Embedding partial Latin squares in Latin squares with many mutually orthogonal mates

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Abstract

In this paper it is shown that any partial Latin square of order n can be embedded in a Latin square of order at most $16n^2$ which has at least $2n$ mutually orthogonal mates. Further, for any $t \geqslant 2$, it is shown that a pair of orthogonal partial Latin squares of order n can be embedded in a set of t mutually orthogonal Latin squares (MOLS) of order a polynomial with respect to n. A consequence of the constructions is that, if $N(n)$ denotes the size of the largest set of MOLS of order n, then $N(n^2) \geqslant N(n) + 2$. In particular, it follows that $N(576) \geqslant 9$, improving the previously known lower bound $N(576) \geqslant 8$.

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1. Introduction

In 1960 Evans [5] showed that it was possible to embed any partial Latin square of order n in some Latin square of order t, for every $t \geqslant 2n$, where $2n$ is a tight bound. In the same paper Evans raised the question of embedding orthogonal partial Latin squares in sets of mutually orthogonal Latin squares.

The importance and relevance of this question is demonstrated by the prevalence and application of orthogonal Latin squares to other areas of mathematics (see [3]). For instance, the existence of a set of $n-1$ mutually orthogonal Latin squares of order n is equivalent to the existence of a projective plane of order n (see [11] for a relevant construction). Thus results on the embedding of orthogonal partial Latin squares provide information on the embedding of sets of partial lines in finite geometries. In addition, early embedding results for partial Steiner triple systems utilised embeddings of partial idempotent Latin squares (see for example [9]). It has also been suggested that embeddings of block designs with block size 4 and embeddings of Kirkman triple systems may make use of embeddings of pairs of orthogonal partial Latin squares (see [7]).

In 1976 Lindner [10] showed that a pair of orthogonal partial Latin squares can always be finitely embedded in a pair of orthogonal Latin squares. However, there was no known method for obtaining an embedding of polynomial order (with respect to the order of the partial arrays). In [7], Hilton *et al.* formulated some necessary conditions for a pair of orthogonal partial Latin squares to be embedded in a pair of orthogonal Latin squares. Then in [8] Jenkins developed a construction for embedding a single partial Latin square of order n in a Latin square of order $4n^2$ for which there exists an orthogonal mate. In 2014, Donovan and Yazıcı [4] developed a construction that verified that a pair of orthogonal partial Latin squares, of order n, can be embedded in a pair of orthogonal Latin squares of order at most $16n^4$.

In 2017, Barber *et al.* [2] established a remarkable result concerning completions of mutually orthogonal partial Latin squares. As a consequence of their Theorem 1.4, it follows that for any $t \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, any set of t mutually orthogonal partial Latin squares of order n can be embedded in a set of t MOLS of order m for every $m \geqslant k_0 n$. That there is such a k_0 is an important existence result because it gives a linear order embedding. However, the proof given in [2] does not yield an estimate for the best (i.e., lowest) value of k_0. For $t = 1$, Evan's result shows that $k_0 = 2$ is the best possible value. For $t \geqslant 2$, the proof given in [2] requires that $k_0 > 10^7(t+2)^3/9$ and, being an existence result, there is little information about the structure of the resulting set of MOLS. For $t = 2$ and small n, certainly $n \leqslant 113$ and possibly much larger, [4] gives a tighter embedding than that of [2], and it more closely specifies the structure of the resulting pair of MOLS.

In the current paper, we provide some new constructions that show that a partial Latin square, of order n, can be embedded in a Latin square, of order at most $16n^2$ with many mutually orthogonal mates. (From here onwards, when we say $\mathcal{B}$ has t mutually orthogonal mates we mean $\mathcal{B}$ together with these t Latin squares form a set of $t+1$ mutually orthogonal Latin squares.) Furthermore, we extend the results of [4] by developing a second construction that takes any pair of orthogonal partial Latin squares of order n and any integer t, and embeds the pair in a set of t MOLS(m), where $m < p_t(n)$ for some polynomial p_t. Also, as a corollary, the construction can be used to increase the best known lower bound for the largest set of MOLS(576). In the literature the existence of 8 MOLS(576) is established. However, we construct 9 MOLS(576).

We preface the discussion of our main result with some necessary definitions.

2. Definitions

Let $I = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ represent a set of n distinct elements. A non-empty subset P of $I \times I \times I$ is said to be a *partial Latin square* (PLS(n)), of order n, if for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in P$ and for all distinct $i, j, k \in \{1, 2, 3\}$,

$$x_i = y_i \text{ and } x_j = y_j \text{ implies } x_k = y_k.$$

We say that P is *indexed* by I. We may think of P as an $n \times n$ array where symbol $e \in I$ occurs in cell (r, c), whenever $(r, c, e) \in P$, and we will write $e = P(r, c)$. We say that cell (r, c) is *empty* in P if, for all $e \in I$, $(r, c, e) \notin P$. The *volume* of P is $|P|$. If $|P| = n^2$, then we say that P is a *Latin square* (LS(n)), of order n. If for all $1 \leqslant i \leqslant n$, $(\alpha_i, \alpha_i, \alpha_i) \in P$, then P is said to be idempotent. The set of elements $\{(x_1, x_2, x_3) \in P \mid x_1 = x_2\}$ forms the main diagonal of P.

Two partial Latin squares P and Q, of the same order n are said to be *orthogonal*, denoted OPLS(n), if they have the same non-empty cells and for all $r_1, c_1, r_2, c_2, x, y \in I$

$$\{(r_1, c_1, x), (r_2, c_2, x)\} \subseteq P \text{ implies } \{(r_1, c_1, y), (r_2, c_2, y)\} \not\subseteq Q.$$

Example 2.1.

0	1	2	
2	0	1	3
3		0	
	2		1

0	2	1	
3	1	0	2
1		2	
	0		3

Figure 1: A pair of orthogonal partial Latin squares of order 4

This definition extends in the obvious way to a pair of *orthogonal Latin squares* of order n. A set of t Latin squares of order n, which are pairwise orthogonal, is said to be a set of t *mutually orthogonal Latin squares*, denoted MOLS(n). $N(n)$ is the maximum number of Latin squares in a set of mutually orthogonal Latin squares of order n.

A set $T \subseteq A$, where A is a Latin square of order n, is said to be a *transversal*, if

- $|T| = n$, and
- for all distinct $(r_1, c_1, x_1), (r_2, c_2, x_2) \in T$, $r_1 \neq r_2$, $c_1 \neq c_2$ and $x_1 \neq x_2$.

Note that a Latin square has an orthogonal mate if and only if it can be partitioned into disjoint transversals.

We say that a partial Latin square P on the set I can be *embedded* in a Latin square L on the set J if there exist one-to-one mappings $f_1^P, f_2^P, f_3^P : I \to J$ such that if $(x_1, x_2, x_3) \in P$ then $(f_1^P(x_1), f_2^P(x_2), f_3^P(x_3)) \in L$. A pair of orthogonal partial Latin squares (P_1, P_2) is said to be embedded in a pair of orthogonal Latin squares (L_1, L_2) if P_1 is embedded in L_1 and P_2 is embedded in L_2 such that $f_1^{P_1} = f_1^{P_2}$ and $f_2^{P_1} = f_2^{P_2}$. A set of mutually orthogonal partial Latin squares $(P_1, P_2, \ldots, P_a)$ is embedded in a set of mutually orthogonal Latin squares $\{L_1, L_2, \ldots, L_b\}$ where $b \geqslant a$ if P_i is embedded in L_i for all $1 \leqslant i \leqslant a$ where $f_1^{P_i} = f_1^{P_j}$ and $f_2^{P_i} = f_2^{P_j}$ for all $1 \leqslant i, j \leqslant a$.

This paper will make extensive use of Evans' embedding result, which is stated as:

Theorem 2.2 ([5]). *A partial Latin square of order n can be embedded in a Latin square of order t, for any $t \geqslant 2n$.*

The following is a similar embedding result for partial idempotent Latin squares.

Theorem 2.3 ([1]). *A partial idempotent Latin square of order n can be embedded in a idempotent Latin square of order t, for any $t \geqslant 2n + 1$.*

It is also worth noting the following well known result which is the culmination of results from a series of papers by many authors, for example [6].

Theorem 2.4. *A pair of orthogonal Latin squares of order n can be embedded in a pair of orthogonal Latin squares of order t if $t \geqslant 3n$, with the bound of $3n$ being best possible.*

3. Embedding a PLS in a set of MOLS

We begin by assuming that there exists a set of t MOLS(n) and show that any Latin square L, of order n, can be embedded in a Latin square $\mathcal{B}$, of order n^2, with the additional property that $\mathcal{B}$ has t mutually orthogonal mates. This result will then allow us to show that any PLS(s) where $s \leqslant n/2$ can be embedded in a Latin square $\mathcal{B}$ of order n^2 such that $\mathcal{B}$ has t orthogonal mates that are also mutually orthogonal. Thus this result, and the associated construction, allows us to generalize Jenkins' result which is stated as:

Theorem 3.1 ([8])**.** *Let L be a Latin square of order n with $n \geqslant 3$ and $n \neq 6$. Then L can be embedded in a Latin square of order n^2 which has an orthogonal mate.*

From here forward when we say $\mathcal{B}$ has t mutually orthogonal mates we mean $\mathcal{B}$ together with t more Latin squares of the same order forms a set of $t+1$ mutually orthogonal Latin squares.

Theorem 3.2. *Let $F_1 = [F_1(r,c)], \dots, F_t = [F_t(r,c)]$ be t mutually orthogonal Latin squares of order n indexed by $[n] = \{0, 1, \dots, n-1\}$. Let $L = [L(r,c)]$ be a Latin square of order n, also indexed by $[n]$. Then the arrays $\mathcal{B}$ and $\mathcal{X}_k$, for $1 \leqslant k \leqslant t$, form a set of $t+1$ mutually orthogonal Latin squares of order n^2 where*

$$\begin{aligned}
\mathcal{X}_k &= \{((p,r),(q,c),(F_k(F_1(p,r),q), F_k(F_1(p,q),c))) \mid 0 \leqslant p,q,r,c \leqslant n-1\},\\
\mathcal{B} &= \{((p,r),(q,c),(F_1(p,q), L(F_1(p,r),c))) \mid 0 \leqslant p,q,r,c \leqslant n-1\}.
\end{aligned}$$

Proof. For completeness we begin by showing these arrays are Latin squares, then that $\mathcal{X}_k$, $1 \leqslant k \leqslant t$, are mutually orthogonal and finally that for each k, $\mathcal{X}_k$ and $\mathcal{B}$ are orthogonal.

Assume that one of $\mathcal{X}_k$, $1 \leqslant k \leqslant t$ or $\mathcal{B}$ is not a Latin square. Then

- for some (p,r), there exists (q,c) and (q',c'), with $(q,c) \neq (q',c')$, such that $(F_k(F_1(p,r),q), F_k(F_1(p,q),c)) = (F_k(F_1(p,r),q'), F_k(F_1(p,q'),c'))$, or $(F_1(p,q), L(F_1(p,r),c)) = (F_1(p,q'), L(F_1(p,r),c'))$;

or

- for some (q,c), there exists (p,r) and (p',r'), with $(p,r) \neq (p',r')$, such that $(F_k(F_1(p,r),q), F_k(F_1(p,q),c)) = (F_k(F_1(p',r'),q), F_k(F_1(p',q),c))$, or $(F_1(p,q), L(F_1(p,r),c)) = (F_1(p',q), L(F_1(p',r'),c))$.

The first case implies

$$F_k(F_1(p,r),q) = F_k(F_1(p,r),q') \text{ and } F_k(F_1(p,q),c) = F_k(F_1(p,q'),c').$$

Thus we may deduce that $q = q'$ and consequently $c = c'$, a contradiction. All the other cases follow in a similar manner and hence $\mathcal{X}_k$, $1 \leqslant k \leqslant t$, and $\mathcal{B}$ are Latin squares of order n^2.

Next assume that $\mathcal{X}_k$ and $\mathcal{X}_\ell$, for $k \neq \ell$ are not orthogonal, and so there exist distinct cells $((p,r),(q,c))$ and $((p',r'),(q',c'))$ such that

$$\begin{aligned}
&(F_k(F_1(p,r),q), F_k(F_1(p,q),c)) = (F_k(F_1(p',r'),q'), F_k(F_1(p',q'),c')) \text{ and}\\
&(F_\ell(F_1(p,r),q), F_\ell(F_1(p,q),c)) = (F_\ell(F_1(p',r'),q'), F_\ell(F_1(p',q'),c')).
\end{aligned}$$

Then

$$F_k(F_1(p,r),q) = F_k(F_1(p',r'),q'), \tag{1}$$
$$F_k(F_1(p,q),c) = F_k(F_1(p',q'),c'), \tag{2}$$
$$F_\ell(F_1(p,r),q) = F_\ell(F_1(p',r'),q'), \tag{3}$$
$$F_\ell(F_1(p,q),c) = F_\ell(F_1(p',q'),c'). \tag{4}$$

But F_k and F_ℓ are orthogonal Latin squares, hence Equations (1) and (3) imply $F_1(p,r) = F_1(p',r')$ and $q = q'$, while Equations (2) and (4) imply $F_1(p,q) = F_1(p',q')$ and $c = c'$. Thus we may deduce that $p = p'$ and hence $r = r'$. So $((p,r),(q,c)) = ((p',r'),(q',c'))$, a contradiction. Hence $\{\mathcal{X}_k \mid 1 \leqslant k \leqslant t\}$, is a set of t MOLS(n^2).

Finally assume that for some $k \in \{1,\ldots,t\}$, $\mathcal{X}_k$ and $\mathcal{B}$ are not orthogonal. Thus there exist distinct cells $((p,r),(q,c))$ and $((p',r'),(q',c'))$ such that

$$\begin{aligned}(F_k(F_1(p,r),q), F_k(F_1(p,q),c)) &= (F_k(F_1(p',r'),q'), F_k(F_1(p',q'),c')) \text{ and}\\ (F_1(p,q), L(F_1(p,r),c)) &= (F_1(p',q'), L(F_1(p',r'),c')).\end{aligned}$$

Then

$$F_k(F_1(p,r),q) = F_k(F_1(p',r'),q'), \tag{5}$$
$$F_k(F_1(p,q),c) = F_k(F_1(p',q'),c'), \tag{6}$$
$$F_1(p,q) = F_1(p',q'), \tag{7}$$
$$L(F_1(p,r),c) = L(F_1(p',r'),c'). \tag{8}$$

Since F_k is a Latin square, Equation (7) substituted into Equation (6) gives $c = c'$. Then Equation (8) gives $F_1(p,r) = F_1(p',r')$ and when substituted into Equation (5) gives $q = q'$. Returning to Equation (7) we get $p = p'$ and consequently $r = r'$. So $((p,r),(q,c)) = ((p',r'),(q',c'))$, a contradiction. Hence for all $1 \leqslant k \leqslant t$, $\mathcal{X}_k$ is orthogonal to $\mathcal{B}$, and the result follows. □

Corollary 3.3. *Let P be a partial Latin square of order n, $n \geqslant 3$. Then P can be embedded in a Latin square $\mathcal{B}$ that has order at most $16n^2$, where $\mathcal{B}$ has at least $2n$ mutually orthogonal mates. Furthermore if P is idempotent then $\mathcal{B}$ can be constructed to be idempotent.*

Proof. We will first embed P in a Latin square L of order m where $2^k = m > 2n \geqslant 2^{k-1}$ which is always possible given Evans' result, Theorem 2.2. We can also assume that L is indexed by $[m] = \{0,1,\ldots,m-1\}$. As is well known, since m is a prime power, there exists a set of $m-1$ mutually orthogonal Latin squares $\{F_1, F_2, \ldots, F_{m-1}\}$ of order m, also indexed by $[m]$ and in standard form (that is, $F_i(0,j) = j$ for each $1 \leqslant i \leqslant m-1$ and $0 \leqslant j \leqslant m-1$). Then the set $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{m-1}, \mathcal{B}\}$ of Theorem 3.2 defined using these F_i is a set of m mutually orthogonal Latin squares of order m^2.

Observe that since $F_1(0,r) = r$, the construction places a copy of P in the sub-array defined by $p = 0$ and $q = 0$ and so P has been embedded in $\mathcal{B}$ which has been shown to have $m-1$ mutually orthogonal mates.

As $2^k = m > 2n \geqslant 2^{k-1}$ we have $2^{k+1} > 4n \geqslant 2^k = m$, so $16n^2 \geqslant m^2$. Hence every partial Latin square of order n embeds in a Latin square of order at most $16n^2$ for which there exists at least $2n$ mutually orthogonal mates.

Now, one can make sure $\mathcal{B}$ is idempotent if P is idempotent. When embedding P, ensure that L is idempotent, which can be guaranteed by Theorem 2.3 because $m \geqslant 2n+1$. Note that F_1 is in standard form and is decomposable into transversals as it has an orthogonal mate. So there exists a transversal of F_1 involving the element $(0,0,0)$. Without loss of generality one can assume that this transversal is on the main diagonal of F_1. So $F_1(p,p) \neq F_1(p',p')$ for $p \neq p'$. Hence, if $p \neq p'$, the cells $((p,r),(p,r))$ and $((p',r),(p',r))$ of $\mathcal{B}$ contain elements with different first coordinates. The second coordinate in cell $((p,r),(p,r))$ of $\mathcal{B}$ is $L(F_1(p,r),r)$. So for each fixed p, these second coordinates form a row-permuted copy of L.

Now consider the subsquare $\mathcal{S}_p$ of $\mathcal{B}$ formed by the cells $((p,r),(p,r'))$ for $0 \leqslant r, r' \leqslant m-1$. The entries in $\mathcal{S}_p$ all have the same first coordinate $F_1(p,p)$, and the second coordinates form a row-permuted copy of L. Since L is idempotent, L has a transversal and by permuting the rows $\{(p,0),(p,1),\ldots,(p,m-1)\}$ of $\mathcal{B}$ we can arrange for this transversal of $\mathcal{S}_p$ to lie on the main diagonal of $\mathcal{B}$. This can be done independently for each $p = 0,1,\ldots,m-1$, and the result is a transversal of $\mathcal{B}$ on its main diagonal. By suitable renaming of the elements of $\mathcal{B}$ we can then arrange for $\mathcal{B}$ to be idempotent. In the case $p = 0$, the original entry in the cell $(0,r),(0,r')$ of $\mathcal{B}$ is $(0, L(r,r'))$, so no permuting of the rows of $\mathcal{S}_0$ or renaming of elements $(0,x)$ is required (strictly speaking we apply the identity permutation and the identity renaming here). Hence $\mathcal{B}$ retains a copy of L in the subsquare $\mathcal{S}_0$. Finally, to complete the proof, we apply the same permutation of the rows and renaming of elements to each $\mathcal{X}_k$ as were applied to $\mathcal{B}$. □

Note that one can increase the number of mutually orthogonal Latin squares that are orthogonal to $\mathcal{B}$ as much as one likes by increasing the order of the embedding Latin square L to guarantee the existence of a larger number of mutually orthogonal Latin squares of the same order as L.

Corollary 3.4. *Let L be a Latin square of order n with $n \geqslant 7$ and $n \neq 10, 18$ or 22. Then L can be embedded in a Latin square $\mathcal{B}$ of order n^2 where $\mathcal{B}$ has at least four mutually orthogonal mates.*

Proof. We know by [3] (Section III.3.6, Table 3.88) and [12] that if $n \geqslant 7$ and $n \neq 10, 18$ or 22, there exist four mutually orthogonal Latin squares of order n. Use these Latin squares to form $\mathcal{B}$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$. □

A *bachelor Latin square* is a Latin square which has no orthogonal mate; equivalently, it is a Latin square with no decomposition into disjoint transversals. A *confirmed bachelor Latin square* is a Latin square that contains an entry through which no transversal passes.

Wanless and Webb [13] have established the existence of confirmed bachelor Latin squares for all possible orders n, $n \notin \{1,3\}$. So it is interesting to note that the above results (including Jenkins' result) established that when one essentially "squares" a bachelor, it is possible to find an orthogonal mate.

4. Embedding a pair of OPLS in a set of MOLS

In this section we make use of the embedding result of Donovan and Yazıcı, [4], to show that a pair of orthogonal partial Latin squares can be embedded in a pair of orthogonal Latin square which have many orthogonal mates.

Theorem 4.1 ([4]). *Let P and Q be a pair of orthogonal partial Latin squares of order n. Then P and Q can be embedded in orthogonal Latin squares of order k^4 and any order greater than or equal to $3k^4$ where $2^a = k \geqslant n > 2^{a-1}$ for some integer a.*

Theorem 4.2. *Let $A_1 = [A_1(i,j)]$, $A_2 = [A_2(i,j)]$ and $B_1 = [B_1(i,j)]$, $B_2 = [B_2(i,j)]$ be pairs of orthogonal Latin squares of order n. Let $C_1 = [C_1(i,j)], \ldots, C_t = [C_t(i,j)]$ be t mutually orthogonal Latin squares of order n. Then the squares*

$$\begin{aligned}
\mathcal{B}_1 &= \{((p,r),(q,c),(A_1(p,q),B_1(r,c)))\},\\
\mathcal{B}_2 &= \{((p,r),(q,c),(A_2(p,q),B_2(r,c)))\},\\
\mathcal{X}_{i,f(i)} &= \{((p,r),(q,c),(C_i(p,B_1(r,c)),C_{f(i)}(q,B_2(r,c)))\},
\end{aligned}$$

where $i \in [t] = \{1,\ldots,t\}$ and $f : [t] \to [t]$ is a bijection, form a set of $t+2$ mutually orthogonal Latin squares of order n^2.

Proof. The arrays $\mathcal{B}_1$ and $\mathcal{B}_2$ may be obtained by taking direct products, so it is clear that they are orthogonal Latin squares.

Assume that the array $\mathcal{X}_{\alpha,\beta}$ is not a Latin square, for some α, β. Then there exists (p,r) such that $(C_\alpha(p,B_1(r,c)), C_\beta(q,B_2(r,c)) = (C_\alpha(p,B_1(r,c')), C_\beta(q',B_2(r,c'))$, for some $(q,c),(q',c')$ with $(q,c) \neq (q',c')$, or there exists (q,c) such that $(C_\alpha(p,B_1(r,c)), C_\beta(q,B_2(r,c))$ $= (C_\alpha(p',B_1(r',c)), C_\beta(q,B_2(r',c))$, for some $(p,r),(p',r')$ with $(p,r) \neq (p',r')$. The former case implies

$$C_\alpha(p,B_1(r,c)) = C_\alpha(p,B_1(r,c')), \tag{9}$$
$$C_\beta(q,B_2(r,c)) = C_\beta(q',B_2(r,c')). \tag{10}$$

By (9) $c = c'$ and so (10) implies $q = q'$, a contradiction. The latter case implies

$$C_\alpha(p,B_1(r,c)) = C_\alpha(p',B_1(r',c)), \tag{11}$$
$$C_\beta(q,B_2(r,c)) = C_\beta(q,B_2(r',c)). \tag{12}$$

But then (12) implies $r = r'$ and by (11) $p = p'$, a contradiction. Hence $\mathcal{X}_{\alpha,\beta}$ is a Latin square.

Next take distinct α and γ, and consequently distinct β and δ, where $\beta = f(\alpha)$ and $\delta = f(\gamma)$. Then assume that for distinct cells $((p,r),(q,c))$ and $((p',r'),(q',c'))$

$$\begin{aligned}
(C_\alpha(p,B_1(r,c)), C_\beta(q,B_2(r,c))) &= (C_\alpha(p',B_1(r',c')), C_\beta(q',B_2(r',c'))),\\
(C_\gamma(p,B_1(r,c)), C_\delta(q,B_2(r,c))) &= (C_\gamma(p',B_1(r',c')), C_\delta(q',B_2(r',c'))).
\end{aligned}$$

Then

$$C_\alpha(p,B_1(r,c)) = C_\alpha(p',B_1(r',c')), \tag{13}$$
$$C_\beta(q,B_2(r,c)) = C_\beta(q',B_2(r',c')), \tag{14}$$
$$C_\gamma(p,B_1(r,c)) = C_\gamma(p',B_1(r',c')), \tag{15}$$
$$C_\delta(q,B_2(r,c)) = C_\delta(q',B_2(r',c')). \tag{16}$$

But C_α is orthogonal to C_γ and so Equations (13) and (15) imply $p = p'$ and $B_1(r, c) = B_1(r', c')$. Further C_β is orthogonal to C_δ and so Equations (14) and (16) imply $q = q'$ and $B_2(r, c) = B_2(r', c')$. Finally B_1 and B_2 are orthogonal and so $r = r'$ and $c = c'$. But this contradicts the assumption that the cells $((p, r), (q, c))$ and $((p', r'), (q', c'))$ are distinct. Hence $\mathcal{X}_{\alpha,\beta}$ and $\mathcal{X}_{\gamma,\delta}$ are orthogonal.

Finally we prove that $\mathcal{B}_1$ and $\mathcal{X}_{\alpha,\beta}$ are orthogonal. Assume this is not the case and that there exist distinct cells $((p, r), (q, c))$ and $((p', r'), (q', c'))$ such that

$$(A_1(p, q), B_1(r, c)) = (A_1(p', q'), B_1(r', c')),$$
$$(C_\alpha(p, B_1(r, c)), C_\beta(q, B_2(r, c))) = (C_\alpha(p', B_1(r', c')), C_\beta(q', B_2(r', c'))).$$

Then

$$A_1(p, q) = A_1(p', q'), \tag{17}$$
$$B_1(r, c) = B_1(r', c'), \tag{18}$$
$$C_\alpha(p, B_1(r, c)) = C_\alpha(p', B_1(r', c')), \tag{19}$$
$$C_\beta(q, B_2(r, c)) = C_\beta(q', B_2(r', c')). \tag{20}$$

Since C_α is a Latin square, substituting Equation (18) into Equation (19) implies $p = p'$. Now since A_1 is a Latin square, Equation (17) implies $q = q'$. Then, since C_β is a Latin square, Equation (20) implies $B_2(r, c) = B_2(r', c')$. But B_1 and B_2 are orthogonal so Equation (18) then gives $r = r'$ and $c = c'$. Consequently $\mathcal{B}_1$ and $\mathcal{X}_{\alpha,\beta}$ are orthogonal. Similarly it can be shown that $\mathcal{B}_2$ and $\mathcal{X}_{\alpha,\beta}$ are orthogonal. $\square$

Note that we can use $A_1, A_2, B_1, B_2 \in \{C_1, C_2, \ldots C_t\}$. So the pairs (A_1, A_2) and (B_1, B_2) do not have to be distinct from the set $\{C_1, C_2, \ldots C_t\}$

Corollary 4.3. *For any $t \geqslant 2$, a pair of mutually orthogonal partial Latin squares of order n can be embedded in a set of t mutually orthogonal Latin squares of polynomial order with respect to n.*

Proof. Let A_1 and A_2 be two orthogonal partial Latin squares of order n. By Theorem 4.1 we can embed them into two orthogonal Latin squares $\mathcal{A}_1$ and $\mathcal{A}_2$ of order k^4 where $2^a = k \geqslant n > 2^{a-1}$. As k is a power of a prime, there are at least $k^4 - 1$ MOLS(k^4). So there are at least $(k^4 - 1 + 2)$ MOLS(k^8) two of which contain the copies of $\mathcal{A}_1$ and $\mathcal{A}_2$. Similarly by choosing the order of $\mathcal{A}_1$ and $\mathcal{A}_2$ larger, one can obtain as many orthogonal mates as one wants at the expense of increasing the order of the squares into which the partial Latin squares are embedded. $\square$

Obviously Theorem 4.2 can also be used to construct mutually orthogonal Latin squares of order n^2 for a given integer n. For example, in the literature only 8 mutually orthogonal Latin squares of order 576 are known to exist, but the following corollaries constructs 9 MOLS(576).

Corollary 4.4. $N(n^2) \geqslant N(n) + 2$.

Corollary 4.5. *There are* 9 *mutually orthogonal Latin squares of order* 576.

Proof. By [3] Table 3.87 there are at least 7 mutually orthogonal Latin squares of order 24. When applied in the construction given in Theorem 4.2, we may obtain $7 + 2 = 9$ mutually orthogonal Latin squares of order $24^2 = 576$. $\square$

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