# Stability in the Erdős–Gallai Theorem on cycles and paths, II\*

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#### Abstract

The Erdős–Gallai Theorem states that for  $k \geq 3$ , any n-vertex graph with no cycle of length at least k has at most  $\frac{1}{2}(k-1)(n-1)$  edges. A stronger version of the Erdős–Gallai Theorem was given by Kopylov: If G is a 2-connected n-vertex graph with no cycle of length at least k, then  $e(G) \leq \max\{h(n,k,2),h(n,k,\lfloor\frac{k-1}{2}\rfloor)\}$ , where  $h(n,k,a) := \binom{k-a}{2} + a(n-k+a)$ . Furthermore, Kopylov presented the two possible extremal graphs, one with h(n,k,2) edges and one with  $h(n,k,\lfloor\frac{k-1}{2}\rfloor)$  edges.

In this paper, we complete a stability theorem which strengthens Kopylov's result. In particular, we show that for  $k \geq 3$  odd and all  $n \geq k$ , every n-vertex 2-connected graph G with no cycle of length at least k is a subgraph of one of the two extremal graphs or  $e(G) \leq \max\{h(n,k,3),h(n,k,\frac{k-3}{2})\}$ . The upper bound for e(G) here is tight.

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## 1 Introduction

One of the basic Turán-type problems is to determine the maximum number of edges in an n-vertex graph with no k-vertex path. Erdős and Gallai [3] in 1959 proved the following fundamental result on this problem.

**Theorem 1.1** (Erdős and Gallai [3]). Fix  $n, k \geq 2$ . If G is an n-vertex graph that does not contain a path with k vertices, then  $e(G) \leq \frac{1}{2}(k-2)n$ .

When n is divisible by k-1, the bound is best possible. Indeed, the n-vertex graph whose every component is the complete graph  $K_{k-1}$  has  $\frac{1}{2}(k-2)n$  edges and no k-vertex paths. Also, if H is an n-vertex graph without a k-vertex path  $P_k$ , then by adding to H a new vertex v adjacent to all vertices of H we obtain an (n+1)-vertex graph H' with e(H) + n edges that contains no cycle of length k+1 or longer. Then Theorem 1.1 follows from another theorem of Erdős and Gallai:

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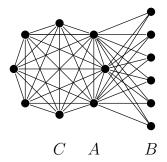


Figure 1:  $H_{14,11,3}$ .

**Theorem 1.2** (Erdős and Gallai [3]). Fix  $n, k \geq 3$ . If G is an n-vertex graph that does not contain a cycle of length at least k, then  $e(G) \leq \frac{1}{2}(k-1)(n-1)$ .

The bound of this theorem is best possible for n-1 divisible by k-2. Indeed, any connected n-vertex graph in which every block is a  $K_{k-1}$  has  $\frac{1}{2}(k-1)(n-1)$  edges and no cycles of length at least k. In the 1970's, some refinements and new proofs of Theorems 1.1 and 1.2 were obtained by Faudree and Schelp [4, 5], Lewin [9], Woodall [10], and Kopylov [8] – see [7] for more details. The strongest version was proved by Kopylov [8]. His result is stated in terms of the following graphs. Let  $n \geq k$  and  $1 \leq a < \frac{1}{2}k$ . The n-vertex graph  $H_{n,k,a}$  is as follows. The vertex set of  $H_{n,k,a}$  is the union of three disjoint sets A, B, and C such that |A| = a, |B| = n - k + a and |C| = k - 2a, and the edge set of  $H_{n,k,a}$  consists of all edges between A and B together with all edges in  $A \cup C$  (Fig. 1 shows  $H_{14,11,3}$ ). Let

$$h(n, k, a) := e(H_{n,k,a}) = {k-a \choose 2} + a(n-k+a).$$

For a graph G, let c(G) denote the length of a longest cycle in G. Observe that  $c(H_{n,k,a}) < k$ : Since  $|A \cup C| = k - a$ , any cycle D of at length at least k has at least a vertices in B. But as B is independent and 2a < k, D also has to contain at least k + 1 neighbors of the vertices in B, while only a vertices in A have neighbors in A. Kopylov [8] showed that the extremal 2-connected n-vertex graphs with no cycles of length at least k are  $G = H_{n,k,2}$  and  $G = H_{n,k,t}$ : the first has more edges for small n, and the second — for large n.

**Theorem 1.3** (Kopylov [8]). Let  $n \ge k \ge 5$  and  $t = \lfloor \frac{1}{2}(k-1) \rfloor$ . If G is an n-vertex 2-connected graph with c(G) < k, then

$$e(G) \le \max\{h(n,k,2),h(n,k,t)\} \tag{1}$$

with equality only if  $G = H_{n,k,2}$  or  $G = H_{n,k,t}$ .

## 2 Main results

#### 2.1 A previous result

Recently, three of the present authors proved in [6] a stability version of Theorems 1.2 and 1.3 for n-vertex 2-connected graphs with  $n \geq 3k/2$ , but the problem remained open for n < 3k/2 when  $k \geq 9$ . The main result of [6] was the following:

**Theorem 2.1** (Füredi, Kostochka, Verstraëte [6]). Let  $t \ge 2$  and  $n \ge 3t$  and  $k \in \{2t+1, 2t+2\}$ . Let G be a 2-connected n-vertex graph c(G) < k. Then  $e(G) \le h(n, k, t-1)$  unless

- (a)  $k = 2t + 1, k \neq 7, \text{ and } G \subseteq H_{n,k,t} \text{ or }$
- (b) k = 2t + 2 or k = 7, and G A is a star forest for some  $A \subseteq V(G)$  of size at most t.

Note that

$$h(n,k,t) - h(n,k,t-1) = \begin{cases} n-t-3 & \text{if } k = 2t+1, \\ n-t-5 & \text{if } k = 2t+2. \end{cases}$$

The paper [6] also describes the 2-connected n-vertex graphs with  $c(G) < k \le 8$  for all  $n \ge k$ .

#### 2.2 The essence of the main result

Together with [6], this paper gives a full description of the 2-connected n-vertex graphs with c(G) < k and 'many' edges for all k and n. Our main result is:

**Theorem 2.2.** Let  $t \ge 4$  and  $k \in \{2t+1, 2t+2\}$ , so that  $k \ge 9$ . If G is a 2-connected graph on  $n \ge k$  vertices and c(G) < k, then either  $e(G) \le \max\{h(n, k, t-1), h(n, k, 3)\}$  or

- (a) k = 2t + 1 and  $G \subseteq H_{n,k,t}$  or
- (b) k = 2t + 2 and G A is a star forest for some  $A \subseteq V(G)$  of size at most t.
- (c)  $G \subseteq H_{n,k,2}$ .

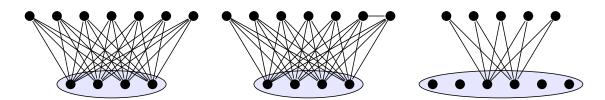


Figure 2:  $H_{n,k,t}(k=2t+1), H_{n,k,t}(k=2t+2), H_{n,k,2}$ ; ovals denote complete subgraphs of order t, t, and k-2 respectively.

Note that the case n < k is trivial and the case  $k \le 8$  was fully resolved in [6].

#### 2.3 A more detailed form of the main result

In order to prove Theorem 2.2, we need a more detailed description of graphs satisfying (b) in the theorem that do not contain 'long' cycles.

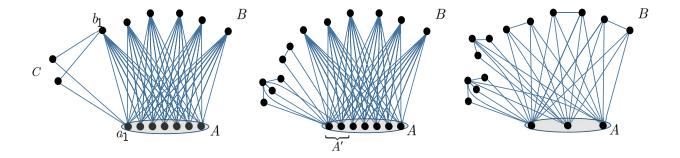


Figure 3: Classes  $\mathcal{G}_2(n,k)$ ,  $\mathcal{G}_3(n,k)$  and  $\mathcal{G}_4(n,10)$ .

Let  $\mathcal{G}_1(n,k) = \{H_{n,k,t}, H_{n,k,2}\}$ . Each  $G \in \mathcal{G}_2(n,k)$  is defined by a partition  $V(G) = A \cup B \cup C$  and two vertices  $a_1 \in A$ ,  $b_1 \in B$  such that |A| = t,  $G[A] = K_t$ , G[B] is the empty graph, G(A,B) is a complete bipartite graph, and  $N(c) = \{a_1,b_1\}$  for every  $c \in C$ . Every member of  $G \in \mathcal{G}_3(n,k)$  is defined by a partition  $V(G) = A \cup B \cup J$  such that |A| = t,  $G[A] = K_t$ , G(A,B) is a complete bipartite graph, and

- G[J] has more than one component,
- all components of G[J] are stars with at least two vertices each,
- there is a 2-element subset A' of A such that  $N(J) \cap (A \cup B) = A'$ ,
- for every component S of G[J] with at least 3 vertices, all leaves of S have degree 2 in G and are adjacent to the same vertex a(S) in A'.

The class  $\mathcal{G}_4(n,k)$  is empty unless k=10. Each graph  $H \in \mathcal{G}_4(n,10)$  has a 3-vertex set A such that  $H[A]=K_3$  and H-A is a star forest such that if a component S of H-A has more than two vertices then all its leaves have degree 2 in H and are adjacent to the same vertex a(S) in A. These classes are illustrated below:

We can refine Theorem 2.2 in terms of the classes  $\mathcal{G}_i(n,k)$  as follows:

**Theorem 2.3.** (Main Theorem) Let  $k \geq 9$ ,  $n \geq k$  and  $t = \lfloor \frac{1}{2}(k-1) \rfloor$ . Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then  $e(G) \leq \max\{h(n, k, t-1), h(n, k, 3)\}$  or G is a subgraph of a graph in G(n, k), where

- (1) if k is odd, then  $G(n,k) = G_1(n,k) = \{H_{n,k,t}, H_{n,k,2}\};$
- (2) if k is even and  $k \neq 10$ , then  $\mathcal{G}(n,k) = \mathcal{G}_1(n,k) \cup \mathcal{G}_2(n,k) \cup \mathcal{G}_3(n,k)$ ;
- (3) if k = 10, then  $\mathcal{G}(n, k) = \mathcal{G}_1(n, 10) \cup \mathcal{G}_2(n, 10) \cup \mathcal{G}_3(n, 10) \cup \mathcal{G}_4(n, 10)$ .

Since every graph in  $\mathcal{G}_2(n,k) \cup \mathcal{G}_3(n,k)$  and many graphs in  $\mathcal{G}_4(n,k)$  have a separating set of size 2 (see Fig. 3), the theorem implies the following simpler statement for 3-connected graphs:

**Corollary 2.4.** Let  $k \in \{2t+1, 2t+2\}$  where  $k \ge 9$ . If G is a 3-connected graph on  $n \ge k$  vertices and c(G) < k, then either  $e(G) \le \max\{h(n, k, t-1), h(n, k, 3)\}$  or  $G \subseteq H_{n,k,t}$  or k = 10 and G is a subgraph of some graph  $H \in \mathcal{G}_4(n, 10)$  such that each component of H - A has at most 2 vertices.

## 3 The proof idea

### 3.1 Small dense subgraphs

First we define some more graph classes. For a graph F and a nonnegative integer s, we denote by  $\mathcal{K}^{-s}(F)$  the family of graphs obtained from F by deleting at most s edges.

Let  $F_0 = F_0(t)$  denote the complete bipartite graph  $K_{t,t+1}$  with partite sets A and B where |A| = t and |B| = t+1. Let  $\mathcal{F}_0 = \mathcal{K}^{-t+3}(F_0)$ , i.e., the family of subgraphs of  $K_{t,t+1}$  with at least t(t+1)-t+3 edges.

Let  $F_1 = F_1(t)$  denote the complete bipartite graph  $K_{t,t+2}$  with partite sets A and B where |A| = t and |B| = t+2. Let  $\mathcal{F}_1 = \mathcal{K}^{-t+4}(F_1)$ , i.e., the family of subgraphs of  $K_{t,t+2}$  with at least t(t+2)-t+4 edges.

Let  $\mathcal{F}_2$  denote the family of graphs obtained from a graph in  $\mathcal{K}^{-t+4}(F_1)$  by subdividing an edge  $a_1b_1$  with a new vertex  $c_1$ , where  $a_1 \in A$  and  $b_1 \in B$ . Note that any member  $H \in \mathcal{F}_2$  has at least |A||B| - (t-3) edges between A and B and the pair  $a_1b_1$  is not an edge.

Let  $F_3 = F_3(t,t')$  denote the complete bipartite graph  $K_{t,t'}$  with partite sets A and B where |A| = t and |B| = t'. Take a graph from  $\mathcal{K}^{-t+4}(F_3)$ , select two non-empty subsets  $A_1, A_2 \subseteq A$  with  $|A_1 \cup A_2| \geq 3$  such that  $A_1 \cap A_2 = \emptyset$  if  $\min\{|A_1|, |A_2|\} = 1$ , add two vertices  $c_1$  and  $c_2$ , join them to each other and add the edges from  $c_i$  to the elements of  $A_i$ , (i = 1, 2). The class of obtained graphs is denoted by  $\mathcal{F}(A, B, A_1, A_2)$ . The family  $\mathcal{F}_3$  consists of these graphs when |A| = |B| = t,  $|A_1| = |A_2| = 2$  and  $A_1 \cap A_2 = \emptyset$ . In particular,  $\mathcal{F}_3(4)$  consists of exactly one graph, call it  $F_3(4)$ .

Graph  $F_4$  has vertex set  $A \cup B$ , where  $A = \{a_1, a_2, a_3\}$  and  $B := \{b_1, b_2, \dots, b_6\}$  are disjoint. Its edges are the edges of the complete bipartite graph K(A, B) and three extra edges  $b_1b_2$ ,  $b_3b_4$ , and  $b_5b_6$  (see Fig. 4 below). Define  $F'_4$  as the (only) member of  $\mathcal{F}(A, B, A_1, A_2)$  such that |A| = |B| = t = 4,  $A_1 = A_2$ , and  $|A_i| = 3$ . Let  $\mathcal{F}_4 := \{F_4, F'_4\}$ , which is defined only for t = 4.

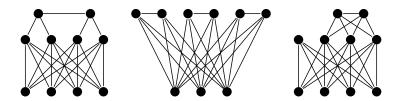


Figure 4: Graphs  $F_3(4)$ ,  $F_4$ , and  $F'_4$ .

Define 
$$\mathcal{F}(k) := \left\{ \begin{array}{ll} \mathcal{F}_0, & \text{if } k \text{ is odd,} \\ \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4, & \text{if } k \text{ is even.} \end{array} \right.$$

#### 3.2 Proof idea

For our proof, it will be easier to use the stronger induction assumption that the graphs in question contain certain dense graphs from  $\mathcal{F}(k)$ . We will prove the following slightly stronger version of Theorem 2.3 which also implies Theorem 2.2.

**Theorem 2.3'** Let  $t \ge 4$ ,  $k \in \{2t+1, 2t+2\}$ , and  $n \ge k$ . Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then  $e(G) \le \max\{h(n, k, t-1), h(n, k, 3)\}$  or

- (a)  $G \subseteq H_{n,k,2}$ , or
- (b) G is contained in a graph in  $\mathcal{G}(n,k) \{H_{n,k,2}\}$ , and G contains a subgraph  $H \in \mathcal{F}(k)$ .

The method of the proof is a variation of that of [6]. Also, when n is close to k, we use Kopylov's disintegration method. We take an n-vertex graph G satisfying the hypothesis of Theorem 2.3', and iteratively contract edges in a certain way so that each intermediate graph still satisfies the hypothesis. We consider the final graph of this process  $G_m$  on m vertices and show that  $G_m$  satisfies Theorem 2.3'. Two results from [6] will be instrumental. The first is:

**Lemma 3.1** (Main lemma on contraction [6]). Let  $k \geq 9$  and suppose F and F' are 2-connected graphs such that F = F'/xy and c(F') < k. If F contains a subgraph  $H \in \mathcal{F}(k)$ , then F' also contains a subgraph  $H' \in \mathcal{F}(k)$ .

This lemma shows that if  $G_m$  contains a subgraph  $H \in \mathcal{F}(k)$ , then the original graph G also contains a subgraph in  $\mathcal{F}(k)$ . The second result (proved in Subsection 4.5 of [6]) is:

**Lemma 3.2** ([6]). Let  $k \geq 9$ , and let G be a 2-connected graph with c(G) < k and e(G) > h(n, k, t-1). If G contains a subgraph  $H \in \mathcal{F}(k)$ , then G is a subgraph of a graph in  $\mathcal{G}(n, k) - \{H_{n,k,2}\}$ .  $\square$ 

We will split the proof into the cases of small n and large n. The following observations can be obtained by simple calculations (for  $t \ge 4$ ):

	k	$h(n,k,3) \ge h(n,k,t-1)$	$h(n,k,2) \ge h(n,k,t-1)$
ſ	2t + 1	If and only if $n \le k + (t-5)/2$	If and only if $n \le k + t/2 - 1$
	2t + 2	If and only if $n \le k + (t-3)/2$	If and only if $n \leq k + t/2$

In the case of large n we will contract an edge such that the new graph still has more than h(n-1,k,t-1) edges. In order to apply induction, we also need the number of edges to be greater than h(n-1,k,3). To guarantee this, we pick the cutoffs for the two cases  $n \le k + (t-1)/2$  and n > k + (t-1)/2 (therefore n-1 > k + (t-3)/2).

#### 4 Tools

#### 4.1 Classical theorems

**Theorem 4.1** (Erdős [2]). Let  $d \ge 1$  and n > 2d be integers, and

$$\ell_{n,d} = \max\left\{ \binom{n-d}{2} + d^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \lfloor \frac{n-1}{2} \rfloor^2 \right\}.$$

Then every n-vertex graph G with  $\delta(G) \geq d$  and  $e(G) > \ell_{n,d}$  is hamiltonian.

**Theorem 4.2** (Chvátal [1]). Let  $n \geq 3$  and G be an n-vertex graph with vertex degrees  $d_1 \leq d_2 \leq \ldots \leq d_n$ . If G is not hamiltonian, then there is some i < n/2 such that  $d_i \leq i$  and  $d_{n-i} < n-i$ .  $\square$ 

**Theorem 4.3** (Kopylov [8]). If G is 2-connected and P is an x, y-path of  $\ell$  vertices, then  $c(G) \ge \min\{\ell, d(x, P) + d(y, P)\}$ .

#### 4.2 Claims on contractions

A helpful tool will be the following lemma from [6] on contraction.

**Lemma 4.4** ([6]). Let  $n \geq 4$  and let G be an n-vertex 2-connected graph. For every  $v \in V(G)$ , there exists  $w \in N(v)$  such that G/vw is 2-connected.

For an edge xy in a graph H, let  $T_H(xy)$  denote the number of triangles containing xy. Let  $T(H) = \min\{T_H(xy) : xy \in E(H)\}$ . When we contract an edge uv in a graph H, the degree of every  $x \in V(H) - u - v$  either does not change or decreases by 1. Also the degree of u \* v in H/uv is at least  $\max\{d_H(u), d_H(v)\} - 1$ . Thus

$$d_{H/uv}(w) \ge d_H(w) - 1$$
 for any  $w \in V(H)$  and  $uv \in E(H)$ . Also  $d_{H/uv}(u * v) \ge d_H(u) - 1$ . (2)

Similarly,

$$T(H/uv) \ge T(H) - 1$$
 for every graph  $H$  and  $uv \in E(H)$ . (3)

We will use the following analog of Lemma 3.3 in [6].

**Lemma 4.5.** Let h be a positive integer. Suppose a 2-connected graph G is obtained from a 2-connected graph G' by contracting edge xy into x \* y chosen using the following rules:

- (i) one of x, y, say x is a vertex of the minimum degree in G';
- (ii)  $T_{G'}(xy)$  is the minimum among the edges xu incident with x such that G'/xu is 2-connected. (Such edges exist by Lemma 4.4). If G has at least h vertices of degree at most h, then either  $G' = K_{h+2}$  or
- (a) G' also has a vertex of degree at most h, and
- (b) G' has at least h+1 vertices of degree at most h+1.

**Proof.** Since G is 2-connected,  $h \ge 2$ . Let  $V_{\le s}(H)$  denote the set of vertices of degree at most s in H. Then by (2), each  $v \in V_{\le h}(G) - x * y$  is also in  $V_{\le h+1}(G')$ . Moreover, then by (i),

$$x \in V_{\leq h+1}(G'). \tag{4}$$

Thus if  $x * y \notin V_{\leq h}(G)$ , then (b) follows. But if  $x * y \in V_{\leq h}(G)$ , then by (2), also  $y \in V_{\leq h+1}(G')$ . So, again (b) holds.

If  $V_{\leq h-1}(G) \neq \emptyset$ , then (a) holds by (2). So, if (a) does not hold, then

each 
$$v \in V_{\leq h}(G) - x * y$$
 has degree  $h + 1$  in  $G'$  and is adjacent to both  $x$  and  $y$  in  $G'$ . (5)

Case 1:  $|V_{\leq h}(G) - x * y| \geq h$ . Then by (4),  $d_{G'}(x) = h+1$ . This in turn yields  $N_{G'}(x) = V_{\leq h}(G) + y$ . Since G' is 2-connected, each  $v \in V_{\leq h}(G) - x * y$  is not a cut vertex. Furthermore,  $\{x, v\}$  is not a cut set. If it was, because y is a common neighbor of all neighbors of x, all neighbors of x must be in the same component as y in G' - x - v. It follows that

for every 
$$v \in V_{\leq h}(G) - x * y$$
,  $G'/vx$  is 2-connected. (6)

If  $uv \notin E(G)$  for some  $u, v \in V_{\leq h}(G)$ , then by (6) and (i), we would contract the edge xu rather

than xy. Thus  $G'[V_{\leq h}(G) \cup \{x,y\}] = K_{h+2}$  and so either  $G' = K_{h+2}$  or y is a cut vertex in G', as claimed.

Case 2:  $|V_{\leq h}(G) - x * y| = h - 1$ . Then  $x * y \in V_{\leq h}(G)$ . This means  $d_{G'}(x) = d_{G'}(y) = h + 1$  and  $N_{G'}[x] = N_{G'}[y]$ . So by (5), there is  $z \in V(G)$  such that  $N_{G'}[x] = N_{G'}[y] = V_{\leq h}(G) \cup \{x, y, z\}$ . Again (6) holds (for the same reason that  $N_{G'}[x] \subseteq N_{G'}[y]$ ). Thus similarly  $vu \in E(G')$  for every  $v \in V_{\leq h}(G) - x * y$  and every  $u \in V_{\leq h}(G) + z$ . Hence  $G'[V_{\leq h}(G) \cup \{x, y, z\}] = K_{h+2}$  and either  $G' = K_{h+2}$  or z is a cut vertex in G', as claimed.

## **4.3** A property of graphs in $\mathcal{F}(k)$

A useful feature of graphs in  $\mathcal{F}(k)$  is the following.

**Lemma 4.6.** Let  $k \geq 9$  and  $n \geq k$ . Let F be an n-vertex graph contained in  $H_{n,k,t}$  with e(F) > h(n,k,t-1). Then F contains a graph in  $\mathcal{F}(k)$ .

*Proof.* Assume the sets A, B, C to be as in the definition of  $H_{n,k,t}$ . We will use induction on n.

Case 1: k = 2t + 1. If n = k, then  $F \in \mathcal{K}^{-t+3}(H_{k,k,t})$  because h(k,k,t) - h(k,k,t-1) - 1 = t - 3. Thus, since  $H_{k,k,t} \supseteq F_0(t)$ , F contains a subgraph in  $\mathcal{F}_0$ . Suppose now the lemma holds for all  $k \le n' < n$ . If  $\delta(F) \ge t$ , then each  $v \in V(F) - A$  is adjacent to every  $u \in A$ . Hence F contains  $K_{t,n-t}$ . If  $\delta(F) < t$ , then since A is dominating and n > 2t, there is  $v \in V(F) - A$  with  $d_F(v) \le t - 1$ . Then  $F - v \subseteq H_{n-1,k,t}$ , and we are done by induction.

Case 2: k = 2t + 2. Let  $C = \{c_1, c_2\}$ . If n = k then as in Case 1,

$$e(H_{k,k,t}) - e(F) \le h(k,k,t) - h(k,k,t-1) - 1 = t - 4,$$

i.e.,  $F \in \mathcal{K}^{-t+4}(H_{k,k,t})$ . Since  $F_1(t) \subseteq H_{k,k,t}$ , F contains a subgraph in  $\mathcal{F}_1$ . Suppose now the lemma holds for all  $k \leq n' < n$ . If  $\delta(F) < t$ , then there is  $v \in V(F) - A$  with  $d_F(v) \leq t - 1$ . Then  $F - v \subseteq H_{n-1,k,t}$ , and we are done by induction.

Finally, suppose  $\delta(F) \geq t$ . So, each  $v \in B$  is adjacent to every  $u \in A$  and each of  $c_1, c_2$  has at least t-1 neighbors in A. Since  $|B \cup \{c_1\}| \geq n-t-1 \geq t+2$ , F contains a member of  $\mathcal{K}^{-1}(F_1(t))$ . Thus F contains a member of  $\mathcal{F}_1$  unless t=4, n=2t+3 and  $c_1$  has a nonneighbor  $x \in A$ . But then  $c_1c_2 \in E(F)$ , and so F contains either  $F_3(4)$  or  $F_4'$ .

## 5 Proof of Theorem 2.3'

#### 5.1 Contraction procedure

If n > k, we iteratively construct a sequence of graphs  $G_n, G_{n-1}, ... G_m$  where  $|V(G_j)| = j$ . In [6], the following **Basic Procedure** (BP) was used:

At the beginning of each round, for some  $j: k \leq j \leq n$ , we have a j-vertex 2-connected graph  $G_j$  with  $e(G_j) > h(j, k, t-1)$ .

- (BP1) If j = k, then we stop.
- (BP2) If there is an edge uv with  $T_{G_j}(uv) \leq t-2$  such that  $G_j/uv$  is 2-connected, choose one such edge so that
  - (i)  $T_{G_i}(uv)$  is minimum, and subject to this
  - (ii) uv is incident to a vertex of minimum possible degree. Then obtain  $G_{i-1}$  by contracting uv.
- (BP3) If (BP2) does not hold,  $j \geq k + t 1$  and there is  $xy \in E(G_j)$  such that  $G_j x y$  has at least 3 components and one of the components, say  $H_1$  is a  $K_{t-1}$ , then let  $G_{j-t+1} = G_j V(H_1)$ .
- (BP4) If neither (BP2) nor (BP3) occurs, then we stop.

**Remark 5.1.** By definition, (BP3) applies only when  $j \ge k+t-1$ . As observed in [6], if  $j \le 3t-2$ , then a j-vertex graph  $G_j$  with a 2-vertex set  $\{x,y\}$  separating the graph into at least 3 components cannot have  $T_{G_j}(uv) \ge t-1$  for every edge uv. It also was calculated there that if  $3t-1 \le j \le 3t$ , then any j-vertex graph G' with such 2-vertex set  $\{x,y\}$  and  $T_{G'}(uv) \ge t-1$  for every edge uv has at most h(j,k,t-1) edges and so cannot be  $G_j$ .

In this paper, we also use a quite similar **Modified Basic Procedure** (MBP): start with a 2-connected, n-vertex graph  $G = G_n$  with e(G) > h(n, k, t - 1) and c(G) < k. Then

- (MBP0) if  $\delta(G_i) \geq t$ , then apply the rules (BP1)–(BP4) of (BP) given above;
- (MBP1) if  $\delta(G_i) \leq t 1$  and j = k, then stop;
- (MBP2) otherwise, pick a vertex v of smallest degree, contract an edge vu with the minimum  $T_{G_j}(vu)$  among the edges vu such that  $G_j/vu$  is 2-connected, and set  $G_{j-1} = G_j/uv$ .

## 5.2 Proof of Theorem 2.3' for the case $n \le k + (t-1)/2$

Apply to G the Modified Basic Procedure (MBP) starting from  $G_n = G$ . By Remark 1, (BP3) was never applied, since k + (t-1)/2 < k+t-1. Therefore at every step, we only contracted an edge. Denote by  $G_m$  the terminating graph of (MBP). Then  $G_j$  is 2-connected and  $c(G_j) \le c(G) < k$  for each  $m \le j \le n$ . By construction, after each contraction, we lose at most t-1 edges. It follows that  $e(G_m) > h(m, k, t-1)$ .

If m > k, then the same argument as in [6] gives us the following structural result:

**Lemma 5.1** ([6]). Let  $m > k \ge 9$  and  $n \ge k$ .

- If  $k \neq 10$ , then  $G_m \subseteq H_{m,k,t}$ .
- If k = 10, then  $G_m \subseteq H_{m,k,t}$  or  $G_m \supseteq F_4$ .

If k = 10 and  $G_m \supseteq F_4$ , then  $G_m$  contains a subgraph in  $\mathcal{F}(k)$ . Otherwise, by Lemma 4.6, again  $G_m$  has a subgraph in  $\mathcal{F}(k)$ . Next, undo the contractions that were used in (MBP). At every step for  $m \le j \le n$ , by Lemma 3.1,  $G_j$  contains some subgraph  $H' \in \mathcal{F}(k)$ . In particular,  $G = G_n$  contains such a subgraph. Thus by Lemma 3.2, we get our result. So, below we assume

$$m = k. (7)$$

Since  $c(G_k) < k$ ,  $G_k$  does not have a hamiltonian cycle. Denote the vertex degrees of  $G_k$   $d_1 \le d_2 \le ... \le d_k$ . By Theorem 4.2, there exists some  $2 \le i \le t$  such that  $d_i \le i$  and  $d_{k-i} < k-i$ . Let  $r = r(G_k)$  be the smallest such i.

Because  $G_k$  has r vertices of degree at most r, similarly to [2],

$$e(G_k) \le r^2 + \binom{k-r}{2}.$$

For k = 2t + 1,  $r^2 + {k-r \choose 2} > h(n, k, t-1)$  only when r = t or r < (t+4)/3, and for k = 2t + 2, when r = t or r < (t+6)/3. If r = t, then repeating the argument in [6] yields:

**Lemma 5.2** ([6]). If  $r(G_k) = t$  then  $G_k \subseteq H_{k,k,t}$ .

Then by Lemma 4.6, Lemma 3.1, and Lemma 3.2,  $G \subseteq H_{n,k,t}$  and contains some subgraph in  $\mathcal{F}(k)$ . So we may assume that

if 
$$k = 2t + 1$$
 then  $r < (t + 4)/3$ , and if  $k = 2t + 2$  then  $r < (t + 6)/3$ . (8)

Our next goal is to show that G contains a large "core", i.e., a subgraph with large minimum degree. For this, we recall the notion of disintegration used by Kopylov [8].

**Definition**: For a natural number  $\alpha$  and a graph G, the  $\alpha$ -disintegration of a graph G is the process of iteratively removing from G the vertices with degree at most  $\alpha$  until the resulting graph has minimum degree at least  $\alpha + 1$ . This resulting subgraph  $H = H(G, \alpha)$  will be called the  $\alpha$ -core of G. It is well known that  $H(G, \alpha)$  is unique and does not depend on the order of vertex deletion.

Claim 5.3. The t-core H(G,t) of G is not empty.

**Proof of Claim 5.3**: We may assume that for all  $m \leq j < n$ , graph  $G_j$  was obtained from  $G_{j+1}$  by contracting edge  $x_j y_j$ , where  $d_{G_{j+1}}(x_j) \leq d_{G_{j+1}}(y_j)$ . By Rule (MBP2),  $d_{G_{j+1}}(x_j) = \delta(G_{j+1})$ , provided that  $\delta(G_{j+1}) \leq t - 1$ .

By definition,  $|V_{\leq r}(G_k)| \geq r$ . So by Lemma 4.5 (applied several times), for each  $k+1 \leq j \leq k+t-r$ , because each  $G_j$  is not a complete graph (otherwise it would have a hamiltonian cycle),

$$\delta(G_j) \le j - k + r - 1 \text{ and } |V_{\le j - k + r}(G_j)| \ge j - k + r.$$
 (9)

To show that

$$\delta(G_j) \le t - 1 \text{ for all } k \le j \le n,$$
 (10)

by (9) and (8), it is enough to observe that

$$\delta(G_j) \le j - k + r - 1 \le (n - k) + r - 1 \le \frac{t - 1}{2} + \frac{t + 6}{3} - 1 = \frac{5t + 3}{6} < t.$$

We will apply a version of t-disintegration in which we first manually remove a sequence of vertices and count the number of edges they cover. By (10) and (MBP2),  $d_{G_n}(x_{n-1}) = \delta(G_n) \leq n - k + r - 1$ . Let  $v_n := x_{n-1}$ . Then  $G - v_n$  is a subgraph of  $G_{n-1}$ . If  $x_{n-2} \neq x_{n-1} * y_{n-1}$  in  $G_{n-1}$ , then let  $v_{n-1} := x_{n-2}$ , otherwise let  $v_{n-1} := y_{n-1}$ . In both cases,  $d_{G-v_n}(v_{n-1}) \leq n - k + r - 2$ . We continue

in this way until j = k: each time we delete from  $G - v_n - \ldots - v_{j+1}$  the unique survived vertex  $v_j$  that was in the preimage of  $x_{j-1}$  when we obtained  $G_{j-1}$  from  $G_j$ . Graph  $G - v_n - \ldots - v_{k+1}$  has  $r \geq 2$  vertices of degree at most r. We additionally delete 2 such vertices  $v_k$  and  $v_{k-1}$ . Altogether, we have lost at most  $(r + n - k - 1) + (r + n - k - 2) + \ldots + r + 2r$  edges in the deletions.

Finally, apply t-disintegration to the remaining graph on  $k-2 \in \{2t-1, 2t\}$  vertices. Suppose that the resulting graph is empty.

Case 1: n = k. Then

$$e(G) \le r + r + t(2t - 1 - t) + {t \choose 2},$$

where r+r edges are from  $v_k$  and  $v_{k-1}$ , and after deleting  $v_k$  and  $v_{k-1}$ , every vertex deleted removes at most t edges, until we reach the final t vertices which altogether span at most  $\binom{t}{2}$  edges.

For k = 2t + 1,

$$h(k,k,t-1) - e(G) \ge \binom{2t+1-(t-1)}{2} + (t-1)^2 - \left[r+r+t(2t-1-t) + \binom{t}{2}\right] = t+2-2r,$$

which is nonnegative for r < (t+3)/3. Therefore  $e(G) \le h(k, k, t-1)$ , a contradiction.

Similarly, if k = 2t + 2,

$$e(G) \le r + r + t(2t - t) + {t \choose 2}$$
, and

$$h(k,k,t-1) - e(G) \ge \binom{2t+2-(t-1)}{2} + (t-1)^2 - [r+r+t(2t-t) + \binom{t}{2}] = t+4-2r,$$

which is nonnegative when r < (t+6)/3.

Case 2:  $k < n \le k + (t-1)/2$ . Then for k = 2t + 1,

$$e(G) \le \left[ (r+n-k-1) + (r+n-k-2) + \ldots + r \right] + 2r + t(2t-1-t) + {t \choose 2}$$

$$\le \left[ (t-1) + (t-1) + \ldots + (t-1) \right] + h(k,k,t-1)$$

$$= (t-1)(n-k) + h(k,k,t-1)$$

$$= h(n,k,t-1),$$

where the last inequality holds because  $r + n - k - 1 \le t - 1$ .

Similarly, for k = 2t + 2,

$$e(G) \le \left[ (r+n-k-1) + (r+n-k-2) + \dots + r \right] + 2r + t(2t-t) + {t \choose 2}$$

$$\le (n-k)(t-1) + h(k,k,t-1)$$

$$= h(n,k,t-1).$$

This contradiction completes the proof of Claim 5.3.

For the rest of the proof of Theorem 2.3', we will follow the method of Kopylov in [8] to show that  $G \subseteq H_{n,k,2}$ . Let  $G^*$  be the k-closure of G. That is, add edges to G until adding any additional

edges creates a cycle of length at least k. In particular, for any non-edge xy of  $G^*$ , there is an (x, y)-path in  $G^*$  with at least k - 1 edges.

Because G has a nonempty t-core, and  $G^*$  contains G as a subgraph,  $G^*$  also has a nonempty t-core (which contains the t-core of G). Let  $H = H(G^*, t)$  denote the t-core of  $G^*$ . We will show that

$$H$$
 is a complete graph.  $(11)$ 

Indeed, suppose there exists a nonedge in H. Choose a longest path P of  $G^*$  whose terminal vertices  $x \in V(H)$  and  $y \in V(H)$  are nonadjacent. By the maximality of P, every neighbor of x in H is in P, similar for y. Hence  $d_P(x) + d_P(y) = d_H(x) + d_H(y) \ge 2(t+1) \ge k$ , and also |P| = k-1 (edges). By Kopylov' Theorem 4.3,  $G^*$  must have a cycle of length at least k, a contradiction.

Therefore H is a complete subgraph of  $G^*$ . Let  $\ell = |V(H)|$ . Because every vertex in H has degree at least t+1,  $\ell \geq t+2$ . Furthermore, if  $\ell \geq k-1$ , then  $G^*$  has a clique K of size at least k-1. Because  $G^*$  is 2-connected, we can extend a (k-1)-cycle of K to include at least one vertex in  $G^* - H'$ , giving us a cycle of length at least k. It follows that

$$t + 2 \le \ell \le k - 2,\tag{12}$$

and therefore  $k-\ell \le t$ . Apply a weaker  $(k-\ell)$ -disintegration to  $G^*$ , and denote by H' the resulting graph. By construction,  $H \subseteq H'$ .

Case 1: There exists  $v \in V(H') - V(H)$ . Since  $v \notin V(H)$ , there exists a nonedge between a vertex in H and a vertex in H' - H. Pick a longest path P with terminal vertices  $x \in V(H')$  and  $y \in V(H)$ . Then  $d_P(x) + d_P(y) \ge (k - \ell + 1) + (\ell - 1) = k$ , and therefore  $c(G^*) \ge k$ .

Case 2: H = H'. Then

$$e(G^*) \le \binom{\ell}{2} + (n-\ell)(k-\ell) = h(n,k,k-\ell).$$

If  $3 \leq (k - \ell) \leq t - 1$ , then  $e(G) \leq \max\{h(n, k, 3), h(n, k, t - 1)\}$ , so by (12),  $k - \ell = 2$ , and H is the complete graph with k - 2 vertices. Let  $D = V(G^*) - V(H)$ . If there is an edge xy in  $G^*[D]$ , then because  $G^*$  is 2-connected, there exist two vertex-disjoint paths,  $P_1$  and  $P_2$ , from  $\{x, y\}$  to H such that  $P_1$  and  $P_2$  only intersect  $\{x, y\} \cup H$  at the beginning and end of the paths. Let a and b be the terminal vertices of  $P_1$  and  $P_2$  respectively that lie in H. Let P be any (a, b)-hamiltonian path of H. Then  $P_1 \cup P \cup P_2 + xy$  is a cycle of length at least k in  $G^*$ , a contradiction.

Therefore D is an independent set, and since  $G^*$  is 2-connected, each vertex of D has degree 2. Suppose there exists  $u, v \in D$  where  $N(u) \neq N(v)$ . Let  $N(u) = \{a, b\}, N(v) = \{c, d\}$  where it is possible that b = c. Then we can find a cycle C of H that covers V(H) which contains edges ab and cd. Then C - ab - cd + ua + ub + vc + vd is a cycle of length k in  $G^*$ . Thus for every  $v \in D$ ,  $N(v) = \{a, b\}$  for some  $a, b \in H$ . I.e.,  $G^* = H_{n,k,2}$ , and thus  $G \subseteq H_{n,k,2}$ .

#### 5.3 Proof of Theorem 2.3' for all n

We use induction on n with the base case  $n \le k + (t-1)/2$ . Suppose  $n \ge k + t/2$  and for all  $k \le n' < n$ , Theorem 2.3' holds. Let G be a 2-connected graph G with n vertices such that

$$e(G) > \max\{h(n, k, t - 1), h(n, k, 3)\} \text{ and } c(G) < k.$$
 (13)

Apply one step of (BP). If (BP4) was applied (so neither (BP2) nor (BP3) applies to G), then  $G_m = G$  (with  $G_m$  defined as in the previous case). By Lemmas 5.1, 4.6, and 3.2, the theorem holds.

Therefore we may assume that either (BP2) or (BP3) was applied. Let  $G^-$  be the resulting graph. Then  $c(G^-) < k$ , and  $G^-$  is 2-connected.

#### Claim 5.4.

$$e(G^{-}) > \max\{h(|V(G^{-})|, k, t-1), h(|V(G^{-})|, k, 3)\}.$$
(14)

*Proof.* If (BP2) was applied, i.e.,  $G^- = G/uv$  for some edge uv, then

$$e(G^{-}) \ge e(G) - (t-1) > h(n-1, k, t-1) \ge h(n-1, k, 3),$$

so (14) holds. Therefore we may assume that (BP3) was applied to obtain  $G^-$ . Then  $n \ge k + t - 1$  and  $e(G) - e(G^-) = {t+1 \choose 2} - 1$ . So by (13),

$$e(G^{-}) > h(n, k, t - 1) - {t+1 \choose 2} + 1.$$
 (15)

The right hand side of (15) equals  $h(n-(t-1),k,t-1)+t^2/2-5t/2+2$  which is at least h(n-(t-1),k,t-1) for  $t \ge 4$ , proving the first part of (14).

We now show that also  $e(G^-) > h(n - (t - 1), k, 3)$ . Indeed, for k = 2t + 1,

$$e(G^{-}) - h(n - (t - 1), k, 3) > {t + 2 \choose 2} + (t - 1)(n - t - 2) - {t + 1 \choose 2} + 1$$
$$- \left[ {2t - 2 \choose 2} + 3(n - (t - 1) - (2t - 2)) \right] \ge 0 \text{ when } n \ge 3t.$$

Similarly, for k = 2t + 2,

$$e(G^{-}) - h(n - (t - 1), k, 3) > {t + 3 \choose 2} + (t - 1)(n - t - 3) - {t + 1 \choose 2} + 1$$
$$- \left[ {2t - 1 \choose 2} + 3(n - (t - 1) - (2t - 1)) \right] > 0 \text{ when } n \ge 3t + 1.$$

Thus if  $n \ge 3t + 1$ , then (14) is proved. But if  $n \in \{3t - 1, 3t\}$  then by Remark 5.1, no graph to which (BP3) applied may have more than h(n, k, t - 1) edges.

By (14), we may apply induction to  $G^-$ . So  $G^-$  satisfies either (a)  $G^- \subseteq H_{|V(G^-)|,n,2}$ , or (b)  $G^-$  is contained in a graph in  $\mathcal{G}(n,k) - H_{|V(G^-)|,k,2}$  and contains a subgraph  $H \in \mathcal{F}(k)$ . Suppose first

that  $G^-$  satisfies (b). If (BP3) was applied to obtain  $G^-$  from G, then because  $G^-$  contains a subgraph  $H \in \mathcal{F}(k)$  and  $G^- \subseteq G$ , G also contains H. If (BP2) was applied, then by Lemma 3.1, G contains a subgraph  $H' \in \mathcal{F}(k)$ . In either case, Lemma 3.2 implies that G is a subgraph of a graph in  $\mathcal{G}(n,k) - H_{n,k,2}$ .

So we may assume that (a) holds, that is,  $G^-$  is a subgraph of  $H_{|V(G^-)|,n,2}$ . Because  $\delta(G^-) \leq 2$ ,  $\delta(G) \leq 3$ , and so G has edges in at most  $2 \leq t-2$  triangles. Therefore (BP2) was applied to obtain  $G^-$ , where  $G/uv = G^-$ . Let D be an independent set of vertices of  $G^-$  of size (n-1) - (k-2) with  $N(D) = \{a, b\}$  for some  $a, b \in V(G^-)$ . Since  $T_{G^-}(xa), T_{G^-}(xb) \leq 1$  for every  $x \in D$ , we have that  $T_G(uv) \leq 2$  with equality only if T(G) = 2 where  $T(G) = \min_{xu \in E(G)} T_G(xy)$ .

We want to show that  $T_G(uv) \leq 1$ . If not, suppose first that  $u * v \in D \subseteq V(G^-)$ . Then there exists  $x \in D - u * v$ , and x and u \* v are not adjacent in  $G^-$ . Therefore x was not in a triangle with u and v in G, and hence  $T_G(xa) = T_{G^-}(xa) \leq 1$ , so the edge xa should have been contracted instead. Otherwise if  $u * v \notin D$ , at least one of  $\{a,b\}$ , say a, is not u \* v. If T(G) = 2, then for every  $x \in D \subseteq V(G)$ ,  $T_G(xa) = 2$ , therefore each such edge xa was in a triangle with uv in G. Then  $T_G(uv) \geq |D| = (n-1) - (k-2) \geq k + t/2 - 1 - k + 2 \geq 3$ , a contradiction.

Thus  $T_G(uv) \le 1$  and  $e(G) \le 2 + e(G^-) \le 2 + h(n-1,k,2) = h(n,k,2)$ . But for  $n \ge k + t/2$ , we have  $h(n,k,t-1) \ge h(n,k,2)$ , a contradiction.

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