A SHORT PROOF THAT ALL LINEAR CODES ARE WEAKLY ALGEBRAIC-GEOMETRIC USING BERTINI THEOREMS OF B. POONEN

SRIMATHY SRINIVASAN

ABSTRACT. In this paper we give a simpler proof of a deep theorem proved by Pellikan, Shen and van Wee that all linear codes are weakly algebraic-geometric using a theorem of B.Poonen.

1. Introduction

Algebraic-geometric codes were first dicovered by Goppa ([Gop81]) and were further developed by Tsfasman,Vladut ([TV91]) and many others along the way. We start by briefly recalling the construction of algebraic-geometric codes. The text [Har77] by Hartshorne is a good reference for all the basic algebraic geometry and notations we use in this paper.

Definition 1. Let X be a smooth projective variety defined over \mathbb{F}_q and let $\mathcal{P} = \{P_1, P_2, \dots P_n\} \subseteq X(\mathbb{F}_q)$. Let D be a divisor on X such that the support of D is disjoint from \mathcal{P} . Define

$$L(D) = \{ f \in \mathbb{F}_q(X)^* | (f) + D \ge 0 \} \cup \{ 0 \}$$

and consider the evaluation map:

$$Ev_{\mathcal{P}}: L(D) \longrightarrow \mathbb{F}_q^n$$

 $f \longmapsto [f(P_1), f(P_2), ..., f(P_n)]$

The image of the map gives a linear code $C = C_L(X, \mathcal{P}, D)$ and we say that C is an algebraic-geometric code realized over X.

Given the data X, \mathcal{P} , D as above, let \mathcal{L} denote the line bundle associated to D and $H^0(X,\mathcal{L})$ denote its global sections. Then we can get a code $C(X,\mathcal{P},\mathcal{L})$ equivalent to $C_L(X,\mathcal{P},D)$ as follows. First note that the local ring \mathcal{L}_{P_i} modulo the maximal ideal of sections vanishing at P_i denoted by $\overline{\mathcal{L}}_{P_i}$ is isomorphic to \mathbb{F}_q by a choice of local trivialization. Then the image of the germ map

$$\alpha_{\mathcal{P}}: H^0(X, \mathcal{L}) \longrightarrow \bigoplus_{i=1}^n \overline{\mathcal{L}}_{P_i} \cong \mathbb{F}_q^n$$

gives a linear code $C(X, \mathcal{P}, \mathcal{L})$ that is same as the code $C_L(X, \mathcal{P}, D)$ upto monomial equivalence.

Remark: In Definition 1, if X is a smooth curve, we get the original construction of Goppa (Goppa codes [Gop81]). In this case, the parameters of the code are easily estimated using the Riemann-Roch theorem. However, it is not so easy for codes over higher dimensional varieties as invoking Riemann-Roch brings higher cohomology groups come into picture.

In the paper [PSvW91], Pellikan, Shen and van Wee define the notion of *weakly algebraic-geometric codes* which we now recall:

Definition 2. A q-ary linear code C is said to be weakly algebraic geometric if there exists a projective non-singular absolutely irreducible curve X defined over \mathbb{F}_q , n distinct points $\mathcal{P} = \{P_1, P_2 \cdots P_n\}$ on X and a divisor D with support disjoint from \mathcal{P} such that $C = C_L(X, \mathcal{P}, D)$.

In their paper, the authors show that every linear code is weakly algebraic-geometric (Theorem 2, [PSvW91]). The goal of this paper is to give a simpler proof of this deep theorem using a theorem of B.Poonen. Although there are articles in the literature such as [Cou11] that apply Poonen's theorem to algebraic-geometric codes, there does not seem to be any literature that state this result.

2. ALL LINEAR CODES ARE WEAKLY ALGEBRAIC-GEOMETRIC

In this section we give a shorter proof of Theorem 2 of Pellikan, Shen, van Wee ([PSvW91]).

We first show that algebraic-geometric codes are ubiquitous in the sense that every linear code can be realized over some smooth variety. In fact we have the following stronger result.

Theorem 1. Let C be a linear code. Then $C = C_L(X, \mathcal{P}, D)$ where X is the blow up of some projective space at finitely many points, \mathcal{P} is a finite set of distinct \mathbb{F}_q -points in X and D is a divisor such that the support of D is disjoint from \mathcal{P} .

Proof. Let \mathcal{C} be a $(n,k,d)_q$ linear code with $k\times n$ generator matrix G. Then the columns $C_1,C_2,\cdots C_n$ of G form (not necessarily distinct) points of \mathbb{A}^k . Then we can find an integer $r\geq 2$ and n distinct points $P_1,P_2,\cdots P_n$ in \mathbb{A}^{r+k} such that the projection map

$$\phi: \mathbb{A}^{r+k} \to \mathbb{A}^k$$
$$[y_1, y_2, \dots, y_r, x_1, x_2, \dots, x_k] \to [x_1, x_2, \dots, x_n]$$

takes P_i to C_i . Let $y_0, y_1, \cdots, y_r, x_1, x_2, \cdots, x_k$ denote the coordinates of \mathbb{P}^{r+k} . Identify \mathbb{A}^{r+k} with the open affine set $y_0=1$ in \mathbb{P}^{r+k} . For $1\leq i\leq r$, let V_i denote the point in \mathbb{P}^{r+k} with $y_0=y_i=1$ and all other coordinates 0. By choosing r large enough we can assume that $V_i\neq P_j \ \forall i,j$. Let X be the smooth geometrically integral variety obtained via the blow up $\pi:X\to\mathbb{P}^{r+k}$ at the points V_i with the corresponding exceptional divisor E_i . Denote by H the hyperplane section $y_0=0$ in \mathbb{P}^{r+k} . Then the global sections of the line bundle associated to the divisor $D=\mathcal{L}(\pi^*H-\sum_i E_i)$ is generated by x_0,x_1,\cdots,x_k . It is easy to see that code $\mathcal{C}=C(X,\mathcal{P},D)$ where \mathcal{P} is the set $\{\pi^{-1}P_1,\pi^{-1}P_2,\cdots,\pi^{-1}P_n\}$.

Let us now restate the results on Bertini theorems over finite fields due to B.Poonen. We refer the reader to Theorem 1.1 in [Poo08] and remarks below Theorem 3.3 in [Poo04] for more details.

Theorem 2 (Poonen). Let X be a smooth, projective geometrically integral variety of \mathbb{P}^n of dimension $m \geq 2$ over \mathbb{F}_q , and let $\mathcal{P} \subset X$ be a finite set of closed points. Then, given any integer d_0 , there exists a hypersurface $H \subset \mathbb{P}^n$ of degree $d \geq d_0$ such that $Y = H \cap X$ is smooth, projective and geometrically integral of dimension m-1 and contains \mathcal{P} .

Example. Consider $X = \mathbb{P}^2$ over \mathbb{F}_2 . Let \mathcal{P} be the set of all $7 \mathbb{F}_2$ -points. Then, the curve $Y = yz^3 + y^3z + xy^3 + x^2z^2 + x^2y^2 + x^3z$ is a smooth curve passing through \mathcal{P} . In fact, one can show that there are 24 smooth curves of degree 4 passing through \mathcal{P} .

Using Theorem 2 we now show that codes realized over higher dimensional varieties can be realized over curves.

Theorem 3. Let $C = C(X, \mathcal{P}, \mathcal{L})$ be a code on a geometrically integral smooth projective variety $X \subseteq \mathbb{P}^k$ of dimension $m \geq 2$ over \mathbb{F}_q . Then C can be realized over a smooth projective geometrically integral curve. In particular, there exists a geometrically integral smooth projective curve Z containing \mathcal{P} such that $C = C(Z, \mathcal{P}, \mathcal{L}|_Z)$.

Proof. Let $Y = H \cap X$ be as Theorem 2 with degree d of H large enough and let $i: Y \hookrightarrow X$ denote the inclusion morphism. Then, we have the following short exact sequence on X.

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0$$

where $\mathcal{I}_Y = \mathcal{O}_X(-d)$. Tensoring with \mathcal{L} we get

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{I}_Y \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes i_* \mathcal{O}_Y \longrightarrow 0$$

(Here tensoring is over \mathcal{O}_X). The above short exact sequence gives rise to a long exact sequence in cohomology on X

$$0 \longrightarrow H^0(\mathcal{L} \otimes \mathcal{I}_Y, X) \longrightarrow H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L} \otimes i_*\mathcal{O}_Y, X) \longrightarrow H^1(\mathcal{L} \otimes \mathcal{I}_Y, X) \longrightarrow \cdots$$

By duality, we have

$$H^{i}(\mathcal{L} \otimes \mathcal{I}_{Y}, X) \simeq H^{m-i}(\omega_{X} \otimes \mathcal{I}_{Y}^{\vee} \otimes \mathcal{L}^{\vee}, X) \simeq H^{m-i}(\omega_{X} \otimes \mathcal{L}^{\vee} \otimes \mathcal{O}_{X}(d), X)$$

where ω_X is the canonical sheaf on X. Since $\mathcal{O}_X(1)$ is ample, for large enough d, $H^0(\mathcal{L} \otimes \mathcal{I}_Y, X)$ and $H^1(\mathcal{L} \otimes \mathcal{I}_Y, X)$ vanishes and we get a canonical isomorphism obtained via restriction

$$H^0(\mathcal{L}) \xrightarrow{\tilde{}} H^0(\mathcal{L} \otimes i_* \mathcal{O}_Y, X) \simeq H^0(\mathcal{L}|_Y \otimes \mathcal{O}_Y, Y) \simeq H^0(\mathcal{L}|_Y, Y)$$

 $f \longmapsto f_{|Y}.$

Inducting the above argument by replacing the m-dimensional variety X with (m-1)-dimensional variety Y and \mathcal{L} with $\mathcal{L}|_{Y}$ we get the result.

Hence, given a code over a geometrically integral smooth projective variety, we have realized it over a geometrically integral smooth projective curve. As a consequence we get:

Corollary 4 (Pellikan, Shen, van Wee). All linear codes are weakly algebraic-geometric.

Proof. This easily follows from Theorem 1 and Theorem 3.

ACKNOWLEDGEMENTS. I would like to thank Patrick Brosnan and Lawrence Washington for their valuable comments and discussions. I would also like to thank Richard Rast for writing a computer program to verify some results.

REFERENCES

- [Cou11] Alain Couvreur. Differential approach for the study of duals of algebraic-geometric codes on surfaces. *J. Théor. Nombres Bordeaux*, 23(1):95–120, 2011.
- [Gop81] V. D. Goppa. Codes on algebraic curves. *Dokl. Akad. Nauk SSSR*, 259(6):1289–1290, 1981.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [Poo04] Bjorn Poonen. Bertini theorems over finite fields. Ann. of Math. (2), 160(3):1099–1127, 2004.
- [Poo08] Bjorn Poonen. Smooth hypersurface sections containing a given subscheme over a finite field. *Math. Res. Lett.*, 15(2):265–271, 2008.
- [PSvW91] R. Pellikaan, B.-Z. Shen, and G. J. M. van Wee. Which linear codes are algebraic-geometric? *IEEE Trans. Inform. Theory*, 37(3, part 1):583–602, 1991.
- [TV91] M. A. Tsfasman and S. G. Vlăduţ. *Algebraic-geometric codes*, volume 58 of *Mathematics and its Applications* (*Soviet Series*). Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the Russian by the authors.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, CO 80309, USA