# A Note on Specializations of Grothendieck Polynomials 

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#### Abstract

Buch and Rimányi proved a formula for a specialization of double Grothendieck polynomials based on the Yang-Baxter equation related to the degenerate Hecke algebra. A geometric proof was found by Yong and Woo by constructing a Gröbner basis for the Kazhdan-Lusztig ideals. In this note, we give an elementary proof for this formula by using only divided difference operators.


## 1 Introduction

Let $S_{n}$ denote the symmetric group of permutations of $\{1,2, \ldots, n\}$. For a permutation $w \in S_{n}$, the double Grothendieck polynomial $\mathfrak{G}_{w}(x ; y)$ introduced by Lascoux and Schützenberger [12] is the polynomial representative of the class of the Schubert variety for $w$ in the equivariant $K$-theory of the flag manifold. Write a permutation $v \in S_{n}$ in one-line notation, that is, write $v=v(1) v(2) \cdots v(n)$. The specialization

$$
\begin{equation*}
\mathfrak{G}_{w}\left(y_{v} ; y\right):=\mathfrak{G}_{w}\left(y_{v(1)}, \ldots, y_{v(n)} ; y\right) \tag{1.1}
\end{equation*}
$$

of $\mathfrak{G}_{w}(x ; y)$ obtained by replacing $x_{i}$ with $y_{v(i)}$ gives the restriction of this class to the fixed point corresponding to $v$. Buch and Rimányi [4] proved a formula for $\mathfrak{G}_{w}\left(y_{v} ; y\right)$ based on the Yang-Baxter equation related to the degenerate Hecke algebra. Buch and Rimányi [4] also pointed out various important applications of this formula. By constructing a Gröbner basis for the Kazhdan-Lusztig ideals, Yong and Woo [15] found a geometric explanation for the Buch-Rimányi formula.

In this note, we give an elementary proof of the Buch-Rimányi formula by using only divided difference operators. As observed by Buch and Rimányi 4, Corollary 2.3], the classical pipe dream (or, RC-graph) formula of $\mathfrak{G}_{w}(x ; y)$ (see for example [10, Corollary 5.4], [13, Theorem $6.3])$ can be directly obtained from the specialization $\mathfrak{G}_{w}\left(y_{v} ; y\right)$. Hence our approach implies that the pipe dream formula for double Grothendieck polynomials can be derived directly from divided difference operators.

## 2 The Buch-Rimányi formula

Fix a nonnegative integer $n$. For $1 \leq i<j \leq n$, let $t_{i j}$ denote the transposition $(i, j)$ in $S_{n}$. So, if $w \in S_{n}$, then $w t_{i j}$ is the permutation obtained from $w$ by interchanging $w(i)$ and $w(j)$, while $t_{i j} w$ is obtained from $w$ by interchanging the values $i$ and $j$. For example, for $w=2143$, we have $w t_{13}=4123$ and $t_{13} w=2341$. Write $s_{i}$ for the adjacent transposition $(i, i+1)$. Each permutation can be written as a product of adjacent transpositions. The length $\ell(w)$ of a permutation $w$ is the minimum $k$ such that $w=s_{i_{1}} s_{i_{2}} \cdots s_{i_{k}}$, and in this case, $\left(s_{i_{1}}, s_{i_{2}}, \ldots, s_{i_{k}}\right)$ is called a reduced word of $w$. It is well known that the length $\ell(w)$ is equal to the number of pairs $(i, j)$ such that $i<j$ and $w(i)>w(j)$ :

$$
\ell(w)=\#\{(i, j): 1 \leq i<j \leq n, w(i)>w(j)\} .
$$

Hence, it is clear that $\ell\left(w s_{i}\right)=\ell(w)+1$ if and only if $w(i)<w(i+1)$, while $\ell\left(w s_{i}\right)=\ell(w)-1$ if and only if $w(i)>w(i+1)$.

Let $\mathbb{Z}\left[x^{ \pm}, y^{ \pm}\right]$denote the ring of Laurent polynomials in the $2 n$ commuting indeterminates $x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}$. For a Laurent polynomial $f(x, y) \in \mathbb{Z}\left[x^{ \pm}, y^{ \pm}\right]$, the divided difference operator $\partial_{i}$ acting on $f(x, y)$ is defined by

$$
\partial_{i} f=\left(f-s_{i} f\right) /\left(x_{i}-x_{i+1}\right),
$$

where $s_{i} f$ is obtained from $f$ by interchanging $x_{i}$ and $x_{i+1}$. It is easy to check that $\partial_{i} f$ is still a Laurent polynomial. Let $w_{0}=n \cdots 21$ be the longest permutation in $S_{n}$. Set

$$
\begin{equation*}
\mathfrak{G}_{w_{0}}(x ; y)=\prod_{i+j \leq n}\left(1-\frac{y_{j}}{x_{i}}\right) . \tag{2.1}
\end{equation*}
$$

For $w \neq w_{0}$, choose an adjacent transposition $s_{i}$ such that $\ell\left(w s_{i}\right)=\ell(w)+1$. Let $\pi_{i}=\partial_{i} x_{i}$ and define

$$
\begin{equation*}
\mathfrak{G}_{w}(x ; y)=\pi_{i} \mathfrak{G}_{w s_{i}}(x ; y)=\frac{x_{i} \mathfrak{G}_{w s_{i}}(x ; y)-x_{i+1} \mathfrak{G}_{w s_{i}}\left(\ldots, x_{i+1}, x_{i}, \ldots ; y\right)}{x_{i}-x_{i+1}} \tag{2.2}
\end{equation*}
$$

The above definition is independent of the choice of $s_{i}$ since the operators $\pi_{i}$ satisfy the Coxeter relations: $\pi_{i} \pi_{j}=\pi_{j} \pi_{i}$ for $|i-j|>1$, and $\pi_{i} \pi_{i+1} \pi_{i}=\pi_{i+1} \pi_{i} \pi_{i+1}$, see for example [14, (2.14)].

We remark that there are other equivalent definitions for double Grothendieck polynomials. The definition adopted here implies that $\mathfrak{G}_{w}(x ; y)$ are Laurent polynomials. The double Grothendieck polynomials $\mathfrak{L}_{w}^{(-1)}(y ; x)$ defined in 5 are legitimate polynomials, which can be obtained from $\mathfrak{G}_{w}(x ; y)$ by replacing $x_{i}$ and $y_{i}$ respectively with $\frac{1}{1-x_{i}}$ and $1-y_{i}$. It should also be noticed that $\mathfrak{G}_{w}\left(x^{-1} ; y^{-1}\right)$ are the double Grothendieck polynomials used in [9], and $\mathfrak{G}_{w}\left(x^{-1} ; y\right)$ are the double Grothendieck polynomials appearing in [10]. It is worth mentioning that the double Schubert polynomial $\mathfrak{S}_{w}(x ; y)$ is the lowest degree homogeneous component of $\mathfrak{L}_{w}^{(-1)}(y ; x)$, see [1,2,6, 7, 11] for combinatorial constructions of Schubert polynomials.

To describe the Buch-Rimányi formula, consider the left-justified array $\Delta_{n}$ with $n-i$ squares in row $i$. Let $w=w(1) w(2) \cdots w(n) \in S_{n}$. For $1 \leq i \leq n$, let

$$
I(w, i)=\{w(j): j>i, w(j)<w(i)\}
$$

be the set of entries in $w$ that are smaller than $w(i)$ but appear to the right of $w(i)$. Set $c(w, i)=|I(w, i)|$. It is clear that $0 \leq c(w, i) \leq n-i$. Let $D(w)$ be the subset of $\Delta_{n}$ consisting of the first $c(w, i)$ squares in the $i$-th row of $\Delta_{n}$, where $1 \leq i \leq n$. Note that $D(w)$ corresponds to the bottom RC-graph of $w$, as defined by Bergeron and Billey [1]. Assume that the values in $I(w, i)$ are

$$
w\left(j_{1}\right)<w\left(j_{2}\right)<\cdots<w\left(j_{c(w, i)}\right) .
$$

For a square $B \in D(w)$ in row $i$ and column $k$, equip $B$ with the weight

$$
\mathrm{wt}(B)=1-\frac{y_{w\left(j_{k}\right)}}{y_{w(i)}},
$$

see Figure 2.1 for an illustration.


Figure 2.1: Weights of squares of $D(w)$ for $w=2157634$.
Given a subset $D$ of $D(w)$, one can generate a word, denoted $\operatorname{word}(D)$, as follows. Label the square of $D(w)$ in row $i$ and column $k$ by the simple transposition $s_{i+k-1}$, see Figure 2.2 for an illustration. Then $\operatorname{word}(D)$ is obtained by reading off the labels of the squares in $D$


Figure 2.2: Labels of the squares of $D(w)$ for $w=2157634$.
along the rows from top to bottom and right to left. For example, for the diagram $D=D(w)$ in Figure 2.2, we have

$$
\operatorname{word}(D)=\left(s_{1}, s_{4}, s_{3}, s_{6}, s_{5}, s_{4}, s_{6}, s_{5}\right)
$$

A word $\left(s_{i_{1}}, s_{i_{2}}, \ldots, s_{i_{m}}\right)$ is called a Hecke word of a permutation $u$ of length $m$ if

$$
\left(\left(\left(s_{i_{1}} * s_{i_{2}}\right) * s_{i_{3}}\right) * \cdots\right) * s_{i_{m}}=u
$$

where, for a permutation $w$, we define $w * s_{i}$ to be $w$ if $\ell\left(w s_{i}\right)<\ell(w)$ and $w s_{i}$ otherwise. For example, $\left(s_{1}, s_{2}, s_{1}, s_{2}\right)$ is a Hecke word of $u=321$ of length 4 since

$$
\left(\left(s_{1} * s_{2}\right) * s_{1}\right) * s_{2}=\left(\left(s_{1} s_{2}\right) * s_{1}\right) * s_{2}=\left(s_{1} s_{2} s_{1}\right) * s_{2}=s_{1} s_{2} s_{1}=321 .
$$

We note in passing that the operation $*$ can be extended to an associative operation on the whole $S_{n}$; this latter operation is the multiplication in the Hecke algebra associated to $S_{n}$ at $q=0$, see [8, Chapter 7.4]. Hence $*$ satisfies the associative property. This means that the set of permutations in $S_{n}$ forms a monoid structure ( 0 -Hecke monoid) under the operation $*$.

Write $\operatorname{Hecke}(D)=u$ if $\operatorname{word}(D)$ is a Hecke word of a permutation $u$. Notice that a Hecke word of $u$ of length $\ell(u)$ is a reduced word of $u$. Note that for any $w \in S_{n}$, the word $\operatorname{word}(D(w))$ is a reduced word of $w$, and therefore, if we multiply the letters of $\operatorname{word}(D(w))$ using either the * product or the usual product of $S_{n}$, then we get $w$. That is, $\operatorname{Hecke}(D(w))=w$.

For any $u, v \in S_{n}$, let

$$
\mathcal{H}(u, v)=\{D \subseteq D(v) \mid \operatorname{Hecke}(D)=u\} .
$$

For a subset $D$ of $D(v)$, let

$$
\begin{equation*}
\mathrm{wt}(D)=\prod_{B \in D} \mathrm{wt}(B) \tag{2.3}
\end{equation*}
$$

Theorem 2.1 (Buch-Rimányi [4, Theorem 2.1]). For permutations $u, v \in S_{n}$, we have

$$
\begin{equation*}
\mathfrak{G}_{u}\left(y_{v} ; y\right)=\sum_{D \in \mathcal{H}(u, v)}(-1)^{|D|-\ell(u)} \mathrm{wt}(D), \tag{2.4}
\end{equation*}
$$

where empty sums are interpreted as 0.
We remark that in [4], formula (2.4) is described in terms of the notation $C\left(\mathfrak{D}_{v}\right)$ and FKgraphs for $u$ with respect to $\mathfrak{D}_{v}$. With the notation in this note, $D(v)$ can be obtained from $C\left(\mathfrak{D}_{v}\right)$ by first reflecting along the main diagonal and then left-justifying the crossing positions. This operation also establishes a weight preserving bijection between the set $\mathcal{H}(u, v)$ and the set of FK-graphs for $u$ with respect to $\mathfrak{D}_{v}$.

## 3 Elementary proof of Theorem 2.1

We need several lemmas which follow directly from the definition of $\mathfrak{G}_{w}(x ; y)$.
Lemma 3.1. Let $v=v^{\prime} s_{i}$ and $\ell(v)>\ell\left(v^{\prime}\right)$. If $\ell\left(u s_{i}\right)<\ell(u)$, then

$$
\begin{equation*}
\mathfrak{G}_{u}\left(y_{v} ; y\right)=\frac{y_{v^{\prime}(i)}}{y_{v^{\prime}(i+1)}} \mathfrak{G}_{u}\left(y_{v^{\prime}} ; y\right)+\left(1-\frac{y_{v^{\prime}(i)}}{y_{v^{\prime}(i+1)}}\right) \mathfrak{G}_{u s_{i}}\left(y_{v^{\prime}} ; y\right) . \tag{3.1}
\end{equation*}
$$

Proof. Applying (2.2) to $w=u s_{i}$ and substituting $x_{j}$ with $y_{v^{\prime}(j)}$, we have

$$
\mathfrak{G}_{u s_{i}}\left(y_{v^{\prime}} ; y\right)=\frac{y_{v^{\prime}(i)} \mathfrak{G}_{u}\left(y_{v^{\prime}} ; y\right)-y_{v^{\prime}(i+1)} \mathfrak{G}_{u}\left(y_{v} ; y\right)}{y_{v^{\prime}(i)}-y_{v^{\prime}(i+1)}},
$$

which is equivalent to (3.1).
Lemma 3.2. Let $v=v^{\prime} s_{i}$. If $\ell\left(u s_{i}\right)>\ell(u)$, then

$$
\begin{equation*}
\mathfrak{G}_{u}\left(y_{v} ; y\right)=\mathfrak{G}_{u}\left(y_{v^{\prime}} ; y\right) . \tag{3.2}
\end{equation*}
$$

Proof. Applying (2.2) to $w=u$ and substituting $x_{j}$ with $y_{v(j)}$ and $y_{v^{\prime}(j)}$ respectively, we see that

$$
\begin{aligned}
\mathfrak{G}_{u}\left(y_{v} ; y\right) & =\frac{y_{v(i)} \mathfrak{G}_{u s_{i}}\left(y_{v} ; y\right)-y_{v(i+1)} \mathfrak{G}_{u s_{i}}\left(y_{v^{\prime}} ; y\right)}{y_{v(i)}-y_{v(i+1)}}, \\
\mathfrak{G}_{u}\left(y_{v^{\prime}} ; y\right) & =\frac{y_{v^{\prime}(i)} \mathfrak{G}_{u s_{i}}\left(y_{v^{\prime}} ; y\right)-y_{v^{\prime}(i+1)} \mathfrak{G}_{u s_{i}}\left(y_{v} ; y\right)}{y_{v^{\prime}(i)}-y_{v^{\prime}(i+1)}},
\end{aligned}
$$

which, together with the fact that $v(i)=v^{\prime}(i+1)$ and $v(i+1)=v^{\prime}(i)$, implies (3.2).
Let $\leq$ denote the (strong) Bruhat order on permutations of $S_{n}$. Recall that the Bruhat order is the closure of the following covering relation: For $u, v \in S_{n}$, we say that $v$ covers $u$ if there exists a transposition $t_{i j}$ such that $v=u t_{i j}$ and $\ell(v)=\ell(u)+1$. The following lemma is known, see [4, Corollary 2.4] and the references therein.

Lemma 3.3. We have $\mathfrak{G}_{u}\left(y_{v} ; y\right)=0$ whenever $u \not \leq v$ in the Bruhat order.
Proof. The idea in the proof of [11, (2.22)] for double Schubert polynomials applies to double Grothendieck polynomials, and we include a proof here for the reader's convenience. Use descending induction on $\ell(u)$. The initial case is $u=w_{0}$. Since $u \not \leq v$, we have $v \neq w_{0}$. It is easily checked from (2.1) that $\mathfrak{G}_{w_{0}}\left(y_{v} ; y\right)=0$.

We now consider the case $u \neq w_{0}$. Choose a position $i$ such that $u(i)<u(i+1)$. Note that $u<u s_{i}$. Since $u \not \leq v$, we must have $u s_{i} \not \leq v$. We further claim that $u s_{i} \not \leq v s_{i}$. This can be seen as follows. We have either $v s_{i}<v$ or $v<v s_{i}$ (depending on which of $\ell\left(v s_{i}\right)$ and $\ell(v)$ is larger). If $v s_{i}<v$, then it is clear that $u s_{i} \not \leq v s_{i}$ since otherwise there would hold $u \leq v$. It remains to verify the case $v<v s_{i}$. Suppose to the contrary that $u s_{i} \leq v s_{i}$. Then $u<v s_{i}$. Since $v s_{i}>v$ and $u s_{i}>u$, applying the Lifting Property (see [3, Proposition 2.2.7]) to $u^{-1}$ and $\left(v s_{i}\right)^{-1}$, we obtain that $u \leq v$, leading to a contradiction. Now, by the definition in (2.2) and by the induction hypothesis,

$$
\mathfrak{G}_{u}\left(y_{v} ; y\right)=\frac{y_{v(i)} \mathfrak{G}_{u s_{i}}\left(y_{v} ; y\right)-y_{v(i+1)} \mathfrak{G}_{u s_{i}}\left(y_{v s_{i}} ; y\right)}{y_{v(i)}-y_{v(i+1)}}=0,
$$

as desired.
Lemma 3.4. Let $u \in S_{n}$ and $u^{\prime}=u s_{i}$ for some $i$ such that $\ell\left(u s_{i}\right)<\ell(u)$. Then,

$$
\mathfrak{G}_{u}\left(y_{u} ; y\right)=\left(1-\frac{y_{u(i+1)}}{y_{u(i)}}\right) \mathfrak{G}_{u^{\prime}}\left(y_{u^{\prime}} ; y\right) .
$$

Proof. Apply Lemma 3.1 to $v=u$ and $v^{\prime}=u^{\prime}$. The first addend on the right side vanishes due to Lemma 3.3 ,

Lemma 3.5 (Buch-Rimányi [4, Corollary 2.6]). For each $u \in S_{n}$, we have

$$
\mathfrak{G}_{u}\left(y_{u} ; y\right)=\prod_{\substack{i<j \\ u(i)>u(j)}}\left(1-\frac{y_{u(j)}}{y_{u(i)}}\right) .
$$

Proof. Make descending induction on $\ell(u)$. The induction base for $u=w_{0}$ is a restatement of (2.1). Assume that $u \neq w_{0}$. Then there exists some $1 \leq k<n$ such that $\ell\left(u s_{k}\right)>\ell(u)$. Let $u^{\prime}=u s_{k}$. It is easy to see that the set

$$
\left\{\left(u^{\prime}(i), u^{\prime}(j)\right) \mid i<j, u^{\prime}(i)>u^{\prime}(j)\right\}
$$

is the union of the two disjoint sets

$$
\{(u(i), u(j)) \mid i<j, u(i)>u(j)\} \cup\{(u(k), u(k+1))\} .
$$

The proof follows by induction together with Lemma 3.4.
Proof of Theorem 2.1. The proof is by induction on $\ell(v)$. Let us first consider the case $\ell(v)=0$, that is, $v$ is the identity permutation $e$. If $u=e$, then it follows from Lemma 3.5 (applied to $u=e$ ) that $\mathfrak{G}_{e}\left(y_{e} ; y\right)=1$. If $u \neq e$, then Lemma 3.3 forces that $\mathfrak{G}_{u}\left(y_{e} ; y\right)=0$. So (2.4) holds for $\ell(v)=0$.

Assume now that $\ell(v)>0$. Let $s_{r}$ be the last descent of $v$, that is, $r$ is the largest index such that $v(r)>v(r+1)$. Write $v=v^{\prime} s_{r}$. Clearly, the bottom row of $D(v)$ lies in row $r$ of $\Delta_{n}$. The leftmost square in the bottom row of $D(v)$, denoted $B_{0}$, has weight

$$
\mathrm{wt}\left(B_{0}\right)=1-\frac{y_{v(r+1)}}{y_{v(r)}}=1-\frac{y_{v^{\prime}(r)}}{y_{v^{\prime}(r+1)}} .
$$

Let $u=u^{\prime} s_{r}$. There are two cases.
Case 1. $s_{r}$ is a descent of $u$. By Lemma 3.1 and by induction hypothesis, we have

$$
\begin{align*}
\mathfrak{G}_{u}\left(y_{v} ; y\right)= & \frac{y_{v^{\prime}(r)}}{y_{v^{\prime}(r+1)}} \mathfrak{G}_{u}\left(y_{v^{\prime}} ; y\right)+\left(1-\frac{y_{v^{\prime}(r)}}{y_{v^{\prime}(r+1)}}\right) \mathfrak{G}_{u^{\prime}}\left(y_{v^{\prime}} ; y\right) \\
= & \left(1-\operatorname{wt}\left(B_{0}\right)\right) \sum_{D \in \mathcal{H}\left(u, v^{\prime}\right)}(-1)^{|D|-\ell(u)} \mathrm{wt}(D)+\operatorname{wt}\left(B_{0}\right) \sum_{D \in \mathcal{H}\left(u^{\prime}, v^{\prime}\right)}(-1)^{|D|-\ell\left(u^{\prime}\right)} \mathrm{wt}(D) \\
= & \sum_{D \in \mathcal{H}\left(u, v^{\prime}\right)}(-1)^{|D|-\ell(u)} \mathrm{wt}(D)-\operatorname{wt}\left(B_{0}\right) \sum_{D \in \mathcal{H}\left(u, v^{\prime}\right)}(-1)^{|D|-\ell(u)} \mathrm{wt}(D) \\
& +\operatorname{wt}\left(B_{0}\right) \sum_{D \in \mathcal{H}\left(u^{\prime}, v^{\prime}\right)}(-1)^{|D|-\ell\left(u^{\prime}\right)} \mathrm{wt}(D) . \tag{3.3}
\end{align*}
$$

To proceed, note that there is an obvious bijection $\phi$ between $D\left(v^{\prime}\right)$ and $D(v) \backslash\left\{B_{0}\right\}$. Since $s_{r}$ is the last descent of $v$, we have $c\left(v^{\prime}, r\right)=0, c\left(v^{\prime}, r+1\right)=c(v, r)-1$, and $c\left(v^{\prime}, i\right)=c(v, i)$
for $i \neq r, r+1$. Let $B \in D\left(v^{\prime}\right)$. If $B$ lies above row $r$, then set $\phi(B)=B$. Assume that $B$ lies in row $r+1$ and column $j$, then let $\phi(B)$ be the square of $D(v) \backslash\left\{B_{0}\right\}$ in row $r$ and column $j+1$. By construction, $B$ and $\phi(B)$ are labeled by the same simple transposition. Moreover, it is easy to see that $\phi$ preserves the weight and words, namely, $\operatorname{wt}(B)=\operatorname{wt}(\phi(B))$ and $\operatorname{word}(\phi(D))=\operatorname{word}(D)$ for all $D \subseteq D\left(v^{\prime}\right)$. Thus $\operatorname{Hecke}(\phi(D))=\operatorname{Hecke}(D)$ for all $D \subseteq D\left(v^{\prime}\right)$.

We claim that $\mathcal{H}(u, v)$ is the disjoint union of the following sets:

$$
\begin{aligned}
& S_{1}=\left\{\phi(D): D \in \mathcal{H}\left(u, v^{\prime}\right)\right\}, \\
& S_{2}=\left\{\phi(D) \cup\left\{B_{0}\right\}: D \in \mathcal{H}\left(u, v^{\prime}\right)\right\}, \\
& S_{3}=\left\{\phi(D) \cup\left\{B_{0}\right\}: D \in \mathcal{H}\left(u^{\prime}, v^{\prime}\right)\right\} .
\end{aligned}
$$

This can be easily seen as follows. Keep in mind that $B_{0}$ is labeled by $s_{r}$. Let $D \in \mathcal{H}(u, v)$. If $B_{0} \notin D$, then $D \in S_{1}$. If $B_{0} \in D$, then $\operatorname{word}(D)$ is obtained from $\operatorname{word}\left(D \backslash\left\{B_{0}\right\}\right)$ by appending the letter $s_{r}$ at the end, and thus we have $\operatorname{Hecke}(D)=\operatorname{Hecke}\left(D \backslash\left\{B_{0}\right\}\right) * s_{r}$, and therefore either $\operatorname{Hecke}\left(D \backslash\left\{B_{0}\right\}\right)=u$ or $\operatorname{Hecke}\left(D \backslash\left\{B_{0}\right\}\right)=u^{\prime}$. Hence either $D \in S_{2}$ or $D \in S_{3}$. Conversely, any $D \in S_{1} \cup S_{2} \cup S_{3}$ belongs to $\mathcal{H}(u, v)$, since $u * s_{r}=u^{\prime} * s_{r}=u$. By the above claim and in view of (3.3), we obtain that

$$
\mathfrak{G}_{u}\left(y_{v} ; y\right)=\sum_{D \in S_{1} \cup S_{2} \cup S_{3}}(-1)^{|D|-\ell(u)} \mathrm{wt}(D)=\sum_{D \in \mathcal{H}(u, v)}(-1)^{|D|-\ell(u)} \mathrm{wt}(D) .
$$

Case 2. $s_{r}$ is not a descent of $u$. Let $D \in \mathcal{H}(u, v)$. We claim that $B_{0} \notin D$. Suppose otherwise that $B_{0} \in D$. Consider $D^{\prime}=D \backslash\left\{B_{0}\right\}$. If $s_{r}$ is a descent of $\operatorname{Hecke}\left(D^{\prime}\right)$, then $\operatorname{Hecke}(D)=$ $\operatorname{Hecke}\left(D^{\prime}\right)$, while if $s_{r}$ is not a descent of $\operatorname{Hecke}\left(D^{\prime}\right)$, then $\operatorname{Hecke}(D)=\operatorname{Hecke}\left(D^{\prime}\right) s_{r}$. In both cases, $s_{r}$ is a descent of $u=\operatorname{Hecke}(D)$, leading to a contradiction. Therefore, we see that $\mathcal{H}(u, v)=\left\{\phi(D) \mid D \in \mathcal{H}\left(u, v^{\prime}\right)\right\}$. By Lemma 3.2 and by induction hypothesis,

$$
\mathfrak{G}_{u}\left(y_{v} ; y\right)=\mathfrak{G}_{u}\left(y_{v^{\prime}} ; y\right)=\sum_{D \in \mathcal{H}\left(u, v^{\prime}\right)}(-1)^{|D|-\ell(u)} \mathrm{wt}(D)=\sum_{D \in \mathcal{H}(u, v)}(-1)^{|D|-\ell(u)} \mathrm{wt}(D) .
$$

This completes the proof.
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