# Minimum supports of eigenfunctions of Johnson graphs* 

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#### Abstract

We study the weights of eigenvectors of the Johnson graphs $J(n, w)$. For any $i \in\{1, \ldots, w\}$ and sufficiently large $n, n \geq n(i, w)$ we show that an eigenvector of $J(n, w)$ with the eigenvalue $\lambda_{i}=(n-w-i)(w-i)-i$ has at least $2^{i}\binom{n-2 i}{w-i}$ nonzeros and obtain a characterization of eigenvectors that attain the bound.


## 1 Introduction

Let $G=(V, E)$ be an undirected graph. A real-valued nonzero function $f$ : $V \rightarrow R$ is called a $\lambda$-eigenfunction of $G$ if the following equality holds for any $x \in V$ :

$$
\lambda f(x)=\sum_{y \in V:(x, y) \in E} f(y) .
$$

In other words, $f$ is a $\lambda$-eigenfunction of $G$ if its vector of values $\bar{f}$ is an eigenvector of the adjacency matrix $A_{G}$ of $G$ with eigenvalue $\lambda$ or $\bar{f}$ is the all-zero vector, i.e. the following holds:

$$
A_{G} \bar{f}=\lambda \bar{f} .
$$

The vertices of the Hamming graph $H(n)$ are the binary vectors of length $n$, where two vectors are adjacent if they differ in exactly one coordinate position. Given a pair of vectors $x$ and $y$ of length $n$ the Hamming distance $d(x, y)$ between a pair of vectors $x$ and $y$ of length $n$ is the Hamming graph distance between

[^0]$x$ and $y$, i.e. the number of positions at which the corresponding symbols are different. The support of a real-valued function (or vector) $f$ is denoted by $\operatorname{supp}(f)$ is the set of nonzeros of $f$. The weight $w t(x)$ of a vector $x$ is the number of nonzero symbols of $x$.

The vertices of the Johnson graph $J(n, w)$ are the binary vectors of length $n$ with $w$ ones, where two vectors are adjacent if they have exactly $w-1$ common ones. Note that the vertices of $J(n, w)$ are vertices of $H(n)$ of weight $w$, with the Johnson graph distance being equal halfed Hamming graph distance.

Various combinatorial objects with extreme characteristics could be defined in terms of eigenfunctions with certain restrictions. In particular, several important notions, such as $(w-1)-(n, w, 1)$-designs (including Steiner triple and quadruple systems), equitable 2 -cell partitions and perfect codes could be defined as eigenfunctions of Johnson graphs [7, 3]. 14]. The symmetric difference of a pair of such objects (for example, Steiner triple systems) is a bitrade [12]. In case of the Johnson graphs, bitrades of small size play an important role in the classification and characterization problems (for example, see [2, [12, [16]) and proved to be a useful constructive tool for Steiner triple and quadruple systems [2], 1]. Moreover, the topic of the current paper is related to the question of existence of 1-perfect codes in different graphs which is one of the most captive problems in combinatorial coding theory. For $n \leq 2^{250}$ it is known that no such codes exist in the Johnson graphs $J(n, w)$, see 9 . The study of bitrades of 1-perfect codes may lead to an improvement of this problem. In this light, the question of finding the size of minimum support of eigenfunctions of Johnson graphs for arbitrary fixed eigenvalues is tempting and intriguing.

For surveys on combinatorial objects connected with eigenfunctions, their bitrades and general theory the reader is referred to the works of Krotov et. al [12, [11, Cho [4, 5] and the book of Colbourn and Dinitz [6].

In the current paper the minimum support question for eigenfunctions of the Johnson graphs $J(n, w)$ with the eigenvalue $(w-i)(n-w-i)-i$ for any $i, w$ and $n, n \geq n(i, w)$ is solved and a characterization of minimum support functions is obtained. The solution for the problem in case of the minimum eigenvalue is $2^{w}$ [10] and the value is attained on a class of so-called Steiner bitrades [10], 12] that include Pasch-configuration.

## 2 Preliminaries

### 2.1 Induced eigenfunctions and eigenvalues of Johnson graphs

Let $f$ be a real-valued function defined on the vertices of the Johnson graph $J(n, i)$. Define the function $I^{i, w}(f)$ on the vertices of $J(n, w)$ as follows:

$$
I^{i, w}(f)(x)=\sum_{y, w t(y)=i, d(x, y)=|w-i|} f(y)
$$

The function $I^{i, w}(f)$ is called induced in $J(n, w)$ by $f$ [3]. The idea of using
induced functions for representation of the Johnson scheme has been exploited since the beginning of its study [8]. In [3] the concept was generalized to a wider class of graphs.

Theorem 1. [3]

1. Let $f$ be $a \lambda$-eigenfunction of $J(n, i)$. Then if $i \leq w$ then $I^{i, w}(f)$ is a $(\lambda+(w-i)(n-i-w))$-eigenfunction of $J(n, w)$.
2. Let $f$ be a real-valued function on the vertices of $J(n, w)$. Then $I^{w, w-1}(f) \equiv 0$ iff $f$ is a $(-w)$-eigenfunction.

Proof. The sketch of the proof is done by induction on $w$. The second and the first statements of the theorem for $i=w-1$ were proven in [3], see Theorem 1. In general case it is easy to see that for any $f$ we have: $(w-i)!I^{i, w}(f)=$ $I^{w-1, w}\left(\ldots\left(I^{i+1, i+2}\left(I^{i, i+1}(f)\right)\right)\right)$, which finishes the proof.

Let $M$ and $M^{\prime}$ be two nonintersecting sets of size $i$ of coordinate positions whose elements are in a one-to-one correspondence ${ }^{\prime}$. For a subset $I$ of $M$ by $I^{\prime}$ denote the set of its images $\left\{m^{\prime}: m \in I\right\} \subseteq M^{\prime}$. Let the function $f^{i, w, n}$ be defined on the vectors of weight $w$ and length $n$ :

$$
\begin{gathered}
f^{i, w, n}(x)=(-1)^{|M \cap \operatorname{supp}(x)|}, \text { if }\left|\operatorname{supp}(x) \cap\left(M \cup M^{\prime}\right)\right|=i \text { and } \\
(\operatorname{supp}(x) \cap M)^{\prime} \cup\left(\operatorname{supp}(x) \cap M^{\prime}\right)=M^{\prime}
\end{gathered}
$$

and $f^{i, w, n}(x)=0$ otherwise. The main result of the current paper is that $f^{i, w, n}$ is the minimum support eigenfunction of the Johnson graphs $J(n, w)$ asymptotically on $n$.

Proposition 1. The function $f^{i, w, n}$ is a $((w-i)(n-w-i)-i)$-eigenfunction of $J(n, w)$ with the support of size $2^{i}\binom{n-2 i}{w-i}$.

Proof. The proof relies on Theorem 1 We show that $f^{i, i, n}$ is a $(-i)$ eigenfunction of the Johnson graph $J(n, i)$ and that $f^{i, w, n}=I^{i, w}\left(f^{i, i, n}\right)$, where the functions $f^{i, w, n}$ and $f^{i, i, n}$ are obtained from the same pair of sets $M$ and $M^{\prime}$ of sizes $i$.

Let $x$ be a binary vector of length $n$ and weight $i-1$. Consider the values of $I^{i, i-1}\left(f^{i, i, n}\right)$ that could be expressed as follows:

$$
I^{i, i-1}\left(f^{i, i, n}\right)(x)=\sum_{y: w t(y)=i, \operatorname{supp}(x) \subset \operatorname{supp}(y)} f^{i, i, n}(y) .
$$

We have several cases. If $\left|(M \cap \operatorname{supp}(x))^{\prime} \cup(M \cap \operatorname{supp}(x))\right|=i-1$ then there are just two elements $m \in M$ and $m^{\prime} \in M^{\prime}$ neither of which belongs to $\operatorname{supp}(x)$. Therefore $I^{i, i-1}\left(f^{i, i, n}\right)(x)$ is zero, since exactly two summands, $f^{i, i, n}(y)$ for $y$ such that $\operatorname{supp}(y)=\operatorname{supp}(x) \cup\{m\}$ or $\operatorname{supp}(y)=\operatorname{supp}(x) \cup\left\{m^{\prime}\right\}$ are equal to -1 and 1 and the other summands are zeros. In the remaining cases every summand $f^{i, i, n}(y)$ is zero, so $I^{i, i-1}\left(f^{i, i, n}\right)$ is the all-zero function and by Theorem 1 the function $f^{i, i, n}$ is $(-i)$-eigenfunction of $J(n, i)$.

We proceed with analogous considerations with the function $I^{i, w}\left(f^{i, i, n}\right)$ :

$$
I^{i, w}\left(f^{i, i, n}\right)(x)=\sum_{y: w t(y)=i, \operatorname{supp}(y) \subseteq \operatorname{supp}(x)} f^{i, i, n}(y) .
$$

Again, we have several cases. If $\operatorname{supp}(x) \cap\left(M \cup M^{\prime}\right)=\operatorname{supp}(y)$ for some $y$ such that $f^{i, i, n}(y) \neq 0$, then the remaining elements of the sum are zeros and we have $I^{i, w}\left(f^{i, i, n}\right)(x)=f^{i, w, n}(x)$. If there is no $y$ such that $f^{i, i, n}(y) \neq 0$ and $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$ then by definition of $f^{i, i, n}$ we have that $I^{i, w}\left(f^{i, i, n}\right)(x)=0=$ $f^{i, w, n}(x)$. The remaining case where there are several $y$ 's such that $\operatorname{supp}(y) \varsubsetneqq$ $\operatorname{supp}(x) \cap\left(M \cup M^{\prime}\right)$ implies that there are the same number of -1 's and 1 's in the sum by definition of $f^{i, i, n}$. Therefore in this case we have that $I^{i, w}\left(f^{i, i, n}\right)(x)=$ $f^{i, w, n}(x)=0$.

Finally, by Theorem 1 we see that $f^{i, w, n}=I^{i, w}\left(f^{i, i, n}\right)$ is a $((w-i)(n-w-$ $i)-i$-eigenfunction of $J(n, i)$ with the support of size $2^{i}\binom{n-2 i}{w-i}$.

Theorem 2. [8] The eigenvalues of $J(n, w)$ are numbers $\lambda_{i}(n, w)=(w-i)(n-$ $w-i)-i, i \in\{0, \ldots, w\}$ with multiplicities $\binom{n}{i}-\binom{n}{i-1}$.

Note that an alternative proof for the previous theorem could be done with the help of Theorem 1 and Proposition 1 by induction on $w$.

The minimum support problem for eigenfunctions of aJohnson graph with the minimum eigenvalues is equivalent to the problem of minimum size of Steiner bitrades of strength $i-1$ with blocks of size $i$, see theorem below.

Theorem 3. [13], [10], [12] The support of $\lambda_{i}(n, i)$-eigenfunction of $J(n, i)$ is at least $2^{i}$ and any function that attains the bound is $f^{i, i, n}$ up to multiplication by a scalar.

The main result of the current paper is that the function $f^{i, w, n}$ is the minimum $((w-i)(n-w-i)-i)$-eigenfunction of $J(n, w)$ asymptotically.

### 2.2 Reduction lemma

Here we describe a way to relate eigenspaces of different Johnson graphs, which can be useful in providing inductive arguments. A similar idea was suggested in [15] for studying minimum support eigenfunctions of q-ary Hamming graphs.

Let $f$ be a real-valued $\lambda_{i}(n, w)$-eigenfunction of $J(n, w)$ for some $i \in\{0,1, \ldots, w\}$ and $j_{1}, j_{2} \in\{1,2, \ldots, n\}, j_{1}<j_{2}$. Define a real-valued function $f_{j_{1}, j_{2}}$ as follows: for any vertex $y=$ $\left(y_{1}, y_{2}, \ldots, y_{j_{1}-1}, y_{j_{1}+1}, \ldots, y_{j_{2}-1}, y_{j_{2}+1}, \ldots, y_{n}\right)$ of $J(n-2, w-1)$

$$
\begin{aligned}
f_{j_{1}, j_{2}}(y) & =f\left(y_{1}, y_{2}, \ldots, y_{j_{1}-1}, 1, y_{j_{1}+1}, \ldots, y_{j_{2}-1}, 0, y_{j_{2}+1}, \ldots, y_{n}\right) \\
& -f\left(y_{1}, y_{2}, \ldots, y_{j_{1}-1}, 0, y_{j_{1}+1}, \ldots, y_{j_{2}-1}, 1, y_{j_{2}+1}, \ldots, y_{n}\right)
\end{aligned}
$$

Lemma 1. If $f$ is $\lambda_{i}(n, w)$-eigenfunction of $J(n, w)$ then $f_{j_{1}, j_{2}}$ is a $\lambda_{i-1}(n-$ $2, w-1)$-eigenfunction of $J(n-2, w-1)$.

Proof. Without loss of generality we have $j_{1}=1, j_{2}=2$. The statement follows from the fact that vertices $\left(0,1, y_{3}, \ldots, y_{n}\right)=(0,1, y)$ and $(1,0, y)$ have common neighbors in the subgraphs of $J(n, w)$ induced by sets of vertices $\{(0,0, z)$ : $w t(z)=w\}$ and $\{(1,1, z): w t(z)=w-2\}$. More precisely, we have the following equalities:

$$
\begin{gathered}
\lambda_{i}(n, w) f(0,1, y)=f(1,0, y)+\sum_{z: w t(z)=w-1, d(z, y)=1} f(0,1, z)+ \\
\sum_{z: w t(z)=w, d(z, y)=1} f(0,0, z)+\sum_{z: w t(z)=w-2, d(z, y)=1} f(1,1, z), \\
\lambda_{i}(n, w) f(0,1, y)=f(1,0, y)+\sum_{z: w t(z)=w-1, d(z, y)=1} f(1,0, z)+ \\
\sum_{z: w t(z)=w, d(z, y)=1} f(0,0, z)+\sum_{z: w t(z)=w-2, d(z, y)=1} f(1,1, z),
\end{gathered}
$$

therefore we have that

$$
\left(\lambda_{i}(n, w)-1\right)(f(0,1, y)-f(1,0, y))=\left(\lambda_{i}(n, w)-1\right) f_{1,2}(y)=\sum_{z: d(z, y)=1} f_{1,2}(z)
$$

In other words, $f_{1,2}$ is $\left(\lambda_{i}(n, w)-1\right)$-eigenfunction which taking into account that $\lambda_{i}(n, w)-1=\lambda_{i-1}(n-2, w-1)$ finishes the proof.

As we see, given an eigenfunction $f$ from the reduction Lemma 1 we obtain the eigenfunctions $f_{j_{1}, j_{2}}$ in a Johnson graph with smaller parameters for every distinct coordinates $j_{1}, j_{2}$. In some cases the resulting function $f_{j_{1}, j_{2}}$ is just allzero function, for example when $f=f^{i, w, n}$ from Proposition 1 with $n \geq 2 w+2$ and $j_{1}, j_{2} \notin M \cup M^{\prime}$.

Lemma 2. Let $f$ be a real-valued function of $J(n, w)$, and $j_{1}, j_{2}, j_{3} \in$ $\{1,2, \ldots, n\}, j_{1}<j_{2}<j_{3}$. If $f_{j_{1}, j_{2}} \equiv 0$ and $f_{j_{1}, j_{3}} \equiv 0$ then $f_{j_{2}, j_{3}} \equiv 0$.

Proof. Without loss of generality we can take $j_{1}=1, j_{2}=2$ and $j_{3}=3$. Let us fix $z=\left(z_{1}, z_{2}, \ldots, z_{n-2}\right) \in J(n-2, w-1)$. In these terms our goal is to prove that

$$
f\left(z_{1}, 1,0, z_{2}, \ldots, z_{n-2}\right)=f\left(z_{1}, 0,1, z_{2}, \ldots, z_{n-2}\right)
$$

Since $f_{1,2} \equiv 0$ and $f_{1,3} \equiv 0$, we have

$$
\begin{aligned}
f\left(0,1,1, z_{2}, \ldots, z_{n-2}\right) & =f\left(1,0,1, z_{2}, \ldots, z_{n-2}\right) \\
f\left(0,1,0, z_{2}, \ldots, z_{n-2}\right) & =f\left(1,0,0, z_{2}, \ldots, z_{n-2}\right) \\
f\left(0,1,1, z_{2}, \ldots, z_{n-2}\right) & =f\left(1,1,0, z_{2}, \ldots, z_{n-2}\right) \\
f\left(0,0,1, z_{2}, \ldots, z_{n-2}\right) & =f\left(1,0,0, z_{2}, \ldots, z_{n-2}\right)
\end{aligned}
$$

Combining the first and the third equalities we obtain

$$
f\left(1,1,0, z_{2}, \ldots, z_{n-2}\right)=f\left(1,0,1, z_{2}, \ldots, z_{n-2}\right)
$$

and combining the second and the fourth equalitues we find that

$$
f\left(0,1,0, z_{2}, \ldots, z_{n-2}\right)=f\left(0,0,1, z_{2}, \ldots, z_{n-2}\right)
$$

so $f_{2,3} \equiv 0$.

Corollary 1. Let $f$ be a real-valued function of $J(n, w)$. Then the set $N=$ $\{1,2, \ldots, n\}$ of coordinates can be partitioned into $t(f)$ sets

$$
N=\bigsqcup_{j=1}^{t(f)} N_{j},\left|N_{j}\right|>0
$$

such that the following properties hold:
for any $j \in\{1,2, \ldots, t(f)\}$ and $j_{1}, j_{2} \in N_{j}$ we have that $f_{j_{1}, j_{2}} \equiv 0$, if there are $j_{1}, j_{2}$ such that $\left(f_{j_{1}, j_{2}} \equiv 0\right)$ then there is $j \in\{1,2, \ldots, t(f)\}: j_{1}, j_{2} \in N_{j}$.

## 3 Main result

Theorem 4. Let $i, w$ be positive integers, $w \geq i$. There is $n_{0}(i, w)$ such that for all $n \geq n_{0}(i, w)$ and any nonzero $\lambda_{i}(n, w)$-eigenfunction $f$ of $J(n, w)$ the following holds:

$$
|\operatorname{supp}(f)| \geq 2^{i}\binom{n-2 i}{w-i}
$$

with equality attained only for the function $f^{i, w, n}$ from Proposition 1 up to multiplication by a scalar.

Proof. The proof is based on the induction on $i$. For $i=0$ the statement is obviously true. We suppose that the statement of the theorem is true for all $i^{\prime}<i$ and we are to prove it for $i$, arbitrary $w \geq i$ and $n$ big enough.

Suppose that the opposite is true, i.e. for some $i$ and $w$ there is a nonzero $\lambda_{i}(n, w)$-eigenfunction with the support of size less then $2^{i}\binom{n-2 i}{w-i}$. According to Corollary 1 the set $N=\{1,2, \ldots, n\}$ of $n$ coordinates can be partitioned. Without loss of generality we may assume that there are exactly $n$ parts $S_{1}, S_{2}, \ldots, S_{n}$ of sizes $t_{1}, t_{2}, \ldots, t_{n}$ correspondingly with $t_{j} \geq 0$ that partition $N: N=\cup_{i \in\{1, \ldots, n\}} S_{i}$, such that $f_{j_{1}, j_{2}} \equiv 0$ iff $j_{1}$ and $j_{2}$ are in one part. Let us take

$$
T=\max _{j=1,2, \ldots, n} t_{j}
$$

so $T$ is the size of the largest part, which may be not unique. Again without loss of generality we can consider $\left|S_{1}\right|=T$. Corollary 1 yields

$$
\begin{equation*}
\left|\left\{\left(j_{1}, j_{2}\right): j_{1}>j_{2}, f_{j_{1}, j_{2}} \not \equiv 0\right\}\right|=\sum_{1 \leq k<l \leq n} t_{k} t_{l} \geq T(n-T) \tag{1}
\end{equation*}
$$

because every pair of coordinates from different parts gives us a non-zero function.

Denote by $X$ the set of pairs of adjacent vertices of $J(n, w)$ where $f$ has distinct values:
$\left\{\left(x_{1}, x_{2}\right): w t\left(x_{1}\right)=w t\left(x_{2}\right)=w,\left|\operatorname{supp}\left(x_{1}\right) \cap \operatorname{supp}\left(x_{2}\right)\right|=w-1, f\left(x_{1}\right) \neq f\left(x_{2}\right)\right\}$.
A pair of adjacent vertices $x_{1}, x_{2}$ is uniquely characterized by the pair of elements $j_{1}=\operatorname{supp}\left(x_{1}\right) \backslash \operatorname{supp}\left(x_{2}\right)$ and $j_{2}=\operatorname{supp}\left(x_{2}\right) \backslash \operatorname{supp}\left(x_{1}\right)$. Therefore the size of $X$ is not less then

$$
\left|\left\{\left(j_{1}, j_{2}\right): j_{1}>j_{2}, f_{j_{1}, j_{2}} \not \equiv 0\right\}\right|\left|\operatorname{supp}\left(f^{i-1, w, n-2}\right)\right| .
$$

On the other hand the size of $X$ obviously does not exceed $w(n-w)|\operatorname{supp}(f)|$, so

$$
\begin{equation*}
w(n-w)|\operatorname{supp}(f)| \geq\left|\left\{\left(j_{1}, j_{2}\right): j_{1}>j_{2}, f_{j_{1}, j_{2}} \not \equiv 0\right\}\right|\left|\operatorname{supp}\left(f^{i-1, w, n-2}\right)\right| . \tag{2}
\end{equation*}
$$

Since $|\operatorname{supp}(f)| \leq 2^{i}\binom{n-2 i}{w-i}$ and by inductive hypothesis for $n$ big enough the inequality $\left|\operatorname{supp}\left(f^{i-1, w, n-2}\right)\right| \geq 2^{i-1}\binom{(n-2)-2(i-1)}{(w-1)-(i-1)}$ holds, we finally obtain

$$
2 w(n-w) \geq T(n-T)
$$

For our following arguments we suppose that

$$
\begin{equation*}
n>2 w^{2}+4 w+1 \tag{3}
\end{equation*}
$$

and it gives us $T \leq 2 w$ or $T \geq n-2 w$.
Let us start with the first case: $T \leq 2 w$. Returning to (1) we have

$$
\left|\left\{\left(j_{1}, j_{2}\right): j_{1}>j_{2}, f_{j_{1}, j_{2}} \not \equiv 0\right\}\right|=\sum_{1 \leq k<l \leq n} t_{k} t_{l}=\frac{1}{2}\left(\left(\sum_{k=1}^{n} t_{k}\right)^{2}-\sum_{k=1}^{n} t_{k}^{2}\right)
$$

Since $n=\sum_{j=1}^{n} t_{j}$, we obtain

$$
\begin{equation*}
\left|\left\{\left(j_{1}, j_{2}\right): j_{1}>j_{2}, f_{j_{1}, j_{2}} \not \equiv 0\right\}\right|=\frac{1}{2}\left(n^{2}-\sum_{k=1}^{n} t_{k}^{2}\right) \tag{4}
\end{equation*}
$$

Providing the same argument as in proving (2) we have that

$$
\left|\left\{\left(j_{1}, j_{2}\right): j_{1}>j_{2}, f_{j_{1}, j_{2}} \not \equiv 0\right\}\right| \leq 2 w(n-w)
$$

As we know, $t_{k}$ 's are real non-negative numbers, which are not greater than $2 w$, therefore we have

$$
\sum_{k=1}^{n} t_{k}^{2} \leq \frac{n}{2 w}(2 w)^{2}
$$

Combining two previous inequalities and (4) we finally get $2 w n \geq$ $n^{2}-4 w n+4 w^{2}$, which is not true for $n>(3+\sqrt{5}) w$ and consequently for $n>2 w^{2}+4 w+1$ too.

Now we consider the case when $T \geq n-2 w$. Let us divide the set of coordinates $N=\{1,2, \ldots, n\}$ into two non-intersecting parts $N_{1}$ and $N_{2}$, such that $N_{2} \subseteq S_{1}$ and $\left|N_{2}\right|=n-2 w$. Without loss of generality we can take $N_{1}=\{1,2, \ldots, 2 w\}$ and $N_{2}=\{2 w+1,2 w+2, \ldots, n\}$. The fact that for any $j_{1}, j_{2} \in N_{2}$ we have that $f_{j_{1}, j_{2}} \equiv 0$ guarantees us that $f(x)$ does not depend on the distribution of ones of vector $x$ in $N_{2}$, only on their number in $N_{2}$. In other words, if $x_{1}, x_{2} \in J(n, w)$ and $\operatorname{supp}\left(x_{1}\right) \cap N_{2}=\operatorname{supp}\left(x_{2}\right) \cap N_{2}$ then $f\left(x_{1}\right)=f\left(x_{2}\right)$. Based on this property let us define a function $h: H(2 w) \rightarrow \mathbb{R}$ as follows:

$$
h(z)= \begin{cases}f(z, \underbrace{1, \ldots, 1}_{w-w t(z)}, & \underbrace{0, \ldots, 0}_{n-3 w+w t(z)}), w t(z) \leq w \\ 0, & \text { otherwise. }\end{cases}
$$

The function $f$ is a $\lambda_{i}(n, w)$-eigenfunction of $J(n, w)$, therefore for every $x \in$ $J(n, w)$

$$
\begin{equation*}
\lambda_{i}(n, w) f(x)=\sum_{y \in J(n, w):|\operatorname{supp}(x) \cap \operatorname{supp}(y)|=w-1} f(y) . \tag{5}
\end{equation*}
$$

Take $x=(z, \underbrace{1, \ldots, 1}_{w-j}, \underbrace{0, \ldots, 0}_{n-3 w+j})$ for arbitrary $z \in H(2 w), w t(z)=j \leq w$ and rewrite (5) in terms of values of $h$ :

$$
\begin{array}{rl}
\lambda_{i}(n, w) h(z)= & +  \tag{6}\\
w t\left(z^{\prime}\right)=j,\left|\operatorname{supp}(z) \cap \operatorname{supp}\left(z^{\prime}\right)\right|=j-1 & h\left(z^{\prime}\right) \\
\sum_{t\left(z^{\prime}\right)=j-1,\left|\operatorname{supp}(z) \cap \operatorname{supp}\left(z^{\prime}\right)\right|=j-1}(n-3 w+j) h\left(z^{\prime}\right) & + \\
\sum_{w t\left(z^{\prime}\right)=j+1,\left|\operatorname{supp}(z) \cap \operatorname{supp}\left(z^{\prime}\right)\right|=j}(w-j) h\left(z^{\prime}\right) & + \\
& (w-j)(n-3 w+j) h(z) .
\end{array}
$$

In the final part of the proof we are focused on properties of the function $h$. The function $f$ is such that $f \not \equiv 0$, so $h \not \equiv 0$. Let $j$ be a minimal integer, such that there is $z \in H(2 w): w t(z)=j$ and $h(z) \neq 0$. There are four different cases:

1. $j=0$. We have that $h(z)=f(\underbrace{0, \ldots, 0}_{2 w}, \underbrace{1, \ldots, 1}_{w}, \underbrace{0, \ldots, 0}_{n-3 w}) \neq 0$. By our previous arguments we can permute zeros and ones in $N_{2}$ without changing
the value of $f$. Then we have at least $\binom{n-2 w}{w}$ non-zero values of $f$. For $n$ big enough it is greater than $2^{i}\binom{n-2 i}{w-i}$ and this leads us to a contradiction.
2. $0<j<i$. Let us take any $z_{0} \in H(2 w): w t\left(z_{0}\right)=j-1$ and rewrite (6) for $z=z_{0}$ :

$$
0=(w-j) \sum_{w t\left(z^{\prime}\right)=j,\left|\operatorname{supp}\left(z_{0}\right) \cap \operatorname{supp}\left(z^{\prime}\right)\right|=j} h\left(z^{\prime}\right) .
$$

Since $j<i \leq w$ and $z_{0}$ is arbitrary, this equation implies that $h$ is a $(-j)$-eigenfunction of $J(2 w, j)$ by Theorem 1 (second item). Therefore by Theorem 3 there are at least $2^{j}$ vectors $z \in J(2 w, j)$, such that $h(z) \neq 0$. As we did in case $j=0$ we can permute the values of coordinates of $z$ in $N_{2}$ without changing the value of $f$. We conclude that there are at least $2^{j}\binom{n-2 w}{w-j}$ non-zeros of function $f$ and for $n$ big enough this value is greater than $2^{i}\binom{n-2 i}{w-i}$, which is a contradiction.
3. $j>i$. In this case we have $z \in H(2 w), w t(z)=j$. Without loss of generality we can take

$$
z=(\underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{2 w-j})
$$

and

$$
\hat{z}=(\underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{2 w-j}, \underbrace{1, \ldots, 1}_{w-j}, \underbrace{0, \ldots, 0}_{n-3 w+j}) .
$$

Consider a function $f_{1,3 w-j+1}$. By Lemma 1 this function is a $\lambda_{i-1}(n-$ $2, w-1$ )-eigenfunction of $J(n-2, w-1)$. After deleting two coordinates from $N$ we obtain the set $\{2,3, \ldots, 3 w-j, 3 w-j+2,3 w-j+3, \ldots, n\}$. Then we repeat this procedure $i$ times more and by $q, q: J(n-2 i-2, w-$ $i-1) \rightarrow \mathbb{R}$ we denote the function such that:

$$
q=\left(\ldots\left(\left(f_{1,3 w-j+1}\right)_{2,3 w-j+2}\right) \ldots\right)_{i+1,3 w-j+i+1} .
$$

By Lemma 1 this function is a $\lambda_{-1}(n-2 i-2, w-i-1)$-eigenfunction of $J(n-2 i-2, w-i-1)$, in other words just a zero-function. On the other hand, $q(\underbrace{1, \ldots, 1}_{j-i-1}, \underbrace{0, \ldots, 0}_{2 w-j}, \underbrace{1, \ldots, 1}_{w-j}, \underbrace{0, \ldots, 0}_{n-3 w+j-i-1})$ equals a linear combination of values of $h$. All vectors except $z$ in this combination have weight less than $j$, so we conclude that $q(\underbrace{1, \ldots, 1}_{j-i-1}, \underbrace{0, \ldots, 0}_{2 w-j}, \underbrace{1, \ldots, 1}_{w-j}, \underbrace{0, \ldots, 0}_{n-3 w+j-i-1})=$ $h(z) \neq 0$, which contradicts the fact that $q$ is all-zero function.
4. $j=i$. Providing the same arguments as in case $0<j<i$ we prove that there are at least $M \geq 2^{i}$ and $M\binom{n-2 w}{w-i} \geq 2^{i}\binom{n-2 w}{w-i}$ non-zero values of $h$
in $J(2 w, i)$ and $f$ in $J(n, w)$ correspondingly. The case $M>2^{i}$ leads us to a contradiction, because

$$
M\binom{n-2 w}{w-i}>2^{i}\binom{n-2 i}{w-i}
$$

for $n$ big enough.
What we are interested now is what one can say about $h$ in case $M=2^{i}$. By Theorem 3 the function $h$ on vertices of $J(2 w, i)$ is $f^{i, i, 2 w}$ up to a permutation of the first $2 w$ coordinates and a multiplication by a scalar, where $M \cup M^{\prime}=\{1,2, \ldots, 2 i\}$. Without loss of generality after dividing by the scalar and applying the permutation to $h$ we can consider that $h$ is equal to $f^{i, i, 2 w}$ on vertices of $J(2 w, i)$. However, we still do not know the values of $h$ in other vertices of $H(2 w)$.
Let us take some $s_{0} \in\{2 i+1,2 i+2, \ldots, 2 w\}$ and consider $f_{s_{0}, 2 w+1}$. Our following goal is to show that $f_{s_{0}, 2 w+1} \equiv 0$. Suppose that the opposite is true and take some $x \in J(n-2, w-1)$ with minimal $m=\left|\operatorname{supp}(x) \cap\left(\{1,2, \ldots 2 w\} \backslash\left\{s_{0}\right\}\right)\right|$ such that $f_{s_{0}, 2 w+1}(x) \neq 0$. By definition $f_{s_{0}, 2 w+1}(x)=f\left(x^{\prime}\right)-f\left(x^{\prime \prime}\right)$, where $x^{\prime}, x^{\prime \prime} \in J(n, w)$ are obtained from $x$ by adding two coordinates $\left(s_{0}\right.$ and $\left.2 w+1\right)$ with values 1 and 0 for $x^{\prime}$ and 0 and 1 for $x^{\prime \prime}$ correspondingly. Particularly, it means that $\left|\operatorname{supp}\left(x^{\prime}\right) \cap\{1,2, \ldots 2 w\}\right|=m+1$ and $\left|\operatorname{supp}\left(x^{\prime \prime}\right) \cap\{1,2, \ldots 2 w\}\right|=m$.
In case $m \leq i-2$ vectors $x^{\prime}$ and $x^{\prime \prime}$ have less than $i$ ones in the first $2 w$ coordinates, so we have $f_{s_{0}, 2 w+1}(x)=0-0=0$, and we reach a contradiction.
In case $m=i-1$ the vector $x^{\prime}$ has exactly $i$ ones in the first $2 w$ coordinates and the vector $x^{\prime \prime}$ has only $i-1$, what gives us $f_{s_{0}, 2 w+1}(x)=f\left(x^{\prime}\right)-0=$ $f\left(x^{\prime}\right)$. However, as we know $f^{i, i, 2 w}$ has nonzero values only on some vectors with ones on the first $2 i$ coordinates. By definition of $s_{0}$ we have $s_{0} \in \operatorname{supp}\left(x^{\prime}\right)$ and $s_{0} \notin\{1,2, \ldots, 2 i\}$, so we conclude that $f_{s_{0}, 2 w+1}(x)=0$ and reach a contradiction.
Consequently, one can claim that $m \geq i$.
Let $\operatorname{supp}(x) \cap\left(\{1,2, \ldots 2 w\} \backslash\left\{s_{0}\right\}\right)=\left\{s_{1}, s_{2}, \ldots, s_{m}\right\}$ and define $\hat{x}$ as the vector obtained from $x$ by deleting coordinates $\left\{s_{1}, s_{2}, \ldots, s_{i}\right\}$.
Consider the function

$$
g=\left(\ldots\left(\left(f_{s_{0}, 2 w+1}\right)_{s_{1}, 2 w+2}\right) \ldots\right)_{s_{i}, 2 w+i+1} .
$$

Similar to the case $j>i$ this function is a $\lambda_{-1}(n-2 i-2, w-i-1)-$ eigenfunction of $J(n-2 i-2, w-i-1)$ by Lemma 1, in other words just the all-zero function. On the other hand, $g(\hat{x})$ equals a linear combination of values of $f_{s_{0}, 2 w+1}$. It is clear, that only one of them is the value of $f_{s_{0}, 2 w+1}$ on the vector with $m$ ones in the first $2 w$ positions (vector $x$ ), and other have less number of ones there. Therefore we conclude that $g(\hat{x})=f_{s_{0}, 2 w+1}(x) \neq 0$ and find a contradiction.

Since $s_{0}$ was an arbitrary element of $\{2 i+1,2 i+2, \ldots, 2 w\}$ we conclude that $\forall j_{1}, j_{2} \in\{2 i+1,2 i+2, \ldots, n\}$ the equality $f_{j_{1}, j_{2}} \equiv 0$ holds by Lemma 2] i.e. $f(x)$ depends only on the distribution of ones of $x$ in the first $2 i$ positions. The knowledge of values of $h$ in $J(2 i, i)$ gives us that there are at least $2^{i}\binom{n-2 i}{w-i}$ non-zero values of $f$ in $J(n, w)$. So we conclude that $h$ is a zero-function outside $J(2 i, i)$ and see that $f=f^{i, w, n}$.

In the proof of the previous theorem we had to take $n$ big enough several times independently, so finding a good lower bound on $n_{0}(i, w)$ is a problem. Even in the case $i=1$ the answer is still unknown.

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