AN INFINITE FAMILY OF CUBIC NONNORMAL CAYLEY GRAPHS ON NONABELIAN SIMPLE GROUPS

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ABSTRACT. We construct a connected cubic nonnormal Cayley graph on A_{2^m-1} for each integer $m \ge 4$ and determine its full automorphism group. This is the first infinite family of connected cubic nonnormal Cayley graphs on nonabelian simple groups.

Key words: nonnormal Cayley graphs; cubic graphs; simple groups

1. INTRODUCTION

In this paper all graphs considered are finite, simple and undirected. Given a group G and an inverse-closed subset S of $G \setminus \{1\}$, the Cayley graph Cay(G, S) on G with respect to S is the graph with vertex set G such that two vertices x and y are adjacent if and only if $yx^{-1} \in S$. Let \widehat{G} be the right regular representation of G. It is easy to see that \widehat{G} is a subgroup of Aut(Cay(G, S)). Moreover, it was shown by Godsil [5] that the normalizer of \widehat{G} in Aut(Cay(G, S)) is $\widehat{G} \rtimes Aut(G, S)$, where Aut(G, S) is the group of automorphisms of G fixing S setwise. In particular, Aut $(Cay(G, S)) = \widehat{G} \rtimes Aut(G, S)$ if and only if \widehat{G} is normal in Aut(Cay(G, S)). Viewing this, Xu in [14] introduced the concept of normal Cayley graphs: a Cayley graph Cay(G, S) is said to be normal if \widehat{G} is normal in Aut(Cay(G, S)). The study of normality of a Cayley graph plays an important role in the study of its automorphism group because once a Cayley graph Cay(G, S) is known to be normal, to determine its full automorphism group one only needs to determine the group Aut(G, S), which is usually much easier. For a survey paper on normality of Cayley graphs we refer the reader to [4].

The normality of cubic Cayley graphs on nonabelian simple groups has received considerable attention. It was proved in [12] that a connected cubic Cayley graph $\operatorname{Cay}(G, S)$ with G nonabelian simple is normal if $\widehat{G} \rtimes \operatorname{Aut}(G, S)$ is transitive on the edge set of $\operatorname{Cay}(G, S)$. A graph is said to be *arc-transitive* if its automorphism group acts transitively on the set of arcs. In [15, 16] it was proved that the only connected arc-transitive cubic nonnormal Cayley graphs on nonabelian simple groups are two Cayley graphs on A_{47} up to isomorphism, and their full automorphism groups are both isomorphic to A_{48} . On the other hand, examples of connected cubic nonnormal Cayley graphs on nonabelian simple groups are very rare. Since the connected arctransitive cubic nonnormal Cayley graphs on nonabelian simple groups are only the above mentioned two graphs on A_{47} , we can concentrate on the non-arc-transitive case. In this context, one has the following theorem combining [2, Theorem 1.1] and [17, Theorem 1.2]. **Theorem 1.1.** ([2, 17]) Let Cay(G, S) be a connected cubic nonnormal Cayley graph on a nonabelian simple group G. If Cay(G, S) is not arc-transitive, then one of the following holds:

- (a) $G = A_{2^m-1}$ with $m \ge 3$;
- (b) G is a simple group of Lie type of even characteristic except $PSL_2(2^e)$, $PSL_3(2^e)$, $PSU_3(2^e)$, $PSp_4(2^e)$, $E_8(2^e)$, $F_4(2^e)$, ${}^2F_4(2^e)'$, $G_2(2^e)$ and $Sz(2^e)$.

Until recently, connected cubic nonnormal Cayley graphs on the groups listed in Theorem 1.1 were only found for A_{15} and A_{31} [9]. In 2008, Feng, Lu and Xu asked the following question in their survey paper [4] on normality of Cayley graphs.

Question 1.2. ([4, Problem 5.9]) Are there infinitely many connected nonnormal Cayley graphs of valency 3 or 4 on nonabelian simple groups?

Question 1.2 in the valency 4 case has been answered by Wang and Feng [13] in the affirmative. In this paper, we answer the question in the remaining case. Our main result is Theorem 1.3, which gives a positive answer to Question 1.2.

Theorem 1.3. For each integer $m \ge 4$, there exists a graph Γ_m satisfying:

- (a) Γ_m is a connected cubic nonnormal Cayley graph on A_{2^m-1} ;
- (b) $\Gamma_m \cong \operatorname{Cay}(A_{2^m-1}, S)$ for some set S of three involutions in A_{2^m-1} such that $\operatorname{Aut}(A_{2^m-1}, S) = 1$;
- (c) $\operatorname{Aut}(\Gamma_m) \cong \mathcal{A}_{2^m}$.

We call a Cayley graph $\operatorname{Cay}(G, S)$ a graphical regular representation (GRR for short) of G if $\operatorname{Aut}(\operatorname{Cay}(G, S)) = \widehat{G}$. Note that a GRR is necessarily a normal Cayley graph, and a necessary condition for $\operatorname{Cay}(G, S)$ to be a GRR is that $\operatorname{Aut}(G, S) =$ 1. In many circumstances it is shown that this condition is also sufficient, see for example [2, 5, 6]. More generally, a problem is posed in [2] to determine the groups G such that a Cayley graph $\operatorname{Cay}(G, S)$ on G is a GRR of G if and only if $\operatorname{Aut}(G, S) = 1$. We remark that our graph Γ_m in Theorem 1.3 as a Cayley graph on $G := \operatorname{A}_{2^m-1}$ is not only nonnormal (and hence not a GRR) but also satisfies the condition $\operatorname{Aut}(G, S) = 1$. It is also worth remarking that, although the graph Γ_m is not arc-transitive, it has local action C_2 so that it corresponds to a tetravalent arc-transitive graph in the standard way described in [11, Section 4.1].

The paper is organized as follows. We shall first give the construction of Γ_m for Theorem 1.3 in Section 2. Then the entirety of section 3 will be devoted to proving the connectivity of Γ_m . Finally in Section 4 we prove the remaining properties of Γ_m described in Theorem 1.3, thus completing the proof of the theorem.

2. Construction of Γ_m

We first introduce some notation that is fixed throughout this paper. Let $m \ge 4$ be an integer,

$$H = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle \times \langle c_1 \rangle \times \langle c_2 \rangle \times \cdots \times \langle c_{m-3} \rangle,$$

where $c_1, c_2, \ldots, c_{m-3}$ are involutions,

$$K = \langle a^2, b, c_1, c_2, \dots, c_{m-3} \rangle = \langle a^2 \rangle \times \langle b \rangle \times \langle c_1 \rangle \times \langle c_2 \rangle \times \dots \times \langle c_{m-3} \rangle$$

and $h = a \prod_{i=0}^{\lfloor (m-5)/2 \rfloor} c_{2i+1}$. Clearly, H is the direct product of a dihedral group D_8 of order 8 and an elementary abelian 2-group of rank m-3, so that $|H| = 2^m$. For the sake of convenience, put $c_{-1} = c_0 = 1$. Define $x \in Aut(H)$ by letting

$$a^{x} = a^{-1}$$
, $b^{x} = ab$, $c^{x}_{2i+1} = c_{2i+1}$ and $c^{x}_{2i+2} = a^{2}c_{2i+1}c_{2i+2}$

for $0 \leq i \leq \lfloor (m-5)/2 \rfloor$ and letting $c_{m-3}^x = a^2 c_{m-3}$ in addition if m is even. Define $\tau \in \operatorname{Aut}(K)$ by letting

$$(a^2)^{\tau} = b, \quad b^{\tau} = a^2, \quad c^{\tau}_{2i+1} = c_{2i-1}c_{2i}c_{2i+2} \quad \text{and} \quad c^{\tau}_{2i+2} = c_{2i-1}c_{2i}c_{2i+1}$$

for $0 \leq i \leq \lfloor (m-5)/2 \rfloor$ and letting $c_{m-3}^{\tau} = c_{m-3}$ in addition if m is even. Note that x and τ are indeed automorphisms of H and K because the images of generators under x and τ satisfy the defining relations for H and K, respectively. Denote the right regular representation of H by R. Let y be the permutation of H such that $k^y = k^{\tau}$ and

$$(hk)^{y} = \begin{cases} hk^{\tau} & \text{if } m \text{ is odd,} \\ hk^{\tau}c_{m-3} & \text{if } m \text{ is even} \end{cases}$$

for $k \in K$. Let

$$z = \begin{cases} R(h)yR(h^{-1}) & \text{if } m \text{ is odd,} \\ R(h)yR(h^{-1}c_{m-3}) & \text{if } m \text{ is even.} \end{cases}$$

We will see that the three permutations x, y and z of H are all involutions in Alt(H).

Lemma 2.1. x, y and z are all involutions.

Proof. It is evident that none of x, y and z is trivial. Since x^2 fixes each of the generators $a, b, c_1, \ldots, c_{m-3}$ of H, we have $x^2 = 1$. Similarly, $\tau^2 = 1$. Let g be an arbitrary element of K. Then $g^{y^2} = (g^{\tau})^y = (g^{\tau})^{\tau} = g^{\tau^2} = g$. If m is odd, then $(hg)^{y^2} = (hg^{\tau})^y = h(g^{\tau})^{\tau} = hg^{\tau^2} = hg$ and so $y^2 = 1$, which in turn implies that $z^2 = R(h)y^2R(h^{-1}) = 1$. Now assume that m is even. Then

$$(hg)^{y^2} = (hg^{\tau}c_{m-3})^y = h(g^{\tau}c_{m-3})^{\tau}c_{m-3} = hg^{\tau^2}c_{m-3}^{\tau}c_{m-3} = hg^{\tau^2} = hg,$$

whence $y^2 = 1$. Moreover,

$$g^{z^{2}} = g^{R(h)yR(c_{m-3})yR(h^{-1}c_{m-3})}$$

= $(hh^{-1}gh)^{yR(c_{m-3})yR(h^{-1}c_{m-3})}$
= $(h(h^{-1}gh)^{\tau})^{yR(h^{-1}c_{m-3})}$
= $h((h^{-1}gh)^{\tau})^{\tau} c_{m-3}h^{-1}c_{m-3}$
= $h(h^{-1}gh)h^{-1}$
= g

and

$$(hg)^{z^{2}} = (hg)^{R(h)yR(c_{m-3})yR(h^{-1}c_{m-3})}$$

= $(hgh)^{yR(c_{m-3})yR(h^{-1}c_{m-3})}$
= $((hgh)^{\tau})^{R(c_{m-3})yR(h^{-1}c_{m-3})}$
= $((hgh)^{\tau}c_{m-3})^{yR(h^{-1}c_{m-3})}$
= $((hgh)^{\tau}c_{m-3})^{\tau}h^{-1}c_{m-3}$
= $hghc_{m-3}h^{-1}c_{m-3}$
= $hg.$

Thus $z^2 = 1$, completing the proof.

Lemma 2.2. $\operatorname{Aut}(H) \leq \operatorname{Alt}(H)$.

Proof. The conclusion for m = 4 is easy to verify. Thus we assume $m \ge 5$ in the following. Since the center $\mathbf{Z}(H) = \langle a^2, c_1, c_2, \ldots, c_{m-3} \rangle$ is a characteristic subgroup of H, each automorphism $\sigma \in \operatorname{Aut}(H)$ induces an automorphism of $H/\mathbf{Z}(H) = \{\mathbf{Z}(H), a\mathbf{Z}(H), b\mathbf{Z}(H), ab\mathbf{Z}(H)\}$. More precisely, there is a homomorphism φ from $\operatorname{Aut}(H)$ to $\operatorname{Aut}(H/\mathbf{Z}(H))$ such that $\varphi(\sigma)$ maps $g\mathbf{Z}(H)$ to $g^{\sigma}\mathbf{Z}(H)$ for all $\sigma \in \operatorname{Aut}(H)$ and $g \in H$.

Take an arbitrary $\sigma \in \operatorname{Aut}(H)$. Note that $a\mathbf{Z}(H)$ contains elements of order 4 while the elements in $b\mathbf{Z}(H)$ and $ab\mathbf{Z}(H)$ are all involutions. We see that if $\varphi(\sigma) \neq 1$ then $\varphi(\sigma)$ must fix the elements $\mathbf{Z}(H)$ and $a\mathbf{Z}(H)$ and swap $b\mathbf{Z}(H)$ and $ab\mathbf{Z}(H)$ in $H/\mathbf{Z}(H)$. Consequently, $\varphi(\sigma) \in \langle \varphi(x) \rangle$, and so $\sigma \in w \langle x \rangle$ for some $w \in \ker(\varphi)$. Since w stabilizes $\mathbf{Z}(H)$, $a\mathbf{Z}(H)$ and $b\mathbf{Z}(H)$, we have

$$a^w = a^{2\lambda+1} \prod_{j=1}^{m-3} c_j^{\lambda_j}, \quad b^w = a^{2\mu} b \prod_{j=1}^{m-3} c_j^{\mu_j} \text{ and } c_i^w = a^{2k_i} \prod_{j=1}^{m-3} c_j^{k_{i,j}}$$

for each *i* with $1 \leq i \leq m-3$, where λ , μ , k_i , λ_j , μ_j and $k_{i,j}$ are all in $\{0, 1\}$. Let w_1, w_2 and w_3 be automorphisms of H such that

$$\begin{aligned} &a^{w_1} = a^{2\lambda+1}, & b^{w_1} = a^{2\mu}b, & c_i^{w_1} = a^{2k_i}c_i, \\ &a^{w_2} = a, & b^{w_2} = b, & c_i^{w_2} = \prod_{j=1}^{m-3} c_j^{k_{i,j}}, \\ &a^{w_3} = a \prod_{j=1}^{m-3} c_j^{\lambda_j}, & b^{w_3} = b \prod_{j=1}^{m-3} c_j^{\mu_j}, & c_i^{w_3} = c_i. \end{aligned}$$

Then w_1 and w_3 are involutions, and $w = w_1 w_2 w_3$.

For each $\rho \in \operatorname{Aut}(H)$, the set of fixed points of ρ is a subgroup of H and thereby has size 2^{ℓ} for some integer ℓ such that $0 \leq \ell \leq m$. Thus, each involution of $\operatorname{Aut}(H)$ with at least four fixed points lies in $\operatorname{Alt}(H)$ as it is a product of $(|H| - 2^{\ell})/2 = 2^{m-1} - 2^{\ell-1}$ transpositions for some integer ℓ such that $2 \leq \ell \leq m - 1$. Since xand w_3 fix every point in $\langle a^2, c_1 \rangle$ and w_1 fixes every point in $\langle a^{\lambda+1}, a^{\mu}b^{\lambda} \rangle$, we then conclude that $x, w_3, w_1 \in \operatorname{Alt}(H)$. Moreover, since $(gv)^{w_2} = gv^{w_2}$ for all $g \in \langle a, b \rangle$ and $v \in \langle c_1, \ldots, c_{m-3} \rangle$, the number of transpositions of w_2 is divisible by $|\langle a, b \rangle| = 8$. In particular, $w_2 \in \operatorname{Alt}(H)$. Now w_1, w_2, w_3 and x are all in $\operatorname{Alt}(H)$. It follows that $\sigma \in \operatorname{Alt}(H)$ due to $\sigma \in w \langle x \rangle = w_1 w_2 w_3 \langle x \rangle$. This shows that $\operatorname{Aut}(H) \leq \operatorname{Alt}(H)$. \Box

The next lemma says that x and y as well as the elements of R(H) are all even permutations on H. Note that this also implies $z \in Alt(H)$ since $z \in \langle y, R(H) \rangle$.

Lemma 2.3. $\langle x, y, R(H) \rangle \leq \operatorname{Alt}(H)$.

Proof. Lemma 2.2 already indicates $x \in Alt(H)$. Let σ be the map from K to hK sending g to hg for all $g \in K$, and t be the permutation on H such that $g^t = g$ and $(hg)^t = hgc_{m-3}^{m-1}$ for all $g \in K$. Then t is the identity permutation if m is odd, and is a product of |K|/2 transpositions if m is even. In particular, $t \in Alt(H)$. From the definition of y one sees that the following diagram commutes.

$$\begin{array}{ccc} K & \xrightarrow{yt} & K \\ \downarrow \sigma & & \downarrow \sigma \\ hK & \xrightarrow{yt} & hK \end{array}$$

Hence $(yt)|_{hK}$ has the same cycle structure as $(yt)|_K$, and so $yt \in Alt(H)$. This in turn gives $y \in Alt(H)$. Finally, as H is a 2-group and not cyclic, we have $R(H) \leq Alt(H)$. Consequently, $\langle x, y, R(H) \rangle \leq Alt(H)$.

Recall the standard construction of the coset graph $\operatorname{Cos}(G, H, HSH)$ given a group G with a subgroup H and a subset S such that $S \cap H = \emptyset$ and HSH is inverseclosed. Such a graph has vertex set [G:H], the set of right cosets of H in G, and edge set $\{\{Hg, Hsg\} \mid g \in G, s \in HSH\}$. It is easy to see that $\operatorname{Cos}(G, H, HSH)$ has valency |HSH|/|H|, and G acts by right multiplication on [G:H] as a group of automorphisms of $\operatorname{Cos}(G, H, HSH)$. Moreover, $\operatorname{Cos}(G, H, HSH)$ is connected if and only if $\langle S, H \rangle = G$.

Now we are in the position to construct the graph Γ_m for Theorem 1.3.

Construction 2.4. For each integer $m \ge 4$, let

$$\Gamma_m = \operatorname{Cos}(\operatorname{Alt}(H), R(H), R(H)\{x, y\}R(H))$$

with H, x and y defined at the beginning of this section.

3. Connectivity of Γ_m

The aim of this section it to prove that Γ_m is connected. According to the construction of Γ_m , it suffices to prove $\langle x, y, R(H) \rangle = \operatorname{Alt}(H)$, and we will achieve this by dealing with the cases m is odd and m is even separately. For a group G, denote the set $G \setminus \{1\}$ by G^* . For a permutation σ of a set Ω and $\alpha, \beta \in \Omega$, we write $\alpha \xrightarrow{\sigma} \beta$ if $\alpha^{\sigma} = \beta$.

3.1. **Technical lemms.** We first establish two technical lemmas that will be needed later in this section.

Lemma 3.1. Let $\ell \ge 2$ be an even integer, and $V = \langle e_1 \rangle \times \langle e_2 \rangle \times \cdots \times \langle e_\ell \rangle$ be a group with involutions e_1, e_2, \ldots, e_ℓ . Denote the right regular representation of V by r. Let $\omega = r(e_{\ell-1}e_\ell)$, $e_{-1} = e_0 = 1$, and χ and ψ be automorphisms of V such that $e_{2i+1}^{\chi} = e_{2i+1}, e_{2i+2}^{\chi} = e_{2i+1}e_{2i+2}, e_{2i+1}^{\psi} = e_{2i-1}e_{2i}e_{2i+2}$ and $e_{2i+2}^{\psi} = e_{2i-1}e_{2i}e_{2i+1}$ for each i with $0 \le i \le (\ell-2)/2$. Then $\langle \chi, \psi, \omega \rangle$ is a transitive subgroup of Sym(V).

Proof. Note that, viewing V as a vector space over \mathbb{F}_2 , the vectors $e_1^{\chi}, e_2^{\chi}, \ldots, e_{\ell}^{\chi}$ form a basis of V. Thus the automorphism χ of V is well-defined. Similarly, ψ is well-defined. Write $N = \langle \chi, \psi, \omega \rangle$. Since χ is an automorphism of V, we have

$$r(e_{\ell}) = r((e_{\ell-1}e_{\ell})^{\chi}) = \chi^{-1}r(e_{\ell-1}e_{\ell})\chi = \chi^{-1}\omega\chi \in N$$

and so $r(e_{\ell-1}) = r(e_{\ell-1}e_{\ell})r(e_{\ell}) = \omega r(e_{\ell}) \in N$. Suppose there exists a nonnegative integer $i \leq (\ell-2)/2$ such that $r(e_{\ell-2i+1}), r(e_{\ell-2i+2}), \ldots, r(e_{\ell-1}), r(e_{\ell})$ are all in N. Note that

$$r(e_{\ell-2i-1}e_{\ell-2i}) = r(e_{\ell-2i-1}e_{\ell-2i}e_{\ell-2i+1})r(e_{\ell-2i+1})$$
$$= r(e_{\ell-2i+2}^{\psi})r(e_{\ell-2i+1}) = \psi^{-1}r(e_{\ell-2i+2})\psi r(e_{\ell-2i+1})$$

since ψ is an automorphism of V. We thereby deduce that $r(e_{\ell-2i-1}e_{\ell-2i}) \in N$. It follows that $r(e_{\ell-2i}) = r((e_{\ell-2i-1}e_{\ell-2i})^{\chi}) = \chi^{-1}r(e_{\ell-2i-1}e_{\ell-2i})\chi \in N$ and thus $r(e_{\ell-2i-1}) = r(e_{\ell-2i-1}e_{\ell-2i})r(e_{\ell-2i}) \in N$. Then by induction one concludes that $r(e_1), r(e_2), \ldots, r(e_{\ell-1}), r(e_{\ell})$ are all in N. Consequently, $r(V) \leq N$ and so N is transitive on V.

The following lemma is a consequence of the classification of doubly transitive permutation groups (see for example [1]).

Lemma 3.2. Suppose that G is a doubly transitive permutation group on 2^m points. Then one of the following holds:

- (i) $G \leq \operatorname{AGL}_m(2)$;
- (ii) $2^m 1 = q$ for some prime power q and $PSL_2(q) \leq G \leq P\Gamma L_2(q)$;

(iii)
$$A_{2^m} \leqslant G \leqslant S_{2^m}$$
.

Remark. In fact, the prime power q in case (ii) of Lemma 3.2 is necessarily a prime by Mihǎilescu's theorem [10]. In particular, m must be odd in case (ii) of Lemma 3.2.

3.2. Odd m. Throughout this subsection, let m be odd, and

$$U = \left\{ \prod_{i=1}^{m-3} c_i^{k_i} \mid \sum_{j=1}^{(m-3)/2} k_{2j} \equiv 0 \pmod{2} \right\}.$$

Note that $\{U, Uc_{m-3}\}$ forms a partition of $\langle c_1, c_2, \ldots, c_{m-3} \rangle$, and x stabilizes U setwise. For each $u \in U$ we have

(1)
$$u \xrightarrow{x} u^x \xrightarrow{y} u^{xy} \xrightarrow{z} u^x \xrightarrow{x} u \xrightarrow{y} u^y \xrightarrow{z} u,$$

(2)
$$au \xrightarrow{x} a^{-1}u^x \xrightarrow{y} abu^{xy}c_{m-4}c_{m-3} \xrightarrow{z} au^x \xrightarrow{x} a^{-1}u \xrightarrow{y} abu^y c_{m-4}c_{m-3} \xrightarrow{z} au,$$

$$(3) \qquad uc_{m-3} \xrightarrow{x} a^{2}u^{x}c_{m-4}c_{m-3} \xrightarrow{y} bu^{xy}c_{m-4}c_{m-3} \xrightarrow{z} bu^{x}c_{m-4}c_{m-3} bu^{x}c_{m-4}c_{m-3} \xrightarrow{x} a^{-1}buc_{m-3} \xrightarrow{y} a^{-1}bu^{y}c_{m-6}c_{m-5}c_{m-3} \xrightarrow{z} a^{-1}buc_{m-3} a^{-1}buc_{m-3} \xrightarrow{x} bu^{x}c_{m-4}c_{m-3} \xrightarrow{y} a^{2}u^{xy}c_{m-4}c_{m-3} \xrightarrow{z} a^{2}bu^{x}c_{m-4}c_{m-3} a^{2}bu^{x}c_{m-4}c_{m-3} \xrightarrow{x} abuc_{m-3} \xrightarrow{y} a^{-1}u^{y}c_{m-6}c_{m-5}c_{m-4} \xrightarrow{z} a^{-1}uc_{m-3} a^{-1}uc_{m-3} \xrightarrow{x} a^{-1}u^{x}c_{m-4}c_{m-3} \xrightarrow{y} abu^{xy} \xrightarrow{z} au^{x}c_{m-4}c_{m-3} au^{x}c_{m-4}c_{m-3} \xrightarrow{x} auc_{m-3} \xrightarrow{y} au^{y}c_{m-6}c_{m-5}c_{m-3} \xrightarrow{z} abuc_{m-3} abuc_{m-3} \xrightarrow{x} a^{2}bu^{x}c_{m-4}c_{m-3} \xrightarrow{y} a^{2}bu^{xy}c_{m-4}c_{m-3} \xrightarrow{z} a^{2}u^{x}c_{m-4}c_{m-3} a^{2}u^{x}c_{m-4}c_{m-3} \xrightarrow{x} uc_{m-3} \xrightarrow{y} u^{y}c_{m-6}c_{m-5}c_{m-4} \xrightarrow{z} uc_{m-3},$$

and

$$(4) \qquad a^{2}u \xrightarrow{x} a^{2}u^{x} \xrightarrow{y} bu^{xy} \xrightarrow{z} bu^{x} \xrightarrow{x} abu \xrightarrow{y} a^{-1}u^{y}c_{m-4}c_{m-3} \xrightarrow{z} a^{-1}u$$

$$a^{-1}u \xrightarrow{x} au^{x} \xrightarrow{y} au^{xy}c_{m-4}c_{m-3} \xrightarrow{z} abu^{x} \xrightarrow{x} bu \xrightarrow{y} a^{2}u^{y} \xrightarrow{z} a^{2}bu$$

$$a^{2}bu \xrightarrow{x} a^{-1}bu^{x} \xrightarrow{y} a^{-1}bu^{xy}c_{m-4}c_{m-3} \xrightarrow{z} a^{-1}bu^{x} \xrightarrow{x} a^{2}bu \xrightarrow{y} a^{2}bu^{y} \xrightarrow{z} a^{2}u.$$

Lemma 3.3. Suppose m is odd. Then the permutation $(xyz)^8$ of H has cycle decomposition

$$(xyz)^8 = \left(\prod_{u \in U} (a^2u, a^{-1}u, a^2bu)\right) \left(\prod_{u \in U} (bu, abu, a^{-1}bu)\right)$$

Proof. Denote $H_{i,j,k} = \{a^i b^j u c_{m-3}^k \mid u \in U\}$, where $i \in \{-1, 0, 1, 2\}$ and $j, k \in \{0, 1\}$. Then $\{H_{i,j,k} \mid -1 \leq i \leq 2, 0 \leq j \leq 1, 0 \leq k \leq 1\}$ forms a partition of H.

By (1) and (2), $u^{(xyz)^2} = u$ and $(au)^{(xyz)^2} = au$ for all $u \in U$. Hence $(xyz)^8$ fixes $H_{0,0,0} \cup H_{1,0,0}$ pointwise. From (3) one sees that $(xyz)^8$ fixes uc_{m-3} , $bu^x c_{m-4} c_{m-3}$, $a^{-1}buc_{m-3}$, $a^2bu^x c_{m-4} c_{m-3}$, $a^{-1}uc_{m-3}$, $au^x c_{m-4} c_{m-3}$, $abuc_{m-3}$ and $a^2u^x c_{m-4} c_{m-3}$ for all $u \in U$. Noting $\{u^x c_{m-4} c_{m-3} \mid u \in U\} = Uc_{m-3}$, we conclude that $(xyz)^8$ fixes

$$H_{0,0,1} \cup H_{0,1,1} \cup H_{-1,1,1} \cup H_{2,1,1} \cup H_{-1,0,1} \cup H_{1,0,1} \cup H_{1,1,1} \cup H_{2,0,1}$$

pointwise. This together with the conclusion that $(xyz)^8$ fixes $H_{0,0,0} \cup H_{1,0,0}$ pointwise shows it suffices to prove that $(xyz)^8$ induces the permutation

$$\left(\prod_{u\in U} (a^2u, a^{-1}u, a^2bu)\right) \left(\prod_{u\in U} (bu, abu, a^{-1}bu)\right)$$

on $H_{2,0,0} \cup H_{-1,0,0} \cup H_{2,1,0} \cup H_{0,1,0} \cup H_{1,1,0} \cup H_{-1,1,0}$. Indeed, (4) implies

$$a^2u \xrightarrow{(xyz)^8} a^{-1}u \xrightarrow{(xyz)^8} a^2bu \xrightarrow{(xyz)^8} a^2u$$

and

$$bu^x \xrightarrow{(xyz)^8} abu^x \xrightarrow{(xyz)^8} a^{-1}bu^x \xrightarrow{(xyz)^8} bu^x$$

for all $u \in U$. This yields the desired conclusion since $x|_U \in Aut(U)$.

Before proving the next lemma, we note that for each $u \in U$,

(5)
$$a^{-1}bu \xrightarrow{xyzx} a^2u \xrightarrow{xyzx} abu \xrightarrow{yz} a^{-1}u \xrightarrow{xyzx} bu \xrightarrow{yz} a^2bu \xrightarrow{zyxzyxyz} au$$

and

(6)
$$a^{-1}uc_{m-3} \xrightarrow{yz} auc_{m-3} \xrightarrow{xyzx} a^2 buc_{m-3} \xrightarrow{yz} a^2 uc_{m-3} \xrightarrow{yz} buc_{m-3}$$

 $auc_{m-3} \xrightarrow{yz} abuc_{m-3} \xrightarrow{xyzx} uc_{m-3} \xrightarrow{xyzx} a^{-1}buc_{m-3}.$

Lemma 3.4. Suppose m is odd. Then for each $g \in H \setminus U$, there exists $\zeta \in \langle x, y, z \rangle$ such that $g^{\zeta} = c_{m-3}$.

Proof. Let $v \in \langle c_1, c_2, \ldots, c_{m-3} \rangle$ such that $g \in \langle a, b \rangle v$, and define χ , ψ and ω as in Lemma 3.1 with $\ell = m - 3$ and $e_i = c_i$. Then χ , ψ and $\psi \omega$ are all involutions in $\operatorname{Sym}(\langle c_1, c_2, \ldots, c_{m-3} \rangle)$, and by Lemma 3.1, there exist $\eta_1, \eta_2, \ldots, \eta_t \in \{\chi, \psi, \psi\omega\}$ such that $v^{\eta_1 \eta_2 \cdots \eta_t} = c_{m-3}$. Let $\eta_0 = 1 \in \operatorname{Sym}(\langle c_1, c_2, \ldots, c_{m-3} \rangle)$ and $\zeta_0 = 1 \in$

 $\langle x, y, z \rangle$. Obviously, $g^{\zeta_0} \in \langle a, b \rangle v^{\eta_0} \setminus U$. We shall prove by induction that there exist $\zeta_0, \zeta_1, \ldots, \zeta_t \in \langle x, y, z \rangle$ with

$$g^{\zeta_0\zeta_1\dots\zeta_t} \in \langle a,b\rangle v^{\eta_0\eta_1\dots\eta_t} \setminus U_{\cdot}$$

By (5), for each $u \in U$ and $\alpha, \beta \in \langle a, b \rangle^*$, there exists $\varepsilon \in \langle x, yz \rangle$ such that $(\alpha u)^{\varepsilon} = \beta u$. By (6), for each $u \in U$ and $\alpha, \beta \in \langle a, b \rangle$, there exists $\varepsilon \in \langle x, yz \rangle$ such that $(\alpha uc_{m-3})^{\varepsilon} = \beta uc_{m-3}$.

Suppose that there exist $\zeta_0, \zeta_1, \ldots, \zeta_{i-1} \in \langle x, y, z \rangle$ with $1 \leq i \leq t$ and

$$g^{\zeta_0\zeta_1\ldots\zeta_{i-1}} \in \langle a,b\rangle v^{\eta_0\eta_1\ldots\eta_{i-1}} \setminus U.$$

If $\eta_i = \chi$, then let $\zeta_i = x$. If $\eta_i = \psi$, then there exists $\varepsilon_i \in \langle x, yz \rangle$ such that $(g^{\zeta_0 \zeta_1 \dots \zeta_{i-1}})^{\varepsilon_i} = a^2 v^{\eta_0 \eta_1 \dots \eta_{i-1}}$ and we let $\zeta_i = \varepsilon_i y$. If $\eta_i = \psi \omega$, then there exists $\varepsilon_i \in \langle x, yz \rangle$ such that $(g^{\zeta_0 \zeta_1 \dots \zeta_{i-1}})^{\varepsilon_i} = a v^{\eta_0 \eta_1 \dots \eta_{i-1}}$ and we let $\zeta_i = \varepsilon_i y$. If follows that

$$g^{\zeta_0\zeta_1\dots\zeta_{i-1}\zeta_i} = (g^{\zeta_0\zeta_1\dots\zeta_{i-1}})^{\zeta_i} \in (\langle a,b\rangle v^{\eta_0\eta_1\dots\eta_{i-1}} \setminus U)^{\zeta_i} \subseteq \langle a,b\rangle v^{\eta_0\eta_1\dots\eta_i} \setminus U.$$

By induction we now have $\zeta_0, \zeta_1, \ldots, \zeta_t \in \langle x, y, z \rangle$ such that

$$g^{\zeta_0\zeta_1...\zeta_t} \in \langle a, b \rangle v^{\eta_0\eta_1...\eta_t} \setminus U = \langle a, b \rangle c_{m-3}$$

Then as there exists $\varepsilon \in \langle x, yz \rangle$ such that $(g^{\zeta_0 \zeta_1 \dots \zeta_t})^{\varepsilon} = c_{m-3}$, we have $g^{\zeta} = c_{m-3}$ with $\zeta := \zeta_0 \zeta_1 \dots \zeta_t \varepsilon \in \langle x, y, z \rangle$.

Lemma 3.5. Suppose that m is odd. Then $\langle x, y, z \rangle$ is transitive on H^* .

Proof. In view of Lemma 3.4, we only need to prove that for each $u \in U^*$, there exists $\varepsilon \in \langle x, y \rangle$ such that $u^{\varepsilon} \in Uc_{m-3}$. Write $u = c_1^{k_1} c_2^{k_2} \cdots c_{m-3}^{k_{m-3}}$ with $k_1, k_2, \ldots, k_{m-3} \in \{0, 1\}$. Denote $V_i = \langle c_i, c_{i+1}, \ldots, c_{m-3} \rangle$ for $1 \leq i \leq m-3$, and set $V_{m-2} = 1$. Let s be the smallest integer in $\{0, 1, \ldots, (m-5)/2\}$ such that $k_{2s+1} + k_{2s+2} > 0$. Taking

$$\varepsilon = x^{k_{2s+1}+k_{2s+2}-1}y(xy)^s,$$

we prove below that $u^{\varepsilon} \in Uc_{m-3}$ by induction on s.

First suppose s = 0. If $k_1 = 0$ and $k_2 = 1$, then $u = c_2u_1$ with $u_1 \in V_3$ and $u_1 \in Uc_{m-3}$ since $u \in U$. In this case, $u_1^y \in Uc_{m-3}$, and it follows that $u^{\varepsilon} = (c_2u_1)^y = c_1u_1^y \in Uc_{m-3}$. If $k_1 = 1$ and $k_2 = 0$, then $u = c_1u_1$ with $u_1 \in V_3$ and $u_1 \in U$ as $u \in U$. In this case, $u_1^y \in U$ and so $u^{\varepsilon} = (c_1u_1)^y = c_2u_1^y \in Uc_{m-3}$. If $k_1 = k_2 = 1$, then $u = c_1c_2u_1$ with $u_1 \in V_3$ and $u_1 \in Uc_{m-3}$ since $u \in U$. In this case, $u_1^x = a^2u_2$ for some $u_2 \in V_3 \cap Uc_{m-3}$, whence $u^{\varepsilon} = (c_1c_2u_1)^{xy} = (a^2c_2u_1^x)^y = (c_2u_2)^y = c_1u_2^y \in Uc_{m-3}$ as $u_2^y \in Uc_{m-3}$.

Next suppose s > 0. If $k_{2s+1} = 0$ and $k_{2s+2} = 1$, then $u = c_{2s+2}u_1$ with $u_1 \in V_{2s+3}$, which implies $u^y = (c_{2s+2}u_1)^y = c_{2s-1}c_{2s}c_{2s+1}u_1^y \in c_{2s-1}c_{2s}V_{2s+1}$. If $k_{2s+1} = 1$ and $k_{2s+2} = 0$, then $u = c_{2s+1}u_1$ with $u_1 \in V_{2s+3}$ and therefore $u^y = (c_{2s+1}u_1)^y = c_{2s-1}c_{2s}c_{2s+2}u_1^y \in c_{2s-1}c_{2s}V_{2s+1}$. If $k_{2s+1} = k_{2s+2} = 1$, then $u = c_{2s+1}c_{2s+2}u_1$ with $u_1 \in V_{2s+3}$ and $u_1 \in Uc_{m-3}$ since $u \in U$. In this case, $u_1^x = a^2u_2$ for some $u_2 \in V_{2s+3}$, and so $u^{xy} = (c_{2s+1}c_{2s+2}u_1)^{xy} = (a^2c_{2s+2}u_1^x)^y = (c_{2s+2}u_2)^y = c_{2s-1}c_{2s}c_{2s+1}u_2^y \in c_{2s-1}c_{2s}V_{2s+1}$. To sum up, we always have $u^{\varepsilon_0} \in c_{2s-1}c_{2s}V_{2s+1}$, where

$$\varepsilon_0 = x^{k_{2s+1} + k_{2s+2} - 1} y.$$

By the inductive hypothesis,

$$(u^{\varepsilon_0})^{(xy)^s} = (u^{\varepsilon_0})^{x^{1+1-1}y(xy)^{s-1}} \in Uc_{m-3}.$$

Consequently,

$$u^{x^{k_{2s+1}+k_{2s+2}-1}y(xy)^s} = (u^{\varepsilon_0})^{(xy)^s} \in Uc_{m-3},$$

completing the proof.

Lemma 3.6. Suppose that m is odd. Then $\langle x, y, R(H) \rangle = Alt(H)$.

Proof. Let $G = \langle x, y, R(H) \rangle$. Notice that x, y and z are all involutions of G fixing 1. By Lemma 3.5, $\langle x, y, z \rangle$ is transitive on H^* , and so is G_1 , the stabilizer of 1 in G. This together with the transitivity of R(H) on H implies that G is doubly transitive on H. Therefore, one of cases (i)–(iii) in Lemma 3.2 holds.

Assume that $G \leq \operatorname{AGL}_m(2)$ as in case (i) of Lemma 3.2. Then $(xyz)^8 \in G_1 \leq \operatorname{GL}_m(2)$ and hence the set of fixed points of $(xyz)^8$ is a vector space over \mathbb{F}_2 . However, Lemma 3.3 shows that the number of fixed points of $(xyz)^8$ is $|H| - 3|U| - 3|U| = 5 \cdot 2^{m-3}$, a contradiction.

Assume that $PSL_2(q) \leq G \leq P\Gamma L_2(q)$ as in case (ii) of Lemma 3.2, where $q = 2^m - 1$ is a prime. Then $R(H) \leq G \leq P\Gamma L_2(q) = PGL_2(q)$. It follows that R(H) is contained in $D_{2(q+1)}$, the Sylow 2-subgroup of $PGL_2(q)$. This is impossible since $R(H) \cong D_8 \times C_2^{m-3}$.

Now $Alt(H) \leq G \leq Sym(H)$. This in conjunction with Lemma 2.3 forces G = Alt(H), which completes the proof.

3.3. Even *m*. Throughout this subsection, let *m* be even, $h_1 = hc_{m-3}$, $H_1 = \langle a, b, c_1, c_2, \ldots, c_{m-4} \rangle$, $K_1 = \langle a^2, b, c_1, c_2, \ldots, c_{m-4} \rangle$ and

$$U = \left\{ \prod_{i=1}^{m-3} c_i^{k_i} \mid \sum_{j=1}^{(m-4)/2} k_{2j} \equiv k_{m-3} \pmod{2} \right\}.$$

Define permutations x_1, y_1 and z_1 on H_1 such that $x_1 = x|_{H_1}, y_1|_{K_1} = y|_{K_1}, y_1|_{h_1K_1} = (yR(c_{m-3}))|_{h_1K_1}, z_1|_{K_1} = z|_{K_1}$ and $z_1|_{h_1K_1} = (zR(c_{m-3}))|_{h_1K_1}$. One can verify readily that $(h_1k_1)^{y_1} = h_1k_1^{y_1}$ for all $k_1 \in K_1, z_1 = (R(h_1)y_1R(h_1^{-1}))|_{H_1}$ and $y_1z_1 = (yz)|_{H_1}$. For each $u \in U \cap H_1$ we have

(7)
$$uc_{m-3} \xrightarrow{xyz} bu^{x}c_{m-3} \xrightarrow{xyz} a^{-1}buc_{m-3} \xrightarrow{xyz} a^{2}bu^{x}c_{m-3} \xrightarrow{xyz} a^{-1}uc_{m-3} a^{-1}uc_{m-3} \xrightarrow{xyz} au^{x}c_{m-3} \xrightarrow{xyz} abuc_{m-3} \xrightarrow{xyz} a^{2}u^{x}c_{m-3} \xrightarrow{xyz} uc_{m-3},$$

(8)
$$uc_{m-4}c_{m-3} \xrightarrow{xyz} u^x c_{m-5}c_{m-4}c_{m-3} \xrightarrow{xyz} uc_{m-4}c_{m-3},$$

(9)
$$auc_{m-4}c_{m-3} \xrightarrow{xyz} au^x c_{m-5}c_{m-4}c_{m-3} \xrightarrow{xyz} auc_{m-4}c_{m-3}$$

and

$$(10) \qquad a^{2}uc_{m-4}c_{m-3} \xrightarrow{xyz} bu^{x}c_{m-5}c_{m-4}c_{m-3} \xrightarrow{xyz} a^{-1}uc_{m-4}c_{m-3}$$
$$a^{-1}uc_{m-4}c_{m-3} \xrightarrow{xyz} abu^{x}c_{m-5}c_{m-4}c_{m-3} \xrightarrow{xyz} a^{2}buc_{m-4}c_{m-3}$$
$$a^{2}buc_{m-4}c_{m-3} \xrightarrow{xyz} a^{-1}bu^{x}c_{m-5}c_{m-4}c_{m-3} \xrightarrow{xyz} a^{2}uc_{m-4}c_{m-3}.$$

Lemma 3.7. Suppose that *m* is even. Then the permutation $(xyz)^8|_{H_{1}c_{m-3}}$ has cycle decomposition

$$(xyz)^{8}|_{H_{1}c_{m-3}} = \left(\prod_{u \in U \cap H_{1}} (a^{2}uc_{m-4}c_{m-3}, a^{-1}uc_{m-4}c_{m-3}, a^{2}buc_{m-4}c_{m-3})\right)$$
$$\times \left(\prod_{u \in U \cap H_{1}} (buc_{m-4}c_{m-3}, abuc_{m-4}c_{m-3}, a^{-1}buc_{m-4}c_{m-3})\right).$$

Proof. Denote $H_{i,j,k} = \{a^i b^j u c_{m-4}^k c_{m-3} \mid u \in U \cap H_1\}$, where $i \in \{-1, 0, 1, 2\}$ and $j, k \in \{0, 1\}$. Then $\{H_{i,j,k} \mid -1 \leq i \leq 2, 0 \leq j \leq 1, 0 \leq k \leq 1\}$ forms a partition of $H_1 c_{m-3}$.

By (8) and (9), $(xyz)^2$ fixes $H_{0,0,1} \cup H_{1,0,1}$ pointwise and so does $(xyz)^8$. From (7) one sees that $(xyz)^8$ fixes uc_{m-3} , $bu^x c_{m-3}$, $a^{-1}buc_{m-3}$, $a^2bu^x c_{m-3}$, $a^{-1}uc_{m-3}$, $au^x c_{m-3}$, $abuc_{m-3}$ and $a^2u^x c_{m-3}$ for all $u \in U \cap H_1$. As $\{u^x \mid u \in U \cap H_1\} = U \cap H_1$, we then conclude that $(xyz)^8$ fixes

$$H_{0,0,0} \cup H_{0,1,0} \cup H_{-1,1,0} \cup H_{2,1,0} \cup H_{-1,0,0} \cup H_{1,0,0} \cup H_{1,1,0} \cup H_{2,0,0}$$

pointwise. This together with the conclusion that $(xyz)^8$ fixes $H_{0,0,1} \cup H_{1,0,1}$ pointwise shows it suffices to prove that $(xyz)^8$ induces the permutation

$$\prod_{e \in U \cap H_1} (a^2 u c_{m-4} c_{m-3}, a^{-1} u c_{m-4} c_{m-3}, a^2 b u c_{m-4} c_{m-3})$$

on $H_{2,0,1} \cup H_{-1,0,1} \cup H_{2,1,1}$ and the permutation

u

u

$$\prod_{\in U\cap H_1} (buc_{m-4}c_{m-3}, abuc_{m-4}c_{m-3}, a^{-1}buc_{m-4}c_{m-3})$$

on $H_{0,1,1} \cup H_{1,1,1} \cup H_{-1,1,1}$. Indeed, (10) implies

$$a^{2}uc_{m-4}c_{m-3} \xrightarrow{(xyz)^{8}} a^{-1}uc_{m-4}c_{m-3} \xrightarrow{(xyz)^{8}} a^{2}buc_{m-4}c_{m-3} \xrightarrow{(xyz)^{8}} a^{2}uc_{m-4}c_{m-3}$$

and

$$bu^{x}c_{m-5}c_{m-4}c_{m-3} \xrightarrow{(xyz)^{8}} abu^{x}c_{m-5}c_{m-4}c_{m-3} \xrightarrow{(xyz)^{8}} a^{-1}bu^{x}c_{m-5}c_{m-4}c_{m-3}$$
$$a^{-1}bu^{x}c_{m-5}c_{m-4}c_{m-3} \xrightarrow{(xyz)^{8}} bu^{x}c_{m-5}c_{m-4}c_{m-3}$$

for all $u \in U \cap H_1$. This yields the desired conclusion since the maps $u \mapsto u^x$ and $u \mapsto u^x c_{m-5}$ are both bijections from $U \cap H_1$ onto itself.

Lemma 3.8. Suppose that m is even. For each $v \in U$ and $\alpha, \beta \in \langle a, b \rangle^*$, there exists $\varepsilon \in \langle x, yz \rangle$ such that $(\alpha v)^{\varepsilon} = \beta v$. For each $v \in \langle c_1, c_2, \ldots, c_{m-3} \rangle \setminus U$ and $\alpha, \beta \in \langle a, b \rangle$, there exists $\varepsilon \in \langle x, yz \rangle$ such that $(\alpha v)^{\varepsilon} = \beta v$.

Proof. First recall from (5) that for each $u \in U \cap H_1$ and $\alpha, \beta \in \langle a, b \rangle^*$, there exist integers $k_1, k_2, \ldots, k_{2r-1}, k_{2r}$ with

$$(\alpha u)^{\prod_{i=1}^{r} x_1^{k_{2i-1}}(y_1 z_1)^{k_{2i}}} = \beta u$$

Then taking $\varepsilon = \prod_{i=1}^r x^{k_{2i-1}} (yz)^{k_{2i}}$, we have $(\alpha u)^{\varepsilon} = \beta u$ since $x|_{H_1} = x_1$ and $(yz)|_{H_1} = y_1 z_1$. Next note that for $u \in U \cap H_1$,

$$buc_{m-4}c_{m-3} \xrightarrow{yz} a^2 buc_{m-4}c_{m-3} \xrightarrow{yz} a^2 uc_{m-4}c_{m-3} \xrightarrow{(xyz)^2} a^{-1}uc_{m-4}c_{m-3}$$
$$auc_{m-4}c_{m-3} \xrightarrow{zy} a^{-1}uc_{m-4}c_{m-3} \xrightarrow{zy} abuc_{m-4}c_{m-3} \xrightarrow{(xyz)^2} a^{-1}buc_{m-4}c_{m-3}.$$

We then conclude that for each $v \in U$ and $\alpha, \beta \in \langle a, b \rangle^*$, there exists $\varepsilon \in \langle x, yz \rangle$ such that $(\alpha v)^{\varepsilon} = \beta v$.

Similarly, one derives from (6) that for each $u \in U \cap H_1$ and $\alpha, \beta \in \langle a, b \rangle$, there exists $\varepsilon \in \langle x, yz \rangle$ with $(\alpha uc_{m-4})^{\varepsilon} = \beta uc_{m-4}$. Moreover,

$$a^{-1}uc_{m-3} \xrightarrow{yz} auc_{m-3} \xrightarrow{xyzx} a^{2}buc_{m-3} \xrightarrow{yz} a^{2}uc_{m-3} \xrightarrow{yz} buc_{m-3}$$
$$auc_{m-3} \xrightarrow{yz} abuc_{m-3} \xrightarrow{xyzx} uc_{m-3} \xrightarrow{xyzx} a^{-1}buc_{m-3}$$

for all $u \in U \cap H_1$. Hence for each $v \in \langle c_1, c_2, \ldots, c_{m-3} \rangle \setminus U$ and $\alpha, \beta \in \langle a, b \rangle$, there exists $\varepsilon \in \langle x, yz \rangle$ such that $(\alpha v)^{\varepsilon} = \beta v$.

Lemma 3.9. Suppose that m is even. Then for each $g \in H \setminus U$, there exists $\zeta \in \langle x, y, z \rangle$ such that $g^{\zeta} = h$.

Proof. Let $v \in \langle c_1, c_2, \ldots, c_{m-4} \rangle$ such that $g \in \langle a, b, c_{m-3} \rangle v$, and define χ , ψ and ω as in Lemma 3.1 with $\ell = m - 4$ and $e_i = c_i$. Then χ , ψ and $\psi\omega$ are all involutions in Sym $(\langle c_1, c_2, \ldots, c_{m-4} \rangle)$, and by Lemma 3.1, there exist $\eta_1, \eta_2, \ldots, \eta_t \in \{\chi, \psi, \psi\omega\}$ such that $v^{\eta_1 \eta_2 \cdots \eta_t} = a^{-1}h_1$. Let $\eta_0 = 1 \in \text{Sym}(\langle c_1, c_2, \ldots, c_{m-4} \rangle)$ and $\zeta_0 = 1 \in \langle x, y, z \rangle$. Obviously, $g^{\zeta_0} \in \langle a, b, c_{m-3} \rangle v^{\eta_0} \setminus U$. We shall prove by induction that there exist $\zeta_0, \zeta_1, \ldots, \zeta_t \in \langle x, y, z \rangle$ with

$$g^{\zeta_0\zeta_1\dots\zeta_t} \in \langle a,b\rangle v^{\eta_0\eta_1\dots\eta_t} \setminus U.$$

Suppose there exist $\zeta_0, \zeta_1, \ldots, \zeta_{i-1} \in \langle x, y, z \rangle$ for some $i \in \{1, \ldots, t\}$ such that

$$g^{\zeta_0\zeta_1\dots\zeta_{i-1}} \in \langle a, b, c_{m-3} \rangle v^{\eta_0\eta_1\dots\eta_{i-1}} \setminus U.$$

If $\eta_i = \chi$, then let $\zeta_i = x$. If $\eta_i = \psi$, then by Lemma 3.8 there exists $\varepsilon_i \in \langle x, yz \rangle$ such that $(g^{\zeta_0 \zeta_1 \dots \zeta_{i-1}})^{\varepsilon_i} \in a^2 \langle c_{m-3} \rangle v^{\eta_0 \eta_1 \dots \eta_{i-1}}$ and we let $\zeta_i = \varepsilon_i y$. If $\eta_i = \psi \omega$, then by Lemma 3.8 there exists $\varepsilon_i \in \langle x, yz \rangle$ such that $(g^{\zeta_0 \zeta_1 \dots \zeta_{i-1}})^{\varepsilon_i} \in a \langle c_{m-3} \rangle v^{\eta_0 \eta_1 \dots \eta_{i-1}}$ and we let $\zeta_i = \varepsilon_i y$. It follows that

$$g^{\zeta_0\zeta_1\ldots\zeta_{i-1}\zeta_i} = (g^{\zeta_0\zeta_1\ldots\zeta_{i-1}})^{\zeta_i} \in (\langle a, b, c_{m-3} \rangle v^{\eta_0\eta_1\ldots\eta_{i-1}} \setminus U)^{\zeta_i} \subseteq \langle a, b, c_{m-3} \rangle v^{\eta_0\eta_1\ldots\eta_i} \setminus U.$$

By induction we now have $\zeta_0, \zeta_1, \ldots, \zeta_t \in \langle x, y, z \rangle$ such that

$$g^{\zeta_0\zeta_1\ldots\zeta_t} \in \langle a, b, c_{m-3} \rangle v^{\eta_0\eta_1\ldots\eta_t} \setminus U = \langle a, b, c_{m-3} \rangle a^{-1}h_1 \setminus U$$

Then by Lemma 3.8 there exists $\varepsilon \in \langle x, yz \rangle$ such that $(g^{\zeta_0 \zeta_1 \dots \zeta_t})^{\varepsilon} \in h_1 \langle c_{m-3} \rangle$. Let $\zeta = \zeta_0 \zeta_1 \dots \zeta_r \varepsilon y$ if $(g^{\zeta_0 \zeta_1 \dots \zeta_r})^{\varepsilon} = h_1$ and let $\zeta = \zeta_0 \zeta_1 \dots \zeta_r \varepsilon$ if $(g^{\zeta_0 \zeta_1 \dots \zeta_r})^{\varepsilon} = h_1 c_{m-3}$. Then $\zeta \in \langle x, y, z \rangle$, and in view of $h = h_1 c_{m-3} = h_1^y$ we have $g^{\zeta} = h$.

Lemma 3.10. Suppose that m is even. Then $\langle x, y, z \rangle$ is transitive on H^* .

Proof. Due to Lemma 3.8, we only need to prove that for each $u \in U^*$, there exists $\varepsilon \in \langle x, y \rangle$ such that $u^{\varepsilon} \in Uc_{m-3}$. Write $u = c_1^{k_1} c_2^{k_2} \cdots c_{m-3}^{k_{m-3}}$ with $k_1, k_2, \ldots, k_{m-3} \in \{0, 1\}$. Denote $V_i = \langle c_i, c_{i+1}, \ldots, c_{m-3} \rangle$ for $1 \leq i \leq m-3$. Since $u \neq c_{m-3}$, there

exists $0 \leq j \leq (m-6)/2$ such that $k_{2j+1} + k_{2j+2} > 0$. Let s be the smallest integer in $\{0, 1, \ldots, (m-6)/2\}$ such that $k_{2s+1} + k_{2s+2} > 0$. Taking

$$\varepsilon = x^{k_{2s+1}+k_{2s+2}-1}y(xy)^s,$$

we prove below that $u^{\varepsilon} \in Uc_{m-3}$ by induction on s.

First suppose s = 0. If $k_1 = 0$ and $k_2 = 1$, then $u = c_2u_1$ with $u_1 \in V_3$ and $u_1 \in Uc_{m-3}$ since $u \in U$. In this case, $u_1^y \in Uc_{m-3}$, and it follows that $u^{\varepsilon} = (c_2u_1)^y = c_1u_1^y \in Uc_{m-3}$. If $k_1 = 1$ and $k_2 = 0$, then $u = c_1u_1$ with $u_1 \in V_3$ and $u_1 \in U$ as $u \in U$. In this case, $u_1^y \in U$ and so $u^{\varepsilon} = (c_1u_1)^y = c_2u_1^y \in Uc_{m-3}$. If $k_1 = k_2 = 1$, then $u = c_1c_2u_1$ with $u_1 \in V_3$ and $u_1 \in Uc_{m-3}$ since $u \in U$. In this case, $u_1^x = a^2u_2$ for some $u_2 \in V_3 \cap Uc_{m-3}$, whence $u^{\varepsilon} = (c_1c_2u_1)^{xy} = (a^2c_2u_1^x)^y = (c_2u_2)^y = c_1u_2^y \in Uc_{m-3}$ as $u_2^y \in Uc_{m-3}$.

Next suppose s > 0. If $k_{2s+1} = 0$ and $k_{2s+2} = 1$, then $u = c_{2s+2}u_1$ with $u_1 \in V_{2s+3}$, which implies $u^y = (c_{2s+2}u_1)^y = c_{2s-1}c_{2s}c_{2s+1}u_1^y \in c_{2s-1}c_{2s}V_{2s+1}$. If $k_{2s+1} = 1$ and $k_{2s+2} = 0$, then $u = c_{2s+1}u_1$ with $u_1 \in V_{2s+3}$ and therefore $u^y = (c_{2s+1}u_1)^y = c_{2s-1}c_{2s}c_{2s+2}u_1^y \in c_{2s-1}c_{2s}V_{2s+1}$. If $k_{2s+1} = k_{2s+2} = 1$, then $u = c_{2s+1}c_{2s+2}u_1$ with $u_1 \in V_{2s+3}$ and $u_1 \in Uc_{m-3}$ since $u \in U$. In this case, $u_1^x = a^2u_2$ for some $u_2 \in V_{2s+3}$, and so $u^{xy} = (c_{2s+1}c_{2s+2}u_1)^{xy} = (a^2c_{2s+2}u_1^x)^y = (c_{2s+2}u_2)^y = c_{2s-1}c_{2s}c_{2s+1}u_2^y \in c_{2s-1}c_{2s}V_{2s+1}$. To sum up, we always have $u^{\varepsilon_0} \in c_{2s-1}c_{2s}V_{2s+1}$, where

$$\varepsilon_0 = x^{k_{2s+1}+k_{2s+2}-1}y$$

By the inductive hypothesis,

$$(u^{\varepsilon_0})^{(xy)^s} = (u^{\varepsilon_0})^{x^{1+1-1}y(xy)^{s-1}} \in Uc_{m-3}.$$

Consequently,

$$u^{x^{k_{2s+1}+k_{2s+2}-1}y(xy)^{s}} = (u^{\varepsilon_{0}})^{(xy)^{s}} \in Uc_{m-3},$$

completing the proof.

Lemma 3.11. Suppose that m is even. Then $\langle x, y, R(H) \rangle = Alt(H)$.

Proof. Let $G = \langle x, y, R(H) \rangle$. Notice that x, y and z are all involutions of G fixing 1. By Lemma 3.10, $\langle x, y, z \rangle$ is transitive on H^* , and so is G_1 , the stabilizer of 1 in G. This together with the transitivity of R(H) on H implies that G is doubly transitive on H. Therefore, either cases (i) or case (iii) in Lemma 3.2 holds since m is even.

Assume that $G \leq \operatorname{AGL}_m(2)$ as in case (i) of Lemma 3.2. Then $(xyz)^8 \in G_1 \leq \operatorname{GL}_m(2)$ and hence the set of fixed points of $(xyz)^8$ is a vector space over \mathbb{F}_2 . Since $x|_{H_1} = x_1$ and $(yz)|_{H_1} = y_1z_1$, we have $(xyz)^8|_{H_1} = (x_1y_1z_1)^8$, and thus Lemma 3.3 shows that the number of fixed points of $(xyz)^8|_{H_1}$ is $|H_1| - 3|U \cap H_1| - 3|U \cap H_1| = 5 \cdot 2^{m-4}$. Furthermore, the number of fixed points of $(xyz)^8|_{H_{1-3}}$ is $|H_{1}c_{m-3}| - 3|U \cap H_1| - 3|U \cap H_1| = 5 \cdot 2^{m-4}$. Hence the number of fixed points of $(xyz)^8|_{S} = 5 \cdot 2^{m-4} + 5 \cdot 2^{m-4} = 5 \cdot 2^{m-3}$, a contradiction.

Now $\operatorname{Alt}(H) \leq G \leq \operatorname{Sym}(H)$. This in conjunction with Lemma 2.3 forces $G = \operatorname{Alt}(H)$, which completes the proof.

4. Proof of Theorem 1.3

Theorem 1.3 will follow directly from Lemmas 4.1–4.4.

Lemma 4.1. Γ_m is a connected cubic graph.

Proof. By Lemmas 3.6 and 3.11 we have for each $m \ge 4$ that

(11)
$$\langle x, y, R(H) \rangle = \operatorname{Alt}(H),$$

which already implies the connectivity of Γ_m . To prove that Γ_m is cubic, we show $|R(H)\{x,y\}R(H)| = 3|R(H)|$ in the following.

It is straightforward to verify that x and y are involutions normalizing R(H) and R(K), respectively. As a consequence, R(H)xR(H) = R(H)x and $yR(H)y \cap R(H) \ge yR(K)y \cap R(K) = R(K)$. Notice that $yR(H)y \cap R(H) \ne R(H)$ for otherwise

$$\langle x, y, R(H) \rangle \leq \mathbf{N}_{\operatorname{Alt}(H)}(R(H)) < \operatorname{Alt}(H),$$

contrary to (11). We derive that $yR(H)y \cap R(H) = R(K)$ as R(K) has index 2 in R(H). Accordingly, $|R(H)yR(H)|/|R(H)| = |R(H)|/|yR(H)y \cap R(H)| = 2$. Moreover, $R(H)xR(H) \cap R(H)yR(H) = \emptyset$ for otherwise $y \in \langle x, R(H) \rangle$, which would cause a contradiction $\langle x, y, R(H) \rangle \leq \langle x, R(H) \rangle \leq \mathbf{N}_{Alt(H)}(R(H)) < Alt(H)$ to (11). Hence

$$|R(H)\{x,y\}R(H)| = |R(H)xR(H)| + |R(H)yR(H)|$$

= |R(H)x| + 2|R(H)| = 3|R(H)|,

as desired.

Lemma 4.2. Cay(Alt(H^*), {x, y, z}) is a nonnormal Cayley graph of Alt(H^*) and is isomorphic to Γ_m by the map $g \mapsto R(H)g$.

Proof. Let $S = \{x, y, z\}$. Consider the map $\varphi : g \mapsto R(H)g$ from $Alt(H^*)$ to the vertex set of Γ_m . We see that φ is injective as $R(H) \cap Alt(H^*) = 1$, and is therefore bijective as $|Alt(H^*)| = |Alt(H)|/|R(H)|$. In particular, R(H)S is a disjoint union of R(H)x, R(H)y and R(H)z. Then since

$$R(H)S = R(H)x \cup R(H)y \cup R(H)z \subseteq R(H)\{x, y\}R(H)$$

and $|R(H)\{x, y\}R(H)| = 3|R(H)|$ by Lemma 4.1, we conclude that

$$R(H)S = R(H)\{x, y\}R(H).$$

For g_1 and g_2 in Alt (H^*) , g_1 is adjacent to g_2 in Cay $(Alt(H^*), S)$ if and only if $g_2g_1^{-1} \in \{x, y, z\}$, which is equivalent to $R(H)g_2g_1^{-1} \in \{R(H)x, R(H)y, R(H)z\}$. This means that g_1 and g_2 is adjacent in Cay $(Alt(H^*), S)$ if and only if

$$R(H)g_2g_1^{-1} \subseteq R(H)S = R(H)\{x, y\}R(H),$$

or equivalently, $R(H)g_1$ is adjacent to $R(H)g_2$ in Γ_m . Therefore, φ is a graph isomorphism from $\operatorname{Cay}(\operatorname{Alt}(H^*), S)$ to Γ_m . Moreover, $\operatorname{Alt}(H)$ acts as a group of automorphisms of Γ_m by right multiplication and $\operatorname{Alt}(H^*)$ is not normal in $\operatorname{Alt}(H)$, whence Γ_m is a nonnormal Cayley graph of $\operatorname{Alt}(H^*)$.

Lemma 4.3. $Aut(Alt(H^*), \{x, y, z\}) = 1.$

Proof. Suppose for a contradiction that there exists $1 \neq \sigma \in \text{Sym}(H^*)$ with

$$\{\sigma^{-1}x\sigma, \sigma^{-1}y\sigma, \sigma^{-1}z\sigma\} = \{x, y, z\}$$

Then the conjugation action of σ induces a nontrivial permutation of $\{x, y, z\}$ as $\langle x, y, z \rangle = \text{Alt}(H^*)$. By Lemma 4.2 and [15, Theorem 1.1] we know that Γ_m is not

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arc-transitive. Note that R(a) interchanges the vertices R(H)y and R(H)z of Γ_m . In view of the isomorphism $g \mapsto R(H)g$ in Lemma 4.2, we have $\sigma^{-1}x\sigma = x$, $\sigma^{-1}y\sigma = z$ and $\sigma^{-1}z\sigma = y$. For $w \in \text{Sym}(H^*)$, denote by Fix(w) the set of fixed points of w on H^* . It follows that

(12)
$$|\operatorname{Fix}(y) \cap \operatorname{Fix}(xyx)| = |\operatorname{Fix}(y)^{\sigma} \cap \operatorname{Fix}(xyx)^{\sigma}| = |\operatorname{Fix}(z) \cap \operatorname{Fix}(xzx)|,$$

and

(13)
$$|\operatorname{Fix}(yxy) \cap \operatorname{Fix}(xyxyx)| = |\operatorname{Fix}(yxy)^{\sigma} \cap \operatorname{Fix}(xyxyx)^{\sigma}| = |\operatorname{Fix}(zxz) \cap \operatorname{Fix}(xzxzx)|.$$

First assume that $m \equiv 1 \pmod{4}$. Let

$$M = \langle a^2 b \rangle \times \langle c_1 c_2 \rangle \times \cdots \times \langle c_{2i-1} c_{2i} \rangle \times \cdots \times \langle c_{m-4} c_{m-3} \rangle.$$

It is easy to verify that $Fix(y) = \{1, h\}M \setminus \{1\}$ and

$$M^{x} = \langle a^{-1}b \rangle \times \langle a^{2}c_{2} \rangle \times \cdots \times \langle a^{2}c_{2i} \rangle \times \cdots \times \langle a^{2}c_{m-3} \rangle.$$

This implies that

$$\begin{aligned} \operatorname{Fix}(y) \cap \operatorname{Fix}(xyx) &= \operatorname{Fix}(y) \cap \operatorname{Fix}(y)^x \\ &= (\{1, h\}M \setminus \{1\}) \cap (\{1, h^x\}M^x \setminus \{1\}) \\ &= (\{1, h\}M \cap \{1, a^2h\}M^x) \setminus \{1\} \\ &= \emptyset. \end{aligned}$$

However, since the element $b \prod_{i=1}^{m-3} c_i$ of H^* is fixed by both z and xzx, we have $|\operatorname{Fix}(z) \cap \operatorname{Fix}(xzx)| > 0$, contrary to (12).

Next assume that $m \equiv 3 \pmod{4}$. Let

$$M = \langle b \rangle \times \langle c_1 c_2 \rangle \times \cdots \times \langle c_{2i-1} c_{2i} \rangle \times \cdots \times \langle c_{m-4} c_{m-3} \rangle.$$

It is easy to verify that $Fix(z) = \{1, a^2h\}M \setminus \{1\}$ and

$$M^{x} = \langle ab \rangle \times \langle a^{2}c_{2} \rangle \times \cdots \times \langle a^{2}c_{2i} \rangle \times \cdots \times \langle a^{2}c_{m-3} \rangle.$$

Hence $\operatorname{Fix}(z) \cap \operatorname{Fix}(xzx) = \operatorname{Fix}(z) \cap \operatorname{Fix}(z)^x = \emptyset$. However, $a^{2b} \prod_{i=1}^{m-3} c_i$ is fixed by both y and xyx, so $|\operatorname{Fix}(y) \cap \operatorname{Fix}(xyx)| > 0$. This again contradicts (12).

Now assume that $m \equiv 2 \pmod{4}$. Let

$$M = \langle c_1 \rangle \times \cdots \times \langle c_{2i-1} \rangle \times \cdots \times \langle c_{m-5} \rangle \times \langle ac_{m-3} \rangle.$$

It is easy to verify that $Fix(x) = M \setminus \{1\}$ and thence

$$\operatorname{Fix}(yxy) \cap \operatorname{Fix}(xyxyx) = \operatorname{Fix}(x)^y \cap \operatorname{Fix}(x)^{yx} = \emptyset.$$

However, $a^2b(\prod_{i=1}^{(m-4)/2} c_{2i})(\prod_{i=0}^{(m-6)/4} c_{4i-1})$ is fixed by both zxz and xzxzx. Thus, $|\operatorname{Fix}(zxz) \cap \operatorname{Fix}(xzxzx)| > 0 = |\operatorname{Fix}(yxy) \cap \operatorname{Fix}(xyxyx)|,$

contrary to (13).

Finally assume that $m \equiv 0 \pmod{4}$. Then in the same vein as above we have $\operatorname{Fix}(zxz) \cap \operatorname{Fix}(xzxzx) = \emptyset$ while the element $b(\prod_{i=0}^{(m-4)/2} c_{2i})(\prod_{i=0}^{(m-4)/4} c_{4i-1})$ of H^* is fixed by both yxy and xyxyx. This causes

$$|\operatorname{Fix}(yxy) \cap \operatorname{Fix}(xyxyx)| > 0 = |\operatorname{Fix}(zxz) \cap \operatorname{Fix}(xzxzx)|,$$

contradicting (13).

In the following lemma we prove that the full automorphism of Γ_m is isomorphic to A_{2^m} . Some of the arguments here were used in the proof of [17, Theorem 1.2].

Lemma 4.4. $\operatorname{Aut}(\Gamma_m) \cong A_{2^m}$.

Proof. Let $A = \operatorname{Aut}(\Gamma_m)$ and v be a vertex of Γ_m . Then by Lemma 4.2, A has a nonnormal vertex-regular subgroup G which is isomorphic to the alternating group A_{2^m-1} . Further, $\mathbf{N}_A(G) = G$ by Lemma 4.3. Note also that Γ_m is connected and cubic as Lemma 4.1 asserts. We derive from [15, Theorem 1.1] that A is not transitive on the arc set of Γ_m , and so A_v is a 2-group. Consequently, $|A|/|G| = |GA_v|/|G| = |A_v|/|G| = |A_v|/|G \cap A_v| = |A_v|$ is a power of 2. Since every nontrivial G-conjugacy class has size greater than 3, it follows from [3, Theorem 1.1] that one of the following two cases occurs:

- (i) Soc(A) is a nonabelian simple group containing G as a proper subgroup;
- (ii) A has a nontrivial normal subgroup N such that N is not transitive on the vertex set of Γ_m and $\operatorname{Soc}(A/N)$ is a nonabelian simple group containing $GN/N \cong G$.

First assume that case (i) occurs. Then as $|\operatorname{Soc}(A)|/|G|$ is a power of 2, we have $\operatorname{Soc}(A) = A_{2^m}$ by [7, Theorem 1], and so $A \cong A_{2^m}$ or S_{2^m} . If $A \cong \operatorname{S}_{2^m}$, then $\mathbf{N}_A(G) \cong \operatorname{S}_{2^m-1}$, contrary to the conclusion that $\mathbf{N}_A(G) = G$. Therefore, $A \cong A_{2^m}$. Next assume that case (ii) occurs. In this case, $N \cap G = 1$ as $GN/N \cong G$. Hence $|N| = |N|/|N \cap G| = |NG|/|G|$ divides |A|/|G|. In particular, N is a 2group. From the construction of Γ_m we know that A has a subgroup B that is isomorphic to A_{2^m} and contains G. Consider the action ϕ of B on N by conjugation. Since B is a simple group, either $\operatorname{ker}(\phi) = 1$ or $\operatorname{ker}(\phi) = B$. If $\operatorname{ker}(\phi) = B$, then B centralizes N and so $N \leq \mathbf{N}_A(G) = G$, contradicting the condition that $N \cap G = 1$. Hence we have $\operatorname{ker}(\phi) = 1$. Then $B \cong A_{2^m}$ is isomorphic to an

irreducible subgroup of $\operatorname{Aut}(N/\Phi(N)) \cong \operatorname{PSL}_d(2)$ for some positive integer d with $2^d \leq |N|$, where $\Phi(N)$ is the Frattini subgroup of N. It follows that $d \geq 2^m - 2$ according to [8, Proposition 5.3.7]. Hence $\nu_2(|N|) \geq 2^m - 2$, where ν_2 is the 2-adic valuation. Moreover, N must be semiregular on the vertex set of Γ_m , for otherwise the quotient graph of Γ_m with respect to N would have valency 2 and so could not admit A/N as a group of automorphisms. Accordingly,

$$\nu_2(|N|) \leq \nu_2(|\mathbf{A}_{2^m-1}|) = \sum_{i=1}^{\infty} \left\lfloor \frac{2^m - 1}{2^i} \right\rfloor - 1 < \sum_{i=1}^{\infty} \frac{2^m - 1}{2^i} - 1 = 2^m - 2.$$

This contradicts the conclusion $\nu_2(|N|) \ge 2^m - 2$, not possible.

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