the choice of y, exactly one of  $u_1, u_2, u_3$  has degree two in H. Then  $e_H(A \cup B, C) = 6$ . Thus  $d_H(v_j) = 2$  for all  $j \in \{4, 5, 6\}$  and by the choice of y, each vertex of B is adjacent to at most one vertex of C in H. Thus  $e_H(A \cup B, C) \leq 5$ , a contradiction. Hence  $d_H(v_3) = 2$ . Now  $d_H(u_i) = 3$  for all  $i \in \{1, 2, 3\}$  because at most one vertex of  $A \cup B$  has degree two in H. We see that  $N(x) \setminus N[y]$  has no vertex of degree two in H but  $N(x) \setminus N[u_3]$  has at least one vertex of degree two in H, contrary to the choice of y.

By Claim 6, we see that for any  $i \in \{1, 2, 3\}, v_i v_j \notin E(G)$  for some  $j \in \{4, 5, 6\}$ . By Lemma 2.4(a), let  $w_i \in V_i \setminus v_i$  be such that  $yw_i \in E(G)$  for all  $i \in \{1, 2, 3\}$ . Let  $D = \{w_1, w_2, w_3\}$ . Then  $N_G(y) \setminus N_G[x] = D$  and  $G[D] = K_3$  because G is claw-free. Clearly, D is not complete to C in G, otherwise  $G[\{y\} \cup D \cup C] = K_7$ , contrary to Proposition 2.1(a). We may assume that  $w_3v_4 \notin E(G)$ . For each  $i \in \{1, 2\}, v_iv_3, v_iu_3 \in E(G)$  by Claim 6. Thus  $v_1w_3, v_2w_3 \notin E(G)$  because G is claw-free. Notice that  $w_3, x, v_1, v_2, v_4 \in N_G(y)$  and  $w_3$  is anti-complete to  $\{x, v_1, v_2, v_4\}$  in G. Thus  $\Delta(\overline{G[N_G(y)]}) \geq 4$ , contrary to Claim 2.

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