# AN INFINITE FAMILY OF CUBIC NONNORMAL CAYLEY GRAPHS ON NONABELIAN SIMPLE GROUPS 

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#### Abstract

We construct a connected cubic nonnormal Cayley graph on $\mathrm{A}_{2^{m}-1}$ for each integer $m \geqslant 4$ and determine its full automorphism group. This is the first infinite family of connected cubic nonnormal Cayley graphs on nonabelian simple groups.


Key words: nonnormal Cayley graphs; cubic graphs; simple groups

## 1. Introduction

In this paper all graphs considered are finite, simple and undirected. Given a group $G$ and an inverse-closed subset $S$ of $G \backslash\{1\}$, the Cayley graph Cay $(G, S)$ on $G$ with respect to $S$ is the graph with vertex set $G$ such that two vertices $x$ and $y$ are adjacent if and only if $y x^{-1} \in S$. Let $\widehat{G}$ be the right regular representation of $G$. It is easy to see that $\widehat{G}$ is a subgroup of $\operatorname{Aut}(\operatorname{Cay}(G, S))$. Moreover, it was shown by Godsil [5] that the normalizer of $\widehat{G}$ in $\operatorname{Aut}(\operatorname{Cay}(G, S))$ is $\widehat{G} \rtimes \operatorname{Aut}(G, S)$, where $\operatorname{Aut}(G, S)$ is the group of automorphisms of $G$ fixing $S$ setwise. In particular, $\operatorname{Aut}(\operatorname{Cay}(G, S))=\widehat{G} \rtimes \operatorname{Aut}(G, S)$ if and only if $\widehat{G}$ is normal in $\operatorname{Aut}(\operatorname{Cay}(G, S))$. Viewing this, Xu in [14] introduced the concept of normal Cayley graphs: a Cayley graph $\operatorname{Cay}(G, S)$ is said to be normal if $\widehat{G}$ is normal in $\operatorname{Aut}(\operatorname{Cay}(G, S))$. The study of normality of a Cayley graph plays an important role in the study of its automorphism group because once a Cayley graph Cay $(G, S)$ is known to be normal, to determine its full automorphism group one only needs to determine the group Aut $(G, S)$, which is usually much easier. For a survey paper on normality of Cayley graphs we refer the reader to 4 .

The normality of cubic Cayley graphs on nonabelian simple groups has received considerable attention. It was proved in [12] that a connected cubic Cayley graph $\operatorname{Cay}(G, S)$ with $G$ nonabelian simple is normal if $\widehat{G} \rtimes \operatorname{Aut}(G, S)$ is transitive on the edge set of Cay $(G, S)$. A graph is said to be arc-transitive if its automorphism group acts transitively on the set of arcs. In [15, 16] it was proved that the only connected arc-transitive cubic nonnormal Cayley graphs on nonabelian simple groups are two Cayley graphs on $A_{47}$ up to isomorphism, and their full automorphism groups are both isomorphic to $\mathrm{A}_{48}$. On the other hand, examples of connected cubic nonnormal Cayley graphs on nonabelian simple groups are very rare. Since the connected arctransitive cubic nonnormal Cayley graphs on nonabelian simple groups are only the above mentioned two graphs on $\mathrm{A}_{47}$, we can concentrate on the non-arc-transitive case. In this context, one has the following theorem combining [2, Theorem 1.1] and [17, Theorem 1.2].

Theorem 1.1. ([2, 17]) Let $\operatorname{Cay}(G, S)$ be a connected cubic nonnormal Cayley graph on a nonabelian simple group $G$. If $\operatorname{Cay}(G, S)$ is not arc-transitive, then one of the following holds:
(a) $G=\mathrm{A}_{2^{m}-1}$ with $m \geqslant 3$;
(b) $G$ is a simple group of Lie type of even characteristic except $\operatorname{PSL}_{2}\left(2^{e}\right)$, $\mathrm{PSL}_{3}\left(2^{e}\right), \mathrm{PSU}_{3}\left(2^{e}\right), \mathrm{PSp}_{4}\left(2^{e}\right), \mathrm{E}_{8}\left(2^{e}\right), \mathrm{F}_{4}\left(2^{e}\right),{ }^{2} \mathrm{~F}_{4}\left(2^{e}\right)^{\prime}, \mathrm{G}_{2}\left(2^{e}\right)$ and $\mathrm{Sz}\left(2^{e}\right)$.

Until recently, connected cubic nonnormal Cayley graphs on the groups listed in Theorem 1.1 were only found for $\mathrm{A}_{15}$ and $\mathrm{A}_{31}$ 9. In 2008, Feng, Lu and Xu asked the following question in their survey paper [4] on normality of Cayley graphs.

Question 1.2. ([4, Problem 5.9]) Are there infinitely many connected nonnormal Cayley graphs of valency 3 or 4 on nonabelian simple groups?

Question 1.2 in the valency 4 case has been answered by Wang and Feng [13] in the affirmative. In this paper, we answer the question in the remaining case. Our main result is Theorem 1.3, which gives a positive answer to Question 1.2.

Theorem 1.3. For each integer $m \geqslant 4$, there exists a graph $\Gamma_{m}$ satisfying:
(a) $\Gamma_{m}$ is a connected cubic nonnormal Cayley graph on $\mathrm{A}_{2^{m}-1}$;
(b) $\Gamma_{m} \cong \operatorname{Cay}\left(\mathrm{~A}_{2^{m}-1}, S\right)$ for some set $S$ of three involutions in $\mathrm{A}_{2^{m}-1}$ such that $\operatorname{Aut}\left(\mathrm{A}_{2^{m}-1}, S\right)=1$;
(c) $\operatorname{Aut}\left(\Gamma_{m}\right) \cong \mathrm{A}_{2^{m}}$.

We call a Cayley graph Cay $(G, S)$ a graphical regular representation ( $G R R$ for short) of $G$ if $\operatorname{Aut}(\operatorname{Cay}(G, S))=\widehat{G}$. Note that a GRR is necessarily a normal Cayley graph, and a necessary condition for $\operatorname{Cay}(G, S)$ to be a $\operatorname{GRR}$ is that $\operatorname{Aut}(G, S)=$ 1. In many circumstances it is shown that this condition is also sufficient, see for example [2, 5, 6]. More generally, a problem is posed in [2] to determine the groups $G$ such that a Cayley graph $\operatorname{Cay}(G, S)$ on $G$ is a GRR of $G$ if and only if $\operatorname{Aut}(G, S)=1$. We remark that our graph $\Gamma_{m}$ in Theorem 1.3 as a Cayley graph on $G:=\mathrm{A}_{2^{m}-1}$ is not only nonnormal (and hence not a GRR) but also satisfies the condition $\operatorname{Aut}(G, S)=1$. It is also worth remarking that, although the graph $\Gamma_{m}$ is not arc-transitive, it has local action $\mathrm{C}_{2}$ so that it corresponds to a tetravalent arc-transitive graph in the standard way described in [11, Section 4.1].

The paper is organized as follows. We shall first give the construction of $\Gamma_{m}$ for Theorem 1.3 in Section 2. Then the entirety of section 3 will be devoted to proving the connectivity of $\Gamma_{m}$. Finally in Section 4 we prove the remaining properties of $\Gamma_{m}$ described in Theorem 1.3, thus completing the proof of the theorem.

## 2. Construction of $\Gamma_{m}$

We first introduce some notation that is fixed throughout this paper. Let $m \geqslant 4$ be an integer,

$$
H=\left\langle a, b \mid a^{4}=b^{2}=(a b)^{2}=1\right\rangle \times\left\langle c_{1}\right\rangle \times\left\langle c_{2}\right\rangle \times \cdots \times\left\langle c_{m-3}\right\rangle,
$$

where $c_{1}, c_{2}, \ldots, c_{m-3}$ are involutions,

$$
K=\left\langle a^{2}, b, c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle=\left\langle a^{2}\right\rangle \times\langle b\rangle \times\left\langle c_{1}\right\rangle \times\left\langle c_{2}\right\rangle \times \cdots \times\left\langle c_{m-3}\right\rangle
$$

and $h=a \prod_{i=0}^{\lceil(m-5) / 2\rceil} c_{2 i+1}$. Clearly, $H$ is the direct product of a dihedral group $\mathrm{D}_{8}$ of order 8 and an elementary abelian 2-group of rank $m-3$, so that $|H|=2^{m}$. For the sake of convenience, put $c_{-1}=c_{0}=1$. Define $x \in \operatorname{Aut}(H)$ by letting

$$
a^{x}=a^{-1}, \quad b^{x}=a b, \quad c_{2 i+1}^{x}=c_{2 i+1} \quad \text { and } \quad c_{2 i+2}^{x}=a^{2} c_{2 i+1} c_{2 i+2}
$$

for $0 \leqslant i \leqslant\lfloor(m-5) / 2\rfloor$ and letting $c_{m-3}^{x}=a^{2} c_{m-3}$ in addition if $m$ is even. Define $\tau \in \operatorname{Aut}(K)$ by letting

$$
\left(a^{2}\right)^{\tau}=b, \quad b^{\tau}=a^{2}, \quad c_{2 i+1}^{\tau}=c_{2 i-1} c_{2 i} c_{2 i+2} \quad \text { and } \quad c_{2 i+2}^{\tau}=c_{2 i-1} c_{2 i} c_{2 i+1}
$$

for $0 \leqslant i \leqslant\lfloor(m-5) / 2\rfloor$ and letting $c_{m-3}^{\tau}=c_{m-3}$ in addition if $m$ is even. Note that $x$ and $\tau$ are indeed automorphisms of $H$ and $K$ because the images of generators under $x$ and $\tau$ satisfy the defining relations for $H$ and $K$, respectively. Denote the right regular representation of $H$ by $R$. Let $y$ be the permutation of $H$ such that $k^{y}=k^{\tau}$ and

$$
(h k)^{y}= \begin{cases}h k^{\tau} & \text { if } m \text { is odd } \\ h k^{\tau} c_{m-3} & \text { if } m \text { is even }\end{cases}
$$

for $k \in K$. Let

$$
z= \begin{cases}R(h) y R\left(h^{-1}\right) & \text { if } m \text { is odd } \\ R(h) y R\left(h^{-1} c_{m-3}\right) & \text { if } m \text { is even }\end{cases}
$$

We will see that the three permutations $x, y$ and $z$ of $H$ are all involutions in $\operatorname{Alt}(H)$.
Lemma 2.1. $x, y$ and $z$ are all involutions.
Proof. It is evident that none of $x, y$ and $z$ is trivial. Since $x^{2}$ fixes each of the generators $a, b, c_{1}, \ldots, c_{m-3}$ of $H$, we have $x^{2}=1$. Similarly, $\tau^{2}=1$. Let $g$ be an arbitrary element of $K$. Then $g^{y^{2}}=\left(g^{\tau}\right)^{y}=\left(g^{\tau}\right)^{\tau}=g^{\tau^{2}}=g$. If $m$ is odd, then $(h g)^{y^{2}}=\left(h g^{\tau}\right)^{y}=h\left(g^{\tau}\right)^{\tau}=h g^{\tau^{2}}=h g$ and so $y^{2}=1$, which in turn implies that $z^{2}=R(h) y^{2} R\left(h^{-1}\right)=1$. Now assume that $m$ is even. Then

$$
(h g)^{y^{2}}=\left(h g^{\tau} c_{m-3}\right)^{y}=h\left(g^{\tau} c_{m-3}\right)^{\tau} c_{m-3}=h g^{\tau^{2}} c_{m-3}^{\tau} c_{m-3}=h g^{\tau^{2}}=h g
$$

whence $y^{2}=1$. Moreover,

$$
\begin{aligned}
g^{z^{2}} & =g^{R(h) y R\left(c_{m-3}\right) y R\left(h^{-1} c_{m-3}\right)} \\
& =\left(h h^{-1} g h\right)^{y R\left(c_{m-3}\right) y R\left(h^{-1} c_{m-3}\right)} \\
& =\left(h\left(h^{-1} g h\right)^{\tau}\right)^{y R\left(h^{-1} c_{m-3}\right)} \\
& =h\left(\left(h^{-1} g h\right)^{\tau}\right)^{\tau} c_{m-3} h^{-1} c_{m-3} \\
& =h\left(h^{-1} g h\right) h^{-1} \\
& =g
\end{aligned}
$$

and

$$
\begin{aligned}
(h g)^{z^{2}} & =(h g)^{R(h) y R\left(c_{m-3}\right) y R\left(h^{-1} c_{m-3}\right)} \\
& =(h g h)^{y R\left(c_{m-3}\right) y R\left(h^{-1} c_{m-3}\right)} \\
& =\left((h g h)^{\tau}\right)^{R\left(c_{m-3}\right) y R\left(h^{-1} c_{m-3}\right)} \\
& =\left((h g h)^{\tau} c_{m-3}\right)^{y R\left(h^{-1} c_{m-3}\right)} \\
& =\left((h g h)^{\tau} c_{m-3}\right)^{\tau} h^{-1} c_{m-3} \\
& =h g h c_{m-3} h^{-1} c_{m-3} \\
& =h g .
\end{aligned}
$$

Thus $z^{2}=1$, completing the proof.
Lemma 2.2. $\operatorname{Aut}(H) \leqslant \operatorname{Alt}(H)$.
Proof. The conclusion for $m=4$ is easy to verify. Thus we assume $m \geqslant 5$ in the following. Since the center $\mathbf{Z}(H)=\left\langle a^{2}, c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle$ is a characteristic subgroup of $H$, each automorphism $\sigma \in \operatorname{Aut}(H)$ induces an automorphism of $H / \mathbf{Z}(H)=$ $\{\mathbf{Z}(H), a \mathbf{Z}(H), b \mathbf{Z}(H), a b \mathbf{Z}(H)\}$. More precisely, there is a homomorphism $\varphi$ from $\operatorname{Aut}(H)$ to $\operatorname{Aut}(H / \mathbf{Z}(H))$ such that $\varphi(\sigma)$ maps $g \mathbf{Z}(H)$ to $g^{\sigma} \mathbf{Z}(H)$ for all $\sigma \in \operatorname{Aut}(H)$ and $g \in H$.

Take an arbitrary $\sigma \in \operatorname{Aut}(H)$. Note that $a \mathbf{Z}(H)$ contains elements of order 4 while the elements in $b \mathbf{Z}(H)$ and $a b \mathbf{Z}(H)$ are all involutions. We see that if $\varphi(\sigma) \neq 1$ then $\varphi(\sigma)$ must fix the elements $\mathbf{Z}(H)$ and $a \mathbf{Z}(H)$ and swap $b \mathbf{Z}(H)$ and $a b \mathbf{Z}(H)$ in $H / \mathbf{Z}(H)$. Consequently, $\varphi(\sigma) \in\langle\varphi(x)\rangle$, and so $\sigma \in w\langle x\rangle$ for some $w \in \operatorname{ker}(\varphi)$. Since $w$ stabilizes $\mathbf{Z}(H), a \mathbf{Z}(H)$ and $b \mathbf{Z}(H)$, we have

$$
a^{w}=a^{2 \lambda+1} \prod_{j=1}^{m-3} c_{j}^{\lambda_{j}}, \quad b^{w}=a^{2 \mu} b \prod_{j=1}^{m-3} c_{j}^{\mu_{j}} \quad \text { and } \quad c_{i}^{w}=a^{2 k_{i}} \prod_{j=1}^{m-3} c_{j}^{k_{i, j}}
$$

for each $i$ with $1 \leqslant i \leqslant m-3$, where $\lambda, \mu, k_{i}, \lambda_{j}, \mu_{j}$ and $k_{i, j}$ are all in $\{0,1\}$. Let $w_{1}, w_{2}$ and $w_{3}$ be automorphisms of $H$ such that

$$
\begin{array}{llll}
a^{w_{1}} & =a^{2 \lambda+1}, & b^{w_{1}}=a^{2 \mu} b, & \\
c_{i}^{w_{1}}=a^{2 k_{i}} c_{i}, \\
a^{w_{2}}=a, & b^{w_{2}}=b, & c_{i}^{w_{2}}=\prod_{j=1}^{m-3} c_{j}^{k_{i, j}}, \\
a^{w_{3}}=a \prod_{j=1}^{m-3} c_{j}^{\lambda_{j}}, & b^{w_{3}}=b \prod_{j=1}^{m-3} c_{j}^{\mu_{j}}, & c_{i}^{w_{3}}=c_{i} .
\end{array}
$$

Then $w_{1}$ and $w_{3}$ are involutions, and $w=w_{1} w_{2} w_{3}$.
For each $\rho \in \operatorname{Aut}(H)$, the set of fixed points of $\rho$ is a subgroup of $H$ and thereby has size $2^{\ell}$ for some integer $\ell$ such that $0 \leqslant \ell \leqslant m$. Thus, each involution of $\operatorname{Aut}(H)$ with at least four fixed points lies in $\operatorname{Alt}(H)$ as it is a product of $\left(|H|-2^{\ell}\right) / 2=$ $2^{m-1}-2^{\ell-1}$ transpositions for some integer $\ell$ such that $2 \leqslant \ell \leqslant m-1$. Since $x$ and $w_{3}$ fix every point in $\left\langle a^{2}, c_{1}\right\rangle$ and $w_{1}$ fixes every point in $\left\langle a^{\lambda+1}, a^{\mu} b^{\lambda}\right\rangle$, we then conclude that $x, w_{3}, w_{1} \in \operatorname{Alt}(H)$. Moreover, since $(g v)^{w_{2}}=g v^{w_{2}}$ for all $g \in\langle a, b\rangle$ and $v \in\left\langle c_{1}, \ldots, c_{m-3}\right\rangle$, the number of transpositions of $w_{2}$ is divisible by $|\langle a, b\rangle|=8$. In particular, $w_{2} \in \operatorname{Alt}(H)$. Now $w_{1}, w_{2}, w_{3}$ and $x$ are all in $\operatorname{Alt}(H)$. It follows that $\sigma \in \operatorname{Alt}(H)$ due to $\sigma \in w\langle x\rangle=w_{1} w_{2} w_{3}\langle x\rangle$. This shows that $\operatorname{Aut}(H) \leqslant \operatorname{Alt}(H)$.

The next lemma says that $x$ and $y$ as well as the elements of $R(H)$ are all even permutations on $H$. Note that this also implies $z \in \operatorname{Alt}(H)$ since $z \in\langle y, R(H)\rangle$.

Lemma 2.3. $\langle x, y, R(H)\rangle \leqslant \operatorname{Alt}(H)$.
Proof. Lemma 2.2 already indicates $x \in \operatorname{Alt}(H)$. Let $\sigma$ be the map from $K$ to $h K$ sending $g$ to $h g$ for all $g \in K$, and $t$ be the permutation on $H$ such that $g^{t}=g$ and $(h g)^{t}=h g c_{m-3}^{m-1}$ for all $g \in K$. Then $t$ is the identity permutation if $m$ is odd, and is a product of $|K| / 2$ transpositions if $m$ is even. In particular, $t \in \operatorname{Alt}(H)$. From the definition of $y$ one sees that the following diagram commutes.


Hence $\left.(y t)\right|_{h K}$ has the same cycle structure as $\left.(y t)\right|_{K}$, and so $y t \in \operatorname{Alt}(H)$. This in turn gives $y \in \operatorname{Alt}(H)$. Finally, as $H$ is a 2 -group and not cyclic, we have $R(H) \leqslant \operatorname{Alt}(H)$. Consequently, $\langle x, y, R(H)\rangle \leqslant \operatorname{Alt}(H)$.

Recall the standard construction of the coset graph $\operatorname{Cos}(G, H, H S H)$ given a group $G$ with a subgroup $H$ and a subset $S$ such that $S \cap H=\emptyset$ and $H S H$ is inverseclosed. Such a graph has vertex set $[G: H]$, the set of right cosets of $H$ in $G$, and edge set $\{\{H g, H s g\} \mid g \in G, s \in H S H\}$. It is easy to see that $\operatorname{Cos}(G, H, H S H)$ has valency $|H S H| /|H|$, and $G$ acts by right multiplication on $[G: H]$ as a group of automorphisms of $\operatorname{Cos}(G, H, H S H)$. Moreover, $\operatorname{Cos}(G, H, H S H)$ is connected if and only if $\langle S, H\rangle=G$.

Now we are in the position to construct the graph $\Gamma_{m}$ for Theorem 1.3,
Construction 2.4. For each integer $m \geqslant 4$, let

$$
\Gamma_{m}=\operatorname{Cos}(\operatorname{Alt}(H), R(H), R(H)\{x, y\} R(H))
$$

with $H, x$ and $y$ defined at the beginning of this section.

## 3. Connectivity of $\Gamma_{m}$

The aim of this section it to prove that $\Gamma_{m}$ is connected. According to the construction of $\Gamma_{m}$, it suffices to prove $\langle x, y, R(H)\rangle=\operatorname{Alt}(H)$, and we will achieve this by dealing with the cases $m$ is odd and $m$ is even separately. For a group $G$, denote the set $G \backslash\{1\}$ by $G^{*}$. For a permutation $\sigma$ of a set $\Omega$ and $\alpha, \beta \in \Omega$, we write $\alpha \xrightarrow{\sigma} \beta$ if $\alpha^{\sigma}=\beta$.
3.1. Technical lemms. We first establish two technical lemmas that will be needed later in this section.
Lemma 3.1. Let $\ell \geqslant 2$ be an even integer, and $V=\left\langle e_{1}\right\rangle \times\left\langle e_{2}\right\rangle \times \cdots \times\left\langle e_{\ell}\right\rangle$ be a group with involutions $e_{1}, e_{2}, \ldots, e_{\ell}$. Denote the right regular representation of $V$ by $r$. Let $\omega=r\left(e_{\ell-1} e_{\ell}\right), e_{-1}=e_{0}=1$, and $\chi$ and $\psi$ be automorphisms of $V$ such that $e_{2 i+1}^{\chi}=e_{2 i+1}, e_{2 i+2}^{\chi}=e_{2 i+1} e_{2 i+2}, e_{2 i+1}^{\psi}=e_{2 i-1} e_{2 i} e_{2 i+2}$ and $e_{2 i+2}^{\psi}=e_{2 i-1} e_{2 i} e_{2 i+1}$ for each $i$ with $0 \leqslant i \leqslant(\ell-2) / 2$. Then $\langle\chi, \psi, \omega\rangle$ is a transitive subgroup of $\operatorname{Sym}(V)$.

Proof. Note that, viewing $V$ as a vector space over $\mathbb{F}_{2}$, the vectors $e_{1}^{\chi}, e_{2}^{\chi}, \ldots, e_{\ell}^{\chi}$ form a basis of $V$. Thus the automorphism $\chi$ of $V$ is well-defined. Similarly, $\psi$ is well-defined. Write $N=\langle\chi, \psi, \omega\rangle$. Since $\chi$ is an automorphism of $V$, we have

$$
r\left(e_{\ell}\right)=r\left(\left(e_{\ell-1} e_{\ell}\right)^{\chi}\right)=\chi^{-1} r\left(e_{\ell-1} e_{\ell}\right) \chi=\chi^{-1} \omega \chi \in N
$$

and so $r\left(e_{\ell-1}\right)=r\left(e_{\ell-1} e_{\ell}\right) r\left(e_{\ell}\right)=\omega r\left(e_{\ell}\right) \in N$. Suppose there exists a nonnegative integer $i \leqslant(\ell-2) / 2$ such that $r\left(e_{\ell-2 i+1}\right), r\left(e_{\ell-2 i+2}\right), \ldots, r\left(e_{\ell-1}\right), r\left(e_{\ell}\right)$ are all in $N$. Note that

$$
\begin{aligned}
r\left(e_{\ell-2 i-1} e_{\ell-2 i}\right) & =r\left(e_{\ell-2 i-1} e_{\ell-2 i} e_{\ell-2 i+1}\right) r\left(e_{\ell-2 i+1}\right) \\
& =r\left(e_{\ell-2 i+2}^{\psi}\right) r\left(e_{\ell-2 i+1}\right)=\psi^{-1} r\left(e_{\ell-2 i+2}\right) \psi r\left(e_{\ell-2 i+1}\right)
\end{aligned}
$$

since $\psi$ is an automorphism of $V$. We thereby deduce that $r\left(e_{\ell-2 i-1} e_{\ell-2 i}\right) \in N$. It follows that $r\left(e_{\ell-2 i}\right)=r\left(\left(e_{\ell-2 i-1} e_{\ell-2 i}\right)^{\chi}\right)=\chi^{-1} r\left(e_{\ell-2 i-1} e_{\ell-2 i}\right) \chi \in N$ and thus $r\left(e_{\ell-2 i-1}\right)=r\left(e_{\ell-2 i-1} e_{\ell-2 i}\right) r\left(e_{\ell-2 i}\right) \in N$. Then by induction one concludes that $r\left(e_{1}\right), r\left(e_{2}\right), \ldots, r\left(e_{\ell-1}\right), r\left(e_{\ell}\right)$ are all in $N$. Consequently, $r(V) \leqslant N$ and so $N$ is transitive on $V$.

The following lemma is a consequence of the classification of doubly transitive permutation groups (see for example [1]).

Lemma 3.2. Suppose that $G$ is a doubly transitive permutation group on $2^{m}$ points. Then one of the following holds:
(i) $G \leqslant \mathrm{AGL}_{m}(2)$;
(ii) $2^{m}-1=q$ for some prime power $q$ and $\operatorname{PSL}_{2}(q) \leqslant G \leqslant \mathrm{P}_{2}(q)$;
(iii) $\mathrm{A}_{2^{m}} \leqslant G \leqslant \mathrm{~S}_{2^{m}}$.

Remark. In fact, the prime power $q$ in case (ii) of Lemma 3.2 is necessarily a prime by Mihǎilescu's theorem [10]. In particular, $m$ must be odd in case (ii) of Lemma 3.2.
3.2. Odd $m$. Throughout this subsection, let $m$ be odd, and

$$
U=\left\{\prod_{i=1}^{m-3} c_{i}^{k_{i}} \mid \sum_{j=1}^{(m-3) / 2} k_{2 j} \equiv 0 \quad(\bmod 2)\right\}
$$

Note that $\left\{U, U c_{m-3}\right\}$ forms a partition of $\left\langle c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle$, and $x$ stabilizes $U$ setwise. For each $u \in U$ we have

$$
\begin{gather*}
u \xrightarrow{x} u^{x} \xrightarrow[\rightarrow]{y} u^{x y} \xrightarrow{z} u^{x} \xrightarrow{x} u \xrightarrow{y} u^{y} \xrightarrow{z} u,  \tag{1}\\
a u \xrightarrow{x} a^{-1} u^{x} \xrightarrow{y} a b u^{x y} c_{m-4} c_{m-3} \xrightarrow{z} a u^{x} \xrightarrow{x} a^{-1} u \xrightarrow{y} a b u^{y} c_{m-4} c_{m-3} \xrightarrow{z} a u,  \tag{2}\\
u c_{m-3} \xrightarrow{x} a^{2} u^{x} c_{m-4} c_{m-3} \xrightarrow{y} b u^{x y} c_{m-4} c_{m-3} \xrightarrow{z} b u^{x} c_{m-4} c_{m-3}  \tag{3}\\
b u^{x} c_{m-4} c_{m-3} \xrightarrow{x} a^{-1} b u c_{m-3} \xrightarrow{y} a^{-1} b u^{y} c_{m-6} c_{m-5} c_{m-3} \xrightarrow{z} a^{-1} b u c_{m-3} \\
a^{-1} b u c_{m-3} \xrightarrow{x} b u^{x} c_{m-4} c_{m-3} \xrightarrow{y} a^{2} u^{x y} c_{m-4} c_{m-3} \xrightarrow{z} a^{2} b u^{x} c_{m-4} c_{m-3} \\
a^{2} b u^{x} c_{m-4} c_{m-3} \xrightarrow{x} a b u c_{m-3} \xrightarrow{y} a^{-1} u^{y} c_{m-6} c_{m-5} c_{m-4} \xrightarrow{z} a^{-1} u c_{m-3} \\
a^{-1} u c_{m-3} \xrightarrow{x} a^{-1} u^{x} c_{m-4} c_{m-3} \xrightarrow{y} a b u^{x y} \xrightarrow{z} a u^{x} c_{m-4} c_{m-3} \\
a u^{x} c_{m-4} c_{m-3} \xrightarrow{x} a u c_{m-3} \xrightarrow{y} a u^{y} c_{m-6} c_{m-5} c_{m-3} \xrightarrow{z} a b u c_{m-3} \\
a b u c_{m-3} \xrightarrow{x} a^{2} b u^{x} c_{m-4} c_{m-3} \xrightarrow{y} a^{2} b u^{x y} c_{m-4} c_{m-3} \xrightarrow{z} a^{2} u^{x} c_{m-4} c_{m-3} \\
a^{2} u^{x} c_{m-4} c_{m-3} \xrightarrow{x} u c_{m-3} \xrightarrow{y} u^{y} c_{m-6} c_{m-5} c_{m-4} \xrightarrow{z} u c_{m-3},
\end{gather*}
$$

and

$$
\begin{align*}
& a^{2} u \xrightarrow{x} a^{2} u^{x} \xrightarrow{y} b u^{x y} \xrightarrow{z} b u^{x} \xrightarrow{x} a b u \xrightarrow{y} a^{-1} u^{y} c_{m-4} c_{m-3} \xrightarrow{z} a^{-1} u  \tag{4}\\
& a^{-1} u \xrightarrow{x} a u^{x} \xrightarrow{y} a u^{x y} c_{m-4} c_{m-3} \xrightarrow{z} a b u^{x} \xrightarrow{x} b u \xrightarrow{y} a^{2} u^{y} \xrightarrow{z} a^{2} b u \\
& a^{2} b u \xrightarrow{x} a^{-1} b u^{x} \xrightarrow{y} a^{-1} b u^{x y} c_{m-4} c_{m-3} \xrightarrow{z} a^{-1} b u^{x} \xrightarrow{x} a^{2} b u \xrightarrow{y} a^{2} b u^{y} \xrightarrow{z} a^{2} u .
\end{align*}
$$

Lemma 3.3. Suppose $m$ is odd. Then the permutation $(x y z)^{8}$ of $H$ has cycle decomposition

$$
(x y z)^{8}=\left(\prod_{u \in U}\left(a^{2} u, a^{-1} u, a^{2} b u\right)\right)\left(\prod_{u \in U}\left(b u, a b u, a^{-1} b u\right)\right)
$$

Proof. Denote $H_{i, j, k}=\left\{a^{i} b^{j} u c_{m-3}^{k} \mid u \in U\right\}$, where $i \in\{-1,0,1,2\}$ and $j, k \in\{0,1\}$. Then $\left\{H_{i, j, k} \mid-1 \leqslant i \leqslant 2,0 \leqslant j \leqslant 1,0 \leqslant k \leqslant 1\right\}$ forms a partition of $H$.

By (11) and (2), $u^{(x y z)^{2}}=u$ and $(a u)^{(x y z)^{2}}=a u$ for all $u \in U$. Hence $(x y z)^{8}$ fixes $H_{0,0,0} \cup H_{1,0,0}$ pointwise. From (3) one sees that $(x y z)^{8}$ fixes $u c_{m-3}, b u^{x} c_{m-4} c_{m-3}$, $a^{-1} b u c_{m-3}, a^{2} b u^{x} c_{m-4} c_{m-3}, a^{-1} u c_{m-3}, a u^{x} c_{m-4} c_{m-3}, a b u c_{m-3}$ and $a^{2} u^{x} c_{m-4} c_{m-3}$ for all $u \in U$. Noting $\left\{u^{x} c_{m-4} c_{m-3} \mid u \in U\right\}=U c_{m-3}$, we conclude that $(x y z)^{8}$ fixes

$$
H_{0,0,1} \cup H_{0,1,1} \cup H_{-1,1,1} \cup H_{2,1,1} \cup H_{-1,0,1} \cup H_{1,0,1} \cup H_{1,1,1} \cup H_{2,0,1}
$$

pointwise. This together with the conclusion that $(x y z)^{8}$ fixes $H_{0,0,0} \cup H_{1,0,0}$ pointwise shows it suffices to prove that $(x y z)^{8}$ induces the permutation

$$
\left(\prod_{u \in U}\left(a^{2} u, a^{-1} u, a^{2} b u\right)\right)\left(\prod_{u \in U}\left(b u, a b u, a^{-1} b u\right)\right)
$$

on $H_{2,0,0} \cup H_{-1,0,0} \cup H_{2,1,0} \cup H_{0,1,0} \cup H_{1,1,0} \cup H_{-1,1,0}$. Indeed, (4) implies

$$
a^{2} u \xrightarrow{(x y z)^{8}} a^{-1} u \xrightarrow{(x y z)^{8}} a^{2} b u \xrightarrow{(x y z)^{8}} a^{2} u
$$

and

$$
b u^{x} \xrightarrow{(x y z)^{8}} a b u^{x} \xrightarrow{(x y z)^{8}} a^{-1} b u^{x} \xrightarrow{(x y z)^{8}} b u^{x}
$$

for all $u \in U$. This yields the desired conclusion since $\left.x\right|_{U} \in \operatorname{Aut}(U)$.
Before proving the next lemma, we note that for each $u \in U$,

$$
\begin{equation*}
a^{-1} b u \xrightarrow{x y z x} a^{2} u \xrightarrow{x y z x} a b u \xrightarrow{y z} a^{-1} u \xrightarrow{x y z x} b u \xrightarrow{y z} a^{2} b u \xrightarrow{z y x z y x y z} a u \tag{5}
\end{equation*}
$$

and

$$
\begin{align*}
& a^{-1} u c_{m-3} \xrightarrow{y z} a u c_{m-3} \xrightarrow{x y z x} a^{2} b u c_{m-3} \xrightarrow{y z} a^{2} u c_{m-3} \xrightarrow{y z} b u c_{m-3}  \tag{6}\\
& a u c_{m-3} \xrightarrow{y z} a b u c_{m-3} \xrightarrow{x y z x} u c_{m-3} \xrightarrow{x y z x} a^{-1} b u c_{m-3} .
\end{align*}
$$

Lemma 3.4. Suppose $m$ is odd. Then for each $g \in H \backslash U$, there exists $\zeta \in\langle x, y, z\rangle$ such that $g^{\zeta}=c_{m-3}$.

Proof. Let $v \in\left\langle c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle$ such that $g \in\langle a, b\rangle v$, and define $\chi, \psi$ and $\omega$ as in Lemma 3.1] with $\ell=m-3$ and $e_{i}=c_{i}$. Then $\chi, \psi$ and $\psi \omega$ are all involutions in $\operatorname{Sym}\left(\left\langle c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle\right)$, and by Lemma 3.1, there exist $\eta_{1}, \eta_{2}, \ldots, \eta_{t} \in\{\chi, \psi, \psi \omega\}$ such that $v^{\eta_{1} \eta_{2} \cdots \eta_{t}}=c_{m-3}$. Let $\eta_{0}=1 \in \operatorname{Sym}\left(\left\langle c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle\right)$ and $\zeta_{0}=1 \in$
$\langle x, y, z\rangle$. Obviously, $g^{\zeta_{0}} \in\langle a, b\rangle v^{\eta_{0}} \backslash U$. We shall prove by induction that there exist $\zeta_{0}, \zeta_{1}, \ldots, \zeta_{t} \in\langle x, y, z\rangle$ with

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{t}} \in\langle a, b\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{t}} \backslash U .
$$

By (5), for each $u \in U$ and $\alpha, \beta \in\langle a, b\rangle^{*}$, there exists $\varepsilon \in\langle x, y z\rangle$ such that $(\alpha u)^{\varepsilon}=$ $\beta u$. By (6), for each $u \in U$ and $\alpha, \beta \in\langle a, b\rangle$, there exists $\varepsilon \in\langle x, y z\rangle$ such that $\left(\alpha u c_{m-3}\right)^{\varepsilon}=\beta u c_{m-3}$.

Suppose that there exist $\zeta_{0}, \zeta_{1}, \ldots, \zeta_{i-1} \in\langle x, y, z\rangle$ with $1 \leqslant i \leqslant t$ and

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}} \in\langle a, b\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}} \backslash U .
$$

If $\eta_{i}=\chi$, then let $\zeta_{i}=x$. If $\eta_{i}=\psi$, then there exists $\varepsilon_{i} \in\langle x, y z\rangle$ such that $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}}\right)^{\varepsilon_{i}}=a^{2} v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}}$ and we let $\zeta_{i}=\varepsilon_{i} y$. If $\eta_{i}=\psi \omega$, then there exists $\varepsilon_{i} \in\langle x, y z\rangle$ such that $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}}\right)^{\varepsilon_{i}}=a v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}}$ and we let $\zeta_{i}=\varepsilon_{i} y$. It follows that

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1} \zeta_{i}}=\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}}\right)^{\zeta_{i}} \in\left(\langle a, b\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}} \backslash U\right)^{\zeta_{i}} \subseteq\langle a, b\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i}} \backslash U .
$$

By induction we now have $\zeta_{0}, \zeta_{1}, \ldots, \zeta_{t} \in\langle x, y, z\rangle$ such that

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{t}} \in\langle a, b\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{t}} \backslash U=\langle a, b\rangle c_{m-3} .
$$

Then as there exists $\varepsilon \in\langle x, y z\rangle$ such that $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{t}}\right)^{\varepsilon}=c_{m-3}$, we have $g^{\zeta}=c_{m-3}$ with $\zeta:=\zeta_{0} \zeta_{1} \ldots \zeta_{t} \varepsilon \in\langle x, y, z\rangle$.
Lemma 3.5. Suppose that $m$ is odd. Then $\langle x, y, z\rangle$ is transitive on $H^{*}$.
Proof. In view of Lemma 3.4, we only need to prove that for each $u \in U^{*}$, there exists $\varepsilon \in\langle x, y\rangle$ such that $u^{\varepsilon} \in U c_{m-3}$. Write $u=c_{1}^{k_{1}} c_{2}^{k_{2}} \cdots c_{m-3}^{k_{m-3}}$ with $k_{1}, k_{2}, \ldots, k_{m-3} \in$ $\{0,1\}$. Denote $V_{i}=\left\langle c_{i}, c_{i+1}, \ldots, c_{m-3}\right\rangle$ for $1 \leqslant i \leqslant m-3$, and set $V_{m-2}=1$. Let $s$ be the smallest integer in $\{0,1, \ldots,(m-5) / 2\}$ such that $k_{2 s+1}+k_{2 s+2}>0$. Taking

$$
\varepsilon=x^{k_{2 s+1}+k_{2 s+2}-1} y(x y)^{s},
$$

we prove below that $u^{\varepsilon} \in U c_{m-3}$ by induction on $s$.
First suppose $s=0$. If $k_{1}=0$ and $k_{2}=1$, then $u=c_{2} u_{1}$ with $u_{1} \in V_{3}$ and $u_{1} \in U c_{m-3}$ since $u \in U$. In this case, $u_{1}^{y} \in U c_{m-3}$, and it follows that $u^{\varepsilon}=\left(c_{2} u_{1}\right)^{y}=c_{1} u_{1}^{y} \in U c_{m-3}$. If $k_{1}=1$ and $k_{2}=0$, then $u=c_{1} u_{1}$ with $u_{1} \in V_{3}$ and $u_{1} \in U$ as $u \in U$. In this case, $u_{1}^{y} \in U$ and so $u^{\varepsilon}=\left(c_{1} u_{1}\right)^{y}=c_{2} u_{1}^{y} \in U c_{m-3}$. If $k_{1}=k_{2}=1$, then $u=c_{1} c_{2} u_{1}$ with $u_{1} \in V_{3}$ and $u_{1} \in U c_{m-3}$ since $u \in U$. In this case, $u_{1}^{x}=a^{2} u_{2}$ for some $u_{2} \in V_{3} \cap U c_{m-3}$, whence $u^{\varepsilon}=\left(c_{1} c_{2} u_{1}\right)^{x y}=\left(a^{2} c_{2} u_{1}^{x}\right)^{y}=$ $\left(c_{2} u_{2}\right)^{y}=c_{1} u_{2}^{y} \in U c_{m-3}$ as $u_{2}^{y} \in U c_{m-3}$.

Next suppose $s>0$. If $k_{2 s+1}=0$ and $k_{2 s+2}=1$, then $u=c_{2 s+2} u_{1}$ with $u_{1} \in V_{2 s+3}$, which implies $u^{y}=\left(c_{2 s+2} u_{1}\right)^{y}=c_{2 s-1} c_{2 s} c_{2 s+1} u_{1}^{y} \in c_{2 s-1} c_{2 s} V_{2 s+1}$. If $k_{2 s+1}=1$ and $k_{2 s+2}=0$, then $u=c_{2 s+1} u_{1}$ with $u_{1} \in V_{2 s+3}$ and therefore $u^{y}=\left(c_{2 s+1} u_{1}\right)^{y}=$ $c_{2 s-1} c_{2 s} c_{2 s+2} u_{1}^{y} \in c_{2 s-1} c_{2 s} V_{2 s+1}$. If $k_{2 s+1}=k_{2 s+2}=1$, then $u=c_{2 s+1} c_{2 s+2} u_{1}$ with $u_{1} \in V_{2 s+3}$ and $u_{1} \in U c_{m-3}$ since $u \in U$. In this case, $u_{1}^{x}=a^{2} u_{2}$ for some $u_{2} \in V_{2 s+3}$, and so $u^{x y}=\left(c_{2 s+1} c_{2 s+2} u_{1}\right)^{x y}=\left(a^{2} c_{2 s+2} u_{1}^{x}\right)^{y}=\left(c_{2 s+2} u_{2}\right)^{y}=c_{2 s-1} c_{2 s} c_{2 s+1} u_{2}^{y} \in$ $c_{2 s-1} c_{2 s} V_{2 s+1}$. To sum up, we always have $u^{\varepsilon_{0}} \in c_{2 s-1} c_{2 s} V_{2 s+1}$, where

$$
\varepsilon_{0}=x^{k_{2 s+1}+k_{2 s+2}-1} y
$$

By the inductive hypothesis,

$$
\left(u^{\varepsilon_{0}}\right)^{(x y)^{s}}=\left(u^{\varepsilon_{0}}\right)^{x^{1+1-1} y(x y)^{s-1}} \in U c_{m-3} .
$$

Consequently,

$$
u^{x^{k_{2 s+1}+k_{2 s+2-1}} y(x y)^{s}}=\left(u^{\varepsilon_{0}}\right)^{(x y)^{s}} \in U c_{m-3}
$$

completing the proof.
Lemma 3.6. Suppose that $m$ is odd. Then $\langle x, y, R(H)\rangle=\operatorname{Alt}(H)$.
Proof. Let $G=\langle x, y, R(H)\rangle$. Notice that $x, y$ and $z$ are all involutions of $G$ fixing 1. By Lemma 3.5, $\langle x, y, z\rangle$ is transitive on $H^{*}$, and so is $G_{1}$, the stabilizer of 1 in $G$. This together with the transitivity of $R(H)$ on $H$ implies that $G$ is doubly transitive on $H$. Therefore, one of cases (i)-(iii) in Lemma 3.2 holds.

Assume that $G \leqslant \mathrm{AGL}_{m}(2)$ as in case (i) of Lemma 3.2. Then $(x y z)^{8} \in G_{1} \leqslant$ $\mathrm{GL}_{m}(2)$ and hence the set of fixed points of $(x y z)^{8}$ is a vector space over $\mathbb{F}_{2}$. However, Lemma 3.3 shows that the number of fixed points of $(x y z)^{8}$ is $|H|-3|U|-3|U|=$ $5 \cdot 2^{m-3}$, a contradiction.

Assume that $\mathrm{PSL}_{2}(q) \leqslant G \leqslant \mathrm{P}_{2}(q)$ as in case (ii) of Lemma 3.2, where $q=$ $2^{m}-1$ is a prime. Then $R(H) \leqslant G \leqslant \mathrm{P}_{2}(q)=\mathrm{PGL}_{2}(q)$. It follows that $R(H)$ is contained in $\mathrm{D}_{2(q+1)}$, the Sylow 2-subgroup of $\mathrm{PGL}_{2}(q)$. This is impossible since $R(H) \cong \mathrm{D}_{8} \times \mathrm{C}_{2}^{m-3}$.

Now $\operatorname{Alt}(H) \leqslant G \leqslant \operatorname{Sym}(H)$. This in conjunction with Lemma 2.3 forces $G=$ $\operatorname{Alt}(H)$, which completes the proof.
3.3. Even $m$. Throughout this subsection, let $m$ be even, $h_{1}=h c_{m-3}, H_{1}=$ $\left\langle a, b, c_{1}, c_{2}, \ldots, c_{m-4}\right\rangle, K_{1}=\left\langle a^{2}, b, c_{1}, c_{2}, \ldots, c_{m-4}\right\rangle$ and

$$
U=\left\{\prod_{i=1}^{m-3} c_{i}^{k_{i}} \mid \sum_{j=1}^{(m-4) / 2} k_{2 j} \equiv k_{m-3} \quad(\bmod 2)\right\}
$$

Define permutations $x_{1}, y_{1}$ and $z_{1}$ on $H_{1}$ such that $x_{1}=\left.x\right|_{H_{1}},\left.y_{1}\right|_{K_{1}}=\left.y\right|_{K_{1}},\left.y_{1}\right|_{h_{1} K_{1}}=$ $\left.\left(y R\left(c_{m-3}\right)\right)\right|_{h_{1} K_{1}},\left.z_{1}\right|_{K_{1}}=\left.z\right|_{K_{1}}$ and $\left.z_{1}\right|_{h_{1} K_{1}}=\left.\left(z R\left(c_{m-3}\right)\right)\right|_{h_{1} K_{1}}$. One can verify readily that $\left(h_{1} k_{1}\right)^{y_{1}}=h_{1} k_{1}^{y_{1}}$ for all $k_{1} \in K_{1}, z_{1}=\left.\left(R\left(h_{1}\right) y_{1} R\left(h_{1}^{-1}\right)\right)\right|_{H_{1}}$ and $y_{1} z_{1}=\left.(y z)\right|_{H_{1}}$. For each $u \in U \cap H_{1}$ we have

$$
\begin{align*}
& u c_{m-3} \xrightarrow{x y z} b u^{x} c_{m-3} \xrightarrow{x y z} a^{-1} b u c_{m-3} \xrightarrow{x y z} a^{2} b u^{x} c_{m-3} \xrightarrow{x y z} a^{-1} u c_{m-3}  \tag{7}\\
& a^{-1} u c_{m-3} \xrightarrow{x y z} a u^{x} c_{m-3} \xrightarrow{x y z} a b u c_{m-3} \xrightarrow{x y z} a^{2} u^{x} c_{m-3} \xrightarrow{x y z} u c_{m-3},
\end{align*}
$$

$$
\begin{gather*}
u c_{m-4} c_{m-3} \xrightarrow{x y z} u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{x y z} u c_{m-4} c_{m-3},  \tag{8}\\
a u c_{m-4} c_{m-3} \xrightarrow{x y z} a u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{x y z} a u c_{m-4} c_{m-3} \tag{9}
\end{gather*}
$$

and

$$
\begin{align*}
& a^{2} u c_{m-4} c_{m-3} \xrightarrow{x y z} b u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{x y z} a^{-1} u c_{m-4} c_{m-3}  \tag{10}\\
& a^{-1} u c_{m-4} c_{m-3} \xrightarrow{x y z} a b u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{x y z} a^{2} b u c_{m-4} c_{m-3} \\
& a^{2} b u c_{m-4} c_{m-3} \xrightarrow{x y z} a^{-1} b u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{x y z} a^{2} u c_{m-4} c_{m-3} .
\end{align*}
$$

Lemma 3.7. Suppose that $m$ is even. Then the permutation $\left.(x y z)^{8}\right|_{H_{1} c_{m-3}}$ has cycle decomposition

$$
\begin{aligned}
\left.(x y z)^{8}\right|_{H_{1} c_{m-3}} & =\left(\prod_{u \in U \cap H_{1}}\left(a^{2} u c_{m-4} c_{m-3}, a^{-1} u c_{m-4} c_{m-3}, a^{2} b u c_{m-4} c_{m-3}\right)\right) \\
& \times\left(\prod_{u \in U \cap H_{1}}\left(b u c_{m-4} c_{m-3}, a b u c_{m-4} c_{m-3}, a^{-1} b u c_{m-4} c_{m-3}\right)\right) .
\end{aligned}
$$

Proof. Denote $H_{i, j, k}=\left\{a^{i} b^{j} u c_{m-4}^{k} c_{m-3} \mid u \in U \cap H_{1}\right\}$, where $i \in\{-1,0,1,2\}$ and $j, k \in\{0,1\}$. Then $\left\{H_{i, j, k} \mid-1 \leqslant i \leqslant 2,0 \leqslant j \leqslant 1,0 \leqslant k \leqslant 1\right\}$ forms a partition of $H_{1} c_{m-3}$.

By (8) and (9), $(x y z)^{2}$ fixes $H_{0,0,1} \cup H_{1,0,1}$ pointwise and so does $(x y z)^{8}$. From (7) one sees that $(x y z)^{8}$ fixes $u c_{m-3}, b u^{x} c_{m-3}, a^{-1} b u c_{m-3}, a^{2} b u^{x} c_{m-3}, a^{-1} u c_{m-3}, a u^{x} c_{m-3}$, $a b u c_{m-3}$ and $a^{2} u^{x} c_{m-3}$ for all $u \in U \cap H_{1}$. As $\left\{u^{x} \mid u \in U \cap H_{1}\right\}=U \cap H_{1}$, we then conclude that $(x y z)^{8}$ fixes

$$
H_{0,0,0} \cup H_{0,1,0} \cup H_{-1,1,0} \cup H_{2,1,0} \cup H_{-1,0,0} \cup H_{1,0,0} \cup H_{1,1,0} \cup H_{2,0,0}
$$

pointwise. This together with the conclusion that $(x y z)^{8}$ fixes $H_{0,0,1} \cup H_{1,0,1}$ pointwise shows it suffices to prove that $(x y z)^{8}$ induces the permutation

$$
\prod_{u \in U \cap H_{1}}\left(a^{2} u c_{m-4} c_{m-3}, a^{-1} u c_{m-4} c_{m-3}, a^{2} b u c_{m-4} c_{m-3}\right)
$$

on $H_{2,0,1} \cup H_{-1,0,1} \cup H_{2,1,1}$ and the permutation

$$
\prod_{u \in U \cap H_{1}}\left(b u c_{m-4} c_{m-3}, a b u c_{m-4} c_{m-3}, a^{-1} b u c_{m-4} c_{m-3}\right)
$$

on $H_{0,1,1} \cup H_{1,1,1} \cup H_{-1,1,1}$. Indeed, (10) implies

$$
a^{2} u c_{m-4} c_{m-3} \xrightarrow{(x y z)^{8}} a^{-1} u c_{m-4} c_{m-3} \xrightarrow{(x y z)^{8}} a^{2} b u c_{m-4} c_{m-3} \xrightarrow{(x y z)^{8}} a^{2} u c_{m-4} c_{m-3}
$$

and

$$
\begin{aligned}
& b u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{(x y z)^{8}} a b u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{(x y z)^{8}} a^{-1} b u^{x} c_{m-5} c_{m-4} c_{m-3} \\
& a^{-1} b u^{x} c_{m-5} c_{m-4} c_{m-3} \xrightarrow{(x y z)^{8}} b u^{x} c_{m-5} c_{m-4} c_{m-3}
\end{aligned}
$$

for all $u \in U \cap H_{1}$. This yields the desired conclusion since the maps $u \mapsto u^{x}$ and $u \mapsto u^{x} c_{m-5}$ are both bijections from $U \cap H_{1}$ onto itself.

Lemma 3.8. Suppose that $m$ is even. For each $v \in U$ and $\alpha, \beta \in\langle a, b\rangle^{*}$, there exists $\varepsilon \in\langle x, y z\rangle$ such that $(\alpha v)^{\varepsilon}=\beta v$. For each $v \in\left\langle c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle \backslash U$ and $\alpha, \beta \in\langle a, b\rangle$, there exists $\varepsilon \in\langle x, y z\rangle$ such that $(\alpha v)^{\varepsilon}=\beta v$.

Proof. First recall from (5) that for each $u \in U \cap H_{1}$ and $\alpha, \beta \in\langle a, b\rangle^{*}$, there exist integers $k_{1}, k_{2}, \ldots, k_{2 r-1}, k_{2 r}$ with

$$
(\alpha u)^{\prod_{i=1}^{r} x_{1}^{k_{2 i-1}}\left(y_{1} z_{1}\right)^{k_{2 i}}}=\beta u .
$$

Then taking $\varepsilon=\prod_{i=1}^{r} x^{k_{2 i-1}}(y z)^{k_{2 i}}$, we have $(\alpha u)^{\varepsilon}=\beta u$ since $\left.x\right|_{H_{1}}=x_{1}$ and $\left.(y z)\right|_{H_{1}}=y_{1} z_{1}$. Next note that for $u \in U \cap H_{1}$,

$$
\begin{aligned}
& b u c_{m-4} c_{m-3} \xrightarrow{y z} a^{2} b u c_{m-4} c_{m-3} \xrightarrow{y z} a^{2} u c_{m-4} c_{m-3} \xrightarrow{(x y z)^{2}} a^{-1} u c_{m-4} c_{m-3} \\
& a u c_{m-4} c_{m-3} \xrightarrow{z y} a^{-1} u c_{m-4} c_{m-3} \xrightarrow{z y} a b u c_{m-4} c_{m-3} \xrightarrow{(x y z)^{2}} a^{-1} b u c_{m-4} c_{m-3} .
\end{aligned}
$$

We then conclude that for each $v \in U$ and $\alpha, \beta \in\langle a, b\rangle^{*}$, there exists $\varepsilon \in\langle x, y z\rangle$ such that $(\alpha v)^{\varepsilon}=\beta v$.

Similarly, one derives from (6) that for each $u \in U \cap H_{1}$ and $\alpha, \beta \in\langle a, b\rangle$, there exists $\varepsilon \in\langle x, y z\rangle$ with $\left(\alpha u c_{m-4}\right)^{\varepsilon}=\beta u c_{m-4}$. Moreover,

$$
\begin{aligned}
& a^{-1} u c_{m-3} \xrightarrow{y z} a u c_{m-3} \xrightarrow{x y z x} a^{2} b u c_{m-3} \xrightarrow{y z} a^{2} u c_{m-3} \xrightarrow{y z} b u c_{m-3} \\
& a u c_{m-3} \xrightarrow{y z} a b u c_{m-3} \xrightarrow{x y z x} u c_{m-3} \xrightarrow{x y z x} a^{-1} b u c_{m-3}
\end{aligned}
$$

for all $u \in U \cap H_{1}$. Hence for each $v \in\left\langle c_{1}, c_{2}, \ldots, c_{m-3}\right\rangle \backslash U$ and $\alpha, \beta \in\langle a, b\rangle$, there exists $\varepsilon \in\langle x, y z\rangle$ such that $(\alpha v)^{\varepsilon}=\beta v$.

Lemma 3.9. Suppose that $m$ is even. Then for each $g \in H \backslash U$, there exists $\zeta \in\langle x, y, z\rangle$ such that $g^{\zeta}=h$.

Proof. Let $v \in\left\langle c_{1}, c_{2}, \ldots, c_{m-4}\right\rangle$ such that $g \in\left\langle a, b, c_{m-3}\right\rangle v$, and define $\chi, \psi$ and $\omega$ as in Lemma 3.1 with $\ell=m-4$ and $e_{i}=c_{i}$. Then $\chi, \psi$ and $\psi \omega$ are all involutions in $\operatorname{Sym}\left(\left\langle c_{1}, c_{2}, \ldots, c_{m-4}\right\rangle\right)$, and by Lemma 3.1, there exist $\eta_{1}, \eta_{2}, \ldots, \eta_{t} \in\{\chi, \psi, \psi \omega\}$ such that $v^{\eta_{1} \eta_{2} \cdots \eta_{t}}=a^{-1} h_{1}$. Let $\eta_{0}=1 \in \operatorname{Sym}\left(\left\langle c_{1}, c_{2}, \ldots, c_{m-4}\right\rangle\right)$ and $\zeta_{0}=1 \in$ $\langle x, y, z\rangle$. Obviously, $g^{\zeta_{0}} \in\left\langle a, b, c_{m-3}\right\rangle v^{\eta_{0}} \backslash U$. We shall prove by induction that there exist $\zeta_{0}, \zeta_{1}, \ldots, \zeta_{t} \in\langle x, y, z\rangle$ with

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{t}} \in\langle a, b\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{t}} \backslash U
$$

Suppose there exist $\zeta_{0}, \zeta_{1}, \ldots, \zeta_{i-1} \in\langle x, y, z\rangle$ for some $i \in\{1, \ldots, t\}$ such that

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}} \in\left\langle a, b, c_{m-3}\right\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}} \backslash U
$$

If $\eta_{i}=\chi$, then let $\zeta_{i}=x$. If $\eta_{i}=\psi$, then by Lemma 3.8 there exists $\varepsilon_{i} \in\langle x, y z\rangle$ such that $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}}\right)^{\varepsilon_{i}} \in a^{2}\left\langle c_{m-3}\right\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}}$ and we let $\zeta_{i}=\varepsilon_{i} y$. If $\eta_{i}=\psi \omega$, then by Lemma 3.8 there exists $\varepsilon_{i} \in\langle x, y z\rangle$ such that $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}}\right)^{\varepsilon_{i}} \in a\left\langle c_{m-3}\right\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}}$ and we let $\zeta_{i}=\varepsilon_{i} y$. It follows that

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1} \zeta_{i}}=\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{i-1}}\right)^{\zeta_{i}} \in\left(\left\langle a, b, c_{m-3}\right\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i-1}} \backslash U\right)^{\zeta_{i}} \subseteq\left\langle a, b, c_{m-3}\right\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{i}} \backslash U .
$$

By induction we now have $\zeta_{0}, \zeta_{1}, \ldots, \zeta_{t} \in\langle x, y, z\rangle$ such that

$$
g^{\zeta_{0} \zeta_{1} \ldots \zeta_{t}} \in\left\langle a, b, c_{m-3}\right\rangle v^{\eta_{0} \eta_{1} \ldots \eta_{t}} \backslash U=\left\langle a, b, c_{m-3}\right\rangle a^{-1} h_{1} \backslash U .
$$

Then by Lemma 3.8 there exists $\varepsilon \in\langle x, y z\rangle$ such that $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{t}}\right)^{\varepsilon} \in h_{1}\left\langle c_{m-3}\right\rangle$. Let $\zeta=\zeta_{0} \zeta_{1} \ldots \zeta_{r} \varepsilon y$ if $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{r}}\right)^{\varepsilon}=h_{1}$ and let $\zeta=\zeta_{0} \zeta_{1} \ldots \zeta_{r} \varepsilon$ if $\left(g^{\zeta_{0} \zeta_{1} \ldots \zeta_{r}}\right)^{\varepsilon}=h_{1} c_{m-3}$. Then $\zeta \in\langle x, y, z\rangle$, and in view of $h=h_{1} c_{m-3}=h_{1}^{y}$ we have $g^{\zeta}=h$.

Lemma 3.10. Suppose that $m$ is even. Then $\langle x, y, z\rangle$ is transitive on $H^{*}$.
Proof. Due to Lemma 3.8, we only need to prove that for each $u \in U^{*}$, there exists $\varepsilon \in\langle x, y\rangle$ such that $u^{\varepsilon} \in U c_{m-3}$. Write $u=c_{1}^{k_{1}} c_{2}^{k_{2}} \cdots c_{m-3}^{k_{m-3}}$ with $k_{1}, k_{2}, \ldots, k_{m-3} \in$ $\{0,1\}$. Denote $V_{i}=\left\langle c_{i}, c_{i+1}, \ldots, c_{m-3}\right\rangle$ for $1 \leqslant i \leqslant m-3$. Since $u \neq c_{m-3}$, there
exists $0 \leqslant j \leqslant(m-6) / 2$ such that $k_{2 j+1}+k_{2 j+2}>0$. Let $s$ be the smallest integer in $\{0,1, \ldots,(m-6) / 2\}$ such that $k_{2 s+1}+k_{2 s+2}>0$. Taking

$$
\varepsilon=x^{k_{2 s+1}+k_{2 s+2}-1} y(x y)^{s},
$$

we prove below that $u^{\varepsilon} \in U c_{m-3}$ by induction on $s$.
First suppose $s=0$. If $k_{1}=0$ and $k_{2}=1$, then $u=c_{2} u_{1}$ with $u_{1} \in V_{3}$ and $u_{1} \in U c_{m-3}$ since $u \in U$. In this case, $u_{1}^{y} \in U c_{m-3}$, and it follows that $u^{\varepsilon}=\left(c_{2} u_{1}\right)^{y}=c_{1} u_{1}^{y} \in U c_{m-3}$. If $k_{1}=1$ and $k_{2}=0$, then $u=c_{1} u_{1}$ with $u_{1} \in V_{3}$ and $u_{1} \in U$ as $u \in U$. In this case, $u_{1}^{y} \in U$ and so $u^{\varepsilon}=\left(c_{1} u_{1}\right)^{y}=c_{2} u_{1}^{y} \in U c_{m-3}$. If $k_{1}=k_{2}=1$, then $u=c_{1} c_{2} u_{1}$ with $u_{1} \in V_{3}$ and $u_{1} \in U c_{m-3}$ since $u \in U$. In this case, $u_{1}^{x}=a^{2} u_{2}$ for some $u_{2} \in V_{3} \cap U c_{m-3}$, whence $u^{\varepsilon}=\left(c_{1} c_{2} u_{1}\right)^{x y}=\left(a^{2} c_{2} u_{1}^{x}\right)^{y}=$ $\left(c_{2} u_{2}\right)^{y}=c_{1} u_{2}^{y} \in U c_{m-3}$ as $u_{2}^{y} \in U c_{m-3}$.

Next suppose $s>0$. If $k_{2 s+1}=0$ and $k_{2 s+2}=1$, then $u=c_{2 s+2} u_{1}$ with $u_{1} \in V_{2 s+3}$, which implies $u^{y}=\left(c_{2 s+2} u_{1}\right)^{y}=c_{2 s-1} c_{2 s} c_{2 s+1} u_{1}^{y} \in c_{2 s-1} c_{2 s} V_{2 s+1}$. If $k_{2 s+1}=1$ and $k_{2 s+2}=0$, then $u=c_{2 s+1} u_{1}$ with $u_{1} \in V_{2 s+3}$ and therefore $u^{y}=\left(c_{2 s+1} u_{1}\right)^{y}=$ $c_{2 s-1} c_{2 s} c_{2 s+2} u_{1}^{y} \in c_{2 s-1} c_{2 s} V_{2 s+1}$. If $k_{2 s+1}=k_{2 s+2}=1$, then $u=c_{2 s+1} c_{2 s+2} u_{1}$ with $u_{1} \in V_{2 s+3}$ and $u_{1} \in U c_{m-3}$ since $u \in U$. In this case, $u_{1}^{x}=a^{2} u_{2}$ for some $u_{2} \in V_{2 s+3}$, and so $u^{x y}=\left(c_{2 s+1} c_{2 s+2} u_{1}\right)^{x y}=\left(a^{2} c_{2 s+2} u_{1}^{x}\right)^{y}=\left(c_{2 s+2} u_{2}\right)^{y}=c_{2 s-1} c_{2 s} c_{2 s+1} u_{2}^{y} \in$ $c_{2 s-1} c_{2 s} V_{2 s+1}$. To sum up, we always have $u^{\varepsilon_{0}} \in c_{2 s-1} c_{2 s} V_{2 s+1}$, where

$$
\varepsilon_{0}=x^{k_{2 s+1}+k_{2 s+2}-1} y
$$

By the inductive hypothesis,

$$
\left(u^{\varepsilon_{0}}\right)^{(x y)^{s}}=\left(u^{\varepsilon_{0}}\right)^{x^{1+1-1} y(x y)^{s-1}} \in U c_{m-3} .
$$

Consequently,

$$
u^{x^{k_{2 s+1}+k_{2 s+2-1}} y(x y)^{s}}=\left(u^{\varepsilon_{0}}\right)^{(x y)^{s}} \in U c_{m-3}
$$

completing the proof.
Lemma 3.11. Suppose that $m$ is even. Then $\langle x, y, R(H)\rangle=\operatorname{Alt}(H)$.
Proof. Let $G=\langle x, y, R(H)\rangle$. Notice that $x, y$ and $z$ are all involutions of $G$ fixing 1. By Lemma 3.10, $\langle x, y, z\rangle$ is transitive on $H^{*}$, and so is $G_{1}$, the stabilizer of 1 in $G$. This together with the transitivity of $R(H)$ on $H$ implies that $G$ is doubly transitive on $H$. Therefore, either cases (i) or case (iii) in Lemma 3.2 holds since $m$ is even.

Assume that $G \leqslant \mathrm{AGL}_{m}(2)$ as in case (i) of Lemma 3.2. Then $(x y z)^{8} \in G_{1} \leqslant$ $\mathrm{GL}_{m}(2)$ and hence the set of fixed points of $(x y z)^{8}$ is a vector space over $\mathbb{F}_{2}$. Since $\left.x\right|_{H_{1}}=x_{1}$ and $\left.(y z)\right|_{H_{1}}=y_{1} z_{1}$, we have $\left.(x y z)^{8}\right|_{H_{1}}=\left(x_{1} y_{1} z_{1}\right)^{8}$, and thus Lemma 3.3 shows that the number of fixed points of $\left.(x y z)^{8}\right|_{H_{1}}$ is $\left|H_{1}\right|-3\left|U \cap H_{1}\right|-3\left|U \cap H_{1}\right|=$ $5 \cdot 2^{m-4}$. Furthermore, the number of fixed points of $\left.(x y z)^{8}\right|_{H_{1} c_{m-3}}$ is $\left|H_{1} c_{m-3}\right|-$ $3\left|U \cap H_{1}\right|-3\left|U \cap H_{1}\right|=5 \cdot 2^{m-4}$. Hence the number of fixed points of $(x y z)^{8}$ is $5 \cdot 2^{m-4}+5 \cdot 2^{m-4}=5 \cdot 2^{m-3}$, a contradiction.

Now $\operatorname{Alt}(H) \leqslant G \leqslant \operatorname{Sym}(H)$. This in conjunction with Lemma 2.3 forces $G=$ $\operatorname{Alt}(H)$, which completes the proof.

## 4. Proof of Theorem 1.3

Theorem 1.3 will follow directly from Lemmas 4.1 4.4.

Lemma 4.1. $\Gamma_{m}$ is a connected cubic graph.
Proof. By Lemmas 3.6 and 3.11 we have for each $m \geqslant 4$ that

$$
\begin{equation*}
\langle x, y, R(H)\rangle=\operatorname{Alt}(H) \tag{11}
\end{equation*}
$$

which already implies the connectivity of $\Gamma_{m}$. To prove that $\Gamma_{m}$ is cubic, we show $|R(H)\{x, y\} R(H)|=3|R(H)|$ in the following.

It is straightforward to verify that $x$ and $y$ are involutions normalizing $R(H)$ and $R(K)$, respectively. As a consequence, $R(H) x R(H)=R(H) x$ and $y R(H) y \cap R(H) \geqslant$ $y R(K) y \cap R(K)=R(K)$. Notice that $y R(H) y \cap R(H) \neq R(H)$ for otherwise

$$
\langle x, y, R(H)\rangle \leqslant \mathbf{N}_{\mathrm{Alt}(H)}(R(H))<\operatorname{Alt}(H)
$$

contrary to (11). We derive that $y R(H) y \cap R(H)=R(K)$ as $R(K)$ has index 2 in $R(H)$. Accordingly, $|R(H) y R(H)| /|R(H)|=|R(H)| /|y R(H) y \cap R(H)|=2$. Moreover, $R(H) x R(H) \cap R(H) y R(H)=\emptyset$ for otherwise $y \in\langle x, R(H)\rangle$, which would cause a contradiction $\langle x, y, R(H)\rangle \leqslant\langle x, R(H)\rangle \leqslant \mathbf{N}_{\operatorname{Alt}(H)}(R(H))<\operatorname{Alt}(H)$ to (11). Hence

$$
\begin{aligned}
|R(H)\{x, y\} R(H)| & =|R(H) x R(H)|+|R(H) y R(H)| \\
& =|R(H) x|+2|R(H)|=3|R(H)|,
\end{aligned}
$$

as desired.
Lemma 4.2. $\operatorname{Cay}\left(\operatorname{Alt}\left(H^{*}\right),\{x, y, z\}\right)$ is a nonnormal Cayley graph of $\operatorname{Alt}\left(H^{*}\right)$ and is isomorphic to $\Gamma_{m}$ by the map $g \mapsto R(H) g$.
Proof. Let $S=\{x, y, z\}$. Consider the map $\varphi: g \mapsto R(H) g$ from $\operatorname{Alt}\left(H^{*}\right)$ to the vertex set of $\Gamma_{m}$. We see that $\varphi$ is injective as $R(H) \cap \operatorname{Alt}\left(H^{*}\right)=1$, and is therefore bijective as $\left|\operatorname{Alt}\left(H^{*}\right)\right|=|\operatorname{Alt}(H)| /|R(H)|$. In particular, $R(H) S$ is a disjoint union of $R(H) x, R(H) y$ and $R(H) z$. Then since

$$
R(H) S=R(H) x \cup R(H) y \cup R(H) z \subseteq R(H)\{x, y\} R(H)
$$

and $|R(H)\{x, y\} R(H)|=3|R(H)|$ by Lemma 4.1, we conclude that

$$
R(H) S=R(H)\{x, y\} R(H)
$$

For $g_{1}$ and $g_{2}$ in $\operatorname{Alt}\left(H^{*}\right), g_{1}$ is adjacent to $g_{2}$ in $\operatorname{Cay}\left(\operatorname{Alt}\left(H^{*}\right), S\right)$ if and only if $g_{2} g_{1}^{-1} \in\{x, y, z\}$, which is equivalent to $R(H) g_{2} g_{1}^{-1} \in\{R(H) x, R(H) y, R(H) z\}$. This means that $g_{1}$ and $g_{2}$ is adjacent in $\operatorname{Cay}\left(\operatorname{Alt}\left(H^{*}\right), S\right)$ if and only if

$$
R(H) g_{2} g_{1}^{-1} \subseteq R(H) S=R(H)\{x, y\} R(H)
$$

or equivalently, $R(H) g_{1}$ is adjacent to $R(H) g_{2}$ in $\Gamma_{m}$. Therefore, $\varphi$ is a graph isomorphism from $\operatorname{Cay}\left(\operatorname{Alt}\left(H^{*}\right), S\right)$ to $\Gamma_{m}$. Moreover, $\operatorname{Alt}(H)$ acts as a group of automorphisms of $\Gamma_{m}$ by right multiplication and $\operatorname{Alt}\left(H^{*}\right)$ is not normal in $\operatorname{Alt}(H)$, whence $\Gamma_{m}$ is a nonnormal Cayley graph of $\operatorname{Alt}\left(H^{*}\right)$.
Lemma 4.3. $\operatorname{Aut}\left(\operatorname{Alt}\left(H^{*}\right),\{x, y, z\}\right)=1$.
Proof. Suppose for a contradiction that there exists $1 \neq \sigma \in \operatorname{Sym}\left(H^{*}\right)$ with

$$
\left\{\sigma^{-1} x \sigma, \sigma^{-1} y \sigma, \sigma^{-1} z \sigma\right\}=\{x, y, z\}
$$

Then the conjugation action of $\sigma$ induces a nontrivial permutation of $\{x, y, z\}$ as $\langle x, y, z\rangle=\operatorname{Alt}\left(H^{*}\right)$. By Lemma 4.2 and [15, Theorem 1.1] we know that $\Gamma_{m}$ is not
arc-transitive. Note that $R(a)$ interchanges the vertices $R(H) y$ and $R(H) z$ of $\Gamma_{m}$. In view of the isomorphism $g \mapsto R(H) g$ in Lemma 4.2, we have $\sigma^{-1} x \sigma=x, \sigma^{-1} y \sigma=z$ and $\sigma^{-1} z \sigma=y$. For $w \in \operatorname{Sym}\left(H^{*}\right)$, denote by $\operatorname{Fix}(w)$ the set of fixed points of $w$ on $H^{*}$. It follows that

$$
\begin{equation*}
|\operatorname{Fix}(y) \cap \operatorname{Fix}(x y x)|=\left|\operatorname{Fix}(y)^{\sigma} \cap \operatorname{Fix}(x y x)^{\sigma}\right|=|\operatorname{Fix}(z) \cap \operatorname{Fix}(x z x)| \tag{12}
\end{equation*}
$$

and

$$
\begin{align*}
|\operatorname{Fix}(y x y) \cap \operatorname{Fix}(x y x y x)| & =\left|\operatorname{Fix}(y x y)^{\sigma} \cap \operatorname{Fix}(x y x y x)^{\sigma}\right|  \tag{13}\\
& =|\operatorname{Fix}(z x z) \cap \operatorname{Fix}(x z x z x)| .
\end{align*}
$$

First assume that $m \equiv 1(\bmod 4)$. Let

$$
M=\left\langle a^{2} b\right\rangle \times\left\langle c_{1} c_{2}\right\rangle \times \cdots \times\left\langle c_{2 i-1} c_{2 i}\right\rangle \times \cdots \times\left\langle c_{m-4} c_{m-3}\right\rangle
$$

It is easy to verify that $\operatorname{Fix}(y)=\{1, h\} M \backslash\{1\}$ and

$$
M^{x}=\left\langle a^{-1} b\right\rangle \times\left\langle a^{2} c_{2}\right\rangle \times \cdots \times\left\langle a^{2} c_{2 i}\right\rangle \times \cdots \times\left\langle a^{2} c_{m-3}\right\rangle .
$$

This implies that

$$
\begin{aligned}
\operatorname{Fix}(y) \cap \operatorname{Fix}(x y x) & =\operatorname{Fix}(y) \cap \operatorname{Fix}(y)^{x} \\
& =(\{1, h\} M \backslash\{1\}) \cap\left(\left\{1, h^{x}\right\} M^{x} \backslash\{1\}\right) \\
& =\left(\{1, h\} M \cap\left\{1, a^{2} h\right\} M^{x}\right) \backslash\{1\} \\
& =\emptyset .
\end{aligned}
$$

However, since the element $b \prod_{i=1}^{m-3} c_{i}$ of $H^{*}$ is fixed by both $z$ and $x z x$, we have $|\operatorname{Fix}(z) \cap \operatorname{Fix}(x z x)|>0$, contrary to (12)).

Next assume that $m \equiv 3(\bmod 4)$. Let

$$
M=\langle b\rangle \times\left\langle c_{1} c_{2}\right\rangle \times \cdots \times\left\langle c_{2 i-1} c_{2 i}\right\rangle \times \cdots \times\left\langle c_{m-4} c_{m-3}\right\rangle
$$

It is easy to verify that $\operatorname{Fix}(z)=\left\{1, a^{2} h\right\} M \backslash\{1\}$ and

$$
M^{x}=\langle a b\rangle \times\left\langle a^{2} c_{2}\right\rangle \times \cdots \times\left\langle a^{2} c_{2 i}\right\rangle \times \cdots \times\left\langle a^{2} c_{m-3}\right\rangle .
$$

Hence $\operatorname{Fix}(z) \cap \operatorname{Fix}(x z x)=\operatorname{Fix}(z) \cap \operatorname{Fix}(z)^{x}=\emptyset$. However, $a^{2} b \prod_{i=1}^{m-3} c_{i}$ is fixed by both $y$ and $x y x$, so $|\operatorname{Fix}(y) \cap \operatorname{Fix}(x y x)|>0$. This again contradicts (12).

Now assume that $m \equiv 2(\bmod 4)$. Let

$$
M=\left\langle c_{1}\right\rangle \times \cdots \times\left\langle c_{2 i-1}\right\rangle \times \cdots \times\left\langle c_{m-5}\right\rangle \times\left\langle a c_{m-3}\right\rangle .
$$

It is easy to verify that $\operatorname{Fix}(x)=M \backslash\{1\}$ and thence

$$
\operatorname{Fix}(y x y) \cap \operatorname{Fix}(x y x y x)=\operatorname{Fix}(x)^{y} \cap \operatorname{Fix}(x)^{y x}=\emptyset
$$

However, $a^{2} b\left(\prod_{i=1}^{(m-4) / 2} c_{2 i}\right)\left(\prod_{i=0}^{(m-6) / 4} c_{4 i-1}\right)$ is fixed by both $z x z$ and $x z x z x$. Thus,

$$
|\operatorname{Fix}(z x z) \cap \operatorname{Fix}(x z x z x)|>0=|\operatorname{Fix}(y x y) \cap \operatorname{Fix}(x y x y x)|,
$$

contrary to (13).
Finally assume that $m \equiv 0(\bmod 4)$. Then in the same vein as above we have $\operatorname{Fix}(z x z) \cap \operatorname{Fix}(x z x z x)=\emptyset$ while the element $b\left(\prod_{i=0}^{(m-4) / 2} c_{2 i}\right)\left(\prod_{i=0}^{(m-4) / 4} c_{4 i-1}\right)$ of $H^{*}$ is fixed by both $y x y$ and $x y x y x$. This causes

$$
|\operatorname{Fix}(y x y) \cap \operatorname{Fix}(x y x y x)|>0=|\operatorname{Fix}(z x z) \cap \operatorname{Fix}(x z x z x)|,
$$

contradicting (13).

In the following lemma we prove that the full automorphism of $\Gamma_{m}$ is isomorphic to $\mathrm{A}_{2^{m}}$. Some of the arguments here were used in the proof of [17, Theorem 1.2].

Lemma 4.4. $\operatorname{Aut}\left(\Gamma_{m}\right) \cong \mathrm{A}_{2^{m}}$.
Proof. Let $A=\operatorname{Aut}\left(\Gamma_{m}\right)$ and $v$ be a vertex of $\Gamma_{m}$. Then by Lemma 4.2, $A$ has a nonnormal vertex-regular subgroup $G$ which is isomorphic to the alternating group $\mathrm{A}_{2^{m}-1}$. Further, $\mathbf{N}_{A}(G)=G$ by Lemma 4.3. Note also that $\Gamma_{m}$ is connected and cubic as Lemma 4.1] asserts. We derive from [15, Theorem 1.1] that $A$ is not transitive on the arc set of $\Gamma_{m}$, and so $A_{v}$ is a 2-group. Consequently, $|A| /|G|=\left|G A_{v}\right| /|G|=$ $\left|A_{v}\right| /\left|G \cap A_{v}\right|=\left|A_{v}\right|$ is a power of 2 . Since every nontrivial $G$-conjugacy class has size greater than 3, it follows from [3, Theorem 1.1] that one of the following two cases occurs:
(i) $\operatorname{Soc}(A)$ is a nonabelian simple group containing $G$ as a proper subgroup;
(ii) $A$ has a nontrivial normal subgroup $N$ such that $N$ is not transitive on the vertex set of $\Gamma_{m}$ and $\operatorname{Soc}(A / N)$ is a nonabelian simple group containing $G N / N \cong G$.
First assume that case (i) occurs. Then as $|\operatorname{Soc}(A)| /|G|$ is a power of 2 , we have $\operatorname{Soc}(A)=\mathrm{A}_{2^{m}}$ by [7, Theorem 1], and so $A \cong \mathrm{~A}_{2^{m}}$ or $\mathrm{S}_{2^{m}}$. If $A \cong \mathrm{~S}_{2^{m}}$, then $\mathbf{N}_{A}(G) \cong \mathrm{S}_{2^{m}-1}$, contrary to the conclusion that $\mathbf{N}_{A}(G)=G$. Therefore, $A \cong \mathrm{~A}_{2^{m}}$.

Next assume that case (ii) occurs. In this case, $N \cap G=1$ as $G N / N \cong G$. Hence $|N|=|N| /|N \cap G|=|N G| /|G|$ divides $|A| /|G|$. In particular, $N$ is a 2 group. From the construction of $\Gamma_{m}$ we know that $A$ has a subgroup $B$ that is isomorphic to $\mathrm{A}_{2^{m}}$ and contains $G$. Consider the action $\phi$ of $B$ on $N$ by conjugation. Since $B$ is a simple group, either $\operatorname{ker}(\phi)=1$ or $\operatorname{ker}(\phi)=B$. If $\operatorname{ker}(\phi)=B$, then $B$ centralizes $N$ and so $N \leqslant \mathbf{N}_{A}(G)=G$, contradicting the condition that $N \cap G=1$. Hence we have $\operatorname{ker}(\phi)=1$. Then $B \cong \mathrm{~A}_{2^{m}}$ is isomorphic to an irreducible subgroup of $\operatorname{Aut}(N / \Phi(N)) \cong \mathrm{PSL}_{d}(2)$ for some positive integer $d$ with $2^{d} \leqslant|N|$, where $\Phi(N)$ is the Frattini subgroup of $N$. It follows that $d \geqslant 2^{m}-2$ according to [8, Proposition 5.3.7]. Hence $\nu_{2}(|N|) \geqslant 2^{m}-2$, where $\nu_{2}$ is the 2-adic valuation. Moreover, $N$ must be semiregular on the vertex set of $\Gamma_{m}$, for otherwise the quotient graph of $\Gamma_{m}$ with respect to $N$ would have valency 2 and so could not admit $A / N$ as a group of automorphisms. Accordingly,

$$
\nu_{2}(|N|) \leqslant \nu_{2}\left(\left|\mathrm{~A}_{2^{m}-1}\right|\right)=\sum_{i=1}^{\infty}\left\lfloor\frac{2^{m}-1}{2^{i}}\right\rfloor-1<\sum_{i=1}^{\infty} \frac{2^{m}-1}{2^{i}}-1=2^{m}-2 .
$$

This contradicts the conclusion $\nu_{2}(|N|) \geqslant 2^{m}-2$, not possible.
Acknowledgements. This research was supported by the National Natural Science Foundation of China (Grant No. 11501011 and 11671030). The authors are very grateful to Prof. Marston Conder for helpful discussion and the anonymous referees for their comments to improve the paper.

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