

Maximum matchings in regular graphs

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Abstract

It was conjectured by Mkrtchyan, Petrosyan, and Vardanyan that every graph G with $\Delta(G) - \delta(G) \leq 1$ has a maximum matching M such that any two M -unsaturated vertices do not share a neighbor. In this note, we confirm the conjecture for all k -regular simple graphs and also k -regular multigraphs with $k \leq 4$.

1 Introduction

Graphs considered in this paper may have multi-edges, but no loops. A graph without multi-edges is called a *simple* graph. A *matching* M of a graph G is a set of independent edges. A vertex is *M -saturated* if it is incident with an edge of M , and *M -unsaturated* otherwise. A matching M is said to be *maximum* if for any other matching M' , $|M| \geq |M'|$. A matching M is *perfect* if it covers all vertices of G . If G has a perfect matching, then every maximum matching is a perfect matching. The maximum and minimum degrees of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Mkrtchyan, Petrosyan and Vardanyan [4, 5] made the following conjecture.

Conjecture 1.1 (Mkrtchyan et. al. [4, 5]). *Let G be a graph with $\Delta(G) - \delta(G) \leq 1$. Then G contains a maximum matching M such that any two M -unsaturated vertices do not share a neighbor.*

This conjecture is verified for subcubic graphs (i.e. $\Delta(G) = 3$) by Mkrtchyan, Petrosyan and Vardanyan [4]. Later, Picouleau [7] find a counterexample to the conjecture, which is a bipartite simple graph with $\delta(G) = 4$ and $\Delta(G) = 5$. Petrosyan [6] constructs counterexamples to the conjecture for all k -regular graphs with $k \geq 7$ and for graphs G with $\Delta(G) - \delta(G) = 1$ and $\Delta(G) \geq 4$. Note that, most of counterexamples of Conjecture 1.1 for graphs G with $\Delta(G) - \delta(G) = 1$ are simple, but all k -regular graphs with $k \geq 7$ given by Petrosyan [6] have multi-edges. As affirmative answer to Conjecture 1.1 is known only for graphs with $\Delta(G) \leq 3$, Mkrtchyan et. al [4] asked whether the conjecture holds for any k -regular graphs with $k \geq 4$.

In this note, we consider the conjecture for both k -regular simple graphs and k -regular graphs with multi-edges. First we show that Conjecture 1.1 does hold for all k -regular simple graphs.

Theorem 1.2. *Let G be a k -regular simple graph. Then G has a maximum matching M such that any two M -unsaturated vertices do not share a neighbor.*

Further, we show that Conjecture 1.1 holds for k -regular graphs with multi-edges for $k \leq 4$.

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Theorem 1.3. *Let G be a k -regular graph with $k \leq 4$. Then G has a maximum matching M such that any two M -unsaturated vertices do not share a neighbor.*

Our results together with examples given by Petrosyan [6] leave Conjecture 1.1 unknown for 5 and 6-regular graphs with multi-edges.

2 Preliminaries

Let G be a graph and v be a vertex of G . The *neighborhood* of v is set of all vertices adjacent to v , denoted by $N(v)$. The degree of v is $d_G(v) = |N(v)|$. If there is no confusion, we use $d(v)$ instead. For $X \subseteq V(G)$, let $\delta(X) := \min\{d(v) | v \in X\}$ and $\Delta(X) := \max\{d(v) | v \in X\}$. The neighborhood of X is defined as $N(X) := \{y | y \text{ is a neighbor of a vertex } x \in X\}$. For two subsets X_1 and X_2 of $V(G)$, use $[X_1, X_2]$ to denote the all edges with one endvertex in X_1 and another endvertex in X_2 . For two subgraphs G_1 and G_2 of G , the symmetric difference of $G_1 \oplus G_2$ is defined as a subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $(E(G_1) \cup E(G_2)) \setminus (E(G_1) \cap E(G_2))$.

A matching of a graph G is a *near-perfect matching* if it covers all vertices except one. If a graph G has a near perfect matching, then G has odd number of vertices. A graph is *factor-critical* if, for any vertex v , the subgraph $G \setminus \{v\}$ has a perfect matching. Every maximum matching of a factor-critical graph is a near-perfect matching.

Let D be the set of all vertices of a graph G which are not covered by at least one maximum matching, and A , the set of all vertices in $V(G) - D$ adjacent to at least one vertex in D . Denote $C = V(G) - A - D$. The graph induced by all vertices in D (resp. A and C) is denoted by $G[D]$ (resp. $G[A]$ and $G[C]$). The following theorem characterizes the structures of maximum matchings of graphs, which is due to Gallai [2] and Edmonds [1].

Theorem 2.1 (Gallai-Edmonds Structure Theorem, Theorem 3.2.1 in [3]). *Let G be a graph, and A , D and C are defined as above. Then:*

- (1) *the components of the subgraph induced by D are factor-critical;*
- (2) *the subgraph induced by C has a perfect matching;*
- (3) *if M is a maximum matching of G , it contains a near-perfect matching of each component of $G[D]$, a perfect matching of $G[C]$ and matches all vertices of A with vertices in distinct components of $G[D]$.*

Contract every component of $G[D]$ to a vertex and let B be the set of all these vertices. Then the graph obtained from $G \setminus C$ by contracting all components of $G[D]$ to a vertex and deleting all generated loops is a bipartite graph, denoted by $G(A, B)$. Because every component of $G[D]$ is factor-critical, a maximum matching of $G(A, B)$ is corresponding to a maximum matching of G , and vice versa. Before processing to prove our main results, we need some results for maximum matchings of bipartite graphs $G(A, B)$.

Theorem 2.2 (Hall's Theorem, Theorem 1.13 in [3]). *Let $G(A, B)$ be a bipartite graph. If $|N(S)| \geq |S|$ for any $S \subseteq A$, then G has a matching M covering all vertices of A .*

The following technical lemma is needed in proof of our main results.

Lemma 2.3. *Let $G(A, B)$ be a bipartite graph such that every maximum matching of $G(A, B)$ covers all vertices of A . Let $W \subseteq B$ such that $\delta(W) \geq \Delta(A)$. Then $G(A, B)$ has a maximum matching M covering all vertices of W .*

Proof. Let M be a maximum matching of $G(A, B)$ such that the number of vertices of W covered by M is maximum. If M covers all vertices of W , the lemma follows. So assume that there exists an M -unsaturated vertex $x \in W$.

For any $U \subseteq W$, we have $\delta(U) \geq \delta(W)$ and $N(U) \subset A$. Further,

$$\delta(W)|U| \leq \delta(U)|U| \leq |[U, N(U)]| \leq \sum_{v \in N(U)} d(v) \leq \Delta(A)|N(U)|.$$

It follows that $|N(U)| \geq |U|$ because $\delta(W) \geq \Delta(A)$. By applying Hall's Theorem on the subgraph induced by W and $N(W)$, it follows that G has a matching M' covering all vertices of W .

Let $M \oplus M'$ be the symmetric difference of M and M' . Every component of $M \oplus M'$ is either a path or a cycle. Since x is not covered by M but is covered by M' , it follows that x is an end-vertex of some path-component P of $M \oplus M'$. Let y be another end-vertex of P . Note that every vertex of A is covered by an edge of M and every vertex of W is covered by an edge of M' . So $y \in B \setminus W$.

Then let $M'' = M \oplus P$. Then M'' is a maximum matching of G which covers x and all vertices covered by M except y . Note that $y \in B \setminus W$ and $x \in W$. Hence M'' covers more vertices of W than M , a contradiction to the maximality of the number of vertices of W covered by M . This completes the proof. \square

3 Proof of main results

Let G be a k -regular graph. Without loss of generality, assume that G is connected. Otherwise, we consider each connected component of G . Let M be a maximum matching of G . If $|M| \geq (|V(G)| - 1)/2$, then G has at most one M -unsaturated vertex. Theorem 1.2 and Theorem 1.3 hold automatically. So in the following, assume $|M| < (|V(G)| - 1)/2$. So $k \geq 3$.

By Gallai-Edmonds Structure Theorem, $V(G)$ can be partitioned into three parts C , A and D such that every maximum matching of G matches all vertices of A with vertices in distinct components of $G[D]$. Let $c(D)$ be the number of components of $G[D]$. Then $|M| = |C|/2 + (|D| - c(D))/2 + |A|$ by Gallai-Edmonds Structure Theorem. So

$$|C|/2 + (|D| - c(D))/2 + |A| = |M| < (|V(G)| - 1)/2 = (|C| + |A| + |D| - 1)/2.$$

It follows that $c(D) \geq 2 + |A|$.

Let $Q_1, Q_2, \dots, Q_{c(D)}$ be all components of $G[D]$. Let $[Q_i, A]$ (resp. $[D, A]$) be the set of all edges joining a vertex of Q_i (resp. D) and a vertex of A . Note that

$$\sum_{i=1}^{c(D)} |[Q_i, A]| = |[D, A]| \leq k|A|$$

because G is k -regular. Let G/Q_i be the graph obtained by contracting Q_i and deleting all loops. Note that Q_i is factor-critical and hence has odd number of vertices, and G/Q_i has even number of vertices of odd degree. So the degree of the new vertex of G/Q_i corresponding to Q_i has the same parity as k . It follows that

$$|[Q_i, A]| \equiv k \pmod{2}.$$

In the following, we always assume that $|[Q_i, A]| \geq |[Q_j, A]|$ for $i \leq j$. Then there exists an integer $t < |A|$ such that $|[Q_i, A]| < k$ for any $i \geq t$. A vertex v of Q_i is **good** if all neighbors of v are contained in Q_i .

Proof of Theorem 1.2. Since G is a simple graph, for each Q_i , we have

$$\frac{k|V(Q_i)| - |[Q_i, A]|}{2} \leq \binom{|V(Q_i)|}{2}.$$

Note that $|[Q_i, A]| < k$ if $i \geq t$. It follows that $|V(Q_i)| > k$ for $i \geq t$. So at least one vertex of Q_i with $i \geq t$ has no neighbors in A . Hence every component of Q_i with $i \geq t$ contains a good vertex. Choose a good vertex v_i from each Q_i with $i \geq t$ and let X be the set of all chosen good vertices v_i . Then any two vertices of X do not share a neighbor because $Q_i \cap Q_j = \emptyset$ if $i \neq j$.

Contract all components Q_i into a vertex q_i , and let $B = \{q_i | i = 1, 2, \dots, c(D)\}$. Let $G(A, B)$ be the bipartite graph with bipartition A and B , and all edges in $[D, A]$ of G . Let $W := \{q_i | q_i \in B \text{ and } d_H(q_i) \geq k\}$, the set of vertices corresponding to such Q_i with $|[Q_i, A]| \geq k$ (i.e., $i < t$). By Gallai-Edmonds Structure Theorem, every maximum matching of $G(A, B)$ covers all vertices of A . By Lemma 2.3, $G(A, B)$ has a maximum matching M covering all vertices of W and all vertices of A . In the graph G , M is a matching which covers all vertices of A , and a vertex from every Q_i with $i < t$, and a vertex from some Q_j with $j \geq t$. For each Q_i , let M_i be a near-perfect matching covering all vertices except the vertex covered by M or the good vertex v_i if the component Q_i has no vertex covered by M .

By Gallai-Edmonds Structure Theorem, $G[C]$ has a perfect matching M_C . Let M' be the union of M , M_C and all M_i 's. Then M' is a maximum matching of G . So all M' -unsaturated vertices belong to X . So any two M' -unsaturated vertices do not share a neighbor. This completes the proof. \square

Now we are going to prove Theorem 1.3.

Proof of Theorem 1.3. Let G be a k -regular graph with multi-edges and $k \leq 4$. Note that, $|[Q_i, A]| \geq k$ if $i < t$ and $|[Q_i, A]| < k$ if $i \geq t$. Since $|[Q_i, A]| \equiv k \pmod{2}$, it follows that $|[Q_i, A]| = k - 2$ for $i \geq t$. Hence Q_i for $i \geq t$ is not a singleton. Further, Q_i with $i \geq t$ has at least three vertices because Q_i is factor-critical. So every component Q_i for $i \geq t$ has a good vertex v_i .

A similar argument as above shows that G has a maximum matching M' which covers all vertices of G except some good vertices from different components Q_i 's of D . Since any two good vertices from different Q_i and Q_j do not share a neighbor, the theorem follows. \square

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