Connectivity keeping stars or double-stars in 2-connected graphs *

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Abstract In [W. Mader, Connectivity keeping paths in k-connected graphs, J. Graph Theory 65 (2010) 61-69.], Mader conjectured that for every positive integer k and every finite tree T with order m, every k-connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ contains a subtree T' isomorphic to T such that G - V(T') is k-connected. In the same paper, Mader proved that the conjecture is true when T is a path. Diwan and Tholiya [A.A. Diwan, N.P. Tholiya, Non-separating trees in connected graphs, Discrete Math. 309 (2009) 5235-5237.] verified the conjecture when k = 1. In this paper, we will prove that Mader's conjecture is true when T is a star or double-star and k = 2.

Keywords: 2-Connected graphs; Stars; Double-stars; Mader's Conjecture

1 Introduction

In this paper, graph always means a finite, undirected graph without multiple edges and without loops. For graph-theoretical terminologies and notation not defined here, we follow [1]. For a graph G, the vertex set, the edge set, the minimum degree and the connectivity number of G are denoted by V(G), E(G), $\delta(G)$ and $\kappa(G)$, respectively. The order of a graph G is the cardinality of its vertex set, denoted by |G|. k and m always denote positive integers.

In 1972, Chartrand, Kaugars, and Lick proved the following well-known result.

Theorem 1.1. [2] Every k-connected graph G of minimum degree $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor$ has a vertex u with $\kappa(G-u) \geq k$.

Fujita and Kawarabayashi proved in [4] that every k-connected graph G with minimum degree at least $\lfloor \frac{3}{2}k \rfloor + 2$ has an edge e such that G - V(e) is still k-connected. They conjectured that there are similar results for the existence of connected subgraphs of prescribed order $m \geq 3$ keeping the connectivity.

Conjecture 1. [4] For all positive integers k, m, there is a (least) non-negative integer $f_k(m)$ such that every k-connected graph G with $\delta(G) \ge \lfloor \frac{3}{2}k \rfloor - 1 + f_k(m)$ contains a connected subgraph W of exact order m such that G - V(W) is still k-connected.

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They also gave examples in [4] showing that $f_k(m)$ must be at least m for all positive integers k, m. In [5], Mader proved that $f_k(m)$ exists and $f_k(m) = m$ holds for all k, m.

Theorem 1.2. [5] Every k-connected graph G with $\delta(G) \ge \lfloor \frac{3}{2}k \rfloor + m - 1$ for positive integers k, m contains a path P of order m such that G - V(P) remains k-connected.

In the same paper, Mader [5] asked whether the result is true for any other tree T instead of a path, and gave the following conjecture.

Conjecture 2. [5] For every positive integer k and every finite tree T, there is a least non-negative integer $t_k(T)$, such that every k-connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor - 1 + t_k(T)$ contains a subgraph $T' \cong T$ with $\kappa(G - V(T')) \geq k$.

Mader showed that $t_k(T)$ exists in [6].

Theorem 1.3. [6] Let G be a k-connected graph with $\delta(G) \geq 2(k-1+m)^2 + m-1$ and let T be a tree of order m for positive integers k, m. Then there is a tree $T' \subseteq G$ isomorphic to T such that G - V(T') remains k-connected.

Mader further conjectured that $t_k(T) = |T|$.

Conjecture 3. [5] For every positive integer k and every tree T, $t_k(T) = |T|$ holds.

Theorem 1.2 showed that Conjecture 3 is true when T is a path. Diwan and Tholiya [3] proved that the conjecture holds when k = 1. In the next section, we will verify that Conjecture 3 is true when T is a star and k = 2. It is proved in the last section that Conjecture 3 is true when T is a double-star and k = 2.

A block of a graph G is a maximal connected subgraph of G that has no cut vertex. Note that any block of a connected graph of order at least two is 2-connected or isomorphic to K_2 .

For a vertex subset U of a graph G, G[U] denotes the subgraph induced by U and G - U is the subgraph induced by V(G) - U. The neighborhood $N_G(U)$ of U is the set of vertices in V(G) - U which are adjacent to some vertex in U. If $U = \{u\}$, we also use G - u and $N_G(u)$ for $G - \{u\}$ and $N_G(\{u\})$, respectively. The degree $d_G(u)$ of u is $|N_G(u)|$. If H is a subgraph of G, we often use H for V(H). For example, $N_G(H)$, $H \cap G$ and $H \cap U$ mean $N_G(V(H))$, $V(H) \cap V(G)$ and $V(H) \cap U$, respectively. If there is no confusion, we always delete the subscript, for example, d(u) for $d_G(u)$, N(u) for $N_G(u)$, N(U) for $N_G(U)$ and so on. A tree is a connected graph without cycles. A star is a tree that has exact one vertex with degree greater than one. A double-star is a tree that has exact two vertices with degree greater than one.

2 Connectivity keeping stars in 2-connected graphs

Theorem 2.1. Let G be a 2-connected graph with minimum degree $\delta(G) \geq m+2$, where m is a positive integer. Then for a star T with order m, G contains a star T' isomorphic to T such that G - V(T') is 2-connected.

Proof. If $m \leq 3$, then T is a path, and the Theorem holds by Theorem 1.2. Thus we assume $m \geq 4$ in the following.

Since $\delta(G) \geq m+2$, there is a star $T' \subseteq G$ with $T' \cong T$. Assume $V(T') = \{u, v_1, \cdots, v_{m-1}\}$ and $E(T') = \{uv_i | 1 \leq i \leq m-1\}$. We say T' is a star rooted at u or with root u. Let G' = G - T'. Let B be a maximum block in G' and let l be the number of components of G' - B. If l = 0, then B = G' is 2-connected. So we may assume that $l \geq 1$. Let H_1, \cdots, H_l be the components of G' - B with $|H_1| \geq \cdots \geq |H_l|$.

Take such a star T' so that

- (**P1**) |B| is as large as possible,
- (**P2**) $(|H_1|, \dots, |H_l|)$ is as large as possible in lexicographic order, subject to (P1).

We will complete the proof by a series of claims.

Claim 1.
$$|N(H_i) \cap B| \le 1$$
 and $|N(H_i) \cap V(T')| \ge 1$ for each $i \in \{1, \dots, l\}$.

Since B is a block of G', we have $|N(H_i) \cap B| \le 1$ for each $i \in \{1, \dots, l\}$. By G is 2-connected, $|N(H_i) \cap V(T')| \ge 1$ for each $i \in \{1, \dots, l\}$.

Claim 2. l = 1.

Assume $l \geq 2$. By Claim 1, there is an edge th between T' and H_1 , where $t \in T'$ and $h \in H_1$. Choose a vertex $x \in H_l$. Since $\delta(G) \geq m+2$ and $|N(H_l) \cap B| \leq 1$ (by Claim 1), we have $|N(x) \setminus (B \cup t)| \geq m+2-1-1=m$. Thus we can choose a star $T'' \cong T$ with root x such that $V(T'') \cap (B \cup t) = \emptyset$. But then either there is a larger block than B in G - T'', or G - T'' - B contains a larger component than H_1 ($H_1 \cup t$ is contained in a component of G - T'' - B), which contradicts to (P1) or (P2).

Claim 3. $|N(t) \cap B| \le 1$ and $|N(t) \cap H_1| \ge 2$ for any vertex $t \in V(T')$.

Assume $|N(t) \cap B| \ge 2$. Choose a vertex $x \in H_1$. Since $\delta(G) \ge m+2$ and $|N(H_1) \cap B| \le 1$, we have $|N(x) \setminus (B \cup t)| \ge m+2-1-1=m$. Thus we can choose a star $T'' \cong T$ with root x such that $V(T'') \cap (B \cup t) = \emptyset$. But G - T'' has a block containing $B \cup t$ as a subset, which contradicts to (P1). Thus $|N(t) \cap B| \le 1$ holds. By $d(t) \ge m+2$ and $|N(t) \cap B| \le 1$, we have $|N(t) \cap H_1| = d(t) - |N(t) \cap B| - |N(t) \cap T'| \ge m+2-1-(m-1)=2$.

Claim 4. For any edge $t_1t_2 \in E(T')$, $|N(\{t_1, t_2\}) \cap B| \le 1$ holds.

By contradiction, assume $|N(\{t_1,t_2\})\cap B| \geq 2$. Because $|N(t_1)\cap B| \leq 1$ and $|N(t_2)\cap B| \leq 1$, we can assume that there are two distinct vertices $b_1,b_2\in B$ such that $t_1b_1,t_2b_2\in E(G)$. Choose a vertex $x\in H_1$. Since $\delta(G)\geq m+2$ and $|N(H_1)\cap B|\leq 1$, we have $|N(x)\setminus (B\cup \{t_1,t_2\})|\geq m+2-1-2=m-1$. Thus we can choose a star $T''\cong T$ with root x such that $V(T'')\cap (B\cup \{t_1,t_2\})=\emptyset$. But then G-T'' has a block containing $B\cup \{t_1,t_2\}$ as a subset, which contradicts to (P1).

Because $|N(H_1) \cap B| \le 1$ and G is 2-connected, we have $|N(T') \cap B| \ge 1$. The following claim further shows that $|N(T') \cap B| = 1$.

Claim 5. $|N(T') \cap B| = 1$.

By contradiction, assume $|N(T') \cap B| \ge 2$. If $N(u) \cap B \ne \emptyset$, say $N(u) \cap B = \{u'\}$, then we have $N(\{v_1, \dots, v_{m-1}\}) \cap B \subseteq \{u'\}$ by Claim 4. That is, $N(T') \cap B = \{u'\}$, a contradiction. Thus $N(u) \cap B = \emptyset$. Assume, without loss of generality, that there are two distinct vertices w

and w' in B such that $v_1w, v_2w' \in E(G)$. If $N(v_3) \cap B = \emptyset$ or $|N(v_3) \cap \{v_1, v_2\}| \leq 1$, then we can choose a star T'' with order m and root v_3 such that $V(T'') \cap (B \cup \{u, v_1, v_2\}) = \emptyset$. But then $B \cup \{u, v_1, v_2\}$ is contained in a block of G - T'', contradicting to (P1). Thus we assume v_3 is adjacent to a vertex y in B and is adjacent to both v_1 and v_2 . Without loss of generality, assume y is distinct from w. Then we can choose a star T'' with order m and root u such that $V(T'') \cap (B \cup \{v_1, v_3\}) = \emptyset$. But $B \cup \{v_1, v_3\}$ is contained in a block of G - T'', contradicting to (P1). Thus $|N(T') \cap B| = 1$.

By Claim 5, $|N(T') \cap B| = 1$. Assume $N(T') \cap B = \{w\}$. Since G is 2-connected, we have $|N(H_1) \cap B| \ge 1$. By Claim 1, $|N(H_1) \cap B| = 1$. Assume $N(H_1) \cap B = \{z\}$. Let P be a shortest path from z to w going through H_1 and T'. Assume $P := p_1 p_2 \cdots p_{q-1} p_q$, where $p_1 = z$, $p_q = w$ and $p_i \in H_1 \cup T'$ for each $i \in \{2, \dots, q-1\}$. Since P is a shortest path, $|N(p_i) \cap P| = 2$ for each $2 \le i \le q-1$. By $N(T') \cap B = \{w\}$ and $N(H_1) \cap B = \{z\}$, $N(p_i) \cap B \subseteq \{w, z\} \subseteq V(P)$ for each $2 \le i \le q-1$. Thus $|N(p_i) \cap (B \cup P)| = 2$ and $|N(p_i) \cap (V(G) \setminus (B \cup P))| \ge m$ for each $2 \le i \le q-1$. This implies $G - B \cup P$ is not empty. For any vertex x in $G - B \cup P$, we have $|N(x) \cap P| \le 3$. For otherwise, we can find a path P' containing x from z to w going through H_1 and T' shorter than P, a contradiction. By $\delta(G) \ge m+2$, $|N(x) \cap (G-B \cup P)| \ge m+2-3=m-1$. Then we can find a star $T'' \cong T$ with root x such that $T'' \cap (B \cup P) = \emptyset$. But then $B \cup P$ is contained in a block of G - T'', a contradiction. The proof is thus complete. \square

3 Connectivity keeping double-stars in 2-connected graphs

Lemma 3.1. Let G be a graph and T be a double-star with order m. If there is an edge $e = uv \in E(G)$ such that $|N(u) \setminus v| \ge \lfloor \frac{m}{2} \rfloor - 1$, $|N(v) \setminus u| \ge m - 3$ and $|N(u) \cup N(v) \setminus \{u, v\}| \ge m - 2$, then there is a double-star $T' \subseteq G$ isomorphic to T.

Proof. By T is a double-star, $m \geq 4$. Assume the double-star T is constructed from an edge e' = u'v' by adding r leaves to u' and s leaves to v', where $1 \leq r \leq s$ and r+s=m-2. Then $1 \leq r \leq \lfloor \frac{m}{2} \rfloor - 1$ and $\lceil \frac{m}{2} \rceil - 1 \leq s \leq m-3$. Since $|N(u) \setminus v| \geq \lfloor \frac{m}{2} \rfloor - 1$, $|N(v) \setminus u| \geq m-3$ and $|N(u) \cup N(v) \setminus \{u,v\}| \geq m-2$, we can find a double-star $T' \cong T$ in G with center-edge e = uv, where u is adjacent to r leaves and v is adjacent to s leaves. \square

The main idea of the proof of Theorem 3.2 is similar to that of Theorem 2.1, with much more complicated and different details.

Theorem 3.2. Let T be a double-star with order m and G be a 2-connected graph with minimum degree $\delta(G) \geq m+2$. Then G contains a double-star T' isomorphic to T such that G-V(T') is 2-connected.

Proof. By T is a double-star, $m \ge 4$. If m = 4, then T is a path, and the Theorem holds by Theorem 1.2. Thus we assume $m \ge 5$ in the following.

Since $\delta(G) \geq m+2$, there is a double-star $T' \subseteq G$ with $T' \cong T$. Assume $V(T') = \{u, v, u_1, \cdots, u_r, v_1, \cdots, v_s\}$ and $E(T') = \{uv\} \cup \{uu_i | 1 \leq i \leq r\} \cup \{vv_j | 1 \leq j \leq s\}$, where $1 \leq r \leq s$ and r+s=m-2. We say T' is a double-star with center-edge uv. Let G' = G - T'. Let B be a maximum block in G' and let l be the number of components of G' - B. If l = 0, then B = G' is 2-connected. So we may assume that $l \geq 1$. Let H_1, \cdots, H_l be the components of G' - B with $|H_1| \geq \cdots \geq |H_l|$.

Take such a double-star T' so that

- $(\mathbf{P1})$ |B| is as large as possible,
- (**P2**) $(|H_1|, \dots, |H_l|)$ is as large as possible in lexicographic order, subject to (P1).

We will complete the proof by a series of claims.

Claim 1. $|N(H_i) \cap B| \le 1$ and $|N(H_i) \cap T'| \ge 1$ for each $i \in \{1, \dots, l\}$.

Since B is a block of G', we have $|N(H_i) \cap B| \le 1$ for each $i \in \{1, \dots, l\}$. By G is 2-connected, $|N(H_i) \cap T'| \ge 1$ for each $i \in \{1, \dots, l\}$.

Claim 2. $|H_i| \ge 2$ for each $i \in \{1, \dots, l\}$.

This claim holds because $|N(h_i) \cap H_i| = d(h_i) - |N(h_i) \cap T'| - |N(h_i) \cap B| \ge m + 2 - m - 1 = 1$ for any vertex $h_i \in H_i$, where $1 \le i \le l$.

Claim 3. l = 1.

Assume $l \geq 2$. By Claim 1, there is an edge th between T' and H_1 , where $t \in T'$ and $h \in H_1$. By Claim 2, we can choose an edge $xy \in E(H_l)$. Since $\delta(G) \geq m+2$ and $|N(H_l) \cap B| \leq 1$ (by Claim 1), we have $|N(x) \setminus (B \cup \{y,t\})| \geq m+2-1-2 = m-1$ and $|N(y) \setminus (B \cup \{x,t\})| \geq m+2-1-2 = m-1$. Thus, by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge xy such that $V(T'') \cap (B \cup t) = \emptyset$. But then either there is a larger block than B in G - T'', or G - T'' - B contains a larger component than H_1 ($H_1 \cup t$ is contained in a component of G - T'' - B), which contradicts to (P1) or (P2).

Claim 4. $|N(t) \cap B| \le 1$ and $|N(t) \cap H_1| \ge 2$ for any vertex $t \in V(T')$.

Assume $|N(t) \cap B| \ge 2$. Choose an edge $xy \in E(H_1)$. Since $\delta(G) \ge m+2$ and $|N(H_1) \cap B| \le 1$, we have $|N(x) \setminus (B \cup \{y,t\})| \ge m+2-1-2 = m-1$ and $|N(y) \setminus (B \cup \{x,t\})| \ge m+2-1-2 = m-1$. Thus, by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge xy such that $V(T'') \cap (B \cup t) = \emptyset$. But then $B \cup t$ is contained in a block of G - T'', which contradicts to (P1). Thus $|N(t) \cap B| \le 1$ holds for any vertex $t \in V(T')$. By $d(t) \ge m+2$ and $|N(t) \cap B| \le 1$, we have $|N(t) \cap H_1| = d(t) - |N(t) \cap B| - |N(t) \cap T'| \ge m+2-1-(m-1) = 2$.

Claim 5. For any edge $t_1t_2 \in E(T')$, $|N(\{t_1, t_2\}) \cap B| \le 1$ holds.

By contradiction, assume $|N(\{t_1,t_2\})\cap B| \geq 2$. Because $|N(t_1)\cap B| \leq 1$ and $|N(t_2)\cap B| \leq 1$, we can assume that there are two distinct vertices $b_1,b_2\in B$ such that $t_1b_1,t_2b_2\in E(G)$. Choose an edge $xy\in E(H_1)$. Since $\delta(G)\geq m+2$ and $|N(H_1)\cap B|\leq 1$, we have $|N(x)\setminus (B\cup\{y,t_1,t_2\})|\geq m+2-1-3=m-2$ and $|N(y)\setminus (B\cup\{x,t_1,t_2\})|\geq m+2-1-3=m-2$. Thus, by Lemma 3.1, we can choose a double-star $T''\cong T$ with center-edge xy such that $V(T'')\cap (B\cup\{t_1,t_2\})=\emptyset$. But then G-T'' has a block containing $B\cup\{t_1,t_2\}$ as a subset, which contradicts to (P1).

Claim 6. For any 3-path $t_1t_2t_3$ in T', $|N(\{t_1, t_2, t_3\}) \cap B| \le 1$ holds.

By contradiction, assume $|N(\{t_1,t_2,t_3\})\cap B|\geq 2$. Then we have $|N(t_2)\cap B|=0$. For otherwise, if $|N(t_2)\cap B|=1$, then we have $|N(\{t_1,t_2,t_3\})\cap B|\leq 1$ by $|N(\{t_1,t_2\})\cap B|\leq 1$ and $|N(\{t_2,t_3\})\cap B|\leq 1$, a contradiction. Because $|N(t_1)\cap B|\leq 1$ and $|N(t_3)\cap B|\leq 1$, we can assume that there are two distinct vertices $b_1,b_3\in B$ such that $t_1b_1,t_3b_3\in E(G)$. Choose any edge $xy\in E(H_1)$. Since $\delta(G)\geq m+2$ and $|N(H_1)\cap B|\leq 1$, we have $|N(x)\setminus (B\cup\{y,t_1,t_2,t_3\})|\geq m+2-1-4=m-3>\lfloor \frac{m}{2}\rfloor-1$ (By $m\geq 5$) and $|N(y)\setminus (B\cup\{x,t_1,t_2,t_3\})|\geq m+2-1-4=m-3$.

If $|N(x)\setminus (B\cup\{y,t_1,t_2,t_3\})| > m-3$ or $|N(y)\setminus (B\cup\{x,t_1,t_2,t_3\})| > m-3$, then by Lemma 3.1, we can choose a double-star $T''\cong T$ with center-edge xy such that $V(T'')\cap (B\cup\{t_1,t_2,t_3\})=\emptyset$. But then G-T'' has a block containing $B\cup\{t_1,t_2,t_3\}$ as a subset, which contradicts to (P1). Thus we assume $|N(x)\setminus (B\cup\{y,t_1,t_2,t_3\})|=m-3$ and $|N(y)\setminus (B\cup\{x,t_1,t_2,t_3\})|=m-3$, which imply $|N(x)\cap B|=1$ and $|N(y)\cap B|=1$. Since $|N(H_1)\cap B|\leq 1$, we can assume $N(x)\cap B=N(y)\cap B=\{z\}$. Without loss of generality, assume $z\neq b_1$.

If $N(x) \setminus y \neq N(y) \setminus x$, then $|N(x) \cup N(y) \setminus (B \cup \{x, y, t_1, t_2, t_3\})| \geq m - 2$. So we can choose a double-star $T'' \cong T$ with center-edge xy disjoint from $B \cup \{t_1, t_2, t_3\}$. But then G - T'' contains a larger block than B, a contradiction. Thus $N(x) \setminus y = N(y) \setminus x$. Because we choose the edge xy in H_1 arbitrarily, we conclude that H_1 is a complete graph and each vertex not in H_1 is adjacent to all vertices in H_1 if it is adjacent to one vertex in H_1 . In particular, every vertex t in T' is adjacent to all vertices in H_1 by Claim 4 and the vertex z in B is adjacent to all vertices in H_1 .

Let t_4h_4 be an edge of graph G, where $t_4 \in V(T') \setminus \{t_1, t_2, t_3\}$ and $h_4 \in V(H_1)$. Let h_1 be a vertex in H_1 distinct from h_4 . Then $t_1h_1, h_1z \in E(G)$. Thus we can choose a double-star $T'' \cong T$ with center-edge t_4h_4 disjoint from $B \cup \{t_1, h_1\}$. But then $B \cup \{t_1, h_1\}$ is contained in a block of G - T'', contradicting to (P1).

Because $|N(H_1) \cap B| \le 1$ and G is 2-connected, we have $|N(T') \cap B| \ge 1$. The following claim further shows that $|N(T') \cap B| = 1$.

Claim 7. $|N(T') \cap B| = 1$.

By contradiction, assume $|N(T') \cap B| \geq 2$. If $N(u) \cap B \neq \emptyset$, say $N(u) \cap B = \{u'\}$, then we have $N(\{u_1, \cdots, u_r, v\}) \cap B \subseteq \{u'\}$ by Claim 5 and $N(\{v_1, \cdots, v_s\}) \cap B \subseteq \{u'\}$ by Claim 6. That is, $N(T') \cap B = \{u'\}$, a contradiction. Thus $N(u) \cap B = \emptyset$. Similarly, we have $N(v) \cap B = \emptyset$. Since $|N(\{u_1, \cdots, u_r\}) \cap B| \leq 1$ and $|N(\{v_1, \cdots, v_s\}) \cap B| \leq 1$ (By Claim 6), we have $|N(T') \cap B| = 2$. Assume, without loss of generality, that there are two distinct vertices w and w' in B such that $u_1w, v_1w' \in E(G)$.

We first show that any vertex x in $\{u_1,\cdots,u_r,v_1,\cdots,v_s\}\setminus\{u_1,v_1\}$ has no neighbors in B. By contradiction, assume there is a vertex in $\{u_1,\cdots,u_r,v_1,\cdots,v_s\}\setminus\{u_1,v_1\}$, say v_j for some $j\in\{2,\cdots,s\}$ (the case u_i for some $i\in\{2,\cdots,r\}$ can be proved similarly), such that $N(v_j)\cap B=\{w'\}$. If v_j is adjacent to u (or u_1), then for any edge vv' (v' is a neighbor of v in H_1), we have $|N(v)\setminus(B\cup\{u,u_1,v_j,v'\})|\geq m+2-4=m-2$ (or $|N(v)\setminus(B\cup\{u_1,v_j,v'\})|\geq m+2-3=m-1$) and $|N(v')\setminus(B\cup\{u,v,u_1,v_j\})|\geq m+2-1-4=m-3$ (or $|N(v')\setminus(B\cup\{v,u_1,v_j\})|\geq m+2-1-3=m-2$). By Lemma 3.1, we can find a double-star $T''\cong T$ with center-edge vv' such that T'' is disjoint from $B\cup\{u,u_1,v_j\}$ (or $B\cup\{u_1,v_j\}$). But then G-T'' contains a larger block than B, a contradiction. Thus neither u nor u_1 is adjacent to v_j . Choose a neighbor v_j' of v_j in H_1 . Since $|N(v_j)\setminus(B\cup\{u,v,u_1,v_1,v_j'\})|\geq m+2-1-3=m-2$ and $|N(v_j')\setminus(B\cup\{u,v,u_1,v_1,v_j\})|\geq m+2-1-5=m-4\geq\lfloor\frac{m}{2}\rfloor-1$ (By $m\geq 5$), we can find a double-star $T''\cong T$ with center-edge v_jv_j' such that T'' is disjoint from $B\cup\{u,v,u_1,v_1\}$. But then G-T'' contains a larger block than B, a contradiction. Thus we have $N(\{u_1,\cdots,u_r,v_1,\cdots,v_s\}\setminus\{u_1,v_1\})\cap B=\emptyset$.

Let $v_2v_2' \in E(G)$, where v_2' is a neighbor of v_2 in H_1 . Since $\delta(G) \geq m+2$ and $N(v_2) \cap B = \emptyset$, we have $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2'\})| \geq m+2-5 = m-3$ and $|N(v_2') \setminus (B \cup \{u, v, u_1, v_1, v_2\})| \geq m+2-1-5 = m-4 \geq \lfloor \frac{m}{2} \rfloor - 1$ (By $m \geq 5$). If $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2'\})| \geq m-2$,

then, by Lemma 3.1, we can find a double-star $T''\cong T$ with center-edge v_2v_2' such that T'' avoids $B\cup\{u,v,u_1,v_1\}$. But then G-T'' contains a larger block than B, a contradiction. Thus assume $|N(v_2)\setminus (B\cup\{u,v,u_1,v_1,v_2'\})|=m-3$, which implies v_2 is adjacent to both u_1 and v_1 . For the edge uv, we can verify that $|N(u)\setminus (B\cup\{v,u_1,v_1,v_2\})|\geq m+2-4=m-2$ and $|N(v)\setminus (B\cup\{u,u_1,v_1,v_2\})|\geq m+2-4=m-2$. By Lemma 3.1, we can find a double-star $T''\cong T$ with center-edge uv such that T'' avoids $B\cup\{u_1,v_1,v_2\}$. But then $B\cup\{u_1,v_1,v_2\}$ is contained in a block of G-T'', contradicting to (P1). Thus Claim 7 holds.

By Claim 7, $|N(T') \cap B| = 1$. Assume $N(T') \cap B = \{w\}$. Since G is 2-connected, we have $|N(H_1) \cap B| = 1$ by Claim 1. Let $N(H_1) \cap B = \{z\}$. Let P be a shortest path from z to w going through H_1 and T''. Assume $P := p_1 p_2 \cdots p_{q-1} p_q$, where $p_1 = z$, $p_q = w$ and $p_i \in H_1 \cup T'$ for each $i \in \{2, \cdots, q-1\}$. Since P is a shortest path, $N(p_i) \cap P = \{p_{i-1}, p_{i+1}\}$ for $2 \le i \le q-1$. Because $\delta(G) \ge m+2$ and $N(p_i) \cap B \subseteq \{w,z\} \subseteq P$ for each $2 \le i \le q-1$, we know p_i has at least m neighbors not in $B \cup P$, that is, $G - B \cup P$ is not empty. For any vertex x in $G - B \cup P$, we have $|N(x) \cap P| \le 3$. For otherwise, we can find a path P' containing x from z to w going through H_1 and T'' shorter than P, a contradiction. By $\delta(G) \ge m+2$, $|N(x) \cap (G - B \cup P)| \ge m+2-3 = m-1$. Choose an edge xy in $G - B \cup P$. Since $|N(x) \setminus (B \cup P \cup y)| \ge m+2-4 = m-2$ and $|N(y) \setminus (B \cup P \cup x)| \ge m+2-4 = m-2$, we can find a double-star $T'' \cong T$ with center-edge xy such that $T'' \cap (B \cup P) = \emptyset$. But then $B \cup P$ is contained in a block of G - T'', a contradiction. The proof is thus complete. \square

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