# Defective 2-colorings of planar graphs without 4-cycles and 5-cycles 

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#### Abstract

Let $G$ be a graph without 4 -cycles and 5 -cycles. We show that the problem to determine whether $G$ is $(0, k)$-colorable is NP-complete for each positive integer $k$. Moreover, we construct non- $(1, k)$-colorable planar graphs without 4 -cycles and 5 -cycles for each positive integer $k$. Finally, we prove that $G$ is $\left(d_{1}, d_{2}\right)$-colorable where $\left(d_{1}, d_{2}\right)=(4,4),(3,5)$, and $(2,9)$.


## 1 Introduction

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A $k$-vertex, a $k^{+}$vertex, and $k^{-}$-vertex are a vertex of degree $k$, at least $k$, and at most $k$, respectively. The similar notation is applied for faces. A $\left(d_{1}, d_{2}, \ldots, d_{k}\right)$-face $f$ is a face of degree $k$ where all vertices on $f$ have degree $d_{1}, d_{2}, \ldots, d_{k}$. If $v$ is not on a 3 -face $f$ but $v$ is adjacent to some 3 -vertex on $f$, then we call $f$ a pendant face of a vertex $v$ and $v$ is a pendant neighbor of a 3 -vertex $v$. A 3 -face (respectively, 2 -vertex) incident to a 2 -vertex (respectively, 3 -face) is called a bad 3-face (respectively, bad 2-vertex). Otherwise, it is a good 3-face (respectively, good 2-vertex).

[^0]A $k$-coloring $c$ (not necessary proper) is a function $c: V(G) \rightarrow\{1, \ldots, k\}$. Define $V_{i}:=$ $\{v \in V(G): c(v)=i\}$. We call $c$ a $\left(d_{1}, d_{2}, \ldots, d_{k}\right)$-coloring if $V_{i}$ is an empty set or the induced subgraph $G\left[V_{i}\right]$ has the maximum degree at most $d_{i}$ for each $i \in\{1, \ldots, k\}$. A graph $G$ is called $\left(d_{1}, d_{2}, \ldots, d_{k}\right)$-colorable if $G$ admits a $\left(d_{1}, d_{2}, \ldots, d_{k}\right)$-coloring Thus the four color theorem [2], 3] can be restated as every planar graphs is ( $0,0,0,0$ )-colorable. For improper 3 -colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is $(2,2,2)$-colorable [10]. Eaton and Hull [11] proved that $(2,2,2)$-colorability is optimal by showing non- $(k, k, 1)$-colorable planar graphs for each $k$.

Grötzsch [12] showed that every planar graph without 3-cycles is $(0,0,0)$-colorable. The famous Steinberg's conjecture proposes that every planar graph without 4 -cycles and 5 -cycles is also $(0,0,0)$-colorable. Recently, this conjecture is disproved by Cohen-Addad et al [1]. One way to relax the conjecture is allowing some color classes to be improper. For every planar graph $G$ without 4-cycles and 5 -cycles, Xu, Miao, and Wang [17] proved that $G$ is $(1,1,0)$-colorable, and Chen et al. [8] proved that $G$ is $(2,0,0)$-colorable.

Many papers investigate $\left(d_{1}, d_{2}\right)$-coloring of planar graphs in various settings. Montassier and Ochem [14] constructed planar graphs of girth 4 that are not $(i, j)$-colorable for each $i, j$. Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed planar graphs of girth 6 that are not $(0, k)$-colorable for each $k$. On the other hand, for every planar graph $G$ of girth 5, Havet and Seren [13] showed that $G$ is $(2,6)$-colorable and (4, 4)-colorable, and Choi and Raspaud [9] showed that $G$ is $(3,5)$-colorable.

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph $G$ is called $\left(d_{1}, d_{2}, \ldots, d_{k}\right)$-colorable if $V(G)$ can be partitioned into sets $V_{1}, V_{2}, \ldots, V_{k}$ such that the induced subgraph $G\left[V_{i}\right]$ for $i \in[k]$ has the maximum degree at most $d_{i}$. Thus the four color theorem [2], [3] can be restated as every planar graphs is ( $0,0,0,0$ )-colorable. For improper 3colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is (2,2,2)-colorable [10]. Eaton and Hull [11] and Škrekovski [15] prove that (2, 2, 2)-colorability is optimal by showing non- $(k, k, 1)$-colorable planar graphs for each $k$.

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planar graph $G$ of girth 5, Havet and Seren [13] showed that $G$ is $(2,6)$-colorable and $(4,4)$ colorable, and Choi and Raspaud [9] showed that $G$ is $(3,5)$-colorable. Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed planar graphs of girth 6 that are not ( $0, k$ )colorable for each $k$. Montassier and Ochem [14] constructed planar graphs of girth 4 that are not $(i, j)$-colorable for any $i, j$.

There are many papers [4, 6, 13, 7, 5, 14] that investigate $\left(d_{1}, d_{2}\right)$-colorability forgraphs with girth length of $g$ for $g \geq 6$; see 14 for the rich history. For example, Borodin, Ivanova, Montassier, Ochem, and Raspaud [4 constructed a graph in $g_{6}$ (and thus also in $g_{5}$ ) that is not $(0, k)$-colorable for any $k$. The question of determining if there exists a finite $k$ where all graphs in $g_{5}$ are ( $1, k$ )-colorable is not yet known and was explicitly asked in [14]. On the other hand, Borodin and Kostochka [6] and Havet and Sereni [13], respectively, proved results that imply graphs in $g_{5}$ are $(2,6)$-colorable and $(4,4)$-colorable.

Let $G$ be a graph without 4 -cycles and 5 -cycles. We show that the problem to determine whether $G$ is $(0, k)$-colorable is NP-complete for each positive integer $k$. Moreover, we construct non- $(1, k)$-colorable planar graphs without 4 -cycles and 5 -cycles for each positive integer $k$. Finally, we prove that $G$ is $\left(d_{1}, d_{2}\right)$-colorable where $\left(d_{1}, d_{2}\right)=(4,4),(3,5)$, and $(2,9)$.

## 2 NP-completeness of (0,k)-colorings

Theorem 1. [14] Let $g_{k, j}$ be the largest integer $g$ such that there exists a planar graph of girth $g$ that is not $(k, j)$-colorable. The problem to determine whether a planar graph with girth $g_{k, j}$ is $(k, j)$-colorable for $(k, j) \neq(0,0)$ is $N P$-complete.

Theorem 2. The problem to determine whether a planar graph without 4-cycles and 5-cycles is $(0, k)$-colorable is NP-complete for each positive integer $k$.

Proof. We use a reduction from the problem in Theorem 1 to prove that $(0, k)$-coloring for planar graph without 4 -cycles and 5 -cycles. From [14], $6 \leq g_{0,1} \leq 10$. Let $G$ be a graph of girth $g_{0,1}$. Take $k-1$ copies of 3-cycles $v_{i} v_{i}^{\prime} v_{i}^{\prime \prime}(i=1, \ldots, k-1)$ for each vertex $v$ of $G$. The graph $H_{k}$ is obtained from $G$ by identifying $v_{i}$ (in a 3 -cycle $v_{i} v_{i}^{\prime} v_{i}^{\prime \prime}$ ) to $v$ for each vertex $v$. The resulting graph $H_{k}$ has neither 4-cycles nor 5 -cycles.

Suppose $G$ has a $(0,1)$-coloring $c$. We extend a coloring to $c\left(v_{i}^{\prime}\right)=1$ and $c\left(v_{i}^{\prime \prime}\right)=2$ for each vertex $v$ and each $i=1, \ldots, k-1$. One can see that $c$ is a $(0, k)$-coloring of $H_{k}$. Suppose $H_{k}$ has a $(0, k)$-coloring $c$. Consider $v \in V(G)$ with $c(v)=2$. By construction, $v$ has at least
$k-1$ neighbors with the same color in $V\left(H_{k}\right)-V(G)$. Thus $v$ has at most one neighbor with the same color in $V\left(H_{k}\right)-V(G)$. It follows that $c$ with restriction to $V(G)$ is a $(0,1)$ coloring of $G$. Hence $G$ is $(0,1)$-colorable if and only if $H_{k}$ is $(0, k)$-colorable. This completes the proof.

## 3 Non-(1, $k$ )-colorable planar graphs without 4-cycles and 5-cycles

We construct a non- $(1, k)$-colorable planar graph $G$ without 4 -cycles and 5 -cycles. Consider the graph $H_{u, v}$ shown in Figure 1.

$H_{u, v}$


Figure 1.

A non- $(k, 1)$-colorable planar graph $G$ without 4 -cycles and 5 -cycles The vertices $a, b, c$, and $d$ cannot receive the same color 1 . Now, we construct the graph $S_{z}$ as follows. Let $z$ be a vertex and $x_{1} x_{2} x_{3}$ be a path. Take $2 k+1$ copies $H_{u_{i}, v_{j}}$ of $H_{u, v}$ with $1 \leq i \leq 2 k+1$ and $1 \leq j \leq 3$. Identify every $u_{i}$ with $z$ and identify $v_{j}$ with $x_{j}$. Finally, we obtain $G$ from three copies $S_{z_{1}}, S_{z_{1}}$, and $S_{z_{3}}$ by adding the edges $z_{1} z_{2}$ and $z_{2} z_{3}$. In every $(1, k)$-coloring of $G$, the path $z_{1} z_{2} z_{3}$ contains a vertex $z$ with color 2 . In the copy of $S_{z}$ corresponding to $z$, the path $x_{1} x_{2} x_{3}$ contains a vertex $x$ with color 2 . Since each of $z$ and $x$ has at most $k$ neighbors colored 2, one of $2 k+1$ copies of $H_{u, v}$ between $z$ and $x$, does not contain a neighbor of $z$ and $x$ colored 2 . This copy is not $(1, k)$-colorable, and thus $G$ is not $(1, k)$-colorable.

## 4 Helpful Tools

Now, we investigate $\left(d_{1}, d_{2}\right)$ such that $G$ is $\left(d_{1}, d_{2}\right)$-colorable for every graph $G$ without 4 -cycles and 5 -cycles. From two previous sections, we have that $d_{1}, d_{2} \geq 2$. First, we present useful proposition and lemmas about a minimal planar graph $G$ that is not $\left(d_{1}, d_{2}\right)$-colorable where $d_{1} \leq d_{2}$.

Proposition 1. (a) Each vertex $v$ of $G$ is a $2^{+}$-vertex.
(b) If $v$ is a $k$-vertex has $\alpha$ incident 3-faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3 -faces, then $\alpha \leq\left\lfloor\frac{k}{2}\right\rfloor$ and $2 \beta+\alpha+\gamma \leq k$

Lemma 2. [9] Let $G$ be $\left(d_{1}, d_{2}\right)$-colorable where $d_{1} \leq d_{2}$.
(a) If $v$ is a $3^{-}$-vertex, then at least two neighbors of $v$ are $\left(d_{1}+2\right)^{+}$-vertices one of which is a $\left(d_{2}+2\right)^{+}$-vertex.
(b) If $v$ is a $\left(d_{1}+d_{2}+1\right)^{-}$-vertex, then at least one neighbor of $v$ is a $\left(d_{1}+2\right)^{+}$-vertex.

Lemma 3. If a 2-vertex $v$ is on a bad 3-face $f$, then the other face $g$ which is incident to $v$ is a $7^{+}$-face.

Proof. Suppose that a face $g$ is a $6^{-}$-face. Let a face $f=u v w$. By condition of $G$, a face $g$ is neither 4,5 -face nor 3 -face, otherwise $G$ contains $C_{4}$. Now we suppose a face $g$ is a 6 -face and let $g=u_{1} u_{2} u_{3} u v w$. Since $u$ is adjacent to $w$, there is a 5 -cycle $=u_{1} u_{2} u_{3} u w$, a contradiction.

Lemma 4. Let $f$ be a $k$-face where $k \geq 7$. Then, $f$ has at most $k-6$ incident bad 2 -vertices.

Proof. By proof of Lemma 3, if a face $f$ is incident to $m$ bad 2-vertices, then there is a cycle $C_{k-m}$ since we can add some edge to $f$ to obtain a new cycle that has the length least than a face $f$.

Lemma 5. Let $(u, v, w)$ be a bad 3-face $f$ where $d(u)=2$. Then at least one of following statements is true.
(S1) A vertex $v$ is a $\left(d_{1}+3\right)^{+}$-vertex which has at least two $\left(d_{2}+2\right)$-neighbors.
(S2) A vertex $w$ is a $\left(d_{2}+3\right)^{+}$-vertex which has at least two $\left(d_{1}+2\right)$-neighbors.
(S3) A vertex $v$ or a vertex $w$ is a $\left(d_{1}+d_{2}+2\right)^{+}$-vertex.

Proof. Assume $c$ is a $\left(d_{1}, d_{2}\right)$-coloring in $G-u$. If two neighbors of $u$ share the same color, then we can color $u$ by $\{1,2\}-\{c(v)\}$. So $c(v) \neq c(w)$. By symmetry let $c(v)=1$ and
$c(w)=2$. By Lemma 2, we have a vertex $v$ is a $\left(d_{1}+2\right)^{+}$and a vertex $w$ is a $\left(d_{2}+2\right)^{+}$. Then $v$ has $d_{1}$ neighbors of color 1 to forbid $u$ from being colored by 1 and $w$ has $d_{2}$ neighbors of color 2 to forbid $u$ from being colored by 2 . Next, to avoid recoloring $v$ by 2 and $w$ by 1 . Then $v$ has one neighbor with color 2 which has $d_{2}$ neighbors of color 2 or $v$ has $d_{2}$ neighbors with color 2 . Otherwise, $w$ has one neighbor with color 1 which has $d_{1}$ neighbors of color 1 or $w$ has $d_{1}$ neighbors with color 1 .

## 5 (4, 4)-coloring

Theorem 3. If $G$ is a planar graph without cycles of length 4 or 5 , then $G$ is (4, 4)-colorable.

Proof. Suppose that $G$ is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex $u$ in $G$ be $\mu(u)=2 d(u)-6$ and the initial charge of a face $f$ in $G$ be $\mu(f)=d(f)-6$. Then by Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$ and by the Handshaking lemma, we have

$$
\sum_{u \in V(G)} \mu(u)+\sum_{f \in F(G)} \mu(f)=-12 .
$$

Now, we establish a new charge $\mu^{*}(x)$ for all $x \in V(G) \cup F(G)$ by transferring charge from one element to another and the summation of new charge $\mu^{*}(x)$ remains -12 . If the final charge $\mu^{*}(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the prove is completed.

The discharging rules are
(R1) Every $6^{+}$-vertex sends charge 1 to each adjacent good 2-vertex.
(R2) Every $6^{+}$-vertex sends charge 2 to each incident 3 -face.
(R3) Every $6^{+}$-vertex sends charge 1 to each adjacent pendant 3 -face.
(R4) Every $7^{+}$-face sends charge 1 to each incident bad 2-vertex.
(R5) Every 4 -vertex or 5 -vertex sends charge 1 to each incident 3 -face.
(R6) Every bad 3-face sends charge 1 to each incident 2-vertex.
It remains to show that resulting $\mu^{*}(x) \geq 0$ for all $x \in V(G) \cup F(G)$.
It is evident that $\mu^{*}(x)=\mu(x)=0$ if $x$ is a 3 -vertex or a 6 -face.
Now, let $v$ be a $k$-vertex.
For $k=2$, a vertex $v$ has two $6^{+}$-neighbors by Lemma 2, If $v$ is a good 2-vertex, then $\mu^{*}(v) \geq \mu(v)+2 \cdot 1=0$ by (R1). If $v$ is a bad 2 -vertex, then $v$ is incident to a $7^{+}$-face by Lemma 3. Thus $\mu^{*}(v) \geq \mu(v)+1+1=0$ by (R4) and (R6).

For $k=4,5$, by Proposition 1 (b), a vertex $v$ is incident to at most two 3-faces. By (R5), $\mu^{*}(v) \geq \mu(v)-2 \cdot 1 \geq 0$.

Consider $k=6^{+}$. Let $v$ have $\alpha$ incident 3 -faces, $\beta$ adjacent good 2 -vertices, and $\gamma$ pendant 3 -faces. By Proposition 1 (b), we have $2 \alpha+\beta+\gamma \leq d(v)$. Moreover, $\mu(v)=2 d(v)-6 \geq d(v)$ if $d(v) \geq 6$. Thus, by (R1), (R2), and (R3), we have $\mu^{*}(v)=\mu(v)-(2 \alpha+\beta+\gamma) \geq 0$.

Now let $f$ be a $k$-face.
For $k=7^{+}$, by Lemma 4, a $k$-face $f$ has at most $k-6$ incident bad 2 -vertices. By (R4), $\mu^{*}(f)=\mu(f)-(k-6) \cdot 1=0$.

Consider $k=3$. If $f$ is a bad 3-face, then we have $f=\left(2,6^{+}, 6^{+}\right)$-face by Lemma 2, Then by (R2) and (R6), $\mu^{*}(f) \geq \mu(f)+2 \cdot 2-1=0$. Now, It remains to consider a good 3 -face. If $f$ is incident to a $4^{+}$-vertex and a $6^{+}$-vertex, then $\mu^{*}(f) \geq \mu(f)+2+1 \geq 0$ by (R2) and (R5). If $f$ is a $\left(3,3,6^{+}\right)$-face, then the pendant neighbor of a 3 -vertex is a $6^{+}$-vertex by Lemma 2. Thus $\mu^{*}(f) \geq \mu(f)+2+1+1 \leq 0$ by (R2) and (R3). Finally, if $f$ is a $\left(4^{+}, 4^{+}, 4^{+}\right)$-face, then $\mu^{*}(f) \geq \mu(f)+3 \cdot 1 \leq 0$ by (R5).

Since $\mu^{*}(x) \geq 0$ for all $x \in V(G) \cup F(G)$, this completes the proof.

## 6 (3, 5)-coloring

Theorem 4. If $G$ is a planar graph without cycles of length 4 or 5 , then $G$ is (3,5)-colorable.

Proof. Suppose that $G$ is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex $u$ in $G$ be $\mu(u)=2 d(u)-6$ and the initial charge of a face $f$ in $G$ be $\mu(f)=d(f)-6$. Then by Euler's formula $|V(G)|-|E(G)|+F(G)=2$ and by the Handshaking lemma, we have

$$
\sum_{u \in V(G)} \mu(u)+\sum_{f \in F(G)} \mu(f)=-12 .
$$

Now, we establish a new charge $\mu^{*}(x)$ for all $x \in V(G) \cup F(G)$ by transferring charge from one element to another and the summation of new charge $\mu^{*}(x)$ remains -12 . If the final charge $\mu^{*}(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the prove is completed.

The discharging rules are
(R1) Every 5-vertex sends charge $\frac{4}{5}$ to each adjacent good 2-vertex.
(R2) Every 5 -vertex sends charge $\frac{8}{5}$ to each incident 3 -face.
(R3) Every 5 -vertex sends charge $\frac{4}{5}$ to each adjacent pendant 3 -face.
(R4) Every 6-vertex sends charge 1 to eeach adjacent good 2-vertex.
(R5) Every 6 -vertex or 7 -vertex sends charge 2 to each incident 3 -face.
(R6) Every 6-vertex sends charge 1 to each adjacent pendant 3-face.
(R7) Every $7^{+}$-vertex sends charge $\frac{6}{5}$ to each adjacent good 2 -vertex.
(R8) Every $8^{+}$-vertex sends charge $\frac{12}{5}$ to each incident 3 -face.
(R9) Every $7^{+}$-vertex sends charge $\frac{6}{5}$ to each adjacent pendant 3 -face.
(R10) Every $7^{+}$-face sends charge 1 to each incident bad 2-vertex.
(R11) Every 4-vertex sends charge 1 to each incident 3 -face.
(R12) Every bad 3 -face sends charge 1 to each incident 2-vertex.
Next, we show that the final charge $\mu^{*}(u)$ is nonnegative.
It is evident that $\mu^{*}(x)=\mu(x)=0$ if $x$ is a 3 -vertex or a 6 -face.
Now, let $v$ be a $k$-vertex.
For $k=2$, a vertex $v$ has two $5^{+}$-neighbors one of which is a $7^{+}$-neighbor by Lemma 2. If $v$ is a good 2-vertex, then $\mu^{*}(v) \geq \mu(v)+\frac{4}{5}+\frac{6}{5}=0$ by (R1) and (R7). If $v$ is a bad 2-vertex, then $v$ is incident to a $7^{+}$-face by Lemma 3. Thus $\mu^{*}(v) \geq \mu(v)+1+1=0$ by (R10) and (R12).

For $k=4$, by Proposition 1 (b), a vertex $v$ is incident to at most two 3 -faces. By (R11), $\mu^{*}(v) \geq \mu(v)-2 \cdot 1 \geq 0$.

Consider $k=5$. Let $v$ have $\alpha$ incident 3 -faces, $\beta$ adjacent good 2 -vertices, and $\gamma$ pendant 3 -faces. By Proposition (b), $2 \alpha+\beta+\gamma \leq d(v)$. Moreover, we have $\frac{8}{5} \alpha+\frac{4}{5} \beta+\frac{4}{5} \gamma=$ $\frac{4}{5}(2 \alpha+\beta+\gamma) \leq \frac{4}{5} d(v)$ and $\mu(v)=2 d(v)-6=\frac{4}{5} d(v)$ if $d(v)=5$. Thus by (R1), (R2), and (R3), we have $\mu^{*}(v)=\mu(v)-\left(\frac{8}{5} \alpha+\frac{4}{5} \beta+\frac{4}{5} \gamma\right) \geq 0$.

Consider $k=6$. Let $v$ have $\alpha$ incident 3 -faces, $\beta$ adjacent good 2 -vertices, and $\gamma$ pendant 3 -faces. By Proposition 1 (b), we have $2 \alpha+\beta+\gamma \leq d(v)$. Moreover, $\mu(v)=2 d(v)-6=d(v)$ if $d(v)=6$. Thus, by (R4), (R5), and (R6), we have $\mu^{*}(v)=\mu(v)-(2 \alpha+\beta+\gamma)=0$.

Consider $k=7$. If $v$ is not incident to a 3 -face, then we have $\mu^{*}(v)=\mu(v)-6 \cdot \frac{6}{5} \geq 0$ by Lemma 2, (R7), and (R9). If $v$ is incident to one 3-face, then we have $\mu^{*}(v)=\mu(v)-$ $\left(2+5 \cdot \frac{6}{5}\right)=0$ by (R5), (R7), and (R9). If $v$ is incident to two 3 -faces, then we have $\mu^{*}(v)=\mu(v)-\left(2 \cdot 2+3 \cdot \frac{6}{5}\right) \geq 0$ by (R5), (R7), and (R9). Finally, if $v$ is incident to three 3 -faces, then we have $\mu^{*}(v)=\mu(v)-\left(3 \cdot 2+\frac{6}{5}\right) \geq 0$ by (R5), (R7) and (R9).

Consider $k=8^{+}$. Let $v$ have $\alpha$ incident 3 -faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3-faces. By Proposition 1 (b), $2 \alpha+\beta+\gamma \leq d(v)$. Moreover, we have $\frac{12}{5} \alpha+\frac{6}{5} \beta+\frac{6}{5} \gamma=$
$\frac{6}{5}(2 \alpha+\beta+\gamma) \leq \frac{6}{5} d(v)$ and $\mu(v)=2 d(v)-6 \geq \frac{6}{5} d(v)$ if $d(v) \geq 8$. Thus by (R7), (R8), and (R9), we have $\mu^{*}(v)=\mu(v)-\left(\frac{12}{5} \alpha+\frac{6}{5} \beta+\frac{6}{5} \gamma\right) \geq 0$.

Now let $f$ be a $k$-face.
For, $k=7^{+}$. By Lemma 4, a $k$-face $f$ has at most $k-6$ incident bad 2 -vertices. By (R11), $\mu^{*}(f)=\mu(f)-(k-6) \cdot 1=0$.

Consider $k=3$. If $f$ is a bad 3 -face, then we have $f$ is $\mathrm{a}\left(2,6^{+}, 6^{+}\right)$-face or $f$ is a $\left(2,5^{+}, 8^{+}\right)$by Lemma 5. Then by (R2), (R5), (R8), and (R12), $\mu^{*}(f) \geq \mu(f)+2 \cdot 2-1=0$ or $\mu^{*}(f) \geq \mu(f)+\frac{8}{5}+\frac{12}{5}-1=0$. Now, it remains to consider a good 3 -face. If $f$ is incident to a $4^{+}$-vertex and a $6^{+}$-vertex, then $\mu^{*}(f) \geq \mu(f)+2+1 \geq 0$ by (R5) and (R11). If $f$ is a $\left(3,3,7^{+}\right)$-face, then the pendant neighbor of a 3 -vertex is a $5^{+}$-vertex by Lemma 2 Thus $\mu^{*}(f) \geq \mu(f)+2 \cdot \frac{4}{5}+2 \geq 0$ by (R3) and (R5). If $f$ is a $\left(3,3,5^{+}\right)$-face, then the pendant neighbor of a 3 -vertex is a $7^{+}$-vertex by Lemma 2. Thus $\mu^{*}(f) \geq \mu(f)+2 \cdot \frac{6}{5}+\frac{8}{5} \geq 0$ by (R2) and (R7). Finally, if $f$ is a $\left(4^{+}, 4^{+}, 4^{+}\right)$-face, then $\mu^{*}(f) \geq \mu(f)+3 \cdot 1 \leq 0$ by (R11).

Since $\mu^{*}(x) \geq 0$ for all $x \in V(G) \cup F(G)$, this completes the proof.

## 7 (2,9)-coloring

Theorem 5. If $G$ is a planar graph without cycles of length 4 or 5 , then $G$ is $(2,9)$-colorable.

Proof. Suppose that $G$ is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex $u$ in $G$ be $\mu(u)=2 d(u)-6$ and the initial charge of a face $f$ in $G$ be $\mu(f)=d(f)-6$. Then by Euler's formula $|V(G)|-|E(G)|+F(G)=2$ and by the Handshaking lemma, we have

$$
\sum_{u \in V(G)} \mu(u)+\sum_{f \in F(G)} \mu(f)=-12 .
$$

Now, we establish a new charge $\mu^{*}(x)$ for all $x \in V(G) \cup F(G)$ by transferring charge from one element to another and the summation of new charge $\mu^{*}(x)$ remains -12 . If the final charge $\mu^{*}(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the prove is completed.

The discharging rules are
(R1) Every $k$-vertex for $4 \leq k \leq 10$ sends charge $\frac{1}{2}$ to each adjacent good 2-vertex.
(R2) Every 4-vertex sends charge 1 to each incident 3-face.
(R3) Every $k$-vertex for $4 \leq k \leq 10$ sends $\frac{1}{2}$ to each adjacent pendant 3 -face.
(R4) Every $k$-vertex for $5 \leq k \leq 10$ sends charge $\frac{3}{2}$ to each incident 3-face.
(R5) Every 11-vertex sends charge $\frac{5}{2}$ to each incident 3 -face.
(R6) Every $11^{+}$-vertex sends charge $\frac{3}{2}$ to each adjacent good 2-vertex.
(R7) Every $12^{+}$-vertex sends charge 3 to each incident 3 -face.
(R8) Every $11^{+}$-vertex sends charge $\frac{3}{2}$ to each adjacent pendant 3 -face.
(R9) Every $7^{+}$-face sends charge 1 to each incident bad 2-vertex.
(R10) Every bad 3-face sends charge 1 to each incident 2-vertex.
Next, we show that the final charge $\mu^{*}(u)$ is nonnegative.
It is evident that $\mu^{*}(x)=\mu(x)=0$ if $x$ is a 3 -vertex or a 6 -face.
Now, let $v$ be a $k$-vertex.
For $k=2$, a vertex $v$ has two $4^{+}$-neighbors one of which is a $11^{+}$-neighbor by Lemma 2. If $v$ is a good 2-vertex, then $\mu^{*}(v) \geq \mu(v)+\frac{1}{2}+\frac{3}{2}=0$ by (R1) and (R6). If $v$ is a bad 2-vertex, then $v$ is incident to a $7^{+}$-face by Lemma 3. Thus $\mu^{*}(v) \geq \mu(v)+1+1=0$ by (R9) and (R10).

Consider $k=4$. Let $v$ have $\alpha$ incident 3 -faces, $\beta$ adjacent good 2 -vertices, and $\gamma$ pendant 3 -faces. By Proposition 1 (b), $2 \alpha+\beta+\gamma \leq d(v)$. Moreover, we have $\alpha+\frac{1}{2} \beta+\frac{1}{2} \gamma=$ $\frac{1}{2}(2 \alpha+\beta+\gamma) \leq \frac{1}{2} d(v)$ and $\mu(v)=2 d(v)-6=\frac{1}{2} d(v)$ if $d(v)=4$. Thus by (R1), (R2), and (R3), we have $\mu^{*}(v)=\mu(v)-\left(\alpha+\frac{1}{2} \beta+\frac{1}{2} \gamma\right) \geq 0$.

Consider $k$ for $5 \leq k \leq 10$. By (R1), (R3), and (R4), we show only the case that $v$ has $\left\lfloor\frac{d(v)}{2}\right\rfloor$ incident 3 -faces because this case has final charge less than the other cases. Consider $\frac{3}{2} \frac{d(v)}{2} \leq 2 d(v)-6$, then we have $d(v) \geq 5$ because two times charge in (R1) or (R3) is less than charge in (R4). Thus we have $\mu^{*}(v) \geq 0$.

Consider $k=11$. By (R5), (R6), and (R8), we show only the case that $v$ is not incident to 3 -face because this case has final charge less than the other cases. we have $\mu^{*}(v)=$ $16-10\left(\frac{3}{2}\right) \geq 0$. If there is one 3 -face, then $\mu^{*}(v)=16-\left(9\left(\frac{3}{2}\right)+\frac{5}{2}\right)=0$.

Now let $f$ be a $k$-face.
For $k=7^{+}$. By Lemma 4, a $k$-face $f$ has at most $k-6$ incident bad 2-vertices. By (R9), $\mu^{*}(f)=\mu(f)-(k-6) \cdot 1=0$.

Consider $k=3$. If $f$ is a bad 3 -face, then we have $f$ is $\mathrm{a}\left(2,4^{+}, 12^{+}\right)$-face or $f$ is a $\left(2,5^{+}, 11^{+}\right)$by Lemma 5. Then by (R2), (R4), (R5), and (R7), $\mu^{*}(f) \geq \mu(f)+1+3-1=0$ or $\mu^{*}(f) \geq \mu(f)+\frac{3}{2}+\frac{5}{2}-1=0$. Now, it remains to consider a good 3 -face. Consider $f$ is incident to exactly one 3 -vertex. If $f$ is not incident to a $11^{+}$-vertex, then pendant neighbor of a 3 -vertex is a $11^{+}$-vertex by Lemma 2. Thus $\mu^{*}(f) \geq \mu(f)+2 \cdot \frac{1}{2}+\frac{3}{2} \geq 0$ by (R2) and (R8). If $f$ is incident to a $4^{+}$-vertex and a $11^{+}$-vertex, then $\mu^{*}(f) \geq \mu(f)+\frac{1}{2}+\frac{5}{2} \geq 0$
by (R2) and (R5). If $f$ is a $\left(3,3,11^{+}\right)$-face, then the pendant neighbor of a 3 -vertex is a $4^{+}$-vertex by Lemma 2. Thus $\mu^{*}(f) \geq \mu(f)+2 \cdot \frac{1}{2}+\frac{5}{2} \geq 0$ by (R3) and (R5). If $f$ is a $\left(3,3,4^{+}\right)$-face, then the pendant neighbor of a 3 -vertex is a $11^{+}$-vertex by Lemma 2. Thus $\mu^{*}(f) \geq \mu(f)+2 \cdot \frac{3}{2}+1 \geq 0$ by (R2) and (R8). Finally, if $f$ is a $\left(4^{+}, 4^{+}, 4^{+}\right)$-face, then $\mu^{*}(f) \geq \mu(f)+3 \cdot 1 \geq 0$ by (R2).

Since $\mu^{*}(x) \geq 0$ for all $x \in V(G) \cup F(G)$, this completes the proof.

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