Equation-regular sets and the Fox–Kleitman conjecture

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Keywords:

Partition regularity

Degree of regularity

Monochromatic solution

Discrete derivative

ABSTRACT

Given $k \geq 1$, the Fox–Kleitman conjecture from 2006 states that there exists a nonzero integer b such that the $2k$-variable linear Diophantine equation

$$\sum_{i=1}^{k}(x_i - y_i) = b$$

is $(2k-1)$-regular. This is best possible, since Fox and Kleitman showed that for all $b \geq 1$, this equation is not $2k$-regular. While the conjecture has recently been settled for all $k \geq 2$, here we focus on the case $k = 3$ and determine the degree of regularity of the corresponding equation for all $b \geq 1$. In particular, this independently confirms the conjecture for $k = 3$. We also briefly discuss the case $k = 4$.

1. Introduction

A Diophantine equation L is said to be *n-regular*, for some positive integer n, if for every n-coloring of $\mathbb{N}_+ = \{1, 2, \ldots\}$, there is a monochromatic solution to L. Further, L is said to be *regular* if it is n-regular for all $n \geq 1$. Of course, $(n+1)$-regularity implies n-regularity. The *degree of regularity* of L, denoted as $\operatorname{dor}(L)$, is defined to be infinite if L is regular, or else, it is the largest n such that L is n-regular [4]. Determining the degree of regularity of a given Diophantine equation is difficult in general, even if it is linear.

In this paper, we focus on a particular linear equation proposed by Fox and Kleitman in [3]. Given positive integers k, b, we shall denote by $L_k(b)$ the $2k$-variable Diophantine equation

$$\sum_{i=1}^{k}(x_i - y_i) = b.$$

Fox and Kleitman [3] showed that this equation is never $2k$-regular, *i.e.*, that

$$\operatorname{dor}(L_k(b)) \leq 2k - 1 \tag{1}$$

for all $b \in \mathbb{N}_+$. Moreover, they conjectured that this upper bound is best possible.

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Conjecture 1.1 ([3]). *Let $k \geq 1$. Then there exists an integer $b \geq 1$, depending on k, such that $dor(L_k(b)) = 2k - 1$.*

The case $k = 2$ of the conjecture was recently settled in [1], where it is shown that $\mathrm{dor}(L_2(b)) = 3$ for all $b \equiv 0 \bmod 6$. More generally, the authors determined $\mathrm{dor}(L_2(b))$ for all $b \geq 1$, as follows:

Theorem 1.2 ([1]). *For all positive integers b, we have*

$$dor(L_2(b)) = \begin{cases} 1 & \text{if } \ b \equiv 1 \bmod 2, \\ 2 & \text{if } \ b \equiv 2, 4 \bmod 6, \\ 3 & \text{if } \ b \equiv 0 \bmod 6. \end{cases}$$

A reduced 3-variable version of the 4-variable equation $L_2(b)$ had already been studied in [2]. Indeed, they considered the equation $x_3 - x_2 = x_2 - x_1 + b$, which can be obtained from the equation $(x_1 - y_1) + (x_2 - y_2) = b$ by setting $y_1 = y_2$ and renaming the resulting three variables x_1, x_2, y_2 as x_1, x_3, x_2, respectively.

Finally, as pointed out by a referee, the conjecture has recently been fully settled, using sophisticated methods of additive combinatorics [5].

1.1. The functions ν and f

We now introduce two functions $\nu, f : \mathbb{N}_+ \to \mathbb{N}_+$ which are somehow related to $\mathrm{dor}(L_k(b))$. In particular, they will provide nice reformulations of the above formulas for $\mathrm{dor}(L_2(b))$. For $n \in \mathbb{N}_+$, define

$$\nu(n) = \text{ the smallest } \textit{non-divisor} \text{ of } n,$$

i.e. the least integer $m \geq 2$ such that m does not divide n. For instance, we have $\nu(n) = 2$ if n is odd, and $\nu((p-1)!) = p$ if p is prime. Still for $n \in \mathbb{N}_+$, define

$$f(n) = \text{ the largest } r \text{ such that } r! \text{ divides } n.$$

Thus for instance, we have $f(n) = 1$ if n is odd, and $f(n) = 2$ if and only if n is even and not divisible by 6, *i.e.* $n \equiv 2, 4 \bmod 6$.

As is easily seen, Theorem 1.2 is equivalent to the formulas

$$\mathrm{dor}(L_2(b)) = \min(3, f(b)) = \min(3, \nu(b) - 1) \tag{2}$$

for all $b \geq 1$. Our purpose in this paper is to similarly determine $\mathrm{dor}(L_3(b))$ for all $b \geq 1$. Indeed, with the upper bound (1) in mind, we shall establish the equality

$$\mathrm{dor}(L_3(b)) = \min(5, f(b)) \tag{3}$$

for all $b \geq 1$. This implies $\mathrm{dor}(L_3(5!)) = 5$, thereby verifying the Fox–Kleitman conjecture for $k = 3$. Unfortunately, in contrast with (2), there is no equality in general between $\mathrm{dor}(L_3(b))$ and $\min(5, \nu(b) - 1)$. For $b = 60$ for instance, we shall see that $\mathrm{dor}(L_3(60)) = 3$, whereas $\min(5, \nu(60) - 1) = 5$. However, the inequality

$$\mathrm{dor}(L_k(b)) \leq \min(2k - 1, \nu(b) - 1) \tag{4}$$

always holds, as will be shown further down. Moreover, for fixed $b \geq 1$, we shall prove that

$$\mathrm{dor}(L_k(b)) = \min(2k - 1, \nu(b) - 1) = \nu(b) - 1 \tag{5}$$

for all sufficiently large k, in fact for all $k \geq b$. As for the function f, can one expect, based on (2), (3) and (4), an equality or at least an inequality between $\mathrm{dor}(L_k(b))$ and $\min(2k - 1, f(b))$ for $k \geq 4$? Here again, the answer turns out to be negative. Indeed, for $k = 4$, we shall establish the following values.

b	$\mathrm{dor}(L_4(b))$	$\min(7, f(b))$
12	4	3
$720 = 6!$	5	6

This indicates that the behavior of $\mathrm{dor}(L_k(b))$ as a function of b, for fixed $k \geq 4$, is much more tricky than what formulas (2) and (3) for $k \leq 3$ might lead one to expect. In the same vein, we shall show that if b is a positive integer satisfying the Fox–Kleitman conjecture for $k = 4$, *i.e.* such that $\mathrm{dor}(L_4(b)) = 7$, then necessarily b must be divisible by $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 = 1141140$. At the time of writing, we do not know whether $\mathrm{dor}(L_4(b))$ reaches 7 for $b = 1141140$.

1.2. Contents

In Section 2, we provide basic lemmas regarding the behavior of $\mathrm{dor}(L_k(b))$. In Section 3 we introduce L-regular and L-singular sets with respect to a Diophantine equation L, and we provide tools to determine such sets for the equation $L_k(b)$. In Section 4 we give a new proof of the above formula for $\mathrm{dor}(L_2(b))$. In Section 5 we determine $\mathrm{dor}(L_3(b))$ for all $b \geq 1$, with some computer assistance in the specific case $b = 120$, and we independently verify the Fox–Kleitman conjecture for $k = 3$. In Section 6 we briefly discuss the case $k = 4$. Finally, in Section 7 we determine $\mathrm{dor}(L_k(b))$ whenever $k \geq b$.

2. Basic lemmas

We shall use the following notation from additive combinatorics. For subsets A, B of an abelian group G, we write $A + B = \{a + b \mid a \in A, b \in B\}$, $-A = \{-a \mid a \in A\}$ and $kA = A + \cdots + A$, the k-fold sum of A with itself. Thus, given a subset $X \subseteq \mathbb{Z}$, we see that *equation $L_k(b)$ has a solution with all entries in X if and only if*

$$b \in k(X - X).$$

(See also Remark 3.5.) This will be used throughout the paper. For instance, using this formulation, here is a brief explanation for Fox and Kleitman's upper bound (1) stating that $L_k(b)$ is never $2k$-regular. For integers $a \leq b$, we shall denote by $[a, b]$ the *integer interval* consisting of all $n \in \mathbb{Z}$ such that $a \leq n \leq b$.

Lemma 2.1 ([3]). *For all integers $k, b \geq 1$, one has $dor(L_k(b)) \leq 2k - 1$.*

Proof. If b is not a multiple of k, then $L_k(b)$ is not even k-regular by the independent Lemma 2.4 below. If $b = kr$ with $r \in \mathbb{N}_+$, we first $2k$-color $[1, 2b]$ as follows. The color class C_1 is given by $C_1 = [1, r]$. For $2 \leq i \leq 2k$, the color class C_i is the translate of C_1 by $(i-1)r$, namely $C_i = C_1 + (i-1)r$. We then extend this coloring to the whole of $\mathbb{N}_+$ by $2b$-periodicity, namely where the color classes are $X_i = C_i + 2b\mathbb{Z} = C_i + 2kr\mathbb{Z}$ for all $1 \leq i \leq 2k$.

We claim that $b \notin k(X_i - X_i)$ for all $1 \leq i \leq 2k$. It suffices to check it for $i = 1$, since $X_i - X_i$ is independent of i. Now $X_1 - X_1 = [-(r-1), r-1] + 2kr\mathbb{Z}$, whence

$$k(X_1 - X_1) = [-k(r-1), k(r-1)] + 2kr\mathbb{Z}.$$

Therefore $b \notin k(X_1 - X_1)$, as claimed. It follows that $L_k(b)$ is not $2k$-regular. □

We now show that $\mathrm{dor}(L_k(b))$ is monotone with respect to k.

Lemma 2.2. *Let b, k_1, k_2 be positive integers such that $k_1 \leq k_2$. Then*

$$dor(L_{k_1}(b)) \leq dor(L_{k_2}(b)).$$

Proof. Let $n = \mathrm{dor}(L_{k_1}(b))$. We claim that $L_{k_2}(b)$ is n-regular. Let c be an n-coloring of $\mathbb{N}_+$. Then there is a c-monochromatic subset $X \subseteq \mathbb{N}_+$ such that $b \in k_1(X - X)$. Now $k_1(X - X) \subseteq k_2(X - X)$ since $0 \in X - X$ and $k_1 \leq k_2$. Hence $b \in k_2(X - X)$. Therefore $L_{k_2}(b)$ is n-regular, as claimed. □

Next, for fixed k, here are two basic lemmas about the behavior of $\mathrm{dor}(L_k(b))$ as a function of b. The first one shows that this function is monotone with respect to multiplication.

Lemma 2.3. *Let b_1, b_2, k be positive integers such that b_1 divides b_2. Then*

$$dor(L_k(b_1)) \leq dor(L_k(b_2)).$$

Proof. By hypothesis, there is an integer $t \in \mathbb{N}_+$ such that $b_2 = tb_1$. Let $r = \mathrm{dor}(L_k(b_1))$. We now show that $L_k(b_2)$ is r-regular. Let c be a r-coloring of $\mathbb{N}_+$. It suffices to establish the existence of a c-monochromatic solution of $L_k(b_2)$. Let c' be the new r-coloring of $\mathbb{N}_+$ defined by $c'(n) = c(tn)$ for all $n \geq 1$. Since $L_k(b_1)$ is r-regular, there is a c'-monochromatic solution $a_1, \ldots, a_k, d_1, \ldots, d_k$ in $\mathbb{N}_+$ of $L_k(b_1)$, *i.e.* satisfying

$$\sum_{i=1}^{k}(a_i - d_i) = b_1.$$

Multiplying this equality by t, we get

$$\sum_{i=1}^{k}(ta_i - td_i) = tb_1 = b_2. \tag{6}$$

Since the a_i's, d_i's are c'-monochromatic, it follows by construction that the ta_i's, td_i's are c-monochromatic. Moreover, by (6), they constitute a solution of $L_k(b_2)$. Hence $r \leq \mathrm{dor}(L_k(b_2))$, as desired. □

On the other hand, here is an obvious upper bound on $\mathrm{dor}(L_k(b))$.

Lemma 2.4. *Let b, m be positive integers such that m does not divide b. Then $dor(L_k(b)) \leq m - 1$.*

Proof. It suffices to find an m-coloring of $\mathbb{N}_+$ for which there is no monochromatic solution of $L_k(b)$. Consider the coloring π_m given by the classes mod m. That is, the subset $X_i \subseteq \mathbb{N}_+$ of elements of color i, for $1 \leq i \leq m$, is defined as $X_i = i + m \cdot \mathbb{N}$. Then $X_i - X_i \subseteq m \cdot \mathbb{Z}$. Since $b \not\equiv 0 \bmod m$, it follows that $b \notin k(X_i - X_i)$. Hence equation $L_k(b)$ admits no π_m-monochromatic solution, as desired. □

Proposition 2.5. *Let $b \geq 1$. Then* $dor(L_k(b)) \leq \min(2k-1, \nu(b)-1)$.

Proof. We have $\mathrm{dor}(L_k(b)) \leq 2k-1$ by Lemma 2.1, and $\mathrm{dor}(L_k(b)) \leq \nu(b)-1$ by Lemma 2.4 since $\nu(b)$ does not divide b. $\square$

3. Equation-regular sets

We now introduce a notion of regularity for sets which is closely linked to the usual notion of partition-regularity for Diophantine equations. This will turn out to be useful to determine, in some cases, the degree of regularity of the Fox–Kleitman equation $L_k(b)$.

Definition 3.1. Given a Diophantine equation L, or a system of such equations, we say that a set $X \subset \mathbb{N}_+$ is *regular* with respect to L, or more shortly L-regular, if it contains a solution of L. We say that X is *singular* if X is not regular.

Here is an easy remark linking these notions of regularity.

Remark 3.2. Let L be a Diophantine equation. Then L is n-regular if and only if every partition of $\mathbb{N}_+$ into n parts admits an L-regular part.

The notion of equation-regular sets moves the focus away from partitions and more towards those properties of a set which will force it to contain a solution of a given Diophantine system L.

For instance, given $k, b \geq 1$, we shall see that any sufficiently dense subset of a sufficiently large integer interval is $L_k(b)$-regular. That is, for sets, density alone implies regularity. This will allow us to determine the degree of regularity of $L_k(b)$ in some instances, and in particular to independently verify the Fox–Kleitman conjecture for $k=3$, as well as to provide a shorter proof of it for $k=2$.

3.1. Block sums

We first need some basic notions to be used throughout the paper. Let $A = (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers.

- A *block* in A is any subsequence A' of consecutive terms in A, *i.e.* of the form
$$A' = (a_i, a_{i+1}, \ldots, a_j)$$
for some indices i, j such that $1 \leq i \leq j \leq r$.
- The empty subsequence of length 0 is also considered to be a block of A.
- We denote by $\sigma(A) = \sum_{i=1}^{r} a_i$ the sum of the elements of A, and by $\mu(A)$ the average of A, *i.e.* $\mu(A) = \sigma(A)/r$.
- Finally, we denote by $\mathrm{bs}(A)$ the set of signed *block sums* in A, *i.e.*
$$\mathrm{bs}(A) = \{\pm\sigma(A') \mid A' \text{ is a block in } A\}.$$

For instance, if $A = (1, 10, 4)$, then $\mathrm{bs}(A) \cap \mathbb{N} = \{0, 1, 4, 10, 11, 14, 15\}$. Note that 5 is *not* a block sum in A.

Note that if A' is a block in A, then $\mathrm{bs}(A') \subseteq \mathrm{bs}(A)$. We will further discuss this property later on. Observe also that $A \subseteq \mathrm{bs}(A)$.

3.2. The discrete derivative

We shall need a variant of the discrete derivative, associating to a subset $X \subset \mathbb{Z}$ of cardinality $r+1$ a sequence of length r, as follows:

Definition 3.3. Let $X \subset \mathbb{Z}$ be a finite subset of cardinality $r+1$. Let the elements of X be $x_0 < x_1 < \cdots < x_r$. We associate to X the sequence
$$\Delta X = (x_1 - x_0,\ x_2 - x_1, \ldots,\ x_r - x_{r-1})$$
of length r, and call ΔX the discrete derivative of X.

Of interest to us here is the fact that $X - X$ can be read off from the signed block sums of ΔX.

Lemma 3.4. *Let $X \subset \mathbb{Z}$ be a nonempty finite subset. Then*
$$X - X = \mathrm{bs}(\Delta X).$$

Proof. Let the elements of X be $x_0 < x_1 < \cdots < x_r$. Then

$$X - X = \{x_t - x_s \mid 0 \le s, t \le r\}.$$

Let now $A = \Delta X = (a_1, \ldots, a_r)$, where $a_i = x_i - x_{i-1}$ for $1 \le i \le r$. Then for indices s, t such that $0 \le s \le t \le r$, we have

$$x_t - x_s = \sum_{i=s+1}^{t} (x_i - x_{i-1}) = \sum_{i=s+1}^{t} a_i.$$

Thus $x_t - x_s = \sigma(A')$, where A' is the block $(a_{s+1}, \ldots, a_t)$ of A. Hence $x_t - x_s \in \mathrm{bs}(A) \cap \mathbb{N}$ for $s \le t$, and $x_t - x_s \in -(\mathrm{bs}(A) \cap \mathbb{N})$ if $s \ge t$. The claim follows. $\square$

3.3. $L_k(b)$-regular sets

We now turn to equation $L_k(b)$ for given integers $k, b \ge 1$. We first recall an informal observation made at the beginning of Section 2.

Remark 3.5. Let L denote equation $L_k(b)$ for some integers $k, b \ge 1$. Let $X \subset \mathbb{N}_+$. Then X is regular with respect to L if and only if $b \in k(X - X)$.

Indeed, we have $b \in k(X - X)$ if and only if there exist $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ such that $b = \sum_{i=1}^{k}(x_i - y_i)$, which therefore constitute a solution of L in X.

This allows us to express the $L_k(b)$-regularity of a set X in terms of the discrete derivative ΔX. The resulting lemma will be tacitly used in the sequel.

Lemma 3.6. *Let $X \subset \mathbb{Z}$ be a nonempty finite subset. Then X is regular with respect to equation $L_k(b)$ if and only if $b \in k\,\mathrm{bs}(\Delta X)$.*

Proof. Directly follows from the equality $X - X = \mathrm{bs}(\Delta X)$ of Lemma 3.4 together with Remark 3.5. $\square$

3.4. Forbidden sequences and subsets

In the sequel, we will try to construct finite subsets $X \subset \mathbb{Z}$ which are *singular* with respect to $L_k(b)$, with the purpose of showing that it is hard to achieve if X is sufficiently dense in a suitable integer interval. Moreover, working with ΔX rather than X is more convenient. This justifies the following terminology.

Definition 3.7. Let L denote equation $L_k(b)$ for some integers $k, b \ge 1$. Let $A = (a_1, \ldots, a_r)$ be a sequence of positive integers. We say that A is *admissible* (with respect to L) if $b \notin k\,\mathrm{bs}(A)$. We say that A is *forbidden* if it is not admissible, *i.e.* if $b \in k\,\mathrm{bs}(A)$.

Remark 3.8. With respect to equation $L_k(b)$, a subset $X \subset \mathbb{N}$ *is regular if and only if its discrete derivative $A = \Delta X$ is forbidden*. Equivalently, X is singular if and only if ΔX is admissible.

The following result is the heart of our approach to evaluate the degree of regularity of the Fox–Kleitman equation $L_k(b)$. Recall that $\mu(A)$ denotes the average of the sequence $A = (a_1, \ldots, a_r)$.

Proposition 3.9. *Let L denote equation $L_k(b)$ for some integers $k, b \ge 1$. Let $d \ge 1$. Assume that there exists an integer $N \ge 1$ such that all positive integer sequences A of length N and average $\mu(A) \le d$ are forbidden with respect to L. Then L is d-regular.*

Proof. Consider an arbitrary d-coloring of the integer interval $[1, dN + 1]$. By the pigeonhole principle, there exists a monochromatic subset $X \subset [1, dN + 1]$ of cardinality $|X| = N + 1$. Let $A = \Delta X$. Then A is of length N. Let $x_0 = \min X$, $x_N = \max X$. Then $x_N - x_0 = \sigma(A)$. Since $X \subset [1, dN + 1]$, it follows that $x_N - x_0 \le dN$, whence $\mu(A) \le d$. It follows from the hypothesis that A is forbidden with respect to L, and hence that $b \in k\,\mathrm{bs}(A)$. Therefore $b \in k(X - X)$, *i.e.* X is L-regular. Since the d-coloring was arbitrary, it follows that L is d-regular, as claimed. $\square$

We end this section by looking at ways to construct new forbidden sequences from a given one.

Definition 3.10. Let $A = (a_1, \ldots, a_r)$ be a sequence of positive integers.

- An *elementary contraction* of A is any sequence $\overline{A}$ obtained by replacing a block A' in A by its sum $\sigma(A')$. That is, if $A' = (a_i, \ldots, a_j)$ for some $1 \le i \le j \le r$, then

$$\overline{A} = (a_1, \ldots, a_{i-1}, \sigma(A'), a_{j+1}, \ldots, a_r).$$

- A *contraction* of A is any sequence obtained from A by successive elementary contractions.

For instance, let $A = (1, 2, 3, 4)$. Then $(3, 3, 4)$, $(6, 4)$ and $(3, 7)$ are contractions of A, the first two ones being elementary.

Definition 3.11. Let $A = (a_1, \dots, a_r)$ be a sequence of positive integers. A *minor* of A is either a block A' in A or a contraction $\overline{A}$ of A.

We now show that if a sequence A has a forbidden minor, then A itself is forbidden.

Proposition 3.12. *Let L denote equation $L_k(b)$ for some integers $k, b \geq 1$. Let A be a finite sequence of positive integers, and let B be a minor of A. If B is forbidden with respect to L, then A also is.*

Proof. We have $b \in k\,\mathrm{bs}(B)$ by hypothesis. Therefore, to prove that A is forbidden, it suffices to show that $\mathrm{bs}(B) \subseteq \mathrm{bs}(A)$. This inclusion clearly holds if B is a block in A, since any block sum in B is a block sum in A. If now B is an elementary contraction of A, then again, any block sum in B is a block sum in A. Therefore, the same holds if B is obtained from A by successive elementary contractions. □

Here is another condition forcing a sequence A to be forbidden.

Proposition 3.13. *Let L denote equation $L_k(b)$ for some integers $k, b \geq 1$, and let A be a finite sequence of positive integers. If $\mathrm{bs}(A)$ contains a subset Z such that $b \in k(Z \cup -Z)$, then A is forbidden with respect to L.*

Proof. If $Z \subseteq \mathrm{bs}(A)$, then $(Z \cup -Z) \subseteq \mathrm{bs}(A)$ since $\mathrm{bs}(A) = -\mathrm{bs}(A)$. Therefore $k(Z \cup -Z) \subseteq k\,\mathrm{bs}(A)$. The hypothesis on b then implies $b \in k\,\mathrm{bs}(A)$, and we are done. □

Observe finally that if $A = (a_1, \dots, a_r)$ is forbidden with respect to $L_k(b)$, then so is the reverse sequence $A' = (a_r, \dots, a_1)$. Indeed, A and A' have identical block sums, *i.e.* $\mathrm{bs}(A) = \mathrm{bs}(A')$. Hence, when looking for admissible or forbidden sequences, we only need do it up to reversal.

4. The case $k = 2$

As an illustration of the method, we provide here a proof of the Fox–Kleitman conjecture for $k = 2$, and more generally of Theorem 1.2, which is shorter than the ones given in [1,5]. The key tool is Proposition 3.9.

Proposition 4.1. *Let L denote equation $L_2(2)$. Every positive integer sequence A of length 1 and average $\mu(A) \leq 2$ is forbidden with respect to L.*

Proof. The only sequences to consider are $A = (a)$ with $a \in \{1, 2\}$. Then $\mathrm{bs}(A) = \{0, a, -a\}$ and $2\,\mathrm{bs}(A) = \{0, a, 2a, -a, -2a\}$. Hence $2 \in 2\,\mathrm{bs}(A)$ in either case, showing that A is forbidden. □

Corollary 4.2. *We have* $dor(L_2(2)) = 2$.

Proof. Propositions 3.9 and 4.1 imply that $L_2(2)$ is 2-regular, *i.e.* $\mathrm{dor}(L_2(2)) \geq 2$. Since $\nu(2) = 3$, the reverse inequality follows from the bound $\mathrm{dor}(L_k(b)) \leq \min(2k - 1, \nu(b) - 1)$ of Proposition 2.5. □

Proposition 4.3. *Let L denote equation $L_2(6)$. Every positive integer sequence A of length 6 and average $\mu(A) \leq 3$ is forbidden with respect to $L_2(6)$.*

Proof. In length 1, the sequences (3) and (6) are clearly forbidden.

Therefore, by Proposition 3.12, any sequence admitting either (3) or (6) as a minor is forbidden. Let us now look at forbidden minors of length 2.

• The sequences $A = (1, a)$ with $2 \leq a \leq 7$ are forbidden. This is clear if $a \in \{2, 3, 6\}$ by the above. If $4 \leq a \leq 5$, then $1, 5 \in \mathrm{bs}(A)$, hence $6 \in 2\,\mathrm{bs}(A)$. Finally, if $a = 7$, then $6 = -1 + 7 \in 2\,\mathrm{bs}(A)$.

• The sequences $A = (2, a)$ with $a \in \{2, 3, 4, 6, 8\}$ are forbidden. This is clear if $a \in \{3, 6\}$ by the above. If $a = 2$ or 4, then $2, 4 \in \mathrm{bs}(A)$, hence $6 \in 2\,\mathrm{bs}(A)$. Finally, if $a = 8$, then $6 = -2 + 8 \in \mathrm{bs}(A)$.

Claim. If $A = (a_1, a_2, a_3)$ is admissible and if $\mu(A) \leq 3$, then $A = (2, 5, 2)$.

Indeed, we have $\sigma(A) \leq 9$ by hypothesis. Let $x = \min A$. Then $x \leq 3$. As A is admissible, it cannot contain 3 for otherwise (3) would be a forbidden minor. Hence $x \leq 2$.

Assume $x = 1$. Since $(1, 1, 1)$ is forbidden, because (3) is a minor of it, there can be at most two 1's in A. Let $a \in A$ with $a \neq 1$. We may assume that $(1, a)$ or $(a, 1)$ is a block in A. As A is admissible, we must have $a \geq 8$ by the length 2 case above, whence $\sigma(A) \geq 10$. But this is impossible since $\sigma(A) \leq 9$ by assumption.

Hence $x = 2$. Let $(2, a)$ or $(a, 2)$ be a block in A. Since A is admissible, then $a \geq 5$ by the length 2 case, whence $\sigma(A) \geq 2 + 2 + a \geq 9$. But since $\sigma(A) \leq 9$, we must have $a = 5$ and $A = (2, 5, 2)$. This settles the claim.

Claim. Let $A = (a_1, \dots, a_6)$ such that $\mu(A) \leq 3$. Then A is forbidden.

Indeed, assume for a contradiction that A is admissible. Then every block in A is admissible, and in particular so are its two halves $A_1 = (a_1, a_2, a_3)$, $A_2 = (a_4, a_5, a_6)$. Since $\mu(A) \leq 3$, we have $\mu(A_i) \leq 3$ for some $i \in \{1, 2\}$, say $\mu(A_1) \leq 3$ up to renumbering. But since A_1 is admissible, we must have $A_1 = (2, 5, 2)$ by the above claim, and in particular $\mu(A_1) = 3$. Therefore $\mu(A_2) \leq 3$ as well, whence $A_2 = A_1$ by the same argument as for A_1. Hence $A = (2, 5, 2, 2, 5, 2)$. But this sequence cannot be admissible, since it contains the forbidden minor $(2, 2)$. This contradiction establishes the claim, and the proof is complete. □

Corollary 4.4. *We have* $\operatorname{dor}(L_2(6)) = 3$.

Proof. Propositions 3.9 and 4.3 imply $\operatorname{dor}(L_2(6)) \geq 3$. The reverse inequality follows from Proposition 2.5 using $\nu(6) = 4$. □

Note that this equality alone settles the Fox–Kleitman conjecture for $k = 2$. We are now ready to prove Theorem 1.2 in the following reformulation.

Theorem 4.5. *We have* $\operatorname{dor}(L_2(b)) = \min(3, \nu(b) - 1)$ *for all* $b \geq 1$.

Proof. If $\nu(b) = 2$, then b is odd, whence $\operatorname{dor}(L_2(b)) = 1$ by Lemma 2.4. If $\nu(b) = 3$, then b is even but not divisible by 3. Since $\operatorname{dor}(L_2(2)) = 2$ and since 2 divides b, it follows from Lemma 2.3 that $\operatorname{dor}(L_2(b)) \geq 2$. The reverse inequality follows from Lemma 2.4. Finally, if $\nu(b) \geq 4$, then 6 divides b. Again, since $\operatorname{dor}(L_2(6)) = 3$, it follows that $\operatorname{dor}(L_2(b)) = 3$ by Lemmas 2.1 and 2.3. □

5. The case $k = 3$

Our purpose here is to determine $\operatorname{dor}(L_3(b))$ for all $b \geq 1$ and, in the process, independently settle the Fox–Kleitman conjecture for $k = 3$. The development is self-contained, except for $b = 120$ where we need to rely on some computer calculations.

The determination of $\operatorname{dor}(L_3(b))$ is achieved in Theorem 5.11. We start with the more challenging case $b \equiv 0 \bmod 4$, treated in the next three sections.

5.1. *On equation $L_3(b)$ when $b \equiv 4 \bmod 8$*

Let L denote equation $L_3(b)$ for some $b \equiv 4 \bmod 8$. Our present purpose is to prove that L is not 4-regular. We shall work with the group $G = \mathbb{Z}/8\mathbb{Z}$. Quite naturally, we shall say that a subset $X \subseteq G$ is *regular* if $b \in 3(X - X)$, *singular* otherwise. We start by partitioning G into four singular subsets of cardinality 2.

Lemma 5.1. *The four subsets* $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}$ *of* $G = \mathbb{Z}/8\mathbb{Z}$ *constitute a partition of* G *into singular subsets.*

Proof. Let $X = \{0, 1\} \subset G$. Then $X - X = \{-1, 0, 1\}$, whence

$$3(X - X) = \{-3, -2, -1, 0, 1, 2, 3\} = G \setminus \{4\}.$$

Thus X is L-singular, as claimed. Since the property of being L-singular is stable under translation, the three translates $X + t$ with $t \in \{2, 4, 6\}$ are also singular. □

Proposition 5.2. *If* $b \not\equiv 0 \bmod 8$, *then equation* $L_3(b)$ *is not 4-regular, i.e.* $\operatorname{dor}(L_3(b)) \leq 3$.

Proof. First, if $b \not\equiv 0 \bmod 4$, *i.e.* if $\nu(b) \leq 4$, then $\operatorname{dor}(L_3(b)) \leq 3$ by Lemma 2.4. Assume now $b \equiv 4 \bmod 8$, the last remaining case. The subset $X = \{0, 1\} + 8\mathbb{Z}$ satisfies $b \notin 3(X - X)$, since in the quotient group $G = \mathbb{Z}/8\mathbb{Z}$, we have $\overline{b} = 4 \notin 3(\overline{X} - \overline{X})$ as seen in the above lemma. Said otherwise, the subset $X \subset \mathbb{Z}$ is singular with respect to equation $L_3(b)$. This property remaining true under translation, the subsets $X + t$ with $t \in \{0, 2, 4, 6\}$ constitute a partition of $\mathbb{Z}$ into four $L_3(b)$-singular sets. This implies $\operatorname{dor}(L_3(b)) \leq 3$ as stated. □

5.2. *On equation $L_3(24)$*

Let L denote equation $L_3(24)$. Our aim here is to prove that L is 4-regular. During research on this paper, our first proof of this followed the same line of reasoning as above, using Proposition 3.9 as the key ingredient. Indeed, the 4-regularity of L directly follows from that tool applied to the following statement.

Proposition 5.3. *Let L denote equation $L_3(24)$. Every positive integer sequence A of length 8 and average $\mu(A) \leq 4$ is forbidden with respect to L.*

Now, our detailed proof of this proposition is several pages long. In its place, we shall present here an alternate shorter proof of the 4-regularity of L with a slightly different and more ad-hoc approach. We start with a lemma on L-singular subsets of $[0, 32]$ which are *constant mod 4*, *i.e.* contained in a single class $a + 4\mathbb{N}$ for some integer a.

Lemma 5.4. *Let $S \subset [0, 32]$ be an L-singular subset of cardinality at least 3 which is constant mod 4. Then S is constant mod 16.*

Proof. We may assume $|S| = 3$, since if the statement is valid in that particular case, its validity automatically extends to the general case $|S| \geq 3$. Let $A = \Delta S = (\delta_1, \delta_2)$ be the discrete derivative of S. We have $\delta_1 + \delta_2 \leq 32$ since $S \subset [0, 32]$, and $\delta_1, \delta_2 \in \{4, 8, 12, 16, 20, 24, 28\}$ since S is constant mod 4. We must show that $\delta_1 = \delta_2 = 16$. Since S is singular, we have $24 \notin 3\,\mathrm{bs}(A)$. Note that $\mathrm{bs}(A) = \pm\{0, \delta_1, \delta_2, \delta_1 + \delta_2\}$.

Let $j \in \{1, 2\}$. Since $3\,\mathrm{bs}(A)$ contains $\pm\{3\delta_j, 2\delta_j, \delta_j\}$ and does not contain 24, it follows that $\delta_j \notin \{8, 12, 24\}$. Further, we may assume $\delta_1 \leq \delta_2$. This is achieved by replacing S by $S' = 32 - S$ if necessary, and noting that S' is singular if and only if S is, since $S' - S' = S - S$. It follows that $\delta_1 \leq 16$.

Let us now show that $\delta_1 \neq 4$. For assume, on the contrary, that $\delta_1 = 4$. Each possible value of $\delta_2 \in \{4, 16, 20, 28\}$ is then excluded by the following table, which in each case would explicitly write 24 as an element of $3\,\mathrm{bs}(A)$, in contradiction with the hypothesis. For that, it suffices to note that $3\,\mathrm{bs}(A)$ contains $3(\delta_1 + \delta_2)$, $2\delta_1 + \delta_2$, $\delta_1 + \delta_2 + 0$ and $\delta_2 - \delta_1 + 0$.

δ_2	$24 =$
4	$3(4+4)$
16	$4+4+16$
20	$4+20+0$
28	$28-4+0$

It follows that $\delta_1 = 16$, whence $\delta_2 = 16$ as well, since $\delta_1 \leq \delta_2$ and $\delta_1 + \delta_2 \leq 32$. $\square$

This lemma will be used below in conjunction with the following obvious remark.

Remark 5.5. The only subset $B \subset [0, 32]$ of cardinality at least 3 which is constant mod 16 is $B = \{0, 16, 32\}$ of cardinality 3.

We are now ready to prove the main result of this section.

Theorem 5.6. *Let L denote equation $L_3(24)$. Every subset $X \subset [0, 32]$ of cardinality $|X| = 9$ is regular with respect to L.*

Proof. Let $X \subset [0, 32]$ be such that $|X| = 9$. Assume for a contradiction that X is singular. Let us partition X according to the class mod 4:

$$X = X_0 \cup X_1 \cup X_2 \cup X_3,$$

where $X_i = X \cap (i + 4\mathbb{N})$ for all $0 \leq i \leq 3$. There are several steps.

Step 1. Since subsets of singular sets are singular, and since X is singular, it follows that X_i is singular for all $0 \leq i \leq 3$.

Step 2. We claim that $X_0 = \{0, 16, 32\}$ and that $|X_j| = 2$ for $1 \leq j \leq 3$. Indeed, since

$$9 = |X| = |X_0| + |X_1| + |X_2| + |X_3|, \tag{7}$$

there is some index $0 \leq i \leq 3$ for which $|X_i| \geq 3$. As X_i is singular and constant mod 4, it follows from Lemma 5.4 that $X_i = B = \{0, 16, 32\}$. Thus $i = 0$ and $X_0 = \{0, 16, 32\}$ as claimed. Further, it now follows from (7) that

$$|X_1| = |X_2| = |X_3| = 2.$$

Step 3. Taking the discrete derivative of X_1, X_2, X_3, let $\Delta X_j = (\delta_j)$ for $j = 1, 2, 3$. Since the X_j's are positive and constant mod 4, it follows that $\delta_j \in 4\mathbb{N}_+$. Moreover, since $0 < \delta_j < 32$ for $1 \leq j \leq 3$, we have

$$\delta_j \in \{4, 8, 12, 16, 20, 24, 28\}.$$

Step 4. We claim that $\delta_1 = \delta_2 = \delta_3 = 16$. Indeed, let $j \in \{1, 2, 3\}$. Since X is singular, we have $24 \notin 3\,(X - X)$. On the other hand, since $X \ni 0$ and $X \supset X_j$, we have $3\,(X - X) \supseteq \{3\delta_j, 2\delta_j, \delta_j\}$. Hence $\delta_j \notin \{8, 12, 24\}$. Further, since

$$24 = 4 + 4 + 16 = 20 + 20 - 16 = 28 + 28 - 32$$

and since $X - X \supseteq \{\pm 16, -32, \delta_j\}$, it follows that $\delta_j \notin \{4, 20, 28\}$. Hence $\delta_j = 16$, as claimed.

Step 5. For $j \in \{1, 2, 3\}$ and $z \in \mathbb{N}$, let us denote

$$Y_j(z) = \{4z + j, 4z + j + 16\}.$$

Since X_j is of class j mod 4 and since $\delta_j = 16$, we have $X_j = Y_j(a_j)$ for some $a_j \in \mathbb{N}$. Further, since $X_j \subset [0, 32]$, we have $a_j \in \{0, 1, 2, 3\}$.

Step 6. Thus X depends on the three parameters a_1, a_2, a_3. We shall then write

$$X = X(a_1, a_2, a_3) = X_0 \cup Y_1(a_1) \cup Y_2(a_2) \cup Y_3(a_3).$$

Since $0 \leq a_j \leq 3$ for $j \in \{1, 2, 3\}$, there are 64 cases to consider. Our task is to show that $X(a_1, a_2, a_3)$ is regular in each one, thereby leading to a contradiction and concluding the proof of the theorem. Fortunately, it turns out that only 8 cases need to be considered.

Step 7. To start with, we may assume $a_2 \in \{0, 2\}$. Indeed, for any subset $Z \subseteq [0, 32]$, denote $Z' = 32 - Z$ as in the proof of the preceding lemma. Then $Z' \subseteq [0, 32]$ and Z' is singular if and only if Z is. Now note that for $0 \leq a \leq 3$, we have

$$Y_2(a)' = Y_2(3 - a)$$

as easily verified. In particular, we have $Y_2(1)' = Y_2(2)$ and $Y_2(3)' = Y_2(0)$. The claim follows by replacing X by X' if a_2 is odd. This reduces the number of cases to consider from 64 to 32.

Step 8. We now see that it suffices to consider the 8 cases given by $(a_2, a_3) \in \{0, 2\} \times \{0, 1, 2, 3\}$, the value of the parameter a_1 being irrelevant. Indeed, for the 8 listed cases, the following table shows that $X = X(a_1, a_2, a_3)$ is regular by explicitly writing 24 as an element of $3\,(X - X)$.

(a_2, a_3)	$X_2 \cup X_3 = Y_2(a_2) \cup Y_3(a_3)$	$24 =$
$(0, 0)$	$\{2, 18, 3, 19\}$	$(18-0)+(3-0)+(3-0)$
$(0, 1)$	$\{2, 18, 7, 23\}$	$(23-2)+(7-2)+(0-2)$
$(0, 2)$	$\{2, 18, 11, 27\}$	$(11-0)+(11-0)+(2-0)$
$(0, 3)$	$\{2, 18, 15, 31\}$	$(15-2)+(15-2)+(0-2)$
$(2, 0)$	$\{10, 26, 3, 19\}$	$(10-3)+(10-3)+(10-0)$
$(2, 1)$	$\{10, 26, 7, 23\}$	$(10-0)+(7-0)+(7-0)$
$(2, 2)$	$\{10, 26, 11, 27\}$	$(27-10)+(27-10)+(0-10)$
$(2, 3)$	$\{10, 26, 15, 31\}$	$(15-0)+(15-0)+(10-16)$

These contradictions conclude the proof of the theorem. □

Corollary 5.7. *We have* $dor(L_3(24)) = 4$.

Proof. Consider an arbitrary 4-coloring of the integer interval $[0, 32]$. Since that interval has cardinality 33, one of the color classes contains a subset X of cardinality 9. By the theorem, X is regular, and hence contains a solution to L, which is monochromatic by construction. This implies $\mathrm{dor}(L_3(24)) \geq 4$. The reverse inequality follows from Lemma 2.4 and the fact that 5 does not divide 24. □

5.3. On equation $L_3(120)$

We establish here that equation $L_3(120)$ is 5-regular, thereby independently settling the Fox–Kleitman conjecture for $k = 3$. We give two different proofs. They both require some computer calculations, but of a very different nature. Here is the first approach.

Proposition 5.8. *Let* $X = -Y \cup \{0\} \cup Y$, *where*

$$\begin{aligned} Y &= [1, 30] \cup 5 \cdot [7, 20] \cup (110 + 10 \cdot [0, 5]) \cup (220 + 60 \cdot [0, 4]) \\ &= \{1, \ldots, 30\} \cup \{35, 40, \ldots, 100\} \cup \{110, 120, \ldots, 160\} \cup \{220, 280, \ldots, 460\}. \end{aligned}$$

Then $|X| = 111$ *and, for every 5-coloring of* X, *there is a monochromatic solution of equation* $L_3(120)$ *in* X.

Proof. By translating this coloring problem as a Boolean satisfiability problem and then feeding it to a SAT solver. The solver `march` reached the conclusion in about 20 s on a standard desktop computer. The set X itself was discovered through a patient and delicate computer-aided purification process using `march`. □

Corollary 5.9. *We have* $dor(L_3(120)) = 5$.

Proof. Proposition 5.8 implies that $L_3(120)$ is 5-regular, *i.e.* $\mathrm{dor}(L_3(120)) \geq 5$. The reverse inequality follows from Lemma 2.1. □

Our second proof of the 5-regularity of $L_3(120)$ uses Proposition 3.9. Here is the precise statement.

Theorem 5.10. *Every positive integer sequence A of length* 80 *and average* $\mu(A) \leq 5$ *is forbidden with respect to* $L_3(120)$. *Moreover,* 80 *is minimal with respect to that property.*

Proof. The first part of this result has been obtained by exhaustive computer search. Combined with Proposition 3.9, it directly implies Corollary 5.9.

The fact that length 80 is minimal for the stated property is witnessed by the following instances. First, for $1 \leq r \leq 39$, the sequence

$$(\underbrace{1, \ldots, 1}_{r})$$

is admissible and of average 1. Next, let

$$A(r) = (\underbrace{1, \ldots, 1}_{r}, 121 + 2r, \underbrace{1, \ldots, 1}_{r})$$

of length $2r + 1$. Then $A(r)$ is admissible for all $r \leq 39$, and it is of average $\mu(A(r)) \leq 5$ for all $r \geq 20$. Since any block of an admissible sequence is admissible, we get admissible sequences of average at most 5 and of any length $41 \leq l \leq 79$. Finally, an admissible sequence of length 40 and average exactly 5 is provided by the sequence $A(20)$ with the last 1 removed. □

5.4. On dor($L_3(b)$) for all b

We are now in a position to determine $\operatorname{dor}(L_3(b))$ for all $b \geq 1$. Recall that $f(b)$ is the largest integer r such that $r!$ divides b.

Theorem 5.11. *We have* $\operatorname{dor}(L_3(b)) = \min(5, f(b))$ *for all* $b \geq 1$.

Proof. If $f(b) = 1$, then b is odd, whence $\operatorname{dor}(L_3(b)) = 1$ by Lemma 2.4. If $f(b) = 2$, then b is even but not divisible by 3. Hence $\nu(b) = 3$ and $\operatorname{dor}(L_2(b)) = 2$ by Theorem 4.5. Therefore $\operatorname{dor}(L_3(b)) \geq 2$ by Lemma 2.2. The reverse inequality follows from Lemma 2.4. If $f(b) = 3$, then b is divisible by 6, but not by 24 and hence not by 8. Then $\operatorname{dor}(L_2(b)) = 3$ by Theorem 4.5, yielding $\operatorname{dor}(L_3(b)) \geq 3$ by Lemma 2.2. The reverse inequality is provided by Proposition 5.2, which applies here since $b \not\equiv 0 \bmod 8$. If $f(b) = 4$, then b is divisible by 24, but not by 120 and hence not by 5. The inequality $\operatorname{dor}(L_3(b)) \geq 4$ follows from Corollary 5.7 and Lemma 2.3, while the reverse one follows from Lemma 2.4. Finally, if $f(b) \geq 5$, then b is a multiple of 120, whence $\operatorname{dor}(L_3(b)) \geq 5$ by Corollary 5.9 and Lemma 2.3. The reverse inequality follows from Lemma 2.1. □

6. The case $k = 4$

On the basis of Theorems 4.5 and 5.11 determining $\operatorname{dor}(L_k(b))$ for $k = 2$ and $k = 3$, respectively, one is led to think that $\operatorname{dor}(L_4(b))$ might follow a similar pattern and coincide with $\min(7, f(b))$, where again $f(b)$ denotes the largest integer r such that $r!$ divides b. However, it turns out that this is far from being the case, as shown in the next three sections.

We start with the case $b = 12$, for which $f(12) = 3$ but where $\operatorname{dor}(L_4(12))$ turns out to be equal to 4.

6.1. On equation $L_4(12)$

We show here that equation $L_4(12)$ is 4-regular.

Proposition 6.1. *Let L denote equation $L_4(12)$. Every positive integer sequence A of length 3 and average $\mu(A) \leq 4$ is forbidden with respect to $L_4(12)$.*

Proof. In length 1, the sequences (3), (4), (6) and (12) are all forbidden. Indeed, let $A = (a)$ with $a \in \{3, 4, 6, 12\}$, and let $t = 12/a \leq 4$. Since $12 = ta$, we have

$$12 \in tA \subseteq t\,\mathrm{bs}(A) \subseteq 4\,\mathrm{bs}(A),$$

where the last inclusion derives from $0 \in \mathrm{bs}(A)$. Thus (a) is indeed forbidden with respect to $L_4(12)$.

By Proposition 3.12, any sequence admitting either (3), (4), (6) or (12) as a minor is forbidden. Let us now look at forbidden minors of length 2.

• The sequences $A = (1, a)$ with $2 \leq a \leq 12$ are forbidden. This is clear if $2 \leq a \leq 6$ by the above. If $a = 7$, then $14 \in 2\,\mathrm{bs}(A)$, whence $12 = -1 - 1 + 14 \in 4\,\mathrm{bs}(A)$. If $8 \leq a \leq 9$, then $9 \in \mathrm{bs}(A)$, whence $12 = 1 + 1 + 1 + 9 \in 4\,\mathrm{bs}(A)$. If $a = 10$, then $12 = 1 + 1 + 10 \in 4\,\mathrm{bs}(A)$. And finally, if $11 \leq a \leq 12$, then (12) is a forbidden minor of A.

• The sequences $A = (2, a)$ with $2 \leq a \leq 8$ are forbidden. Indeed, if $2 \leq a \leq 4$, then either (3) or (4) is a forbidden minor. If $a = 5$, then $12 = 2 + 5 + 5 \in 4\,\mathrm{bs}(A)$. If $a = 6$, then (6) is a forbidden minor. If $a = 7$, then $12 = -2 + 7 + 7 \in 4\,\mathrm{bs}(A)$. And finally, if $a = 8$, then $12 = 2 + 2 + 8 \in 4\,\mathrm{bs}(A)$.

Let now $A = (a_1, a_2, a_3)$, and assume $\mu(A) \leq 4$, *i.e.* $\sigma(A) \leq 12$. Let $x = \min A$. Then $x \leq 4$. If $x = 3$ or 4, then A is forbidden. Assume now $x \leq 2$.

Let y, z be the other two members of A. We may assume that y is a neighbor of x in A, so that either (x, y) or (y, x) is a block in A. We have $x \leq y, z$ by hypothesis.

If $x = 2$, then $y \leq 8$ since $\sigma(A) \leq 12$, whence A is forbidden since $(2, y)$ or $(y, 2)$ is a forbidden minor as seen above.

Finally, if $x = 1$, then $y \leq 10$ since $\sigma(A) \leq 12$, whence A is forbidden since $(1, y)$ or $(y, 1)$ is a forbidden minor.

In all cases, we conclude that A is forbidden, as claimed. □

Corollary 6.2. *We have* $\operatorname{dor}(L_4(12)) = 4$.

Proof. Follows from Lemma 2.4 and from Proposition 3.9 applied to the above statement. □

6.2. On equation $L_4(6!)$

Our second instance of discrepancy between $\mathrm{dor}(L_4(b))$ and $\min(7, f(b))$ is for $b = 6! = 720$. The equality $f(6!) = 6$ is obvious.

Theorem 6.3. *We have* $\mathrm{dor}(L_4(6!)) = 5$.

Proof. Let L denote equation $L_4(6!)$. On the one hand, by successively applying Corollary 5.9, Lemmas 2.2 and 2.3, we have

$$5 = \mathrm{dor}(L_3(5!)) \leq \mathrm{dor}(L_4(5!)) \leq \mathrm{dor}(L_4(6!)).$$

It remains to show $\mathrm{dor}(L_4(6!)) < 6$. For that, it suffices to exhibit a partition of $\mathbb{Z}$ into 6 subsets which are singular with respect to L. Denote $X_0 = \{0, 1\} + 11\mathbb{Z}$, and $X_i = X_0 + 2i$ for $1 \leq i \leq 4$, and finally $X_5 = \{10\} + 11\mathbb{Z}$. The sets $X_0, X_1, X_2, X_3, X_4, X_5$ constitute a partition of $\mathbb{Z}$. We claim that they are all L-singular. To this end, it suffices to show that X_0 is L-singular, since the other five sets are either translates of X_0 or subsets thereof.

Thus, it remains to show that $720 \notin 4(X_0 - X_0)$. To do this, let us apply the canonical morphism from $\mathbb{Z}$ to the quotient group $G = \mathbb{Z}/11\mathbb{Z}$. Let $X = \{0, 1\} \subset G$. Then X_0 is mapped to X, and 720 is mapped to 5 since $720 = 11 \cdot 65 + 5$. It remains to show that $5 \notin 4(X - X)$ in G. But this is obvious, since $X - X = \{-1, 0, 1\}$ and hence $4(X - X) = \{-4, -3, \dots, 3, 4\}$. $\square$

6.3. Is $\mathrm{dor}(L_4(b)) = 7$ realizable?

That the answer to this question is positive is precisely the Fox–Kleitman conjecture for $k = 4$. We show here, quite surprisingly, that any integer b which would satisfy $\mathrm{dor}(L_4(b)) = 7$ must be much bigger than $7! = 5040$. This is our third instance of discrepancy between $\mathrm{dor}(L_4(b))$ and $\min(7, f(b))$. For $b = 18!$ for instance, we have $f(18!) = 18$ and so $\min(7, f(18!)) = 7$, whereas the following result implies $\mathrm{dor}(L_4(18!)) < 7$.

Theorem 6.4. *If* $b \not\equiv 0 \bmod 3 \cdot 4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19$, *then* $\mathrm{dor}(L_4(b)) < 7$.

Proof. To start with, if $b \not\equiv 0 \bmod 3 \times 4 \times 5 \times 7$ then $\nu(b) \leq 7$, and Proposition 2.5 implies $\mathrm{dor}(L_4(b)) < 7$. In order to treat the remaining prime factors 11, 13 and 19, we proceed as in the proof of Theorem 6.3.

First, let $p \in \{11, 13\}$. Let $X_0 = \{0, 1\} + p\mathbb{Z}$, and consider its image $X = \{0, 1\}$ in the quotient group $G = \mathbb{Z}/p\mathbb{Z}$. Then $4(X - X) = \{-4, -3, \dots, 3, 4\} \neq G$. For instance, $5 \notin 4(X - X)$. Let $b \in \mathbb{N}_+$ such that $b \not\equiv 0 \bmod p$. Since b is invertible mod p, there exists $c \in \mathbb{N}_+$ such that $bc \equiv 5 \bmod p$. Then $bc \notin 4(X_0 - X_0)$, since $5 \notin 4(X - X)$ in G. Hence X_0 is singular with respect to $L_4(bc)$. The same is true for its translates $t + X_0$ with $t \in \{2, 4, 6, 8, 10, 12\}$. It follows that equation $L_4(bc)$ is not 7-regular. By Lemma 2.3, we get

$$\mathrm{dor}(L_4(b)) \leq \mathrm{dor}(L_4(bc)) < 7.$$

Assume now $p = 19$. The proof proceeds in an analogous way, except that here we need consider $X_0 = \{0, 1, 2\} + p\mathbb{Z}$ and its image $X = \{0, 1, 2\}$ in the quotient group $G = \mathbb{Z}/p\mathbb{Z}$. Then $4(X - X) = \{-8, -7, \dots, 7, 8\} \neq G$. For instance, $9 \notin 4(X - X)$. Let $b \in \mathbb{N}_+$ such that $b \not\equiv 0 \bmod p$. Since b is invertible mod p, there exists $c \in \mathbb{N}_+$ such that $bc \equiv 9 \bmod p$. Hence X_0 is singular with respect to $L_4(bc)$, and the same is true for its translates $t + X_0$ with $t \in \{3, 6, 9, 12, 15, 18\}$. It follows that equation $L_4(bc)$ is not 7-regular, and by Lemma 2.3 again we conclude $\mathrm{dor}(L_4(b)) < 7$. $\square$

The arguments in the above proof cannot be extended to prime factors greater than 19. We may thus ask whether the equality $\mathrm{dor}(L_4(19!)) = 7$ might hold. For the time being, we can neither prove nor disprove it.

7. The case $k \geq b$

While for $k \geq 1$ fixed, it looks hard to determine $\mathrm{dor}(L_k(b))$ as a function of b, we now show that, for $b \geq 1$ fixed, it is easy to determine $\mathrm{dor}(L_k(b))$ for all sufficiently large k, in fact for all $k \geq b$. The answer again involves the function ν.

Proposition 7.1. *Let* $b \geq 1$. *Then, for all* $k \geq b$, *we have* $\mathrm{dor}(L_k(b)) = \nu(b) - 1$.

Proof. Let $m = \nu(b)$. We have $\mathrm{dor}(L_k(b)) \leq m - 1$ by Proposition 2.5. For the reverse inequality, we shall invoke Proposition 3.9 at $N = 1$. So let $A = (a)$ be any positive integer sequence of length 1 and average $\mu(A) \leq m - 1$. Thus $a \leq m - 1$, whence a divides b by definition of $m = \nu(b)$. We then have

$$b = (b/a)a \in (b/a)\,\mathrm{bs}(A) \subseteq k\,\mathrm{bs}(A)$$

since $b/a \leq k$, whence A is forbidden with respect to $L_k(b)$. It then follows from Proposition 3.9 that $L_k(b)$ is $(m-1)$-regular, *i.e.* $\mathrm{dor}(L_k(b)) \geq m - 1$. $\square$

Acknowledgments

The authors wish to thank the referees for their very careful reading of a preliminary version of this paper, their sharp comments correcting a few dubious points, and for pointing out [5]. Part of this work was carried out with support from the *Instituto de Matemáticas* Antonio de Castro Brzezicki *de la Universidad de Sevilla*.

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