# Improving bounds on the diameter of a polyhedron in high dimensions 

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#### Abstract

In 1992, Kalai and Kleitman proved that the diameter of a $d$-dimensional polyhedron with $n$ facets is at most $n^{2+\log _{2} d}$. In 2014, Todd improved the Kalai-Kleitman bound to $(n-d)^{\log _{2} d}$. We improve the Todd bound to $(n-d)^{-1+\log _{2} d}$ for $n \geq d \geq 7,(n-d)^{-2+\log _{2} d}$ for $n \geq d \geq 37$, and $(n-d)^{-3+\log _{2} d+O(1 / d)}$ for $n \geq d \geq 1$.


keyword: Diameter; polyhedra; high dimension; computer-assisted method

## 1 Introduction

The diameter $\delta(P)$ of a polyhedron $P$ is the smallest integer $k$ such that every pair of vertices of $P$ can be connected by a path using at most $k$ edges of $P$. The diameter is a fundamental feature of a polyhedron and is closely related to the theoretical complexity of the simplex algorithm; the number of pivots needed, in the worst case, by the simplex algorithm to solve a linear program on a polyhedron $P$ is bounded from below by $\delta(P)$.

One of the outstanding open problems in the areas of polyhedral combinatorics and operations research is to understand the behavior of $\Delta(d, n)$, the maximum possible diameter of a $d$ dimensional polyhedron with $n$ facets. In 1957, Warren M. Hirsch asked whether $\Delta(d, n) \leq n-d$. While this inequality was shown to hold for $d \leq 3$ [14, 15, 16, Klee and Walkup 17] disproved it for unbounded polyhedra when $d \geq 4$ in 1967, and Santos 26 finally disproved it for bounded polyhedra, i.e., for polytopes, in 2012. Santos' lower bound, later refined by Matschke, Santos, and Weibel [24], however, violates $n-d$ by only 5 percent. For the history of the Hirsch conjecture, see 27.

The first subexponential upper bound on $\Delta(d, n)$ is due to Kalai and Kleitman 12 who proved in 1992 that $\Delta(d, n)$ is at $\operatorname{most} n^{2+\log _{2} d}$. The key ingredient for their proof is a recursive inequality on $\Delta(d, n)$, which we call the Kalai-Kleitman inequality. The Kalai-Kleitman inequality was later extended to more general settings such as connected layer families by Eisenbrand et al. [8], and subset partition graphs by Gallagher and Kim 9. For the corresponding lower bounds, we refer to [8, 13 .

Refining Kalai and Kleitman's approach, in [29], Todd showed in 2014 that $\Delta(d, n) \leq(n-$ $d)^{\log _{2} d}$ for $n \geq d \geq 1$. The Todd bound is tight for $d \leq 2$ and coincides with the true value $\Delta(d, d)$, i.e., 0 , when $n=d$. Sukegawa and Kitahara 28] slightly improved the Todd bound to $(n-d)^{\log _{2}(d-1)}$ for $n \geq d \geq 3$. We note that their bound is no longer valid for $d \leq 2$, however, it coincides with the Hirsch bound of $n-d$, and is tight for $d=3$. On the other hand, Gallagher and Kim [10] proved that the same bound holds for the diameter of normal simplicial complexes, and also improved it for polytopes.

[^0]
### 1.1 Main results

In this paper, we improve the Todd bound in high dimensions as follows:

## Theorem 1.

(a) $\Delta(d, n) \leq(n-d)^{\log _{2}(d / 2)}=(n-d)^{-1+\log _{2} d}$ for $n \geq d \geq 7$,
(b) $\Delta(d, n) \leq(n-d)^{\log _{2}(d / 4)}=(n-d)^{-2+\log _{2} d}$ for $n \geq d \geq 37$, and
(c) $\Delta(d, n) \leq(n-d)^{\log _{2}(16+d / 8)}=(n-d)^{-3+\log _{2} d+\mathcal{O}(1 / d)}$ for $n \geq d \geq 1$.

Inequalities (a) and (b) hold for, respectively, $d \geq 7$ and $d \geq 37$, and improve the Todd bound by, respectively, one and two orders of magnitude. Inequality $(c)$ holds for any $d$, and improves the Todd bound for $d \geq 19$. Note that $\log _{2}\left(16+\frac{d}{8}\right)=\log _{2}(d)-3+O\left(\frac{1}{d}\right)$ since $\log _{e}(1+x) \leq x$ for $x \geq 0$. Thus, Inequality (c) improves the Todd bound by roughly three orders of magnitude for sufficiently large $d$.

### 1.2 Our approach

As in 12, 28, 29, each inequality stated in Theorem 1 will be proved via an induction on $d$ based on the Kalai-Kleitman inequality. In contrast to [12, 28, 29, we introduce a way of strengthening Todd's analysis for the inductive step in high dimensions. In this approach, on the other hand, we need to check a large number of pairs $(d, n)$ for the base case. To address this issue, we devise a computer-assisted method which is based on two previously known upper bounds on $\Delta(d, n)$ :
(i) $\tilde{\Delta}(d, n)$, an implicit upper bound on $\Delta(d, n)$ computed recursively from the Kalai-Kleitman inequality,
(ii) the generalized Larman bound implying $\Delta(d, n) \leq 2^{d-3} n$.

The Larman bound of $2^{d-3} n$ was originally proved for bounded polyhedra [20], and improved to $\frac{2 n}{3} 2^{d-3}$ by Barnette [1]. Considering a more generalized setting, Eisenbrand et al. [8 proved a bound of $2^{d-1} n$ in 2010, before Labbé, Manneville, and Santos 19 established in 2015 an upper bound on the diameter of simplicial complexes implying $\Delta(d, n) \leq 2^{d-3} n$.

### 1.3 Related work

It should be noted that although this paper deals with only the two parameters $d$ and $n$, i.e., the dimension and the number of facets of a polyhedron, there have been studies on other parameters.

A well-known example is the maximum integer coordinate of lattice polytopes. In 18, Kleinschmidt and Onn proved that the diameter of a lattice polytope whose vertices are drawn from $\{0,1, \ldots, k\}^{d}$ is at most $k d$. This is an extension of Naddef [25] showing that the diameter of a $0-1$ polytope is at most $d$. In 2015, Del Pia and Michini [4 improved the Kleinschmidt-Onn bound to $k d-\left\lceil\frac{d}{2}\right\rceil$ for $k \geq 2$ and showed that it is tight for $k=2$, before Deza and Pournin [6] further improved the bound to $k d-\left\lceil\frac{2 d}{3}\right\rceil-(k-3)$ for $k \geq 3$. On the other hand, considering Minkowski sums of primitive lattice vectors, in [5], Deza, Manoussakis, and Onn provided a lower bound of $\left\lfloor\frac{(k+1) d}{2}\right\rfloor$ for $k<d$.

Another well-studied parameter would be $\Delta_{A}$ which is defined as the largest absolute value of a subdeterminant of the constraint matrix $A$ associated to a polyhedron. Bonifas et al. [2] strengthened and extended the Dyer and Frieze upper bound 7 holding for totally unimodular case; i.e., when $\Delta_{A}=1$. Complexity analyses based on $\Delta_{A}$ for the shadow vertex algorithm and the primal-simplex based Tardos' algorithm were proposed by Dadush and Hähnle [3, and Mizuno, Sukegawa, and Deza [22, 23], respectively.

We also note that there are studies that attempt to understand the behavior of $\Delta(d, n)$ when the number of facets is sufficiently large. Gallagher and Kim [10] provided an upper bound on the diameter of a normal simplicial complex and showed the tail-polynomiality; more specifically, they showed that the diameter is bounded from above by a polynomial in $n$ when $n$ is sufficiently large. An alternative simpler proof for such tail-polynomial upper bounds can be found in Mizuno and Sukegawa [21]. In contrast, in this paper, we assume that $d$ is large, and try to utilize this assumption to strengthen the previous results.

## 2 Preliminaries

A polyhedron $P \subseteq \mathbb{R}^{d}$ is an intersection of a finite number of closed halfspaces, and $\operatorname{dim}(P)$ denotes the dimension of the affine hull of $P$. For a polyhedron $P$, an inequality $a^{\top} x \leq \beta$ is said to be valid for $P$ if it is satisfied by every $x \in P$. We say that $F$ is a face of $P$ if there is a valid inequality $a^{\top} x \leq \beta$ for $P$ which satisfies $F=P \cap\left\{x \in \mathbb{R}^{d}: a^{\top} x=\beta\right\}$. In particular, $0-$, $1-$, and $(\operatorname{dim}(P)-1)$-dimensional faces are, respectively, referred to as vertices, edges, and facets.

The diameter $\delta(P)$ of a polyhedron $P$ is the smallest integer $k$ such that every pair of vertices of $P$ can be connected by a path using at most $k$ edges of $P$. In this paper, we are concerned with upper bounds on $\Delta(d, n)$, the maximum possible diameter of a $d$-dimensional polyhedron with $n$ facets. Lemma 1 states the Kalai-Kleitman inequality on which our approach is based.
Lemma 1 (Kalai-Kleitman inequality [12]). For $\left\lfloor\frac{n}{2}\right\rfloor \geq d \geq 2$,

$$
\Delta(d, n) \leq \Delta(d-1, n-1)+2 \Delta\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)+2
$$

### 2.1 Basic idea of our proof

We consider upper bounds of the form:

$$
f_{\alpha, \beta}(d, n)=(n-d)^{\log _{2}(\beta+d / \alpha)}
$$

where $(\alpha, \beta) \in S=\left\{(\alpha, \beta) \in \mathbb{Z}^{2}: \alpha>0, \beta \geq 0\right\}$ is a pair of integers controlling the quality of upper bounds. Note that the Todd bound is $f_{1,0}(d, n)$. The upper bounds appearing in Inequalities $(a)$, $(b)$, and $(c)$ stated in Theorem 1 correspond, respectively, to $f_{2,0}(d, n), f_{4,0}(d, n)$, and $f_{8,16}(d, n)$.

As mentioned earlier, we prove Inequalities $(a),(b)$, and $(c)$ stated in Theorem 1 via an induction on $d$ based on the Kalai-Kleitman inequality. The following lemma is the key ingredient for the inductive step; see Section 3.1 for a proof.
Lemma 2. If $(\alpha, \beta) \in S$, then there exists $d(\alpha, \beta)$ such that $d \geq d(\alpha, \beta), n \geq 2 d$, and $n \geq d+2^{2 \alpha+1}$ imply

$$
\begin{equation*}
f_{\alpha, \beta}(d-1, n-1)+2 f_{\alpha, \beta}\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)+2 \leq f_{\alpha, \beta}(d, n) \tag{1}
\end{equation*}
$$

### 2.1.1 Inductive step

Assume $d \geq d(\alpha, \beta)$ and $\mathrm{P}_{d-1}: \Delta(d-1, n) \leq f_{\alpha, \beta}(d-1, n)$ for $n \geq d-1$, as the induction hypothesis on $d$. In what follows, by induction on $n$, we prove $\mathrm{P}_{d}: \Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $n \geq d$. First, let us consider the case $n<2 d$. In this case, the claim, i.e., the desired inequality $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$, follows from the following fundamental proposition; for a proof, see, e.g., [29].
Proposition 1. $\Delta(d, n) \leq \Delta(d-1, n-1)$ for $n<2 d$.
From Proposition 1 and $\mathrm{P}_{d-1}$, for $n<2 d$,

$$
\Delta(d, n) \leq \Delta(d-1, n-1) \leq f_{\alpha, \beta}(d-1, n-1) \leq f_{\alpha, \beta}(d, n)
$$

where the last inequality follows since $\alpha>0$.
Now, suppose that $n \geq 2 d$. First, let us consider the case $n<d+2^{2 \alpha+1}$. We observe that the number of integers $n$ satisfying the condition, i.e., $2 d \leq n<d+2^{2 \alpha+1}$, is finite for fixed $d$, and becomes zero for $d \geq 2^{2 \alpha+1}$. We therefore verify $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for these pairs as a part of the base case. Next, let us consider the case $n \geq d+2^{2 \alpha+1}$. In this case, we apply the Kalai-Kleitman inequality to yield

$$
\begin{aligned}
\Delta(d, n) & \leq \Delta(d-1, n-1)+2 \Delta\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)+2 \\
& \leq f_{\alpha, \beta}(d-1, n-1)+2 f_{\alpha, \beta}\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)+2
\end{aligned}
$$

where the second inequality follows from the induction hypotheses on $d$ and $n$. Note that Lemma 2 applies to this case, which yields the desired inequality $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$.

### 2.1.2 Base case

Proposition 2. Let $(\alpha, \beta) \in S$. If there exists $l$ satisfying
$\left(B_{0}\right)$ case $d=l: \Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $n \geq d$,
$\left(B_{1}\right)$ case $l<d<d(\alpha, \beta): \Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $n \geq 2 d$,
$\left(B_{2}\right)$ case $d(\alpha, \beta) \leq d<2^{2 \alpha+1}: \Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $n$ with $2 d \leq n<d+2^{2 \alpha+1}$,
then $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $d \geq l$
Proof. By similar arguments used in the inductive step in Section 2.1.1, $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $l<d<d(\alpha, \beta)$ if $\left(B_{0}\right)$ and $\left(B_{1}\right)$ hold. Similarly, $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $d \geq d(\alpha, \beta)$ if $\left(B_{0}\right),\left(B_{1}\right)$, and $\left(B_{2}\right)$ hold.

In this study, we devise a computer-assisted method to test whether $\left(B_{0}\right),\left(B_{1}\right)$, and $\left(B_{2}\right)$ hold or not in a finite process. To this end, we
$\left(I_{1}\right)$ make the number of pairs $(d, n)$ to be checked in $\left(B_{0}\right)$ and $\left(B_{1}\right)$ finite, and
$\left(I_{2}\right)$ establish an upper bound $\tilde{\Delta}(d, n)$ on $\Delta(d, n)$ which enables us to ensure $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ via the relationship $\Delta(d, n) \leq \tilde{\Delta}(d, n) \leq f_{\alpha, \beta}(d, n)$.
When $d \leq 3$, one can set, for example, $\tilde{\Delta}(d, n):=n-d$ in $\left(I_{2}\right)$. However, for large $d$, previously known upper bounds on $\Delta(d, n)$, including the Todd bound, are of course greater than $f_{\alpha, \beta}(d, n)$, and therefore cannot be used for deriving the desired inequality, i.e., $\tilde{\Delta}(d, n) \leq f_{\alpha, \beta}(d, n)$. This is the reason why we need a computer-assisted method.

### 2.1.3 Strategy to $\left(I_{1}\right)$

We first explain our strategy to $\left(I_{1}\right)$, i.e., how to make the number of pairs $(d, n)$ to be checked in $\left(B_{0}\right)$ and $\left(B_{1}\right)$ finite.
Assumption 1. The choice of $(\alpha, \beta) \in S$ is such that $f_{\alpha, \beta}(d, n)=(n-d)^{\log _{2}(\beta+d / \alpha)}$ is superlinear in $n$ for fixed $d$ when $d \geq l$; i.e., $\alpha$ and $\beta$ satisfy $\log _{2}\left(\beta+\frac{d}{\alpha}\right)>1$ for $d \geq l$.
Observation 1. Suppose that $(\alpha, \beta) \in S$ satisfies Assumption 1. For fixed $d$, if we let $n_{L}(d)$ be the smallest integer $n$ such that $2^{d-3} n \leq f_{\alpha, \beta}(d, n)$ holds, then for $n \geq n_{L}(d)$,

$$
\Delta(d, n) \leq 2^{d-3} n \leq f_{\alpha, \beta}(d, n)
$$

Proof. Direct consequence of the generalized Larman bound.
Thus, with Assumption 1 if
$\left(B_{0}^{\prime}\right)$ case $d=l: \Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $d \leq n \leq n_{L}(d)$, $\left(B_{1}^{\prime}\right)$ case $l<d<d(\alpha, \beta): \Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $2 d \leq n \leq n_{L}(d)$, are satisfied, then $\left(B_{0}\right)$ and $\left(B_{1}\right)$ are satisfied.
Proposition 3. Let $(\alpha, \beta) \in S$. If there exists l satisfying Assumption 11 and $\left(B_{0}^{\prime}\right),\left(B_{1}^{\prime}\right)$, and $\left(B_{2}\right)$, then $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $d \geq l$.
The total number of pairs $(d, n)$ to be checked in $\left(B_{0}^{\prime}\right),\left(B_{1}^{\prime}\right)$, and $\left(B_{2}\right)$ is finite as illustrated in Table 1. In the table, we assume that we have already found a dimension $l$ satisfying ( $B_{0}^{\prime}$ ) and that $l+1<d(\alpha, \beta)$. If $l+1 \geq d(\alpha, \beta)$, then there is no pair $(d, n)$ to be checked for $\left(B_{1}^{\prime}\right)$. Also, if $d(\alpha, \beta) \geq 2^{2 \alpha+1}$, then the table will be much simpler since there is no pair $(d, n)$ to be checked for $\left(B_{1}^{\prime}\right)$ and $\left(B_{2}\right)$. The pairs $(d, n)$ with $n<2 d$ are omitted as the desired inequalities hold inductively. The meanings of the symbols are as follows:

- : corresponds to a pair $(d, n)$ to which $\tilde{\Delta}(d, n) \leq f_{\alpha, \beta}(d, n)$ must be ensured,
$\triangleleft$ : corresponds to a pair $(d, n)$ with $n=n_{L}(d)$, and
- : corresponds to a pair $(d, n)$ with $n=d+2^{2 \alpha+1}$.

Remark 1 (How to compute $n_{L}(d)$ ). In practice, we do not need to compute the value of $n_{L}(d)$ in advance. It suffices to check if $2^{d-3} n \leq f_{\alpha, \beta}(d, n)$ for $n=2 d, 2 d+1, \ldots$, for each fixed $d$. If $2^{d-3} n \leq f_{\alpha, \beta}(d, n)$ holds for the first time for some pair $\left(d, n^{\prime}\right)$, then $n_{L}(d)=n^{\prime}$.

Table 1: The pairs $(d, n)$ for which $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ should be ensured for Proposition 3


### 2.1.4 Strategy to $\left(I_{2}\right)$

We now explain our strategy to $\left(I_{2}\right)$, i.e., how to establish an upper bound $\tilde{\sim}(d, n)$ on $\Delta(d, n)$ which enables us to ensure $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ via the relationship $\Delta(d, n) \leq \tilde{\Delta}(d, n) \leq f_{\alpha, \beta}(d, n)$. We note that our strategy is based on Todd [29]. Specifically, we define $\tilde{\Delta}(d, n)$ as a value recursively computed via:

$$
\tilde{\Delta}(d, n)= \begin{cases}n-3 & \text { if } d=3 \text { and } n \geq d \\ \tilde{\Delta}(d-1, n-1) & \text { if } d>3 \text { and } d \leq n<2 d \\ \tilde{\Delta}(d-1, n-1)+2 \tilde{\Delta}\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)+2 & \text { if } d>3 \text { and } n \geq 2 d\end{cases}
$$

For example,

$$
\begin{aligned}
\tilde{\Delta}(5,13) & =\tilde{\Delta}(4,12)+2 \tilde{\Delta}(5,6)+2 \\
& =[\tilde{\Delta}(3,11)+2 \tilde{\Delta}(4,6)+2]+2 \tilde{\Delta}(4,5)+2 \\
& =[(11-3)+2 \tilde{\Delta}(3,5)+2]+2 \tilde{\Delta}(3,4)+2 \\
& =[10+2(5-3)]+2(4-3)+2=18
\end{aligned}
$$

Then, by the validity of the Kalai-Kleitman inequality, Proposition 1 and the correct inequality $\Delta(3, n) \leq n-3$, we have $\Delta(d, n) \leq \tilde{\Delta}(d, n)$ for every pair $(d, n)$ with $n \geq d \geq 3$. Therefore, for the pairs $(d, n)$ indicated by "-" in Table we check if $\tilde{\Delta}(d, n) \leq f_{\alpha, \beta}(d, n)$ instead of $\Delta(d, n) \leq$ $f_{\alpha, \beta}(d, n)$.

## 3 Proof Method

The section is devoted to the detailed description of the computer-assisted method for verifying the base case. We show our code in the programming language C and its execution results in Appendix A

## BASECASEChECKER

Input: $(\alpha, \beta) \in S$, and nonnegative integers $d(\alpha, \beta)$ and $l$ with $l \geq 3$
Output: either success or failure

Step $0\left(B_{0}^{\prime}\right)$ : If $\tilde{\Delta}(l, n)>f_{\alpha, \beta}(l, n)$ holds for some pair $(l, n)$ with $n<n_{L}(l)$, then output failure and stop. Otherwise, go to Step 1 if $l+1<d(\alpha, \beta)$, go to Step 2 if $d(\alpha, \beta)<2^{2 \alpha+1}$, and output success otherwise.
Step $1\left(B_{1}^{\prime}\right)$ : If $\tilde{\Delta}(d, n)>f_{\alpha, \beta}(d, n)$ holds for some pair $(d, n)$ with $l+1 \leq d<d(\alpha, \beta)$ and $2 d \leq n<n_{L}(d)$, then output failure and stop. Otherwise, go to Step 2.
Step $2\left(B_{2}\right)$ : If $\tilde{\Delta}(d, n)>f_{\alpha, \beta}(d, n)$ holds for some $(d, n)$ with $d(\alpha, \beta) \leq d<2^{2 \alpha+1}$ and $2 d \leq n<$ $d+2^{2 \alpha+1}$, then output failure and stop. Otherwise, output success.
(End)

Remark 2. The parameter $l$ can be excluded from the list of inputs by adding an outer-loop for $l$; i.e., starting from $l=3$, if BASECASECHECKER outputs success, then we are done; otherwise, incrementing $l$ by one, we feed it to BASECASECHECKER and repeat the same procedure.

Remark 3. The computation of $d(\alpha, \beta)$ is not included in the procedure, and hence should be done in advance; see Claim 1 for the sufficient condition for $d(\alpha, \beta)$, and also Remark 4 and Section 4 for how to compute $d(\alpha, \beta)$ in practice based on the condition.

Proposition 4. If BASECASEChECKER outputs success, then $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $n \geq d \geq$ $l$.

### 3.1 Correctness: proof of Lemma 2

Recall that Lemma 2 states that for given $(\alpha, \beta) \in S$, there exists $d(\alpha, \beta)$ such that $d \geq d(\alpha, \beta)$, $n \geq 2 d$, and $n \geq d+2^{2 \alpha+1}$ imply Inequality (1). Recall that Inequality (1) is

$$
f_{\alpha, \beta}(d-1, n-1)+2 f_{\alpha, \beta}\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)+2 \leq f_{\alpha, \beta}(d, n) .
$$

Since $n \geq 2 d$, we have either $\left\lfloor\frac{n}{2}\right\rfloor=d$ or $\left\lfloor\frac{n}{2}\right\rfloor>d$.
Suppose that $\left\lfloor\frac{n}{2}\right\rfloor=d$. In this case, the second term $2 f_{\alpha, \beta}\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)$ of the left hand side of Inequality (11) vanishes. As a matter of fact, a proof for the corresponding inequality immediately follows from the proof for $\left\lfloor\frac{n}{2}\right\rfloor>d$ which will be given in the following.

Suppose that $\left\lfloor\frac{n}{2}\right\rfloor>d$. Observe that in general, $a^{\log _{2} b}=b^{\log _{2} a}$ for $a, b>0$. Hence, letting $d(\alpha, \beta)$ be sufficiently large so that $\beta+\frac{d}{\alpha}>0$, since $n \geq d+2^{2 \alpha+1}$,

$$
f_{\alpha, \beta}(d, n)=(n-d)^{\log _{2}(\beta+d / \alpha)}=\left(\beta+\frac{d}{\alpha}\right)^{\log _{2}(n-d)}
$$

Using this, (11) can be rewritten as

$$
\begin{equation*}
\left(\beta+\frac{d-1}{\alpha}\right)^{\log _{2}(n-d)}+2\left(\beta+\frac{d}{\alpha}\right)^{\log _{2}(\lfloor n / 2\rfloor-d)}+2 \leq\left(\beta+\frac{d}{\alpha}\right)^{\log _{2}(n-d)} . \tag{2}
\end{equation*}
$$

Note that by the integrality of $\left\lfloor\frac{n}{2}\right\rfloor$ and $d$, we have $\left\lfloor\frac{n}{2}\right\rfloor-d \geq 1$. It is easily seen that

$$
\left(\beta+\frac{d}{\alpha}\right)^{\log _{2}(\lfloor n / 2\rfloor-d)} \leq\left(\beta+\frac{d}{\alpha}\right)^{\log _{2}(n / 2-d / 2)}=\left(\beta+\frac{d}{\alpha}\right)^{-1+\log _{2}(n-d)}
$$

Now, observe that $n \geq d+2^{2 \alpha+1}$ implies $\log _{2}(n-d) \geq 2 \alpha+1$, and also that for $d \geq 2$,

$$
0<\left(1-\frac{\frac{1}{\alpha}}{\beta+\frac{d}{\alpha}}\right)=\left(1-\frac{1}{d+\alpha \beta}\right)<1
$$

because $(\alpha, \beta) \in S$ implies that $\alpha \geq 1$ and $\beta \geq 0$. Then, the left-hand side of (2) is bounded from above by

$$
\begin{aligned}
& \left(\beta+\frac{d-1}{\alpha}\right)^{\log _{2}(n-d)}+2\left(\beta+\frac{d}{\alpha}\right)^{-1+\log _{2}(n-d)}+2 \\
= & f_{\alpha, \beta}(d, n)\left[\left(1-\frac{\frac{1}{\alpha}}{\beta+\frac{d}{\alpha}}\right)^{\log _{2}(n-d)}+\frac{2}{\beta+\frac{d}{\alpha}}\right]+2 \\
\leq & f_{\alpha, \beta}(d, n)\left[\left(1-\frac{\frac{1}{\alpha}}{\beta+\frac{d}{\alpha}}\right)^{2 \alpha+1}+\frac{2}{\beta+\frac{d}{\alpha}}\right]+2 \\
\leq & f_{\alpha, \beta}(d, n)\left[1-\frac{2}{f_{\alpha, \beta}(d, n)}\right]+2 \\
= & f_{\alpha, \beta}(d, n)
\end{aligned}
$$

where the second inequality follows from Claim 1 below.
Claim 1. For a given $(\alpha, \beta) \in S$, there exists $d(\alpha, \beta)$ such that $d \geq d(\alpha, \beta), n \geq 2 d$, and $n \geq d+2^{2 \alpha+1}$ imply

$$
\left(1-\frac{\frac{1}{\alpha}}{\beta+\frac{d}{\alpha}}\right)^{2 \alpha+1}+\frac{2}{\beta+\frac{d}{\alpha}} \leq 1-\frac{2}{f_{\alpha, \beta}(d, n)}
$$

Proof. Since $n \geq d+2^{2 \alpha+1}$ implies $\log _{2}(n-d) \geq 2 \alpha+1$, we have $f_{\alpha, \beta}(d, n) \geq\left(\beta+\frac{d}{\alpha}\right)^{2 \alpha+1}$, and hence it suffices to show that

$$
\begin{equation*}
\left(1-\frac{\frac{1}{\alpha}}{\beta+\frac{d}{\alpha}}\right)^{2 \alpha+1}+\frac{2}{\beta+\frac{d}{\alpha}}+2\left(\frac{1}{\beta+\frac{d}{\alpha}}\right)^{2 \alpha+1} \leq 1 \tag{3}
\end{equation*}
$$

Letting $D=\beta+\frac{d}{\alpha}$, the left-hand side of (3) can be rewritten as

$$
\left(1-\frac{\frac{1}{\alpha}}{D}\right)^{2 \alpha+1}+\frac{2}{D}+2\left(\frac{1}{D}\right)^{2 \alpha+1}=1+\frac{\sum_{k=0}^{2 \alpha} c(k) D^{k}}{D^{2 \alpha+1}}
$$

where $c(1), c(2), \ldots, c(2 \alpha)$ are coefficients independent from $D$. In particular, the coefficient $c(2 \alpha)$ of the term of maximum degree with respect to $D$ is strictly negative:

$$
c(2 \alpha)=\binom{2 \alpha+1}{1} \cdot\left(-\frac{1}{\alpha}\right)+2=-2-\frac{1}{\alpha}+2=-\frac{1}{\alpha}<0 .
$$

Therefore, when $D$ is sufficiently large, the numerator $\sum_{k=0}^{2 \alpha} c(k) D^{k}$ is strictly negative. Since $\alpha>0$, one can conclude that there exists $d(\alpha, \beta)$ satisfying the desired condition, which completes the proof of Claim

Remark 4 (How to calculate $d(\alpha, \beta)$ in practice). To compute $d(\alpha, \beta)$ satisfying the conditions of Claim 1, it is enough to determine the largest root $D^{*}$ of the numerator $f(D)=\sum_{k=0}^{2 \alpha} c(k) D^{k}$ and simply set $d(\alpha, \beta)=\left\lceil\alpha\left(D^{*}-\beta\right)\right\rceil$. In this paper, we compute an upper bound on $\left\lceil\alpha\left(D^{*}-\beta\right)\right\rceil$ by elementary calculus; see Section 4 for the details.

## 4 Numerical Examples

This section explains how BaseCaseChecker works using the cases $(\alpha, \beta) \in\{(2,0),(4,0),(8,16)\}$, which yield the inequalities stated in Theorem 1

### 4.1 Case $(\alpha, \beta)=(2,0)$

As indicated in the proof of Claim it suffices to find $d(2,0)$ such that $d \geq d(2,0)$ implies Inequality (3) with $(\alpha, \beta)=(2,0)$, i.e.,

$$
\begin{equation*}
\left(1-\frac{\frac{1}{2}}{\frac{d}{2}}\right)^{5}+\frac{2}{\frac{d}{2}}+2\left(\frac{2}{d}\right)^{5} \leq 1 \tag{4}
\end{equation*}
$$

Observation 2. Inequality (4) holds for $d \geq 10$, hence, we can set $d(2,0):=10$.
Proof. Observe that Inequality (4) is equivalent to $(d-1)^{5}+4 d^{4}+2 \cdot 2^{5} \leq d^{5}$, which can be rewritten as

$$
\begin{equation*}
-d^{4}+10 d^{3}-10 d^{2}+5 d+63 \leq 0 . \tag{5}
\end{equation*}
$$

For $d \geq 10$,

- $-d^{4}+10 d^{3} \leq-10 d^{3}+10 d^{3} \leq 0$,
- $-10 d^{2}+5 d \leq-100 d+5 d \leq-95 d$.

Therefore, for $d \geq 10$, the left-hand side of (5) is bounded from above by

$$
-10 d^{3}+10 d^{3}-100 d+5 d+63 \leq-95 d+63
$$

which is negative for $d \geq 10$.
For the case $(\alpha, \beta)=(2,0), l$ must be at least four because the exponent $\log _{2}\left(\frac{d}{2}\right)$ is smaller than 1 for $d \leq 3$. It was verified that for each of $d$ with $d \in\{4,5,6\}$, there exists a pair $(d, n)$ such that $n<n_{L}(d)$ while $\tilde{\Delta}(d, n)>f_{\alpha, \beta}(d, n)$ :

$$
\begin{aligned}
& f_{2,0}(4,8)=(8-4)^{\log _{2}(4 / 2)}=4.00<6.00=\tilde{\Delta}(4,8) \\
& f_{2,0}(5,10)=(10-5)^{\log _{2}(5 / 2)}<8.40<9.00=\tilde{\Delta}(5,10) \\
& f_{2,0}(6,24)=(24-6)^{\log _{2}(6 / 2)}<97.63<98.00=\tilde{\Delta}(6,24)
\end{aligned}
$$

Hence, our approach cannot ensure $\Delta(d, n) \leq f_{\alpha, \beta}(d, n)$ for $d \leq 6$ although it can be true. On the other hand, BASECASECHECKER outputs success for $l=7$. In what follows, we provide a few details.

Execution results on $\left(B_{0}^{\prime}\right)$ Figure 1 shows the values of $f_{2,0}(7, n), \tilde{\Delta}(7, n)$, and the generalized Larman bound for $d=7$. As we see from Figure [1 it was verified that for $d=7 \leq n \leq 45$, we have $\tilde{\Delta}(7, n) \leq f_{2,0}(7, n)$, which implies $\Delta(7, n) \leq f_{2,0}(7, n)$. Also, for $n=46$, the value of $f_{2,0}(7, n)$ is at most 750.96 while that of the generalized Larman bound is 736 , hence, $n_{L}(7)=46$. Therefore, $\Delta(d, n) \leq f_{2,0}(d, n)$ for $d=7$, namely, $\left(B_{0}^{\prime}\right)$ holds.

Execution results on $\left(B_{1}^{\prime}\right)$ and $\left(B_{2}\right)$ Furthermore, it was verified that $\left(B_{1}^{\prime}\right)$ and $\left(B_{2}\right)$ hold, where $n_{L}(8)=47$ and $n_{L}(9)=51$ when verifying $\left(B_{1}^{\prime}\right)$, and $d$ ranges from 10 to $2^{2 \cdot 2+1}=32$ when verifying $\left(B_{2}\right)$. To sum up, by Proposition 4 ,

$$
\Delta(d, n) \leq f_{2,0}(d, n)=(n-d)^{\log _{2}(d / 2)}=(n-d)^{-1+\log _{2} d} \text { for } n \geq d \geq 7
$$

which yields Inequality (a) stated in Theorem 1

### 4.2 Case $(\alpha, \beta)=(4,0)$

In this case, it suffices to find $d(4,0)$ such that $d \geq d(4,0)$ implies

$$
\begin{equation*}
\left(1-\frac{\frac{1}{4}}{\frac{d}{4}}\right)^{9}+\frac{2}{\frac{d}{4}}+2\left(\frac{1}{\frac{d}{4}}\right)^{9} \leq 1 . \tag{6}
\end{equation*}
$$



Figure 1: Values of $f_{2,0}(7, n), \tilde{\Delta}(7, n)$, and the generalized Larman bound for $d=7$, i.e, $2^{4} n$

Observation 3. Inequality (6) holds for $d \geq 36$, hence, we can set $d(4,0):=36$.
Proof. Observe that Inequality (6) is equivalent to $(d-1)^{9}+8 d^{8}+2 \cdot 4^{9} \leq d^{9}$, which can be rewritten as

$$
\begin{equation*}
-d^{8}+36 d^{7}-84 d^{6}+126 d^{5}-126 d^{4}+84 d^{3}-36 d^{2}+9 d+524287 \leq 0 \tag{7}
\end{equation*}
$$

For $d \geq 36$,

- $-d^{8}+36 d^{7} \leq-36 d^{7}+36 d^{7} \leq 0$,
- $-84 d^{6}+126 d^{5} \leq-3024 d^{5}+126 d^{5}=-2898 d^{5}$,
- $-126 d^{4}+84 d^{3} \leq-4536 d^{3}+126 d^{3}=-4452 d^{3}$,
- $-36 d^{2}+9 d \leq-1296 d+9 d=-1287 d$.

Therefore, for $d \geq 36$, the left-hand side of (7) is bounded from above by

$$
-2898 d^{5}-4452 d^{3}-1287 d+524287
$$

which is strictly negative for $d \geq 36$.
Since $l \geq d(4,0)=36$, BaseCaseChecker skips Step 1 and goes to Step 2 after verifying in Step 0 that $\left(B_{0}^{\prime}\right)$ holds with $l=37$, where $n_{L}(37)=42946$. BASECASECHECKER verified that $\left(B_{2}\right)$ also holds and outputs success, which implies that

$$
\Delta(d, n) \leq f_{4,0}(d, n)=(n-d)^{\log _{2}(d / 4)}=(n-d)^{-2+\log _{2} d} \text { for } n \geq d \geq 37
$$

This is Inequality (b) stated in Theorem 1

### 4.3 Case $(\alpha, \beta)=(8,16)$

As a matter of fact, BASECASECHECKER runs out of computational memory for the case $(\alpha, \beta)=$ $(8,0)$. This is because $n_{L}(d)$ exceeded the limitation on the array length in our circumstance.

For achieving an upper bound with $\alpha=8$, one can increase the value of $\beta$. This makes $n_{L}(d)$ relatively small. For example, BASECASECHECKER outputs success for the case $(\alpha, \beta)=(8,16)$
with $l=4$. It is not difficult to see that we can set $d(8,16)=8$; see Appendix B for the details. Since for $d \leq 3$,

$$
\Delta(d, n) \leq n-d \leq(n-d)^{4} \leq(n-d)^{\log _{2}(16+d / 8)}=f_{8,16}(d, n)
$$

we conclude that

$$
\Delta(d, n) \leq f_{8,16}(d, n)=(n-d)^{\log _{2}(16+d / 8)}=(n-d)^{-3+\log _{2} d+\mathcal{O}(1 / d)} \text { for } n \geq d \geq 1
$$

which yields Inequality (c) stated in Theorem 1
Remark 5. A further improved upper bound of the form $f_{\alpha, \beta}(d, n)$ with $\alpha>8$ may be proven by making $\beta$ larger. The resulting upper bound is, however, still in the form of $(n-d)^{\log _{2} \mathcal{O}(d)}$.

Remark 6. Since our proof method is based on only the Kalai-Kleitman inequality and the generalized Larman bound, one can easily apply it to a more generalized setting where we have the similar results; see, e.g., [10] who proved an improved upper bound on the diameter of normal simplicial complexes by extending the proof of [28], a special case of this study.

Remark 7. Although not surprising, our approach cannot yield any polynomial bound. Specifically, for an arbitrarily given polynomial function $p(d, n)$, there are infinitely many pairs $(d, n)$ such that Inequality (1), the inequality which needs to shown in the inductive step, does not hold. Proof. See Appendix C

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## A A C Code for BaseCaseChecker and Its Execution Results

In what follows, we show our code for BASECASECHECKER in the programming language C. The values of the parameters are those used for the case $(\alpha, \beta)=(2,0)$. Note that $d(2,0)=10$. It accepts the value of $l$ from the standard input.

## A C Code for BaseCaseChecker

```
#include<stdio.h>
#include<math.h>
#include<stdlib.h>
#define A 2 //alpha
#define B 0 //beta
#define D_AB 10 //d(alpha,beta)
#define N 1000000 //array length
double T[N]={0}, U[N]={0}; //T: tilde_D(d-1,n), U: tilde_D(d,n)
double bound_Larman(int,int); //return n*2^{d-3}
double bound_Ours(int,int); //return f_{A,B}(d,n)
void initialize(void); //initialize T and U
void Update(int); //update U using T
void check(int,int); //check if tilde_D(d,n)<=f(d,n)
int main(void)
    int d,n,d_max=(int) pow(2.0,2.0*A+1); //d, n, and the maximum of d
    int flag = 0; //takes 1 if tilde_D(d,n)>f(d,n) holds
    initialize();
    int l=3;
    printf("Enter l (>=3): ");
    scanf("%d",&l);
    //Compute tilde_D(d,n) for d=l
    for(d=3; d<l; d++){
        Update (d+1);
    }
    //B0
    n=l; printf("\n");
    while(n<N && bound_Larman(d,n)>bound_Ours(d,n)){
        check(d,n);
        n++;
    }
    printf("- n_L(%d) = %d\n",d,n);
    printf("(BO) OK\n");
    Update(d+1);
    d++;
    //B1
    while(d<D_AB){
        n=2*d;
        while(bound_Larman(d,n)>bound_Ours(d,n)){
            check(d,n);
            n++;
        }
        printf("- n_L(%d) = %d\n",d,n);
        Update(d+1);
        d++;
    }
    printf("(B1) OK\n");
    //B2
    while(d<d_max){
        n=2*d;
        int count = 0;
        while(n<d+d_max){
            check(d,n);
            n++;
            count++;
        }
        printf("- # pairs (%d,n) checked = %d\n",d,count);
        Update(d+1);
        d++;
    }
    printf("(B2) OK\n");
    printf("\n****** SUCCESS ******");
    return 0;
```

```
}
double bound_Larman(int d, int n){return n*pow(2.0,d-3);}
double bound_Ours(int d, int n){return pow(1.0*(n-d),log(1.0*d/A+B)/log(2));}
void initialize(void)
{
    int i;
    for(i=0; i<N; i++){
        int n = i+3; //i = n - 3
        U[i] = 1.0*(n-3); //use the Hirsch bound for d=3
    }
}
void Update(int d)
{
    int i;
    for(i=0; i<N; i++) T[i] = U[i];
    for(i=0; i<N; i++){
            int n = i+d; //i = n - d
            if(n < 2*d){U[i] = T[i];} //tilde_D(d,n)=tilde_D(d-1,n-1)
            else{U[i] = T[i]+2*U[n/2-d]+2;} //tilde_D(d,n)=tilde_D(d-1,n-1)+2*tilde_D(d,n/2)+2
    }
}
void check(int d,int n)
{
    int i=n-d;
    if(bound_Ours(d,n)<U[i]){
        printf("Error: %.1f [Ours] < %.1f [tilde] (%d,%d)\n",bound_Ours(d,n),U[i],d,n);
        printf("\n****** FAILURE ******"); exit(1);
    }else if (n==N-1){
        printf("Error: Out of Memory\n");
        printf("\n****** FAILURE ******"); exit(1);
    }
}
```

Next, we show some execution results of the above code.
Case $(\alpha, \beta)=(2,0)$ with $l=7$ and $d(2,0)=10$
Enter 1 (>=3): 7


Case $(\alpha, \beta)=(2,0)$ with $l=6$ and $d(2,0)=10$

Enter 1 (>=3): 6
Error: 97.6 [Ours] < 98.0 [tilde] $(6,24)$
****** FAILURE ******

Case $(\alpha, \beta)=(4,0)$ with $l=37$ and $d(4,0)=36$

Enter l (>=3): 37
$-n_{-} L(37)=42946$
(B0) OK
(B1) OK

- \# pairs $(38, n)$ checked $=474$

```
# pairs (39,n) checked = 473
- # pairs (40,n) checked = 472
//snip
# pairs (509,n) checked = 3
_ # pairs (510,n) checked = 2
# pairs (511,n) checked = 1
(B2) OK
```

Case $(\alpha, \beta)=(4,0)$ with $l=36$ and $d(4,0)=36$ Enter 1 ( $>=3$ ): 36
Error: 1469828390203.3 [Ours] < 1469922992914.0 [tilde] $(36,6928)$
****** FAILURE ******

## B The computation of $d(8,16)$

It suffices to find $d(8,16)$ such that $d \geq d(8,16)$ implies

$$
\begin{equation*}
\left(1-\frac{\frac{1}{8}}{16+\frac{d}{8}}\right)^{17}+\frac{2}{16+\frac{d}{8}}+2\left(\frac{1}{16+\frac{d}{8}}\right)^{17} \leq 1 \tag{8}
\end{equation*}
$$

For notational simplicity, set $D:=\frac{d}{8}+16$, and rewrite Inequality (8) as

$$
\begin{equation*}
\left(D-\frac{1}{8}\right)^{17}+2 D^{16}+2 \leq D^{17} \tag{9}
\end{equation*}
$$

We prove that Inequality (9) holds for $D \geq 17$. If this is true, then Inequality (8) is satisfied for $\frac{d}{8}+16 \geq 17$, i.e., for $d \geq 8$. By simple calculus, Inequality (9) is rewritten as

$$
\begin{aligned}
& D^{17}-\frac{17}{8} D^{16}+2 D^{16}+2-D^{17}+\sum_{i=2}^{17}\binom{17}{i} D^{17-i}\left(-\frac{1}{8}\right)^{i} \leq 0 \\
\Longleftrightarrow & -\frac{1}{8} D^{16}+\binom{17}{2} \frac{1}{8^{2}} D^{15}-\binom{17}{3} \frac{1}{8^{3}} D^{14}+\binom{17}{4} \frac{1}{8^{4}} D^{13}-\cdots-\binom{17}{16} \frac{1}{8^{16}} D-\binom{17}{17} \frac{1}{8^{17}}+2 \leq 0 .
\end{aligned}
$$

We observe that for $D \geq 17$,

$$
\begin{gather*}
-\frac{1}{8} D^{16}+\binom{17}{2} \frac{1}{8^{2}} D^{15} \leq-\frac{17}{8} D^{15}+\frac{17 \cdot 16}{2 \cdot 1} \frac{1}{8^{2}} D^{15}=0 \\
-\binom{17}{3} \frac{1}{8^{3}} D^{14}+\binom{17}{4} \frac{1}{8^{4}} D^{13} \leq D^{13} \frac{1}{8^{3}}\binom{17}{3}\left(-17+\frac{14}{4} \cdot \frac{1}{8}\right) \leq 0  \tag{10}\\
-\binom{17}{5} \frac{1}{8^{5}} D^{12}+\binom{17}{6} \frac{1}{8^{6}} D^{11} \leq D^{11} \frac{1}{8^{5}}\binom{17}{5}\left(-17+\frac{12}{6} \cdot \frac{1}{8}\right) \leq 0 \\
\vdots \\
-\binom{17}{15} \frac{1}{8^{15}} D^{2}+\binom{17}{16} \frac{1}{8^{16}} D \leq D \frac{1}{8^{15}}\binom{17}{15}\left(-17+\frac{2}{16} \cdot \frac{1}{8}\right) \leq 0
\end{gather*}
$$

We remark that here, the tiny term of $-\binom{17}{17} \frac{1}{8^{17}}+2$, which is at most 2 , was ignored. It can be included, for example, in Inequality (10) because for $D \geq 17$,

$$
D^{13} \frac{1}{8^{3}}\binom{17}{3}\left(-17+\frac{14}{4} \cdot \frac{1}{8}\right) \leq 17^{13} \frac{1}{17^{3}}(-1) \ll-2 .
$$

This completes the proof.

## C Proof of Remark 7

We want to prove that for an arbitrarily given polynomial function $p(d, n)$, there are infinitely many pairs $(d, n)$ such that the inequality, which needs to be proved in the inductive step of our proof method, does not hold; i.e.,

$$
p(d-1, n-1)+2 p\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)+2>p(d, n)
$$

holds, even when $d$ and $n$ are sufficiently large.
The polynomial function $p(d, n)$ can be rewritten as

$$
p(d, n)=\sum_{i=0}^{k} g_{i}(d) n^{i}
$$

for some nonnegative integer $k$, where $g_{i}(d)$ is a polynomial function of $d$ for each $i \in\{0,1, \ldots, k\}$.
We can assume that for any $d, g_{k}(d) \geq 0$; otherwise $p(d, n)$ cannot be a valid upper bound on $\Delta(d, n)$ because, in this case, $p(d, n)$ is negative for sufficiently large $n$. In what follows, we consider the case when $d$ is larger than the maximum root of $g_{k}$. Note that in this case, $g_{k}(d)>0$. It is easily seen that

$$
\frac{p(d, n)}{g_{k}(d) n^{k}}=\sum_{i=0}^{k} \frac{g_{i}(d)}{g_{k}(d)} \cdot \frac{1}{n^{k-i}}=1+\sum_{i=0}^{k-1} \frac{g_{i}(d)}{\left.g_{k} d\right)} \cdot \frac{1}{n^{k-i}}
$$

Therefore, for any $\epsilon>0$, when $n$ is sufficiently large,

$$
(1-\epsilon) g_{k}(d) n^{k} \leq p(d, n) \leq(1+\epsilon) g_{k}(d) n^{k} .
$$

By similar arguments,

$$
\begin{aligned}
(1-\epsilon) g_{k}(d-1)(n-1)^{k} & \leq p(d-1, n-1) \leq(1+\epsilon) g_{k}(d-1)(n-1)^{k} \\
(1-\epsilon) g_{k}(d)\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{k} & \leq p\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right) \leq(1+\epsilon) g_{k}(d)\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{k}
\end{aligned}
$$

Using these relations,

$$
\frac{p(d-1, n-1)}{p(d, n)} \geq \frac{1-\epsilon}{1+\epsilon} \cdot \frac{g_{k}(d-1)}{g_{k}(d)} \cdot\left(1-\frac{1}{n}\right)^{k}
$$

Since $g_{k}$ is a polynomial function of $d$, the ratio of $\frac{g_{k}(d-1)}{g_{k}(d)}$ gets arbitrarily close to 1 by taking $d$ sufficiently large. Therefore, the right-hand side gets arbitrarily close to 1 by taking $d$ and $n$ sufficiently large, and $\epsilon$ sufficiently small.

On the other hand, assuming that $n$ is even for convenience,

$$
\frac{p\left(d,\left\lfloor\frac{n}{2}\right\rfloor\right)}{p(d, n)} \geq \frac{1-\epsilon}{1+\epsilon}\left(\frac{1}{2}\right)^{k}
$$

The right-hand side is bounded from below by a positive constant factor, say, $\frac{1}{3}\left(\frac{1}{2}\right)^{k}$ when $\epsilon \leq \frac{1}{2}$. To sum up, if both $d$ and $n$ are sufficiently large, then

$$
\frac{p(d-1, n-1)}{p(d, n)}+\frac{p(d,\lfloor n / 2\rfloor)}{p(d, n)}>1
$$

which implies $p(d-1, n-1)+p(d,\lfloor n / 2\rfloor)+2>p(d, n)$.


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