# Maximum scattered $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1,q^4)$

Bence Csajbók and Corrado Zanella*

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**Abstract**

There are two known families of maximum scattered $\mathbb{F}_q$-linear sets in $\mathrm{PG}(1,q^t)$: the linear sets of pseudoregulus type and for $t \geq 4$ the scattered linear sets found by Lunardon and Polverino. For $t=4$ we show that these are the only maximum scattered $\mathbb{F}_q$-linear sets and we describe the orbits of these linear sets under the groups $\mathrm{PGL}(2,q^4)$ and $\mathrm{P\Gamma L}(2,q^4)$.

## 1 Introduction

Recent investigations on linear sets in a finite projective line $\mathrm{PG}(1,q^t)$ of rank $t$ concerned: the hypersurface obtained from the linear sets of pseudoregulus type by applying field reduction [12]; a geometric characterization of the linear sets of pseudoregulus type [9]; a characterization of the clubs, that is, the linear sets of rank $r$ with a point of weight $r-1$ [13]; a generalization of clubs in order to construct KM-arcs [10]; a condition for the equivalence of two linear sets [8, 18]; the definition and study of the class of a linear set in order to study their equivalence [7]; a construction method which yields MRD-codes from maximum scattered linear sets of $\mathrm{PG}(1,q^t)$ [17]. Furthermore, the linear sets in $\mathrm{PG}(1,q^t)$ coincide with the so-called splashes of subgeometries [13]. The results of such investigations make it reasonable to attempt to classify the linear sets in $\mathrm{PG}(1,q^t)$ of rank $t$ for small $t$.

A point in $\mathrm{PG}(1,q^t)$ is the $\mathbb{F}_{q^t}$-span $\langle \mathbf{v}\rangle_{\mathbb{F}_{q^t}}$ of a nonzero vector $\mathbf{v}$ in a two-dimensional vector space, say $W$, over $\mathbb{F}_{q^t}$. If $U$ is a subspace over $\mathbb{F}_q$ of

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$W$, then $L_U = \{\langle \mathbf{v}\rangle_{\mathbb{F}_{q^t}} : \mathbf{v} \in U\setminus\{\mathbf{0}\}\}$ denotes the associated $\mathbb{F}_q$-*linear set* (or simply *linear set*) in $\mathrm{PG}(1,q^t)$. The *rank* of such a linear set is $r = \dim_{\mathbb{F}_q} U$. Any linear set in $\mathrm{PG}(1,q^t)$ of rank greater than $t$ coincides with the whole projective line. The *weight* of a point $P = \langle \mathbf{v}\rangle_{\mathbb{F}_{q^t}}$ is $w(P) = \dim_{\mathbb{F}_q}(U\cap P)$. If the rank and the size of $L_U$ are $r$ and $(q^r-1)/(q-1)$, respectively, then $L_U$ is *scattered*. Equivalently, $L_U$ is scattered if and only if all its points have weight one. A scattered $\mathbb{F}_q$-linear set of rank $t$ in $\mathrm{PG}(1,q^t)$ is *maximum scattered*. An example of maximum scattered $\mathbb{F}_q$-linear set in $\mathrm{PG}(1,q^t)$ is $L_V$ with $V = \{(u,u^q) \colon u \in \mathbb{F}_{q^t}\}$. Any subset of $\mathrm{PG}(1,q^t)$ projectively equivalent to this $L_V$ is called *linear set of pseudoregulus type*. See [9] for a geometric description, and [7] or the survey [16] for further background on linear sets. Note that for any $\varphi \in \Gamma\mathrm{L}(2,q^t)$ with related collineation $\tilde{\varphi} \in \mathrm{P}\Gamma\mathrm{L}(2,q^t)$ and any $\mathbb{F}_q$-linear set $L_U$, $L_{U^\varphi} = (L_U)^{\tilde{\varphi}}$. In [7, Theorem 4.5] it is proved that if $t = 4$ and $L_U$ has *maximum field of linearity* $\mathbb{F}_q$, that is, $L_U$ is not an $\mathbb{F}_{q^s}$-linear set for $s > 1$, then any linear set in the same orbit of $L_U$ under the action of $\mathrm{P}\Gamma\mathrm{L}(2,q^4)$ is of type $L_{U^\varphi}$ with $\varphi \in \Gamma\mathrm{L}(2,q^4)$. Note that this is not true if $t > 4$. In [14], Lunardon and Polverino construct a class of maximum scattered linear sets:

**Theorem 1.1** ([14])**.** *Let $q$ be a prime power, $t \geq 4$ an integer, $b \in \mathbb{F}_{q^t}$ such that the norm $\mathrm{N}_{q^t/q}(b)$ of $b$ over $\mathbb{F}_q$ is distinct from one, and*

$$U(b,t) = \{(u, bu^q + u^{q^{t-1}}) : u \in \mathbb{F}_{q^t}\}. \tag{1}$$

*If $b \neq 0$ then $L_{U(b,t)}$ is a maximum scattered $\mathbb{F}_q$-linear set in $\mathrm{PG}(1,q^t)$ and if $q > 3$, then it is not of pseudoregulus type.*

It can be directly seen that $L_{U(0,t)}$ is maximum scattered of pseudoregulus type. For $t = 4$, Theorem 1.1 can be extended to $q = 3$, as it can be checked by using the package `FinInG` of `GAP` [3]. In the following $t = 4$ is assumed. For all $b \in \mathbb{F}_{q^4}$ define

$$U(b) = U(b,4) = \{(x, bx^q + x^{q^3}) : x \in \mathbb{F}_{q^4}\}. \tag{2}$$

In section 2 it is shown that $\mathrm{N}_{q^4/q}(b) \neq 1$ is a necessary condition to obtain scattered linear sets of $\mathrm{PG}(1,q^4)$ and the case $\mathrm{N}_{q^4/q}(b) = 1$ is dealt with. In this case, $L_{U(b)}$ contains either one or $q+1$ points of weight two, and the remaining points have weight one.

The main result in section 3 is that if $L$ is a maximum scattered linear set in $\mathrm{PG}(1,q^4)$, then $L$ is projectively equivalent to $L_{U(b)}$ for some $b \in \mathbb{F}_{q^4}$ with $\mathrm{N}_{q^4/q}(b) \neq 1$ (cf. Theorem 3.4).

In section 4 the orbits of the $\mathbb{F}_q$-linear sets of rank four in $\mathrm{PG}(1,q^4)$ of type $L_{U(b)}$, under the actions of both $\mathrm{PGL}(2,q^4)$ and $\mathrm{P\Gamma L}(2,q^4)$, are completely characterized. Such orbits only depend on the norm $b^{q^2+1}$ of $b$ over $\mathbb{F}_{q^2}$. In particular, $\mathrm{PG}(1,q^4)$ contains precisely $q(q-1)/2$ maximum scattered linear sets up to projective equivalence (Theorem 4.5), one of them is of pseudoregulus type, the others are as in Theorem 1.1.

## 2 Classification

This section is devoted to the classification of all $L_{U(b)}$ for $b \in \mathbb{F}_{q^4}$, where $U(b)$ is as in (2).

**Theorem 2.1.** *For $b \in \mathbb{F}_{q^4}$ the following holds.*

1. *If $\mathrm{N}_{q^4/q}(b) \neq 1$, then $L_{U(b)}$ is scattered.*
2. *If $\mathrm{N}_{q^4/q^2}(b) = 1$, then $L_{U(b)}$ has a unique point with weight two, the point $\langle(1,0)\rangle_{\mathbb{F}_{q^4}}$, and all other with weight one.*
3. *If $\mathrm{N}_{q^4/q^2}(b) \neq 1$ and $\mathrm{N}_{q^4/q}(b) = 1$, then $L_{U(b)}$ has $q+1$ points with weight two and all other with weight one.*

*Proof.* Put $f_b(x) = bx^q + x^{q^3}$. For $x \in \mathbb{F}_{q^4}^*$ the point $P_x := \langle(x, f_b(x))\rangle_{\mathbb{F}_{q^4}}$ of $L_{U(b)}$ has weight more than one if and only if there exists $y \in \mathbb{F}_{q^4}^*$ and $\lambda \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ such that $\lambda(x, f_b(x)) = (y, f_b(y))$. This holds if and only if $y = \lambda x$ and

$$\lambda b x^q + \lambda x^{q^3} - \lambda^q b x^q - \lambda^{q^3} x^{q^3} = 0. \tag{3}$$

For a given $x$ the solutions in $\lambda$ of (3) form an $\mathbb{F}_q$-subspace whom rank equals to the weight of the point $P_x$. Since $q$-polynomials over $\mathbb{F}_{q^4}$ of rank 1 are of the form $\alpha \operatorname{Tr}_{q^4/q}(\beta x) \in \mathbb{F}_{q^4}[x]$, it is clear that the kernel of the $\mathbb{F}_q$-linear map in the variable $\lambda$ at the left-hand side of (3) has dimension at most two and hence the weight of each point of $L_{U(b)}$ is at most two. If $(\lambda, x)$ is a solution of (3) for some $\lambda \in \mathbb{F}_{q^4}$ and $x \in \mathbb{F}_{q^4}^*$, then $(\lambda', x')$ is also a solution for each $\lambda' \in \langle 1, \lambda\rangle_{\mathbb{F}_q}$ and $x' \in \langle x\rangle_{\mathbb{F}_{q^2}}$ and hence for each $\mu \in \mathbb{F}_{q^2}^*$ if $P_x$ has weight two, then $P_{\mu x} := \langle(\mu x, f_b(\mu x))\rangle_{\mathbb{F}_{q^4}}$ has weight two as well. Note that $P_{\mu x} = \langle(1, \mu^{q-1}(b x^{q-1} + x^{q^3-1}))\rangle_{\mathbb{F}_{q^4}}$ and hence if $P_x \neq \langle(1,0)\rangle_{\mathbb{F}_{q^4}}$ has weight two, then $\{P_{\mu x} \colon \mu \in \mathbb{F}_{q^2}^*\}$ is a set of $q+1$ distinct points with weight 2.

The function $f_b(x)$ is not $\mathbb{F}_{q^2}$-linear and hence the maximum field of linearity of $L_{U(b)}$ is $\mathbb{F}_q$. It follows (cf. [7, Proposition 2.2])) that $L_{U(b)}$ has at least one point with weight one, say $\langle(x_0, f_b(x_0))\rangle_{\mathbb{F}_{q^4}}$. Then the line of $\mathrm{AG}(2,q^4)$ with equation $x_0Y = f_b(x_0)X$ meets the graph of $f_b(x)$, that is, $\{(x, f_b(x)) \colon x \in \mathbb{F}_{q^4}\}$, in exactly $q$ points. It follows from [1, 2], see also [6], that the number of directions determined by $f_b(x)$ is at least $q^3+1$, and hence also $|L_{U(b)}| \geq q^3+1$. Denote by $w_1$ and $w_2$ the number of points of $L_{U(b)}$ with weight one and two, respectively. Then

$$w_1 + w_2 = |L_{U(b)}| \geq q^3 + 1, \tag{4}$$

$$w_1(q-1) + w_2(q^2-1) = q^4 - 1. \tag{5}$$

Subtracting (4) $(q-1)$-times from (5) gives $w_2(q^2-q) \leq q^3 - q$ and hence $w_2 \leq q+1$. At this point it is clear that in $L_{U(b)}$ there is either one point with weight two, the point $\langle(1,0)\rangle_{\mathbb{F}_{q^4}}$, or there are exactly $q+1$ of them and $\langle(1,0)\rangle_{\mathbb{F}_{q^4}}$ is not one of them.

If $\mathrm{N}_{q^4/q}(b) \neq 1$, then Theorem 1.1 states that $L_{U(b)}$ is scattered. We show that $\langle(1,0)\rangle_{q^4}$ has weight two if and only if $\mathrm{N}_{q^4/q^2}(b) = 1$. Note that the weight of this point is the dimension of the kernel of $f_b(x)$. If $f_b(x) = 0$ for some $x \in \mathbb{F}^*_{q^4}$, then $b = -x^{q^3-q}$ and hence, by taking $(q^2+1)$-th powers at both sides, $\mathrm{N}_{q^4/q^2}(b) = 1$. On the other hand, if $\mathrm{N}_{q^4/q^2}(b) = 1$, then $b = w^{q^2-1}$ for some $w \in \mathbb{F}^*_{q^4}$. Let $\varepsilon$ be a non-zero element of $\mathbb{F}_{q^4}$ such that $\varepsilon^{q^2} + \varepsilon = 0$. Then it is easy to check that the kernel of $f_b(x)$ is $\langle(\varepsilon w)^{q^3}\rangle_{\mathbb{F}_{q^2}}$ which has dimension two over $\mathbb{F}_q$ and hence $\langle(1,0)\rangle_{q^4}$ has weight two.

It remains to prove that if $\mathrm{N}_{q^4/q}(b) = 1$ and $\mathrm{N}_{q^4/q^2}(b) \neq 1$, then there is at least one point (hence precisely $q+1$ points) of weight two. After rearranging in (3), we obtain

$$(\lambda - \lambda^q)^{q^3-1} = bx^{q-q^3}. \tag{6}$$

By taking $(q^2+1)$-th powers on both sides we can eliminate $x$, obtaining

$$(\lambda - \lambda^q)^{(q^3-1)(q^2+1)} = (\lambda - \lambda^q)^{(q-1)(q^2+1)} = b^{q^2+1}. \tag{7}$$

It is clear that we can find $\lambda \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ satisfying (7) if and only if there exists $\epsilon \in \mathbb{F}^*_{q^4}$ such that

$$(\lambda - \lambda^q)^{q^3-1}/b = \epsilon^{q^2-1}. \tag{8}$$

Then $x \in \langle \epsilon^q \rangle_{\mathbb{F}_{q^2}}$ with $y = \lambda x$ satisfies our initial conditions in (3).

Now use $\mathrm{N}_{q^4/q}(b)=1$ and put $b=\mu^{q-1}$ for some $\mu\in\mathbb{F}_{q^4}^*$. Then (7) can be written as

$$\left(\frac{\lambda-\lambda^q}{\mu}\right)^{(q-1)(q^2+1)}=1. \tag{9}$$

We can solve (9) if and only if there exists $\delta\in\mathbb{F}_{q^4}^*$ such that

$$\left(\frac{\lambda-\lambda^q}{\mu}\right)^{q-1}=\delta^{q^2-1}, \tag{10}$$

or, equivalently,

$$\left\langle\frac{\lambda-\lambda^q}{\mu}\right\rangle_{\mathbb{F}_q}=\langle\delta^{q+1}\rangle_{\mathbb{F}_q}. \tag{11}$$

Now we will continue in $\mathrm{PG}(\mathbb{F}_{q^4},\mathbb{F}_q)=\mathrm{PG}(3,q)$. At the left-hand side of (11) we can see a point of the hyperplane $\mathcal{H}_\mu$ defined as

$$\mathcal{H}_\mu=\{\langle z\rangle_{\mathbb{F}_q}\colon\ \mathrm{Tr}_{q^4/q}(\mu z)=0\},$$

while on the right-hand side we can see a point of the elliptic quadric $\mathcal{Q}$ defined as

$$\mathcal{Q}=\{\langle z\rangle_{\mathbb{F}_q}\colon z^{(q-1)(q^2+1)}=1\}.$$

For a proof that $\mathcal{Q}$ is an elliptic quadric see [5, Theorem 3.2]. Since $\mathcal{Q}\cap\mathcal{H}_\mu\neq\emptyset$ it follows that we can always find $\lambda\in\mathbb{F}_{q^4}\setminus\mathbb{F}_q$ satisfying (8) and hence $L_{U(b)}$ is not scattered.

□

**Remark 2.2.** *The linear sets in Theorem 2.1 are of sizes $q^3+q^2+q+1$, $q^3+q^2+1$, or $q^3+1$. The linear set associated with $\{(x,\mathrm{Tr}_{q^4/q}(x)):x\in\mathbb{F}_{q^4}\}$ is of size $q^3+1$ as well. As it turns out from [4] the projective line* $\mathrm{PG}(1,q^4)$ *also contains $\mathbb{F}_q$-linear sets of size $q^3+q^2-q+1$.*

## 3 The canonical form

In this section $\mathbb{L}$ denotes a maximum scattered $\mathbb{F}_q$-linear set in $\mathrm{PG}(1,q^4)$, not of pseudoregulus type. In particular, this implies $q>2$. By [15], $\mathbb{L}$ is a projection $p_\ell(\Sigma)$, where the vertex $\ell$ is a line and $\Sigma$ is a $q$-order canonical subgeometry[1] in $\mathrm{PG}(3,q^4)$, with $\ell\cap\Sigma=\emptyset$. The axis of the projection

[1]Let $\mathrm{PG}(V,\mathbb{F}_{q^t})=\mathrm{PG}(n-1,q^t)$, let $U$ be an $n$-dimensional $\mathbb{F}_q$-vector subspace of $V$, and $\Sigma=\{\langle\mathbf{u}\rangle_{\mathbb{F}_{q^t}}:\mathbf{u}\in U\setminus\{\mathbf{0}\}\}$. If $\langle\Sigma\rangle=\mathrm{PG}(n-1,q^t)$, then $\Sigma$ is a *(q-order) canonical subgeometry* of $\mathrm{PG}(n-1,q^t)$. Here and in the following, angle brackets $\langle-\rangle$ without a subscript denote projective span in $\mathrm{PG}(n-1,q^t)$, that is, $\mathrm{PG}(3,q^4)$ in our case.

is immaterial and can be chosen by convenience. Let $\sigma$ be a generator of the subgroup of order four of $\mathrm{P\Gamma L}(4,q^4)$ fixing pointwise $\Sigma$. Let $M$ be a $k$-dimensional subspace of $\mathrm{PG}(3,q^4)$. We say that $M$ is a *subspace of* $\Sigma$ if $M\cap\Sigma$ is a $k$-dimensional subpsace of $\Sigma$, which happens exactly when $M^\sigma=M$.

**Proposition 3.1.** *Let $\Sigma'$ be the unique $q^2$-order canonical subgeometry of* $\mathrm{PG}(3,q^4)$ *containing $\Sigma$, that is, the set of all points fixed by $\sigma^2$. Then the intersection of $\ell$ and $\Sigma'$ is empty.*

*Proof.* Assume the contrary, that is, there exists a point $P$ in $\ell\cap\Sigma'$. Then $P^{\sigma^2}=P$, the subspace $\ell_P=\langle P,P^\sigma\rangle$ is a line, and satisfies $\ell_P^\sigma=\ell_P$, whence $\ell_P$ is a line of $\Sigma$. This implies that $p_\ell(\ell_P)$ is a point, and $\mathbb{L}$ is not scattered. $\square$

Let $\mathcal{K}$ and $\mathcal{K}'$ be the Klein quadrics representing – via the Plücker embedding $\wp$ – the lines of $\Sigma$ and $\Sigma'$. In order to precisely define $\wp$, take coordinates in $\mathrm{PG}(3,q^4)$ such that $\Sigma$ (resp. $\Sigma'$) is the set of all points with coordinates rational over $\mathbb{F}_q$ (resp. $\mathbb{F}_{q^2}$), and define the image $r^\wp$ of any line $r$ through minors of order two in the usual way. Then $\mathcal{K}=\mathcal{K}'\cap\mathrm{PG}(5,q)$ by considering $\mathrm{PG}(5,q)$ as a subset of $\mathrm{PG}(5,q^2)$. The only nontrivial element of the subgroup of $\mathrm{P\Gamma L}(6,q^2)$ fixing $\mathrm{PG}(5,q)$ pointwise is

$$\tau:\langle(x_0,x_1,x_2,x_3,x_4,x_5)\rangle_{\mathbb{F}_{q^2}}\mapsto\langle(x_0^q,x_1^q,x_2^q,x_3^q,x_4^q,x_5^q)\rangle_{\mathbb{F}_{q^2}}. \tag{12}$$

Then $\mathcal{K}_2^\tau=\mathcal{K}_2$, and $\sigma\wp=\wp\tau$.

**Proposition 3.2.** *Let $S$ be a solid in* $\mathrm{PG}(5,q^2)$ *such that $(i)$ $S\cap\mathcal{K}'\cong Q^-(3,q^2)$, $(ii)$ $S\cap\mathcal{K}=\emptyset$. Then $S\cap S^\tau\cap\mathcal{K}'$ is a set of two distinct points forming an orbit of $\tau$.*

*Proof.* If $\dim(S\cap S^\tau)\geq 2$, then $S\cap S^\tau$ contains a plane of $\mathrm{PG}(5,q)$. Each plane of $\mathrm{PG}(5,q)$ meets $\mathcal{K}$ in at least one point of $\mathrm{PG}(5,q)$, contradicting $(ii)$. Then $r=S\cap S^\tau$ is a line fixed by $\tau$, so it is a line of $\mathrm{PG}(5,q)$. This $r$ is external to the Klein quadric $\mathcal{K}$ by $(ii)$, hence it meets $\mathcal{K}'$ in two points. Since both of $\mathcal{K}'$ and $r$ are fixed by $\tau$ the assertion follows. $\square$

**Proposition 3.3.** *There is a line $r$ in* $\mathrm{PG}(3,q^4)$*, such that $r$ and $r^\sigma$ are skew lines both meeting $\ell$, and $r^{\sigma^2}=r$.*

*Proof.* Let $\Sigma$ and $\Sigma'$ be as in Proposition 3.1. Since $\ell\cap\Sigma'=\emptyset$, $\ell$ defines a regular (Desarguesian) spread $\mathcal{F}$ of $\Sigma'$. The lines of $\mathcal{F}$ are all lines $\langle P,P^{\sigma^2}\rangle\cap\Sigma'$ where $P\in\ell$. The image $\mathcal{F}^\wp$ under the Plücker embedding of $\mathcal{F}$ is an elliptic quadric $S \cap \mathcal{K}' \cong Q^-(3,q^2)$ in $\mathrm{PG}(5,q^2)$, $S$ a solid. Since $\mathbb{L}$ is scattered, there is no line of $\mathcal{F}$ fixed by $\sigma$, whence $S \cap \mathcal{K} = \emptyset$. Then the assertion follows from Proposition 3.2. $\square$

**Theorem 3.4.** *Any maximum scattered linear $\mathbb{F}_q$-linear set in $\mathrm{PG}(1,q^4)$ is projectively equivalent to $L_{U(b)}$ for some $b \in \mathbb{F}_{q^4}$, $\mathrm{N}_{q^4/q}(b) \neq 1$.*

*Proof.* The set $L_{U(0)}$ is a linear set of pseudoregulus type. Now assume that $\mathbb{L} = p_\ell(\Sigma)$ is maximum scattered, not of pseudoregulus type. Coordinates $X_0, X_1, X_2, X_3$ in $\mathrm{PG}(3,q^4)$ can be chosen such that

$$\Sigma = \{\langle (u, u^q, u^{q^2}, u^{q^3}) \rangle_{\mathbb{F}_{q^4}} : u \in \mathbb{F}_{q^4}^*\}, \tag{13}$$

and a generator of the subgroup of $\mathrm{P\Gamma L}(4,q^4)$ fixing $\Sigma$ pointwise is

$$\sigma : \langle (x_0, x_1, x_2, x_3) \rangle_{\mathbb{F}_{q^4}} \mapsto \langle (x_3^q, x_0^q, x_1^q, x_2^q) \rangle_{\mathbb{F}_{q^4}}. \tag{14}$$

Define $C = \ell \cap r$, where $r$ is as in Proposition 3.3. The points $C$ and $C^{\sigma^2}$ lie on $r$, as well as the points $C^\sigma$ and $C^{\sigma^3}$ lie on $r^\sigma$. By Proposition 3.1, $C \neq C^{\sigma^2}$ and $C^\sigma \neq C^{\sigma^3}$. This implies $\ell \subset \langle C, C^\sigma, C^{\sigma^3} \rangle$, and $\langle C, C^\sigma, C^{\sigma^2}, C^{\sigma^3} \rangle = \mathrm{PG}(3,q^4)$. Since the stabilizer of $\Sigma$ in $\mathrm{PGL}(4,q^4)$ acts transitively on the points $C$ of $\mathrm{PG}(3,q^4)$ such that $\langle C, C^\sigma, C^{\sigma^2}, C^{\sigma^3} \rangle = \mathrm{PG}(3,q^4)$ [4, Proposition 3.1], it may be assumed that $C = \langle (0,0,1,0) \rangle_{\mathbb{F}_{q^4}}$, whence

$$\ell = \langle (0,0,1,0), (0,a,0,-b) \rangle_{\mathbb{F}_{q^4}},$$

for some $a, b \in \mathbb{F}_{q^4}$, not both of them zero. If $a = 0$, then $\mathbb{L}$ is of pseudoregulus type [9, Theorem 2.3], so $a = 1$ may be assumed. For any point $P_u = \langle (u, u^q, u^{q^2}, u^{q^3}) \rangle_{\mathbb{F}_{q^4}}$ in $\Sigma$, the plane containing $\ell$ and $P_u$ has coordinates $[u^{q^3} + bu^q, -bu, 0, -u]$, and this leads to the desired form for the coordinates of $\mathbb{L}$. $\square$

# 4 Orbits

Analogously to the definition of the $\Gamma$L-class of linear sets (cf. Definition 2.4 in [7]) we define the GL-class, which will be needed to study $\mathrm{PGL}(2,q^4)$-equivalence. Note that for any scattered $\mathbb{F}_q$-linear set the maximum field of linearity is $\mathbb{F}_q$.

**Definition 4.1.** *Let $L_U$ be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(1,q^t)$ of rank $t$ with maximum field of linearity $\mathbb{F}_q$. We say that $L_U$ is of* $\Gamma$L-class *$s$ [resp.*

GL-class $s$] *if $s$ is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1$, $U_2$, ..., $U_s$ of $\mathbb{F}_{q^t}^2$ with $L_{U_i} = L_U$ for $i \in \{1, 2, \dots, s\}$ and there is no $\varphi \in \Gamma\mathrm{L}(2, q^t)$ [resp. $\varphi \in \mathrm{GL}(2, q^t)$] such that $U_i = U_j^{\varphi}$ for each $i \neq j$, $i, j \in \{1, 2, \dots, s\}$.*

The first part of the following result is [7, Theorem 4.5], while the second part follows from its proof. We briefly summarize the main steps of the proof from [7].

**Theorem 4.2.** *[7, Theorem 4.5] Each $\mathbb{F}_q$-linear set of rank four in $\mathrm{PG}(1, q^4)$, with maximum field of linearity $\mathbb{F}_q$, is of $\Gamma\mathrm{L}$-class one. More precisely, if $L_U = L_V$ for some 4 dimensional $\mathbb{F}_q$-subspaces $U$, $V$ of $\mathbb{F}_{q^4}^2$, then there exists $\varphi \in \Gamma\mathrm{L}(2, q^4)$ such that $U^{\varphi} = V$. Also, $\varphi$ can be chosen such that it has companion automorphism either the identity, or $x \mapsto x^{q^2}$.*

*Proof.* Assume $L_U = L_V$. We may assume $\langle (0, 1) \rangle_{\mathbb{F}_{q^4}} \notin L_U$. Then $U = U_f = \{(x, f(x)) \colon x \in \mathbb{F}_{q^4}\}$ and $V = V_g = \{(x, g(x)) \colon x \in \mathbb{F}_{q^4}\}$ for some $q$-polynomials $f$ and $g$ over $\mathbb{F}_{q^4}$. By [7, Proposition 4.2], either $g(x) = f(\lambda x)/\lambda$, or $g(x) = \hat{f}(\lambda x)/\lambda$ for some $\lambda \in \mathbb{F}_{q^4}^*$, where here $\hat{f}$ denotes the adjoint map of $f$ with respect to the bilinear form $< x, y > := \mathrm{Tr}_{q^4/q}(xy)$. The $\mathbb{F}_{q^4}$-linear map $\mathbf{v} \mapsto \lambda \mathbf{v}$ maps $U_g$ to one of $U_f$, or $U_{\hat{f}}$. In the proof of [7, Theorem 4.5], a $\kappa \in \Gamma\mathrm{L}(2, q^4)$ with companion automorphism the identity, or $x \mapsto x^{q^2}$ is determined such that $U_f^{\kappa} = U_{\hat{f}}$. $\square$

**Theorem 4.3.** *For any $b \in \mathbb{F}_{q^4}$, $L_{U(b)}$ is of* GL*-class one.*

*Proof.* By Theorem 4.2, if $L_{U(b)} = L_V$, then there exists $\varphi \in \Gamma\mathrm{L}(2, q^4)$ such that $U(b)^{\varphi} = V$ and the companion automorphism of $\varphi$ is $x \mapsto x^{q^2}$, or the identity. In order to prove the statement it is enough to show that $U(b)$ and $U(b)^{q^2} = \{(x^{q^2}, y^{q^2}) \colon (x, y) \in U(b)\}$ lie on the same orbit of $\mathrm{GL}(2, q^4)$. If $b = 0$, then $U(b) = U(b)^{q^2}$. If $b \neq 0$, then for any $u \in \mathbb{F}_{q^4}$,

$$\begin{pmatrix} b^{q^3} & 0 \\ 0 & b^{q^2} \end{pmatrix} \begin{pmatrix} u \\ bu^q + u^{q^3} \end{pmatrix} = \begin{pmatrix} b^q u^{q^2} \\ b\left(b^q u^{q^2}\right)^q + \left(b^q u^{q^2}\right)^{q^3} \end{pmatrix}^{q^2} = \begin{pmatrix} v \\ bv^q + v^{q^3} \end{pmatrix}^{q^2},$$

with $v = b^q u^{q^2}$. $\square$

**Corollary 4.4.** *Let $b, c \in \mathbb{F}_{q^4}$. The linear sets $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent if and only if $U(b)$ and $U(c)$ are in the same orbit under the action of $\mathrm{GL}(2, q^4)$.*

*Proof.* The "if" part is obvious, so assume that $L_{U(b)}^{\tilde{\kappa}} = L_{U(c)}$ where $\kappa \in \mathrm{GL}(2, q^4)$. Then $L_{U(b)^\kappa} = L_{U(c)}$ and by Theorem 4.3 there is $\kappa' \in \mathrm{GL}(2, q^4)$ such that $U(b)^{\kappa\kappa'} = U(c)$. $\square$

It follows that in order to classify the $\mathbb{F}_q$-linear sets $L_{U(b)}$ up to $\mathrm{PGL}(2, q^4)$ and $\mathrm{P\Gamma L}(2, q^4)$-equivalence, it is enough to determine the orbits of the subspaces $U(b)$ under the actions of $\Gamma\mathrm{L}(2, q^4)$ and $\mathrm{GL}(2, q^4)$.

**Theorem 4.5.** *Let $q$ be a power of a prime $p$.*

*(i) For any $b, c \in \mathbb{F}_{q^4}$, $L_{U(b)}$ and $L_{U(c)}$ are equivalent up to an element of $\mathrm{P\Gamma L}(2, q^4)$ if and only if $c^{q^2+1} = b^{\pm p^s(q^2+1)}$ for some integer $s \geq 0$.*

*(ii) For any $b, c \in \mathbb{F}_{q^4}$, the linear sets $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent if and only if $c^{q^2+1} = b^{q^2+1}$ or $c^{q^2+1} = b^{-q(q^2+1)}$.*

*(iii) All linear sets described in 2. of Theorem 2.1 are projectively equivalent.*

*(iv) There are precisely $q(q-1)/2$ distinct linear sets up to projective equivalence in the family described in 1. of Theorem 2.1, and these are the only maximum scattered linear sets of $\mathrm{PG}(1, q^4)$.*

*(v) There are precisely $q$ distinct linear sets up to projective equivalence in the family described in 3. of Theorem 2.1.*

*Proof.* Take $b \in \mathbb{F}_{q^4}^*$. If $L_{U(b)}$ is not scattered, then it clearly cannot be equivalent to $L_{U(0)}$ (the scattered linear set of pseudoregulus type), while if $L_{U(b)}$ is scattered, then it follows from Theorem 1.1 (and from a computer search when $q = 3$) that $U(b)$ and $U(0)$ yield projectively inequivalent linear sets. Since the automorphic collineations $(x, y) \mapsto (x^{p^s}, y^{p^s})$ fix $U(0)$, it also follows that $L_{U(0)}$ and $L_{U(b)}$ lie on different orbits of $\mathrm{P\Gamma L}(2, q^4)$. Thus (i) and (ii) are true when one of $b$ or $c$ is zero, so from now on we may assume $b \neq 0$ and $c \neq 0$.

The sets $L_{U(b)}$ and $L_{U(c)}$ are equivalent up to elements of $\mathrm{P\Gamma L}(2, q^4)$ if and only for some $\psi = p^k$, $k \in \mathbb{N}$ and some $A, B, C, D \in \mathbb{F}_{q^4}$ such that $AD - BC \neq 0$ the following holds:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u^\psi \\ b^\psi u^{\psi q} + u^{\psi q^3} \end{pmatrix} : u \in \mathbb{F}_{q^4} \right\} = \left\{ \begin{pmatrix} v \\ cv^q + v^{q^3} \end{pmatrix} : v \in \mathbb{F}_{q^4} \right\}. \quad (15)$$

Furthermore, by Corollary 4.4, $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent if, and only if, (15) has a solution with $\psi = 1$. This leads to a polynomial in

$u^{\psi}$ of degree at most $q^3$ which is identically zero. Equating its coefficients to zero,

$$\left\{\begin{array}{rcl} A^{q^3}-D & = & 0 \\ B^q b^{\psi q}c+B^{q^3} & = & 0 \\ A^q c-Db^{\psi} & = & 0 \\ B^q c+B^{q^3}b^{\psi q^3}-C & = & 0. \end{array}\right. \tag{16}$$

Assume that $L_{U(b)}$ and $L_{U(c)}$ are in the same orbit of $\mathrm{P\Gamma L}(2,q^4)$, and take $\psi=1$ in case they are also projectively equivalent. If $D\neq 0$, then the first and third equations imply $b^{\psi}=D^{q^2-1}c$ and so $c^{q^2+1}=b^{\psi(q^2+1)}$. If $D=0$, then $BC\neq 0$; from the second equation, $(b^{\psi q}c)^{q^2+1}=1$, hence $c^{q^2+1}=b^{-\psi q(q^2+1)}$. This proves the only if parts of (i) and (ii).

Conversely, if $c^{q^2+1}=b^{p^s(q^2+1)}$ for some $s\in\mathbb{N}$, then $b^{p^s}c^{-1}=\delta^{q^2-1}$ for some $\delta\in\mathbb{F}_{q^4}^*$. The quadruple $A=\delta^q$, $B=C=0$, $D=\delta$ with $\psi=p^s$ is a solution of (16) with $AD-BC\neq 0$. This proves the if part of (i) when $c^{q^2+1}=b^{p^s(q^2+1)}$ and the if part of (ii) when $c^{q^2+1}=b^{q^2+1}$. If $b^{q^2+1}=c^{q^2+1}=1$, i.e. when $U(b)$ and $U(c)$ define linear sets described in 2. of Theorem 2.1, then the above condition holds, thus (iii) follows. From now on we may assume $b^{q^2+1}\neq 1$ and $c^{q^2+1}\neq 1$.

Assume $c^{q^2+1}=b^{-p^s(q^2+1)}$ for some $s\in\mathbb{N}$, i.e. $b^{p^s}c=\varepsilon^{q^2-1}$ for some $\varepsilon\in\mathbb{F}_{q^4}^*$. Define $\psi=p^sq^3$. A $\rho\in\mathbb{F}_{q^4}^*$ exists such that $\rho^{q^2-1}=-1$. Take $A=D=0$, $B=(\rho\varepsilon)^{q^3}$, $C=\varepsilon\rho c(1-b^{p^s(q^2+1)})$. If $C=0$, then $b^{q^2+1}=1$, a contradiction. So $AD-BC\neq 0$ and (16) has a solution. If $p^s=q$, then $\psi=1$, hence in this case $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent. This finishes the proofs of (i) and (ii).

Now we prove (iv). Note that $\mathrm{N}_{q^4/q}(b)=(b^{q^2+1})^{q+1}$ for any $b\in\mathbb{F}_q$, therefore, $L_{U(b)}$ is a maximum scattered $\mathbb{F}_q$-linear set not of pseudoregulus type if, and only if, $b^{q^2+1}$ is an element of the set

$$S=\{x\in\mathbb{F}_{q^2}^*\colon x^{q+1}\neq 1\}.$$

The orbits of point sets of type $L_{U(b)}$, $b\neq 0$, under the action of $\mathrm{PGL}(2,q^4)$ are as many as the pairs $\{x,x^{-q}\}$ of elements in $S$. Since all such pairs are made of distinct elements, adding one for the linear set of pseudoregulus type, one obtains

$$1+\frac{q^2-q-2}{2}=\frac{q(q-1)}{2}.$$

Finally we prove (v). $L_{U(b)}$ is an $\mathbb{F}_q$-linear set described in 3. of Theorem 2.1 if, and only if, $b^{q^2+1}$ is an element of the set

$$Z=\{x\in\mathbb{F}_{q^2}\setminus\{1\}\colon x^{q+1}=1\}.$$

The orbits of point sets of this type under the action of $\mathrm{PGL}(2,q^4)$ are as many as the pairs $\{x, x^{-q}\}$ of elements in $Z$. Since for each $x \in Z$ we have $x = x^{-q}$, this number is $q$. □

**Remark 4.6.** *The number of orbits of maximum scattered linear sets under the action of* $\mathrm{P\Gamma L}(2,q^4)$ *depends on the exponent* $e$ *in* $q = p^e$. *A general formula is not provided here. For* $e = 1$ *each orbit which does not arise from the linear set of pseudoregulus type is related to two or four norms over* $\mathbb{F}_{q^2}$, *according to whether* $\mathrm{N}_{q^4/q^2}(b) \in \mathbb{F}_q \setminus \{0, 1, -1\}$ *or not. This leads (including now the linear set of pseudoregulus type) to a total number of* $(q^2-1)/4$ *orbits for odd* $q$.


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Bence Csajbók
MTA–ELTE Geometric and Algebraic Combinatorics Research Group,
Eötvös Loránd University,

H–1117 Budapest, Pázmány Péter Sétány 1/C, Hungary
*csajbok.bence@gmail.com*

Corrado Zanella
Dipartimento di Tecnica e Gestione dei Sistemi Industriali,
Università di Padova, Stradella S. Nicola, 3, I-36100 Vicenza, Italy
*corrado.zanella@unipd.it*