Block-avoiding point sequencings of Mendelsohn triple systems

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September 20, 2019

Abstract

A cyclic ordering of the points in a Mendelsohn triple system of order v (or $\mathsf{MTS}(v)$) is called a *sequencing*. A sequencing $\mathcal{D}$ is *ℓ-good* if there does not exist a triple (x, y, z) in the $\mathsf{MTS}(v)$ such that

1. the three points x, y, and z occur (cyclically) in that order in $\mathcal{D}$; and
2. $\{x, y, z\}$ is a subset of ℓ cyclically consecutive points of $\mathcal{D}$.

In this paper, we prove some upper bounds on ℓ for $\mathsf{MTS}(v)$ having ℓ-good sequencings and we prove that any $\mathsf{MTS}(v)$ with $v \geq 7$ has a 3-good sequencing. We also determine the optimal sequencings of every $\mathsf{MTS}(v)$ with $v \leq 10$.

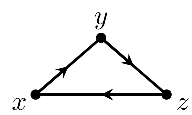
1 Introducton

There has been considerable recent interest in different kinds of block-avoiding sequencings of Steiner triple systems (or $\mathsf{STS}(v)$). See for example, [1, 2, 4, 5, 9]. A similar problem, in the setting of directed triple systems (or $\mathsf{DTS}(v)$), was introduced in [6]. In this paper, we initiate a study of sequencings of Mendelsohn triple systems, or $\mathsf{MTS}(v)$.

A *cyclic triple* is an ordered triple (x, y, z), where x, y, z are distinct. This triple contains the directed edges (or ordered pairs) (x, y), (y, z) and (z, x) (we might also write these directed edges as xy, yz and zx, respectively). Note that

*D.R. Stinson's research is supported by NSERC discovery grant RGPIN-03882.

(x, y, z), (y, z, x) and (z, x, y) are "equivalent" when considered as cyclic triples, i.e., they all contain the same three directed edges. The cyclic triple (x, y, z) can be depicted as follows:



Let X be a set of v points (or vertices) and let $\vec{K_v}$ denote the complete directed graph on vertex set X. This graph has $v(v-1)$ directed edges. A *Mendelsohn triple system of order* v (see [8]) is a pair $(X, \mathcal{T})$, where X is a set of v *points* and $\mathcal{T}$ is a set of cyclic triples (or more simply, *triples*) whose elements are members of X, such that every directed edge in $\vec{K_v}$ occurs in exactly one triple in $\mathcal{T}$. In graph-theoretic language, we are decomposing the complete directed graph into directed cycles of length three.

We will abbreviate the phrase "Mendelsohn triple system of order v" to $\mathsf{MTS}(v)$. It is well-known that an $\mathsf{MTS}(v)$ contains exactly $v(v-1)/3$ triples, and an $\mathsf{MTS}(v)$ exists if and only if $v \equiv 0, 1 \bmod 3$, $v \neq 6$. Various results on $\mathsf{MTS}(v)$ can be found in [3].

Suppose $(X, \mathcal{T})$ is an $\mathsf{MTS}(v)$, where, for convenience, $X = \{1, \ldots, v\}$. Suppose we arrange the points in X in a directed cycle, say $\mathcal{D} = (i_1\ i_2\ \cdots\ i_v)$. We will refer to such a directed cycle as a *sequencing*. Clearly, any cyclic shift of the sequencing $\mathcal{D}$ is equivalent to $\mathcal{D}$.

A *cyclic ordering* can be defined as a ternary relation as follows. Given a sequencing $\mathcal{D} = (i_1\ i_2\ \cdots\ i_v)$, we first define the associated total ordering $i_1 < i_2 < \cdots < i_v$. Then we define the induced ternary relation $\mathcal{C}(\mathcal{D})$ as follows

$$[x, y, z] \in \mathcal{C}(\mathcal{D}) \text{ if and only if } x < y < z \text{ or } y < z < x \text{ or } z < x < y.$$

Observe that any cyclic shift of $\mathcal{D}$ gives rise to the same ternary relation.

This definition can be explained informally as follows: In order to determine if a triple $[x, y, z] \in \mathcal{C}(\mathcal{D})$, we start at x and proceed around the directed cycle $\mathcal{D}$. Then $[x, y, z] \in \mathcal{C}(\mathcal{D})$ if and only if we encounter y before we encounter z. From this, it is obvious that exactly one of $[x, y, z]$ or $[x, z, y]$ is in $\mathcal{C}(\mathcal{D})$.

We say that a cyclic triple $T = (x, y, z)$ is *contained* in a sequencing $\mathcal{D} = (i_1\ i_2 \cdots i_v)$ if $[x, y, z] \in \mathcal{C}(\mathcal{D})$. For an integer $\ell \geq 3$, we say that the sequencing $\mathcal{D}$ is ℓ-*good* if there does not exist a triple $(x, y, z) \in \mathcal{T}$ such that

1. (x, y, z) is contained in $\mathcal{D}$, and
2. $\{x, y, z\}$ is a subset of ℓ cyclically consecutive points of $\mathcal{D}$.

Of course an ℓ-good sequencing is automatically κ-good for all κ such that $3 \leq \kappa \leq \ell - 1$.

The basic questions we address in this paper are as follows:

- Given a particular $\mathsf{MTS}(v)$, say $(X, \mathcal{T})$, what is the largest integer ℓ such that $(X, \mathcal{T})$ has an ℓ-good sequencing?

- Given a positive integer $v \equiv 0, 1 \bmod 3$, $v \neq 6$, what is the largest integer ℓ such that
  - there exists an $\mathsf{MTS}(v)$ that has an ℓ-good sequencing, or
  - every $\mathsf{MTS}(v)$ has an ℓ-good sequencing?

Example 1.1. *The triples* $(0, 1, 3)$ *and* $(0, 3, 2)$*, developed modulo* 7*, yield an* $\mathsf{MTS}(7)$*. It is not hard to see that* $\mathcal{D} = (0\ 1\ 2\ 3\ 4\ 5\ 6)$ *is a* 3*-good sequencing for this* $\mathsf{MTS}(7)$*. This follows because:*

1. *none of the seven triples obtained from* $(0, 3, 2)$ *are contained in* $\mathcal{D}$*, and*
2. *the seven triples obtained from* $(0, 1, 3)$ *are contained in* $\mathcal{D}$*, but none of these triples is a subset of three cyclically consecutive points of* D*.*

However, this sequencing is not 4*-good, because each of the triples obtained from* $(0, 1, 3)$ *is a subset of four cyclically consecutive points of* $\mathcal{D}$*.*

The rest of this paper is organized as follows. In Section 2, we prove an $\mathsf{MTS}(v)$ has an ℓ-good sequencing only if $\ell \leq \lfloor \frac{v-1}{2} \rfloor$. In Section 3, we summarize the results of computer searches we used to determine the optimal sequencings of every $\mathsf{MTS}(v)$ with $v \leq 10$. In Section 4, we prove that any $\mathsf{MTS}(v)$ with $v \geq 7$ has a 3-good sequencing. Finally, in Section 5, we conclude with a few comments.

2 Necessary Conditions

Theorem 2.1. *Suppose* v *is even. Then no* $\mathsf{MTS}(v)$ *has an* ℓ*-good sequencing if* $\ell \geq v/2$*.*

Proof. Without loss of generality, we assume the sequencing is

$$\mathcal{D} = (0\ 1\ \cdots\ v-1).$$

We will show that there is no $\mathsf{MTS}(v)$, say $(X, \mathcal{T})$, where $X = \{0, 1, \ldots, v-1\}$ and for which $\mathcal{D}$ is a $v/2$-good sequencing. In what follows, all arithmetic is modulo v.

There must be a triple $(0, 1, x) \in \mathcal{T}$, where $x \in \{2, \ldots, v-1\}$. If $2 \leq x < v/2$, then $\{0, 1, x\}$ is a subset of the first $v/2$ points of $\mathcal{D}$, namely, $0, 1, \ldots, v/2 - 1$. Similarly, if $2 + v/2 \leq x \leq v - 1$, then $\{0, 1, x\}$ a subset of $v/2$ cyclically consecutive points of $\mathcal{D}$, namely, $2 + v/2, \ldots, v-1, 0, 1$. Hence, $x = v/2$ or $x = 1 + v/2$ and thus either

$$R_0 = \left(0, 1, \frac{v}{2}\right) \quad \text{or} \quad S_0 = \left(0, 1, 1 + \frac{v}{2}\right)$$

is a triple in $\mathcal{T}$.

Similarly, it follows for each $i \in \mathbb{Z}_v$ that exactly one of

$$R_i = \left(i, i+1, i + \frac{v}{2}\right) \quad \text{or} \quad S_i = \left(i, i+1, i + 1 + \frac{v}{2}\right)$$

is a triple in $\mathcal{T}$. Let $\mathcal{R} = \{R_i : i \in \mathbb{Z}_v\}$ and let $\mathcal{S} = \{S_i : i \in \mathbb{Z}_v\}$.

Suppose $R_i \in \mathcal{T}$. This triple contains the ordered pair $(i+v/2, i)$. The triple

$$S_{i-1+v/2} = \Big(i - 1 + \frac{v}{2}, i + \frac{v}{2}, i\Big)$$

also contains the ordered pair $(i + v/2, i)$, so $S_{i-1+v/2} \notin \mathcal{T}$. Then it must be the case that $R_{i-1+v/2} \in \mathcal{T}$. Similarly, the triples R_i and

$$S_{i+v/2} = \Big(i + \frac{v}{2}, i + 1 + \frac{v}{2}, i + 1\Big)$$

both contain the ordered pair $(i + 1, i + v/2)$. Therefore $S_{i+v/2} \notin \mathcal{T}$ and hence $R_{i+v/2} \in \mathcal{T}$. In summary, if $R_i \in \mathcal{T}$, then $R_{i-1+v/2} \in \mathcal{T}$ and $R_{i+v/2} \in \mathcal{T}$.

Now, using the fact that $R_{i-1+v/2} \in \mathcal{T}$, we see that $R_{i-1+v/2+v/2} = R_{i-1} \in \mathcal{T}$. Therefore, if $R_i \in \mathcal{T}$, we have that $R_{i-1} \in \mathcal{T}$. From this, it follows easily that $\mathcal{R} \subseteq \mathcal{T}$ or $\mathcal{S} \subseteq \mathcal{T}$. We consider the following two cases.

Case 1 : $\mathcal{R} \subseteq \mathcal{T}$ and $\mathcal{S} \cap \mathcal{T} = \emptyset$.

The triples in $\mathcal{R}$ cover all ordered pairs having differences 1, $v/2-1$ and $v/2$, where the *difference* of a pair (a, b) is $(b - a) \bmod v$. Now consider the ordered pair $(0, 2)$, which has difference 2. There must be a triple $(0, 2, x) \in \mathcal{T}$.

If $3 \le x < v/2$, then $\{0, 2, x\}$ is a subset of the first $v/2$ points of $\mathcal{D}$, namely, $0, 1, \ldots, v/2-1$. Similarly, if $3+v/2 \le x \le v-1$, then $\{0, 2, x\}$ is again a subset of $v/2$ cyclically consecutive points of $\mathcal{D}$, namely, $3 + v/2, \ldots, v - 1, 0, 1, 2$.

If $x \in \{v/2, 1 + v/2, 2 + v/2\}$, then we have have two pairs with difference $v/2 - 1$ or $v/2$, because

$$\begin{aligned} 0 - \frac{v}{2} &= \frac{v}{2}, \\ 1 + \frac{v}{2} - 2 &= \frac{v}{2} - 1, \quad \text{and} \\ 2 + \frac{v}{2} - 2 &= \frac{v}{2}. \end{aligned}$$

Therefore, $x = 1$.

It follows in a similar manner that all the triples of the form $(i, i + 2, i + 1)$ are in $\mathcal{T}$. But this is impossible because $(0, 2, 1)$ and $(1, 3, 2)$ both contain the ordered pair $(2, 1)$.

Case 2 : $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{R} \cap \mathcal{T} = \emptyset$.

The triples in $\mathcal{S}$ also cover all ordered pairs having differences 1, $v/2-1$ and $v/2$. Therefore the proof is identical to case 1.

□

Now we turn to the case of odd v.

Theorem 2.2. *Suppose v is odd. Then no MTS(v) has an ℓ-good sequencing if $\ell \ge (v + 1)/2$.*

Proof. The proof is similar to that of Theorem 2.1. We assume the sequencing is $\mathcal{D} = (0\ 1\ \cdots\ v-1)$. We will show that there is no $\mathsf{MTS}(v)$, say $(X, \mathcal{T})$, where $X = \{0, 1, \ldots, v-1\}$ and for which $\mathcal{D}$ is a $(v+1)/2$-good sequencing. There must be a triple $(0, 1, x) \in \mathcal{T}$, where $x \in \{2, \ldots, v-1\}$. If $2 \leq x \leq (v-1)/2$, then $\{0, 1, x\}$ is a subset of the first $(v+1)/2$ points of $\mathcal{D}$, namely, $0, 1, \ldots, (v-1)/2$. Similarly, if $(v+3)/2 \leq x \leq v-1$, then $\{0, 1, x\}$ is a also a subset of $(v+1)/2$ cyclically consecutive points of $\mathcal{D}$, namely, $(v+3)/2, \ldots, v-1, 0, 1$. Hence, it must be the case that $x = (v+1)/2$, i.e., $(0, 1, (v+1)/2) \in \mathcal{T}$.

An identical argument shows that $\mathcal{T}$ must contain all of the triples

$$\left(i, i+1, i+\frac{v+1}{2}\right),$$

where arithmetic is moduli v and $0 \leq i \leq v-1$. In particular, $\mathcal{T}$ contains the triples

$$\left(0, 1, \frac{v+1}{2}\right) \quad \text{and} \quad \left(\frac{v+1}{2}, \frac{v+3}{2}, 1\right).$$

But these two triples both contain the ordered pair $(1, (v+1)/2)$, so we have a contradiction. $\square$

Combining Theorems 2.1 and 2.2, we obtain the following.

Corollary 2.3. *If an* $\mathsf{MTS}(v)$ *has an* ℓ*-good sequencing, then* $\ell \leq \lfloor \frac{v-1}{2} \rfloor$.

3 Sequencings of $\mathsf{MTS}(v)$ for Small Values of v

We have determined the optimal sequencings for all $\mathsf{MTS}(v)$ with $v \leq 10$. The results are given in Table 1. This table lists the number of nonisomorphic $\mathsf{MTS}(v)$ for each v, along with the number of designs that have 3-good and 4-good sequencings. None of these designs have 5-good sequencings, by Corollary 2.3.

We present the three $\mathsf{MTS}(9)$ that have 4-good sequencings, as well as the five $\mathsf{MTS}(10)$ that do not have have 4-good sequencings, in the Appendices. Additional details can be found in the technical report [7].

We noticed one particularly interesting fact concerning the five nonisomorphic $\mathsf{MTS}(10)$ that do not have a 4-good sequencing. If any triple is removed from one of these five $\mathsf{MTS}(10)$, then the resulting "partial" $\mathsf{MTS}(10)$ having 29 triples turns out to have a 4-good sequencing (we verified this fact by computer). So these $\mathsf{MTS}(10)$ "almost" have 4-good sequencings. In fact, we know from these results that any "partial" $\mathsf{MTS}(10)$ having 29 triples has a 4-good sequencing. This is because such a partial $\mathsf{MTS}(10)$ can automatically be completed to an $\mathsf{MTS}(10)$, and therefore any partial $\mathsf{MTS}(10)$ having 29 triples arises from the deletion of a triple from an $\mathsf{MTS}(10)$. Clearly, if we delete a triple from an $\mathsf{MTS}(10)$ that has a 4-good sequencing, then the resulting partial $\mathsf{MTS}(10)$ also has a 4-good sequencing.

Table 1: Sequencings of $\mathsf{MTS}(v)$ with $v \leq 10$

v	Nonisomorphic $\mathsf{MTS}(v)$	ℓ-good sequencings $\ell = 3$	$\ell = 4$
3	1	0	0
4	1	0	0
6	0	0	0
7	3	3	0
9	18	18	3
10	143	143	138

4 Constructing 3-good Sequencings

Charlie Colbourn proved that any $\mathsf{STS}(v)$ has a 3-good sequencing. His method is described in [4]; it is based on examining the triples that contain a particular point x and then relabelling the points in a suitable way. We have adapted this approach to obtain 3-good sequencings of $\mathsf{MTS}(v)$; however, it turned out to be quite a bit more complicated to obtain the desired result for $\mathsf{MTS}(v)$ that it did for $\mathsf{STS}(v)$.

Suppose $(X, \mathcal{T})$ is an $\mathsf{MTS}(v)$ and fix a particular point $x \in X$. Construct a directed graph G_x on vertex set $X \setminus \{x\}$ as follows. For every triple $(x, y, z) \in \mathcal{T}$ (or a cyclic rotation of this triple), include the directed edge (y, z) in G_x. It is not hard to see that G_x consists of a vertex-disjoint union of one or more directed cycles (note that some of these directed cycles could have length two). Suppose the directed cycles are named $C_1, C_2, \ldots, C_s$. We can construct a (cyclic) sequencing $\mathcal{D}_x$ of $X \setminus \{x\}$ by writing out the cycles $C_1, C_2, \ldots, C_s$ in order. For each of the cycles C_i, we can arbitrarily pick any vertex in the cycle as a starting point.

It is easy to see that no three cyclically consecutive vertices of the sequencing $\mathcal{D}_x$ comprise a triple. Consider three consecutive vertices, say x_i, x_j and x_k. At least one of (x_i, x_j) or (x_j, x_k) is an edge in G_x. In the first case, $(x, x_i, x_j) \in \mathcal{T}$ so $(x_i, x_j, x_k) \notin \mathcal{T}$, and in the second case, $(x, x_j, x_k) \in \mathcal{T}$ so again $(x_i, x_j, x_k) \notin \mathcal{T}$.

The difficulty is that, if we insert x into $\mathcal{D}_x$ in any position, the sequencing is no longer 3-good. So we need to modify $\mathcal{D}_x$ at the same time that we insert x. We illustrate how this can be done, in various situations, in the rest of this section.

4.1 A Directed Cycle of Length at least Six

Let's suppose that C_1 is a directed cycle of length $\tau \geq 6$, say $(1, 2, \ldots, \tau)$. This means that the following five triples are in $\mathcal{T}$:

$$(x, 1, 2) \quad (x, 2, 3) \quad (x, 3, 4) \quad (x, 4, 5) \quad (x, 5, 6).$$

Suppose that $\mathcal{D}_x = (1\ 2\ \cdots\ \tau\ \cdots\ v)$. Replace the four vertices 1 2 3 4 by 3 4 1 x 2, obtaining a sequencing $\mathcal{D}$ of X. Notice that v is the vertex preceding 3 in $\mathcal{D}$. We check that there are no triples comprising three consecutive vertices of the modified sequencing $\mathcal{D}$:

$$\begin{array}{ll}
(v,3,4) \notin \mathcal{T} & \text{because } (x,3,4) \in \mathcal{T} \\
(3,4,1) \notin \mathcal{T} & \text{because } (x,3,4) \in \mathcal{T} \\
(4,1,x) \notin \mathcal{T} & \text{because } (x,4,5) \in \mathcal{T} \\
(1,x,2) \notin \mathcal{T} & \text{because } (x,2,3) \in \mathcal{T} \\
(x,2,5) \notin \mathcal{T} & \text{because } (x,2,3) \in \mathcal{T} \\
(2,5,6) \notin \mathcal{T} & \text{because } (x,5,6) \in \mathcal{T}.
\end{array}$$

4.2 Two Directed Cycles, Each of Length at Least Three

Suppose that G_x contains two directed cycles of length at least three, say $(y, 3, 4, 5, \dots)$ (note that it is possible that $y = 5$, if this cycle has length three) and $(1, 2, z, \dots)$. We can assume that y 3 4 1 2 z are consecutive vertices in $\mathcal{D}_x$.

The following five triples are in $\mathcal{T}$:

$$(x,1,2) \quad (x,2,z) \quad (x,y,3) \quad (x,3,4) \quad (x,4,5).$$

Delete the two consecutive vertices 4 1 from $\mathcal{D}_x$ and replace them by 1 x 4, obtaining a sequencing $\mathcal{D}$ of X. We check that there are no triples comprising three consecutive vertices of the modified sequencing $\mathcal{D}$:

$$\begin{array}{ll}
(y,3,1) \notin \mathcal{T} & \text{because } (y,3,x) \in \mathcal{T} \\
(3,1,x) \notin \mathcal{T} & \text{because } (x,3,4) \in \mathcal{T} \\
(1,x,4) \notin \mathcal{T} & \text{because } (x,4,5) \in \mathcal{T} \\
(x,4,2) \notin \mathcal{T} & \text{because } (x,4,5) \in \mathcal{T} \\
(4,2,z) \notin \mathcal{T} & \text{because } (x,2,z) \in \mathcal{T}.
\end{array}$$

4.3 Two Directed Cycles of Length Two

Suppose that $v \geq 7$ and there are two directed cycles in G_x having length two, say $(1, 2)$ and $(3, 4)$. We assume that 3 4 1 2 are consecutive vertices in $\mathcal{D}_x$. The following four triples are in $\mathcal{T}$:

$$(x,1,2) \quad (x,2,1) \quad (x,3,4) \quad (x,4,3).$$

Choose $z \geq 5$ such that $(4, 2, z) \notin \mathcal{T}$ and choose $y \geq 5$ such that $(3, 1, y) \notin \mathcal{T}$ and $y \neq z$. We also require that

- y and z are in different directed cycles in G_x, or
- if G_x contains only three directed cycles, then y immediately precedes z in a directed cycle in G_x (this can be done if $v \geq 7$).

Then we can assume that y immediately precedes 3 in $\mathcal{D}_x$ and z immediately follows 2 in $\mathcal{D}_x$. Now, delete the two consecutive vertices 1 4 from $\mathcal{D}_x$ and replace them by 4 1 x, obtaining a sequencing $\mathcal{D}$ of X.

We check that there are no triples comprising three consecutive vertices of the modified sequencing $\mathcal{D}$:

$$\begin{array}{ll}
(y,3,1) \notin \mathcal{T} & \text{by the choice of } y \\
(3,1,x) \notin \mathcal{T} & \text{because } (x,3,4) \in \mathcal{T} \\
(1,x,4) \notin \mathcal{T} & \text{because } (x,4,3) \in \mathcal{T} \\
(x,4,2) \notin \mathcal{T} & \text{because } (x,4,3) \in \mathcal{T} \\
(4,2,z) \notin \mathcal{T} & \text{by the choice of } z.
\end{array}$$

4.4 Two Directed Cycles, One of Length Two

Suppose that $v \geq 6$ and G_x consists of exactly two directed cycles, one of length two, say $(1,2)$, and one of length $v-2$, say $(3,4,\ldots,v)$. This implies that the following triples are in $\mathcal{T}$:

$$(x,1,2) \quad (x,2,1) \quad (x,3,4) \quad (x,4,5) \quad (x,5,6) \quad (x,v-1,v) \quad (x,v,3).$$

We assume that $\mathcal{D}_x = (1\ 2\ 3\ 4\ \cdots\ v-1\ v)$.

Now, if $(5,4,2) \in \mathcal{T}$, then $(5,4,1) \notin \mathcal{T}$. Therefore by interchanging 1 and 2 if necessary, we can assume that $(5,4,2) \notin \mathcal{T}$.

Replace the four vertices 1 2 3 4 in $\mathcal{D}_x$ by 3 1 x 4 2 to construct the sequencing $\mathcal{D}$, so We check that there are no triples comprising three consecutive vertices of the sequencing $\mathcal{D}$:

$$\begin{array}{ll}
(v-1,v,3) \notin \mathcal{T} & \text{because } (x,v-1,v) \in \mathcal{T} \\
(v,3,1) \notin \mathcal{T} & \text{because } (x,v,3) \in \mathcal{T} \\
(3,1,x) \notin \mathcal{T} & \text{because } (x,3,4) \in \mathcal{T} \\
(1,x,4) \notin \mathcal{T} & \text{because } (x,4,5) \in \mathcal{T} \\
(x,4,2) \notin \mathcal{T} & \text{because } (x,4,5) \in \mathcal{T} \\
(4,2,5) \notin \mathcal{T} & \text{by assumption} \\
(2,5,6) \notin \mathcal{T} & \text{because } (x,5,6) \in \mathcal{T}.
\end{array}$$

4.5 The Main Theorem

We can now show that the four cases we have considered cover all possibilities. Suppose that $v \geq 7$ and we classify G_x according to the number of directed cycles of length two that it contains.

- If G_x has at least two directed cycles of length two, use the construction in Section 4.3.
- If G_x has exactly one directed cycle of length two, then either
  - G_x contains at least two directed cycles of length at least three, in which case we can use the construction in Section 4.2, or

- G_x consists of exactly two directed cycles, one of length two and one of length at least five, so we can use the construction in Section 4.4.

- If G_x has no directed cycles of length two, then either
  - G_x contains at least two directed cycles of length at least three, in which case we can use the construction in Section 4.2, or
  - G_x consists of a single directed cycle of length at least seven, so we can use the construction in Section 4.1.

Therefore, we have the following result.

Theorem 4.1. *Any* $\mathsf{MTS}(v)$ *with* $v \geq 7$ *has a* 3*-good sequencing.*

5 Comments

Recent papers have considered ℓ-good sequencings for $\mathsf{STS}(v)$, $\mathsf{DTS}(v)$ and $\mathsf{MTS}(v)$ (however, we should note that the definition of ℓ-good sequencing is slightly different in each case). It is interesting to compare the results obtained for these three types of triple systems.

- For $\mathsf{STS}(v)$, [9] establishes that an ℓ-good sequencing exists only if $\ell \leq (v+2)/3$. But there are only a few small examples known where this bound is met with equality. In fact, it is currently unknown if there is an infinite class of $\mathsf{STS}(v)$ that have (cv)-good sequencings, for any positive constant c. Proving this for $c \approx 1/2$ would be the best possible result in light of current knowledge, but it would still be of interest if we could establish this result for some smaller value of c, say $c = 1/4$. It is also known that every $\mathsf{STS}(v)$ has a 3-good sequencing (see [4]); every $\mathsf{STS}(v)$ with $v > 71$ has a 4-good sequencing (see [4]); and every $\mathsf{STS}(v)$ with $v \geq \ell^6/16$ has an ℓ-good sequencing (see [9]).

- For $\mathsf{DTS}(v)$ (i.e., directed triple systems of order v), it is possible that a v-good sequencing exists. In fact, there is a $\mathsf{DTS}(v)$ having a v-good sequencing for all permissible values of v (see [6]). It is also shown in [6] that there is a $\mathsf{DTS}(v)$ that does not have a v-good sequencing, for all $v \equiv 0, 1 \bmod 3$, $v \geq 7$.

- In this paper, we showed that $\ell \leq \lfloor \frac{v-1}{2} \rfloor$ is a necessary condition for the existence of an ℓ-good sequencing of an $\mathsf{MTS}(v)$. We showed that this bound is met with equality for $v = 7, 9, 10$ and we also proved that every $\mathsf{MTS}(v)$ has a 3-good sequencing.

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A The three $\mathsf{MTS}(9)$ that have 4-good sequencings

$\mathcal{M}_9$1.1

$(0,2,1)$ $(0,1,6)$ $(0,3,2)$ $(0,7,3)$ $(0,4,7)$ $(0,8,4)$ $(0,6,5)$ $(0,5,8)$
$(1,2,7)$ $(1,3,6)$ $(1,8,3)$ $(1,5,4)$ $(1,4,8)$ $(1,7,5)$ $(2,3,8)$ $(2,4,6)$
$(2,6,4)$ $(2,5,7)$ $(2,8,5)$ $(3,4,5)$ $(3,7,4)$ $(3,5,6)$ $(6,7,8)$ $(6,8,7)$

Lexicographic least 4-good sequencing : 023471856

Number of 4-good sequencings found: 18

$\mathcal{M}_9$3.1

$(0,2,1)$ $(0,1,3)$ $(0,6,2)$ $(0,3,8)$ $(0,4,6)$ $(0,7,4)$ $(0,5,7)$ $(0,8,5)$
$(1,2,7)$ $(1,8,3)$ $(1,6,4)$ $(1,4,8)$ $(1,5,6)$ $(1,7,5)$ $(2,6,3)$ $(2,3,7)$
$(2,4,5)$ $(2,8,4)$ $(2,5,8)$ $(3,5,4)$ $(3,4,7)$ $(3,6,5)$ $(6,7,8)$ $(6,8,7)$

Lexicographic least 4-good sequencing : 047563812

Number of 4-good sequencings found: 36

$\mathcal{M}_9$7.1

$(0,1,2)$ $(0,2,1)$ $(0,6,3)$ $(0,3,8)$ $(0,4,6)$ $(0,7,4)$ $(0,5,7)$ $(0,8,5)$
$(1,3,6)$ $(1,7,3)$ $(1,4,7)$ $(1,8,4)$ $(1,6,5)$ $(1,5,8)$ $(2,3,7)$ $(2,8,3)$
$(2,6,4)$ $(2,4,8)$ $(2,5,6)$ $(2,7,5)$ $(3,4,5)$ $(3,5,4)$ $(6,7,8)$ $(6,8,7)$

Lexicographic least 4-good sequencing : 031485726

Number of 4-good sequencings found: 324

B The five $\mathsf{MTS}(10)$ that do not have 4-good sequencings

$\mathcal{M}_{10}116.1$

$(0,1,8)$ $(0,9,1)$ $(0,5,2)$ $(0,2,7)$ $(0,4,3)$ $(0,3,6)$
$(0,7,4)$ $(0,6,5)$ $(0,8,9)$ $(1,2,3)$ $(1,3,2)$ $(1,4,5)$
$(1,5,4)$ $(1,6,7)$ $(1,7,6)$ $(1,9,8)$ $(2,6,4)$ $(2,4,8)$
$(2,5,9)$ $(2,8,6)$ $(2,9,7)$ $(3,4,9)$ $(3,7,5)$ $(3,5,8)$
$(3,9,6)$ $(3,8,7)$ $(4,6,8)$ $(4,7,9)$ $(5,6,9)$ $(5,7,8)$

$\mathcal{M}_{10}116.2$

$(0,1,8)$ $(0,9,1)$ $(0,5,2)$ $(0,2,7)$ $(0,4,3)$ $(0,3,6)$
$(0,7,4)$ $(0,6,5)$ $(0,8,9)$ $(1,2,3)$ $(1,3,2)$ $(1,4,5)$
$(1,5,4)$ $(1,6,7)$ $(1,7,6)$ $(1,9,8)$ $(2,6,4)$ $(2,4,8)$
$(2,5,9)$ $(2,8,6)$ $(2,9,7)$ $(3,4,9)$ $(3,5,7)$ $(3,8,5)$
$(3,9,6)$ $(3,7,8)$ $(4,6,8)$ $(4,7,9)$ $(5,6,9)$ $(5,8,7)$

$\mathcal{M}_{10}118.1$

$(0,1,8)$ $(0,9,1)$ $(0,4,2)$ $(0,2,7)$ $(0,5,3)$ $(0,3,6)$
$(0,7,4)$ $(0,6,5)$ $(0,8,9)$ $(1,2,3)$ $(1,3,2)$ $(1,4,5)$
$(1,5,4)$ $(1,6,7)$ $(1,7,6)$ $(1,9,8)$ $(2,4,6)$ $(2,8,5)$
$(2,5,9)$ $(2,6,8)$ $(2,9,7)$ $(3,8,4)$ $(3,4,9)$ $(3,5,7)$
$(3,9,6)$ $(3,7,8)$ $(4,8,6)$ $(4,7,9)$ $(5,6,9)$ $(5,8,7)$

$\mathcal{M}_{10}134.1$

$(0,1,8)$ $(0,9,1)$ $(0,5,2)$ $(0,2,7)$ $(0,4,3)$ $(0,3,6)$
$(0,6,4)$ $(0,7,5)$ $(0,8,9)$ $(1,2,3)$ $(1,3,2)$ $(1,4,5)$
$(1,5,4)$ $(1,6,7)$ $(1,7,6)$ $(1,9,8)$ $(2,4,8)$ $(2,8,4)$
$(2,5,7)$ $(2,6,9)$ $(2,9,6)$ $(3,4,6)$ $(3,5,9)$ $(3,9,5)$
$(3,7,8)$ $(3,8,7)$ $(4,7,9)$ $(4,9,7)$ $(5,6,8)$ $(5,8,6)$

$\mathcal{M}_{10}134.2$

$(0,1,8)$ $(0,9,1)$ $(0,2,5)$ $(0,7,2)$ $(0,4,3)$ $(0,3,6)$
$(0,6,4)$ $(0,5,7)$ $(0,8,9)$ $(1,2,3)$ $(1,3,2)$ $(1,4,5)$
$(1,5,4)$ $(1,6,7)$ $(1,7,6)$ $(1,9,8)$ $(2,4,8)$ $(2,8,4)$
$(2,7,5)$ $(2,6,9)$ $(2,9,6)$ $(3,4,6)$ $(3,5,9)$ $(3,9,5)$
$(3,7,8)$ $(3,8,7)$ $(4,7,9)$ $(4,9,7)$ $(5,6,8)$ $(5,8,6)$