Defective 2-colorings of planar graphs without 4-cycles and 5-cycles

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Abstract

Let G be a graph without 4-cycles and 5-cycles. We show that the problem to determine whether G is (0, k)-colorable is NP-complete for each positive integer k. Moreover, we construct non-(1, k)-colorable planar graphs without 4-cycles and 5-cycles for each positive integer k. Finally, we prove that G is (d_1, d_2) -colorable where $(d_1, d_2) = (4, 4), (3, 5),$ and (2, 9).

1 Introduction

Let G be a graph with the vertex set V(G) and the edge set E(G). A k-vertex, a k^+ vertex, and k^- -vertex are a vertex of degree k, at least k, and at most k, respectively. The similar notation is applied for faces. A (d_1, d_2, \ldots, d_k) -face f is a face of degree k where all vertices on f have degree d_1, d_2, \ldots, d_k . If v is not on a 3-face f but v is adjacent to some 3-vertex on f, then we call f a pendant face of a vertex v and v is a pendant neighbor of a 3-vertex v. A 3-face (respectively, 2-vertex) incident to a 2-vertex (respectively, 3-face) is called a bad 3-face (respectively, bad 2-vertex). Otherwise, it is a good 3-face (respectively, good 2-vertex).

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A k-coloring c (not necessary proper) is a function $c : V(G) \to \{1, \ldots, k\}$. Define $V_i := \{v \in V(G) : c(v) = i\}$. We call c a (d_1, d_2, \ldots, d_k) -coloring if V_i is an empty set or the induced subgraph $G[V_i]$ has the maximum degree at most d_i for each $i \in \{1, \ldots, k\}$. A graph G is called (d_1, d_2, \ldots, d_k) -colorable if G admits a (d_1, d_2, \ldots, d_k) -coloring Thus the four color theorem [2],[3] can be restated as every planar graphs is (0, 0, 0, 0)-colorable. For improper 3-colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is (2, 2, 2)-colorable [10]. Eaton and Hull [11] proved that (2, 2, 2)-colorability is optimal by showing non-(k, k, 1)-colorable planar graphs for each k.

Grötzsch [12] showed that every planar graph without 3-cycles is (0, 0, 0)-colorable. The famous Steinberg's conjecture proposes that every planar graph without 4-cycles and 5-cycles is also (0, 0, 0)-colorable. Recently, this conjecture is disproved by Cohen-Addad et al [1]. One way to relax the conjecture is allowing some color classes to be improper. For every planar graph G without 4-cycles and 5-cycles, Xu, Miao, and Wang [17] proved that G is (1, 1, 0)-colorable, and Chen et al. [8] proved that G is (2, 0, 0)-colorable.

Many papers investigate (d_1, d_2) -coloring of planar graphs in various settings. Montassier and Ochem [14] constructed planar graphs of girth 4 that are not (i, j)-colorable for each i, j. Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed planar graphs of girth 6 that are not (0, k)-colorable for each k. On the other hand, for every planar graph G of girth 5, Havet and Seren [13] showed that G is (2, 6)-colorable and (4, 4)-colorable, and Choi and Raspaud [9] showed that G is (3, 5)-colorable.

Let G be a graph with the vertex set V(G) and the edge set E(G). A graph G is called (d_1, d_2, \ldots, d_k) -colorable if V(G) can be partitioned into sets V_1, V_2, \ldots, V_k such that the induced subgraph $G[V_i]$ for $i \in [k]$ has the maximum degree at most d_i . Thus the four color theorem [2],[3] can be restated as every planar graphs is (0, 0, 0, 0)-colorable. For improper 3-colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is (2, 2, 2)-colorable [10]. Eaton and Hull [11] and Škrekovski [15] prove that (2, 2, 2)-colorability is optimal by showing non-(k, k, 1)-colorable planar graphs for each k.

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planar graph G of girth 5, Havet and Seren [13] showed that G is (2, 6)-colorable and (4, 4)colorable, and Choi and Raspaud [9] showed that G is (3, 5)-colorable. Borodin, Ivanova,
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are not (i, j)-colorable for any i, j.

There are many papers [4, 6, 13, 7, 5, 14] that investigate (d_1, d_2) -colorability forgraphs with girth length of g for $g \ge 6$; see [14] for the rich history. For example, Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed a graph in g_6 (and thus also in g_5) that is not (0, k)-colorable for any k. The question of determining if there exists a finite k where all graphs in g_5 are (1, k)-colorable is not yet known and was explicitly asked in [14]. On the other hand, Borodin and Kostochka [6] and Havet and Sereni [13], respectively, proved results that imply graphs in g_5 are (2, 6)-colorable and (4, 4)-colorable.

Let G be a graph without 4-cycles and 5-cycles. We show that the problem to determine whether G is (0, k)-colorable is NP-complete for each positive integer k. Moreover, we construct non-(1, k)-colorable planar graphs without 4-cycles and 5-cycles for each positive integer k. Finally, we prove that G is (d_1, d_2) -colorable where $(d_1, d_2) = (4, 4), (3, 5),$ and (2, 9).

2 NP-completeness of (0, k)-colorings

Theorem 1. [14] Let $g_{k,j}$ be the largest integer g such that there exists a planar graph of girth g that is not (k, j)-colorable. The problem to determine whether a planar graph with girth $g_{k,j}$ is (k, j)-colorable for $(k, j) \neq (0, 0)$ is NP-complete.

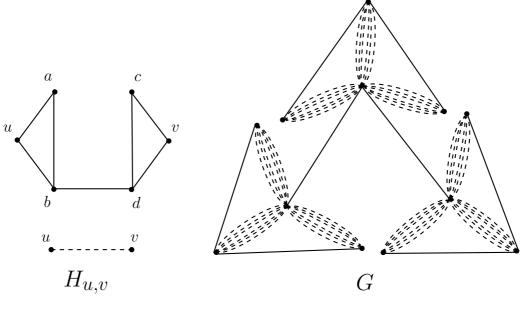
Theorem 2. The problem to determine whether a planar graph without 4-cycles and 5-cycles is (0, k)-colorable is NP-complete for each positive integer k.

Proof. We use a reduction from the problem in Theorem 1 to prove that (0, k)-coloring for planar graph without 4-cycles and 5-cycles. From [14], $6 \leq g_{0,1} \leq 10$. Let G be a graph of girth $g_{0,1}$. Take k - 1 copies of 3-cycles $v_i v'_i v''_i$ (i = 1, ..., k - 1) for each vertex v of G. The graph H_k is obtained from G by identifying v_i (in a 3-cycle $v_i v'_i v''_i$) to v for each vertex v. The resulting graph H_k has neither 4-cycles nor 5-cycles.

Suppose G has a (0, 1)-coloring c. We extend a coloring to $c(v'_i) = 1$ and $c(v''_i) = 2$ for each vertex v and each i = 1, ..., k - 1. One can see that c is a(0, k)-coloring of H_k . Suppose H_k has a (0, k)-coloring c. Consider $v \in V(G)$ with c(v) = 2. By construction, v has at least k - 1 neighbors with the same color in $V(H_k) - V(G)$. Thus v has at most one neighbor with the same color in $V(H_k) - V(G)$. It follows that c with restriction to V(G) is a (0, 1)coloring of G. Hence G is (0, 1)-colorable if and only if H_k is (0, k)-colorable. This completes the proof.

3 Non-(1, k)-colorable planar graphs without 4-cycles and 5-cycles

We construct a non-(1, k)-colorable planar graph G without 4-cycles and 5-cycles. Consider the graph $H_{u,v}$ shown in Figure 1.





A non-(k, 1)-colorable planar graph G without 4-cycles and 5-cycles The vertices a, b, c, and d cannot receive the same color 1. Now, we construct the graph S_z as follows. Let zbe a vertex and $x_1x_2x_3$ be a path. Take 2k + 1 copies H_{u_i,v_j} of $H_{u,v}$ with $1 \le i \le 2k + 1$ and $1 \le j \le 3$. Identify every u_i with z and identify v_j with x_j . Finally, we obtain G from three copies S_{z_1}, S_{z_1} , and S_{z_3} by adding the edges z_1z_2 and z_2z_3 . In every (1, k)-coloring of G, the path $z_1z_2z_3$ contains a vertex z with color 2. In the copy of S_z corresponding to z, the path $x_1x_2x_3$ contains a vertex x with color 2. Since each of z and x has at most k neighbors colored 2, one of 2k + 1 copies of $H_{u,v}$ between z and x, does not contain a neighbor of zand x colored 2. This copy is not (1, k)-colorable, and thus G is not (1, k)-colorable.

4 Helpful Tools

Now, we investigate (d_1, d_2) such that G is (d_1, d_2) -colorable for every graph G without 4-cycles and 5-cycles. From two previous sections, we have that $d_1, d_2 \ge 2$. First, we present useful proposition and lemmas about a minimal planar graph G that is not (d_1, d_2) -colorable where $d_1 \le d_2$.

Proposition 1. (a) Each vertex v of G is a 2^+ -vertex.

(b) If v is a k-vertex has α incident 3-faces, β adjacent good 2-vertices, and γ pendant 3-faces, then $\alpha \leq \lfloor \frac{k}{2} \rfloor$ and $2\beta + \alpha + \gamma \leq k$

Lemma 2. [9] Let G be (d_1, d_2) -colorable where $d_1 \leq d_2$.

(a) If v is a 3⁻-vertex, then at least two neighbors of v are $(d_1 + 2)^+$ -vertices one of which is a $(d_2 + 2)^+$ -vertex.

(b) If v is a $(d_1 + d_2 + 1)^-$ -vertex, then at least one neighbor of v is a $(d_1 + 2)^+$ -vertex.

Lemma 3. If a 2-vertex v is on a bad 3-face f, then the other face g which is incident to v is a 7⁺-face.

Proof. Suppose that a face g is a 6⁻-face. Let a face f = uvw. By condition of G, a face g is neither 4, 5-face nor 3-face, otherwise G contains C_4 . Now we suppose a face g is a 6-face and let $g = u_1u_2u_3uvw$. Since u is adjacent to w, there is a 5-cycle $= u_1u_2u_3uw$, a contradiction.

Lemma 4. Let f be a k-face where $k \ge 7$. Then, f has at most k-6 incident bad 2-vertices.

Proof. By proof of Lemma 3, if a face f is incident to m bad 2-vertices, then there is a cycle C_{k-m} since we can add some edge to f to obtain a new cycle that has the length least than a face f.

Lemma 5. Let (u, v, w) be a bad 3-face f where d(u) = 2. Then at least one of following statements is true.

(S1) A vertex v is a $(d_1+3)^+$ -vertex which has at least two (d_2+2) -neighbors.

(S2) A vertex w is a $(d_2+3)^+$ -vertex which has at least two (d_1+2) -neighbors.

(S3) A vertex v or a vertex w is a $(d_1 + d_2 + 2)^+$ -vertex.

Proof. Assume c is a (d_1, d_2) -coloring in G - u. If two neighbors of u share the same color, then we can color u by $\{1, 2\} - \{c(v)\}$. So $c(v) \neq c(w)$. By symmetry let c(v) = 1 and

c(w) = 2. By Lemma 2, we have a vertex v is a $(d_1+2)^+$ and a vertex w is a $(d_2+2)^+$. Then v has d_1 neighbors of color 1 to forbid u from being colored by 1 and w has d_2 neighbors of color 2 to forbid u from being colored by 2. Next, to avoid recoloring v by 2 and w by 1. Then v has one neighbor with color 2 which has d_2 neighbors of color 2 or v has d_2 neighbors with color 2. Otherwise, w has one neighbor with color 1 which has d_1 neighbors of color 1.

5 (4, 4)-coloring

Theorem 3. If G is a planar graph without cycles of length 4 or 5, then G is (4, 4)-colorable.

Proof. Suppose that G is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex u in G be $\mu(u) = 2d(u) - 6$ and the initial charge of a face f in G be $\mu(f) = d(f) - 6$. Then by Euler's formula |V(G)| - |E(G)| + |F(G)| = 2 and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.$$

Now, we establish a new charge $\mu^*(x)$ for all $x \in V(G) \cup F(G)$ by transferring charge from one element to another and the summation of new charge $\mu^*(x)$ remains -12. If the final charge $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the prove is completed.

The discharging rules are

- (R1) Every 6⁺-vertex sends charge 1 to each adjacent good 2-vertex.
- (R2) Every 6^+ -vertex sends charge 2 to each incident 3-face.
- (R3) Every 6⁺-vertex sends charge 1 to each adjacent pendant 3-face.
- (R4) Every 7⁺-face sends charge 1 to each incident bad 2-vertex.
- (R5) Every 4-vertex or 5-vertex sends charge 1 to each incident 3-face.
- (R6) Every bad 3-face sends charge 1 to each incident 2-vertex.

It remains to show that resulting $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$.

It is evident that $\mu^*(x) = \mu(x) = 0$ if x is a 3-vertex or a 6-face.

Now, let v be a k-vertex.

For k = 2, a vertex v has two 6⁺-neighbors by Lemma 2. If v is a good 2-vertex, then $\mu^*(v) \ge \mu(v) + 2 \cdot 1 = 0$ by (R1). If v is a bad 2-vertex, then v is incident to a 7⁺-face by Lemma 3. Thus $\mu^*(v) \ge \mu(v) + 1 + 1 = 0$ by (R4) and (R6).

For k = 4, 5, by Proposition 1 (b), a vertex v is incident to at most two 3-faces. By (R5), $\mu^*(v) \ge \mu(v) - 2 \cdot 1 \ge 0.$

Consider $k = 6^+$. Let v have α incident 3-faces, β adjacent good 2-vertices, and γ pendant 3-faces. By Proposition 1 (b), we have $2\alpha + \beta + \gamma \leq d(v)$. Moreover, $\mu(v) = 2d(v) - 6 \geq d(v)$ if $d(v) \geq 6$. Thus, by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (2\alpha + \beta + \gamma) \geq 0$. Now let f be a k-face.

For $k = 7^+$, by Lemma 4, a k-face f has at most k - 6 incident bad 2-vertices. By (R4), $\mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0.$

Consider k = 3. If f is a bad 3-face, then we have $f = (2, 6^+, 6^+)$ -face by Lemma 2. Then by (R2) and (R6), $\mu^*(f) \ge \mu(f) + 2 \cdot 2 - 1 = 0$. Now, It remains to consider a good 3-face. If f is incident to a 4⁺-vertex and a 6⁺-vertex, then $\mu^*(f) \ge \mu(f) + 2 + 1 \ge 0$ by (R2) and (R5). If f is a $(3, 3, 6^+)$ -face, then the pendant neighbor of a 3-vertex is a 6⁺-vertex by Lemma 2. Thus $\mu^*(f) \ge \mu(f) + 2 + 1 + 1 \le 0$ by (R2) and (R3). Finally, if f is a $(4^+, 4^+, 4^+)$ -face, then $\mu^*(f) \ge \mu(f) + 3 \cdot 1 \le 0$ by (R5).

Since $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$, this completes the proof.

6 (3, 5)-coloring

Theorem 4. If G is a planar graph without cycles of length 4 or 5, then G is (3, 5)-colorable.

Proof. Suppose that G is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex u in G be $\mu(u) = 2d(u) - 6$ and the initial charge of a face f in G be $\mu(f) = d(f) - 6$. Then by Euler's formula |V(G)| - |E(G)| + F(G) = 2 and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.$$

Now, we establish a new charge $\mu^*(x)$ for all $x \in V(G) \cup F(G)$ by transferring charge from one element to another and the summation of new charge $\mu^*(x)$ remains -12. If the final charge $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the prove is completed.

The discharging rules are

- (R1) Every 5-vertex sends charge $\frac{4}{5}$ to each adjacent good 2-vertex.
- (R2) Every 5-vertex sends charge $\frac{8}{5}$ to each incident 3-face.
- (R3) Every 5-vertex sends charge $\frac{4}{5}$ to each adjacent pendant 3-face.

- (R4) Every 6-vertex sends charge 1 to eeach adjacent good 2-vertex.
- (R5) Every 6-vertex or 7-vertex sends charge 2 to each incident 3-face.
- (R6) Every 6-vertex sends charge 1 to each adjacent pendant 3-face.
- (R7) Every 7⁺-vertex sends charge $\frac{6}{5}$ to each adjacent good 2-vertex.
- (R8) Every 8⁺-vertex sends charge $\frac{12}{5}$ to each incident 3-face.
- (R9) Every 7⁺-vertex sends charge $\frac{6}{5}$ to each adjacent pendant 3-face.
- (R10) Every 7^+ -face sends charge 1 to each incident bad 2-vertex.
- (R11) Every 4-vertex sends charge 1 to each incident 3-face.
- (R12) Every bad 3-face sends charge 1 to each incident 2-vertex.
 - Next, we show that the final charge $\mu^*(u)$ is nonnegative.

It is evident that $\mu^*(x) = \mu(x) = 0$ if x is a 3-vertex or a 6-face.

Now, let v be a k-vertex.

For k = 2, a vertex v has two 5⁺-neighbors one of which is a 7⁺-neighbor by Lemma 2. If v is a good 2-vertex, then $\mu^*(v) \ge \mu(v) + \frac{4}{5} + \frac{6}{5} = 0$ by (R1) and (R7). If v is a bad 2-vertex, then v is incident to a 7⁺-face by Lemma 3. Thus $\mu^*(v) \ge \mu(v) + 1 + 1 = 0$ by (R10) and (R12).

For k = 4, by Proposition 1 (b), a vertex v is incident to at most two 3-faces. By (R11), $\mu^*(v) \ge \mu(v) - 2 \cdot 1 \ge 0.$

Consider k = 5. Let v have α incident 3-faces, β adjacent good 2-vertices, and γ pendant 3-faces. By Proposition 1 (b), $2\alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\frac{8}{5}\alpha + \frac{4}{5}\beta + \frac{4}{5}\gamma = \frac{4}{5}(2\alpha + \beta + \gamma) \leq \frac{4}{5}d(v)$ and $\mu(v) = 2d(v) - 6 = \frac{4}{5}d(v)$ if d(v) = 5. Thus by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (\frac{8}{5}\alpha + \frac{4}{5}\beta + \frac{4}{5}\gamma) \geq 0$.

Consider k = 6. Let v have α incident 3-faces, β adjacent good 2-vertices, and γ pendant 3-faces. By Proposition 1 (b), we have $2\alpha + \beta + \gamma \leq d(v)$. Moreover, $\mu(v) = 2d(v) - 6 = d(v)$ if d(v) = 6. Thus, by (R4), (R5), and (R6), we have $\mu^*(v) = \mu(v) - (2\alpha + \beta + \gamma) = 0$.

Consider k = 7. If v is not incident to a 3-face, then we have $\mu^*(v) = \mu(v) - 6 \cdot \frac{6}{5} \ge 0$ by Lemma 2, (R7), and (R9). If v is incident to one 3-face, then we have $\mu^*(v) = \mu(v) - (2 + 5 \cdot \frac{6}{5}) = 0$ by (R5), (R7), and (R9). If v is incident to two 3-faces, then we have $\mu^*(v) = \mu(v) - (2 \cdot 2 + 3 \cdot \frac{6}{5}) \ge 0$ by (R5), (R7), and (R9). Finally, if v is incident to three 3-faces, then we have $\mu^*(v) = \mu(v) - (3 \cdot 2 + \frac{6}{5}) \ge 0$ by (R5), (R7) and (R9).

Consider $k = 8^+$. Let v have α incident 3-faces, β adjacent good 2-vertices, and γ pendant 3-faces. By Proposition 1 (b), $2\alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\frac{12}{5}\alpha + \frac{6}{5}\beta + \frac{6}{5}\gamma =$

 $\frac{6}{5}(2\alpha + \beta + \gamma) \leq \frac{6}{5}d(v)$ and $\mu(v) = 2d(v) - 6 \geq \frac{6}{5}d(v)$ if $d(v) \geq 8$. Thus by (R7), (R8), and (R9), we have $\mu^*(v) = \mu(v) - (\frac{12}{5}\alpha + \frac{6}{5}\beta + \frac{6}{5}\gamma) \ge 0.$

Now let f be a k-face.

For, $k = 7^+$. By Lemma 4, a k-face f has at most k - 6 incident bad 2-vertices. By (R11), $\mu^*(f) = \mu(f) - (k-6) \cdot 1 = 0.$

Consider k = 3. If f is a bad 3-face, then we have f is $a(2, 6^+, 6^+)$ -face or f is a $(2, 5^+, 8^+)$ by Lemma 5. Then by (R2), (R5), (R8), and (R12), $\mu^*(f) \ge \mu(f) + 2 \cdot 2 - 1 = 0$ or $\mu^*(f) \ge \mu(f) + \frac{8}{5} + \frac{12}{5} - 1 = 0$. Now, it remains to consider a good 3-face. If f is incident to a 4⁺-vertex and a 6⁺-vertex, then $\mu^*(f) \ge \mu(f) + 2 + 1 \ge 0$ by (R5) and (R11). If f is a $(3, 3, 7^+)$ -face, then the pendant neighbor of a 3-vertex is a 5⁺-vertex by Lemma 2. Thus $\mu^*(f) \ge \mu(f) + 2 \cdot \frac{4}{5} + 2 \ge 0$ by (R3) and (R5). If f is a $(3,3,5^+)$ -face, then the pendant neighbor of a 3-vertex is a 7⁺-vertex by Lemma 2. Thus $\mu^*(f) \ge \mu(f) + 2 \cdot \frac{6}{5} + \frac{8}{5} \ge 0$ by (R2) and (R7). Finally, if f is a $(4^+, 4^+, 4^+)$ -face, then $\mu^*(f) \ge \mu(f) + 3 \cdot 1 \le 0$ by (R11).

Since $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$, this completes the proof.

(2,9)-coloring 7

Theorem 5. If G is a planar graph without cycles of length 4 or 5, then G is (2, 9)-colorable.

Proof. Suppose that G is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex u in G be $\mu(u) = 2d(u) - 6$ and the initial charge of a face f in G be $\mu(f) = d(f) - 6$. Then by Euler's formula |V(G)| - |E(G)| + F(G) = 2 and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.$$

Now, we establish a new charge $\mu^*(x)$ for all $x \in V(G) \cup F(G)$ by transferring charge from one element to another and the summation of new charge $\mu^*(x)$ remains -12. If the final charge $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the prove is completed.

The discharging rules are

- (R1) Every k-vertex for $4 \le k \le 10$ sends charge $\frac{1}{2}$ to each adjacent good 2-vertex.
- (R2) Every 4-vertex sends charge 1 to each incident 3-face.
- (R3) Every k-vertex for $4 \le k \le 10$ sends $\frac{1}{2}$ to each adjacent pendant 3-face.

- (R4) Every k-vertex for $5 \le k \le 10$ sends charge $\frac{3}{2}$ to each incident 3-face.
- (R5) Every 11-vertex sends charge $\frac{5}{2}$ to each incident 3-face.
- (R6) Every 11⁺-vertex sends charge $\frac{3}{2}$ to each adjacent good 2-vertex.
- (R7) Every 12^+ -vertex sends charge 3 to each incident 3-face.
- (R8) Every 11⁺-vertex sends charge $\frac{3}{2}$ to each adjacent pendant 3-face.
- (R9) Every 7⁺-face sends charge 1 to each incident bad 2-vertex.
- (R10) Every bad 3-face sends charge 1 to each incident 2-vertex.
 Next, we show that the final charge μ*(u) is nonnegative.
 It is evident that μ*(x) = μ(x) = 0 if x is a 3-vertex or a 6-face.
 Now, let v be a k-vertex.

For k = 2, a vertex v has two 4⁺-neighbors one of which is a 11⁺-neighbor by Lemma 2. If v is a good 2-vertex, then $\mu^*(v) \ge \mu(v) + \frac{1}{2} + \frac{3}{2} = 0$ by (R1) and (R6). If v is a bad 2-vertex, then v is incident to a 7⁺-face by Lemma 3. Thus $\mu^*(v) \ge \mu(v) + 1 + 1 = 0$ by (R9) and (R10).

Consider k = 4. Let v have α incident 3-faces, β adjacent good 2-vertices, and γ pendant 3-faces. By Proposition 1 (b), $2\alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = \frac{1}{2}(2\alpha + \beta + \gamma) \leq \frac{1}{2}d(v)$ and $\mu(v) = 2d(v) - 6 = \frac{1}{2}d(v)$ if d(v) = 4. Thus by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma) \geq 0$.

Consider k for $5 \le k \le 10$. By (R1), (R3), and (R4), we show only the case that v has $\lfloor \frac{d(v)}{2} \rfloor$ incident 3-faces because this case has final charge less than the other cases. Consider $\frac{3}{2}\frac{d(v)}{2} \le 2d(v) - 6$, then we have $d(v) \ge 5$ because two times charge in (R1) or (R3) is less than charge in (R4). Thus we have $\mu^*(v) \ge 0$.

Consider k = 11. By (R5), (R6), and (R8), we show only the case that v is not incident to 3-face because this case has final charge less than the other cases. we have $\mu^*(v) = 16 - 10(\frac{3}{2}) \ge 0$. If there is one 3-face, then $\mu^*(v) = 16 - (9(\frac{3}{2}) + \frac{5}{2}) = 0$.

Now let f be a k-face.

For $k = 7^+$. By Lemma 4, a k-face f has at most k - 6 incident bad 2-vertices. By (R9), $\mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0.$

Consider k = 3. If f is a bad 3-face, then we have f is $a(2, 4^+, 12^+)$ -face or f is a $(2, 5^+, 11^+)$ by Lemma 5. Then by (R2), (R4), (R5), and (R7), $\mu^*(f) \ge \mu(f) + 1 + 3 - 1 = 0$ or $\mu^*(f) \ge \mu(f) + \frac{3}{2} + \frac{5}{2} - 1 = 0$. Now, it remains to consider a good 3-face. Consider f is incident to exactly one 3-vertex. If f is not incident to a 11^+ -vertex, then pendant neighbor of a 3-vertex is a 11^+ -vertex by Lemma 2. Thus $\mu^*(f) \ge \mu(f) + 2 \cdot \frac{1}{2} + \frac{3}{2} \ge 0$ by (R2) and (R8). If f is incident to a 4^+ -vertex and a 11^+ -vertex, then $\mu^*(f) \ge \mu(f) + \frac{1}{2} + \frac{5}{2} \ge 0$

by (R2) and (R5). If f is a $(3,3,11^+)$ -face, then the pendant neighbor of a 3-vertex is a 4^+ -vertex by Lemma 2. Thus $\mu^*(f) \ge \mu(f) + 2 \cdot \frac{1}{2} + \frac{5}{2} \ge 0$ by (R3) and (R5). If f is a $(3,3,4^+)$ -face, then the pendant neighbor of a 3-vertex is a 11^+ -vertex by Lemma 2. Thus $\mu^*(f) \ge \mu(f) + 2 \cdot \frac{3}{2} + 1 \ge 0$ by (R2) and (R8). Finally, if f is a $(4^+, 4^+, 4^+)$ -face, then $\mu^*(f) \ge \mu(f) + 3 \cdot 1 \ge 0$ by (R2).

Since $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$, this completes the proof.

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