A Note on Specializations of Grothendieck Polynomials

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Abstract

Buch and Rimányi proved a formula for a specialization of double Grothendieck polynomials based on the Yang-Baxter equation related to the degenerate Hecke algebra. A geometric proof was found by Yong and Woo by constructing a Gröbner basis for the Kazhdan-Lusztig ideals. In this note, we give an elementary proof for this formula by using only divided difference operators.

1 Introduction

Let S_n denote the symmetric group of permutations of $\{1, 2, ..., n\}$. For a permutation $w \in S_n$, the double Grothendieck polynomial $\mathfrak{G}_w(x; y)$ introduced by Lascoux and Schützenberger [12] is the polynomial representative of the class of the Schubert variety for w in the equivariant K-theory of the flag manifold. Write a permutation $v \in S_n$ in one-line notation, that is, write $v = v(1)v(2)\cdots v(n)$. The specialization

$$\mathfrak{G}_w(y_v; y) := \mathfrak{G}_w(y_{v(1)}, \dots, y_{v(n)}; y) \tag{1.1}$$

of $\mathfrak{G}_w(x;y)$ obtained by replacing x_i with $y_{v(i)}$ gives the restriction of this class to the fixed point corresponding to v. Buch and Rimányi [4] proved a formula for $\mathfrak{G}_w(y_v;y)$ based on the Yang-Baxter equation related to the degenerate Hecke algebra. Buch and Rimányi [4] also pointed out various important applications of this formula. By constructing a Gröbner basis for the Kazhdan-Lusztig ideals, Yong and Woo [15] found a geometric explanation for the Buch-Rimányi formula.

In this note, we give an elementary proof of the Buch-Rimányi formula by using only divided difference operators. As observed by Buch and Rimányi [4, Corollary 2.3], the classical pipe dream (or, RC-graph) formula of $\mathfrak{G}_w(x;y)$ (see for example [10, Corollary 5.4], [13, Theorem 6.3]) can be directly obtained from the specialization $\mathfrak{G}_w(y_v;y)$. Hence our approach implies that the pipe dream formula for double Grothendieck polynomials can be derived directly from divided difference operators.

2 The Buch-Rimányi formula

Fix a nonnegative integer n. For $1 \leq i < j \leq n$, let t_{ij} denote the transposition (i, j) in S_n . So, if $w \in S_n$, then wt_{ij} is the permutation obtained from w by interchanging w(i) and w(j), while $t_{ij}w$ is obtained from w by interchanging the values i and j. For example, for w = 2143, we have $wt_{13} = 4123$ and $t_{13}w = 2341$. Write s_i for the adjacent transposition (i, i + 1). Each permutation can be written as a product of adjacent transpositions. The length $\ell(w)$ of a permutation w is the minimum k such that $w = s_{i_1}s_{i_2}\cdots s_{i_k}$, and in this case, $(s_{i_1}, s_{i_2}, \ldots, s_{i_k})$ is called a reduced word of w. It is well known that the length $\ell(w)$ is equal to the number of pairs (i, j) such that i < j and w(i) > w(j):

$$\ell(w) = \#\{(i,j) \colon 1 \le i < j \le n, \ w(i) > w(j)\}.$$

Hence, it is clear that $\ell(ws_i) = \ell(w) + 1$ if and only if w(i) < w(i+1), while $\ell(ws_i) = \ell(w) - 1$ if and only if w(i) > w(i+1).

Let $\mathbb{Z}[x^{\pm}, y^{\pm}]$ denote the ring of Laurent polynomials in the 2n commuting indeterminates $x_1, \ldots, x_n, y_1, \ldots, y_n$. For a Laurent polynomial $f(x, y) \in \mathbb{Z}[x^{\pm}, y^{\pm}]$, the divided difference operator ∂_i acting on f(x, y) is defined by

$$\partial_i f = (f - s_i f) / (x_i - x_{i+1}),$$

where $s_i f$ is obtained from f by interchanging x_i and x_{i+1} . It is easy to check that $\partial_i f$ is still a Laurent polynomial. Let $w_0 = n \cdots 21$ be the longest permutation in S_n . Set

$$\mathfrak{G}_{w_0}(x;y) = \prod_{i+j \le n} \left(1 - \frac{y_j}{x_i}\right).$$
(2.1)

For $w \neq w_0$, choose an adjacent transposition s_i such that $\ell(ws_i) = \ell(w) + 1$. Let $\pi_i = \partial_i x_i$ and define

$$\mathfrak{G}_{w}(x;y) = \pi_{i} \mathfrak{G}_{ws_{i}}(x;y) = \frac{x_{i} \mathfrak{G}_{ws_{i}}(x;y) - x_{i+1} \mathfrak{G}_{ws_{i}}(\dots, x_{i+1}, x_{i}, \dots; y)}{x_{i} - x_{i+1}}.$$
(2.2)

The above definition is independent of the choice of s_i since the operators π_i satisfy the Coxeter relations: $\pi_i \pi_j = \pi_j \pi_i$ for |i - j| > 1, and $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$, see for example [14, (2.14)].

We remark that there are other equivalent definitions for double Grothendieck polynomials. The definition adopted here implies that $\mathfrak{G}_w(x;y)$ are Laurent polynomials. The double Grothendieck polynomials $\mathfrak{L}_w^{(-1)}(y;x)$ defined in [5] are legitimate polynomials, which can be obtained from $\mathfrak{G}_w(x;y)$ by replacing x_i and y_i respectively with $\frac{1}{1-x_i}$ and $1-y_i$. It should also be noticed that $\mathfrak{G}_w(x^{-1};y^{-1})$ are the double Grothendieck polynomials used in [9], and $\mathfrak{G}_w(x^{-1};y)$ are the double Grothendieck polynomials appearing in [10]. It is worth mentioning that the double Schubert polynomial $\mathfrak{S}_w(x;y)$ is the lowest degree homogeneous component of $\mathfrak{L}_w^{(-1)}(y;x)$, see [1,2,6,7,11] for combinatorial constructions of Schubert polynomials.

To describe the Buch-Rimányi formula, consider the left-justified array Δ_n with n-i squares in row *i*. Let $w = w(1)w(2)\cdots w(n) \in S_n$. For $1 \le i \le n$, let

$$I(w,i) = \{w(j) : j > i, w(j) < w(i)\}$$

be the set of entries in w that are smaller than w(i) but appear to the right of w(i). Set c(w,i) = |I(w,i)|. It is clear that $0 \le c(w,i) \le n-i$. Let D(w) be the subset of Δ_n consisting of the first c(w,i) squares in the *i*-th row of Δ_n , where $1 \le i \le n$. Note that D(w) corresponds to the bottom RC-graph of w, as defined by Bergeron and Billey [1]. Assume that the values in I(w,i) are

$$w(j_1) < w(j_2) < \dots < w(j_{c(w,i)})$$

For a square $B \in D(w)$ in row i and column k, equip B with the weight

$$\operatorname{wt}(B) = 1 - \frac{y_{w(j_k)}}{y_{w(i)}},$$

see Figure 2.1 for an illustration.

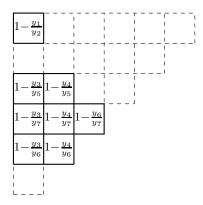


Figure 2.1: Weights of squares of D(w) for w = 2157634.

Given a subset D of D(w), one can generate a word, denoted word(D), as follows. Label the square of D(w) in row i and column k by the simple transposition s_{i+k-1} , see Figure 2.2 for an illustration. Then word(D) is obtained by reading off the labels of the squares in D

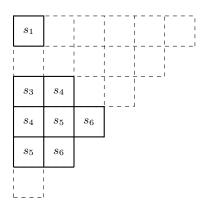


Figure 2.2: Labels of the squares of D(w) for w = 2157634.

along the rows from top to bottom and right to left. For example, for the diagram D = D(w) in Figure 2.2, we have

$$word(D) = (s_1, s_4, s_3, s_6, s_5, s_4, s_6, s_5).$$

A word $(s_{i_1}, s_{i_2}, \ldots, s_{i_m})$ is called a Hecke word of a permutation u of length m if

$$(((s_{i_1} * s_{i_2}) * s_{i_3}) * \cdots) * s_{i_m} = u,$$

where, for a permutation w, we define $w * s_i$ to be w if $\ell(ws_i) < \ell(w)$ and ws_i otherwise. For example, (s_1, s_2, s_1, s_2) is a Hecke word of u = 321 of length 4 since

$$((s_1 * s_2) * s_1) * s_2 = ((s_1 s_2) * s_1) * s_2 = (s_1 s_2 s_1) * s_2 = s_1 s_2 s_1 = 321.$$

We note in passing that the operation * can be extended to an associative operation on the whole S_n ; this latter operation is the multiplication in the Hecke algebra associated to S_n at q = 0, see [8, Chapter 7.4]. Hence * satisfies the associative property. This means that the set of permutations in S_n forms a monoid structure (0-Hecke monoid) under the operation *.

Write $\operatorname{Hecke}(D) = u$ if $\operatorname{word}(D)$ is a Hecke word of a permutation u. Notice that a Hecke word of u of length $\ell(u)$ is a reduced word of u. Note that for any $w \in S_n$, the word $\operatorname{word}(D(w))$ is a reduced word of w, and therefore, if we multiply the letters of $\operatorname{word}(D(w))$ using either the * product or the usual product of S_n , then we get w. That is, $\operatorname{Hecke}(D(w)) = w$.

For any $u, v \in S_n$, let

$$\mathcal{H}(u, v) = \{ D \subseteq D(v) \,|\, \text{Hecke}(D) = u \}.$$

For a subset D of D(v), let

$$\operatorname{wt}(D) = \prod_{B \in D} \operatorname{wt}(B).$$
(2.3)

Theorem 2.1 (Buch-Rimányi [4, Theorem 2.1]). For permutations $u, v \in S_n$, we have

$$\mathfrak{G}_u(y_v; y) = \sum_{D \in \mathcal{H}(u,v)} (-1)^{|D| - \ell(u)} \mathrm{wt}(D), \qquad (2.4)$$

where empty sums are interpreted as 0.

We remark that in [4], formula (2.4) is described in terms of the notation $C(\mathfrak{D}_v)$ and FKgraphs for u with respect to \mathfrak{D}_v . With the notation in this note, D(v) can be obtained from $C(\mathfrak{D}_v)$ by first reflecting along the main diagonal and then left-justifying the crossing positions. This operation also establishes a weight preserving bijection between the set $\mathcal{H}(u, v)$ and the set of FK-graphs for u with respect to \mathfrak{D}_v .

3 Elementary proof of Theorem 2.1

We need several lemmas which follow directly from the definition of $\mathfrak{G}_w(x;y)$.

Lemma 3.1. Let $v = v's_i$ and $\ell(v) > \ell(v')$. If $\ell(us_i) < \ell(u)$, then

$$\mathfrak{G}_{u}(y_{v};y) = \frac{y_{v'(i)}}{y_{v'(i+1)}} \mathfrak{G}_{u}(y_{v'};y) + \left(1 - \frac{y_{v'(i)}}{y_{v'(i+1)}}\right) \mathfrak{G}_{us_{i}}(y_{v'};y).$$
(3.1)

Proof. Applying (2.2) to $w = us_i$ and substituting x_j with $y_{v'(j)}$, we have

$$\mathfrak{G}_{us_i}(y_{v'};y) = \frac{y_{v'(i)}\mathfrak{G}_u(y_{v'};y) - y_{v'(i+1)}\mathfrak{G}_u(y_v;y)}{y_{v'(i)} - y_{v'(i+1)}}$$

which is equivalent to (3.1).

Lemma 3.2. Let $v = v's_i$. If $\ell(us_i) > \ell(u)$, then

$$\mathfrak{G}_u(y_v; y) = \mathfrak{G}_u(y_{v'}; y). \tag{3.2}$$

Proof. Applying (2.2) to w = u and substituting x_j with $y_{v(j)}$ and $y_{v'(j)}$ respectively, we see that

$$\begin{split} \mathfrak{G}_{u}(y_{v};y) &= \frac{y_{v(i)}\mathfrak{G}_{us_{i}}(y_{v};y) - y_{v(i+1)}\mathfrak{G}_{us_{i}}(y_{v'};y)}{y_{v(i)} - y_{v(i+1)}}, \\ \mathfrak{G}_{u}(y_{v'};y) &= \frac{y_{v'(i)}\mathfrak{G}_{us_{i}}(y_{v'};y) - y_{v'(i+1)}\mathfrak{G}_{us_{i}}(y_{v};y)}{y_{v'(i)} - y_{v'(i+1)}}, \end{split}$$

which, together with the fact that v(i) = v'(i+1) and v(i+1) = v'(i), implies (3.2).

Let \leq denote the (strong) Bruhat order on permutations of S_n . Recall that the Bruhat order is the closure of the following covering relation: For $u, v \in S_n$, we say that v covers u if there exists a transposition t_{ij} such that $v = ut_{ij}$ and $\ell(v) = \ell(u) + 1$. The following lemma is known, see [4, Corollary 2.4] and the references therein.

Lemma 3.3. We have $\mathfrak{G}_u(y_v; y) = 0$ whenever $u \leq v$ in the Bruhat order.

Proof. The idea in the proof of [11, (2.22)] for double Schubert polynomials applies to double Grothendieck polynomials, and we include a proof here for the reader's convenience. Use descending induction on $\ell(u)$. The initial case is $u = w_0$. Since $u \not\leq v$, we have $v \neq w_0$. It is easily checked from (2.1) that $\mathfrak{G}_{w_0}(y_v; y) = 0$.

We now consider the case $u \neq w_0$. Choose a position *i* such that u(i) < u(i+1). Note that $u < us_i$. Since $u \not\leq v$, we must have $us_i \not\leq v$. We further claim that $us_i \not\leq vs_i$. This can be seen as follows. We have either $vs_i < v$ or $v < vs_i$ (depending on which of $\ell(vs_i)$ and $\ell(v)$ is larger). If $vs_i < v$, then it is clear that $us_i \not\leq vs_i$ since otherwise there would hold $u \leq v$. It remains to verify the case $v < vs_i$. Suppose to the contrary that $us_i \leq vs_i$. Then $u < vs_i$. Since $vs_i > v$ and $us_i > u$, applying the Lifting Property (see [3, Proposition 2.2.7]) to u^{-1} and $(vs_i)^{-1}$, we obtain that $u \leq v$, leading to a contradiction. Now, by the definition in (2.2) and by the induction hypothesis,

$$\mathfrak{G}_{u}(y_{v};y) = \frac{y_{v(i)}\mathfrak{G}_{us_{i}}(y_{v};y) - y_{v(i+1)}\mathfrak{G}_{us_{i}}(y_{vs_{i}};y)}{y_{v(i)} - y_{v(i+1)}} = 0,$$

as desired.

Lemma 3.4. Let $u \in S_n$ and $u' = us_i$ for some i such that $\ell(us_i) < \ell(u)$. Then,

$$\mathfrak{G}_u(y_u;y) = \left(1 - \frac{y_{u(i+1)}}{y_{u(i)}}\right) \mathfrak{G}_{u'}(y_{u'};y).$$

Proof. Apply Lemma 3.1 to v = u and v' = u'. The first addend on the right side vanishes due to Lemma 3.3.

Lemma 3.5 (Buch-Rimányi [4, Corollary 2.6]). For each $u \in S_n$, we have

$$\mathfrak{G}_u(y_u; y) = \prod_{\substack{i < j \\ u(i) > u(j)}} \left(1 - \frac{y_{u(j)}}{y_{u(i)}} \right).$$

Proof. Make descending induction on $\ell(u)$. The induction base for $u = w_0$ is a restatement of (2.1). Assume that $u \neq w_0$. Then there exists some $1 \leq k < n$ such that $\ell(us_k) > \ell(u)$. Let $u' = us_k$. It is easy to see that the set

$$\{(u'(i), u'(j)) \mid i < j, \ u'(i) > u'(j)\}$$

is the union of the two disjoint sets

$$\{(u(i), u(j)) \mid i < j, \ u(i) > u(j)\} \cup \{(u(k), u(k+1))\}.$$

The proof follows by induction together with Lemma 3.4.

Proof of Theorem 2.1. The proof is by induction on $\ell(v)$. Let us first consider the case $\ell(v) = 0$, that is, v is the identity permutation e. If u = e, then it follows from Lemma 3.5 (applied to u = e) that $\mathfrak{G}_e(y_e; y) = 1$. If $u \neq e$, then Lemma 3.3 forces that $\mathfrak{G}_u(y_e; y) = 0$. So (2.4) holds for $\ell(v) = 0$.

Assume now that $\ell(v) > 0$. Let s_r be the last descent of v, that is, r is the largest index such that v(r) > v(r+1). Write $v = v's_r$. Clearly, the bottom row of D(v) lies in row r of Δ_n . The leftmost square in the bottom row of D(v), denoted B_0 , has weight

wt(B₀) = 1 -
$$\frac{y_{v(r+1)}}{y_{v(r)}} = 1 - \frac{y_{v'(r)}}{y_{v'(r+1)}}.$$

Let $u = u's_r$. There are two cases.

Case 1. s_r is a descent of u. By Lemma 3.1 and by induction hypothesis, we have

$$\mathfrak{G}_{u}(y_{v};y) = \frac{y_{v'(r)}}{y_{v'(r+1)}} \mathfrak{G}_{u}(y_{v'};y) + \left(1 - \frac{y_{v'(r)}}{y_{v'(r+1)}}\right) \mathfrak{G}_{u'}(y_{v'};y)
= (1 - \operatorname{wt}(B_{0})) \sum_{D \in \mathcal{H}(u,v')} (-1)^{|D| - \ell(u)} \operatorname{wt}(D) + \operatorname{wt}(B_{0}) \sum_{D \in \mathcal{H}(u',v')} (-1)^{|D| - \ell(u')} \operatorname{wt}(D)
= \sum_{D \in \mathcal{H}(u,v')} (-1)^{|D| - \ell(u)} \operatorname{wt}(D) - \operatorname{wt}(B_{0}) \sum_{D \in \mathcal{H}(u,v')} (-1)^{|D| - \ell(u)} \operatorname{wt}(D)
+ \operatorname{wt}(B_{0}) \sum_{D \in \mathcal{H}(u',v')} (-1)^{|D| - \ell(u')} \operatorname{wt}(D).$$
(3.3)

To proceed, note that there is an obvious bijection ϕ between D(v') and $D(v) \setminus \{B_0\}$. Since s_r is the last descent of v, we have c(v', r) = 0, c(v', r+1) = c(v, r) - 1, and c(v', i) = c(v, i)

for $i \neq r, r+1$. Let $B \in D(v')$. If B lies above row r, then set $\phi(B) = B$. Assume that B lies in row r+1 and column j, then let $\phi(B)$ be the square of $D(v) \setminus \{B_0\}$ in row r and column j+1. By construction, B and $\phi(B)$ are labeled by the same simple transposition. Moreover, it is easy to see that ϕ preserves the weight and words, namely, wt $(B) = \text{wt}(\phi(B))$ and word $(\phi(D)) = \text{word}(D)$ for all $D \subseteq D(v')$. Thus $\text{Hecke}(\phi(D)) = \text{Hecke}(D)$ for all $D \subseteq D(v')$.

We claim that $\mathcal{H}(u, v)$ is the disjoint union of the following sets:

$$S_{1} = \{\phi(D) : D \in \mathcal{H}(u, v')\},\$$

$$S_{2} = \{\phi(D) \cup \{B_{0}\} : D \in \mathcal{H}(u, v')\},\$$

$$S_{3} = \{\phi(D) \cup \{B_{0}\} : D \in \mathcal{H}(u', v')\}.\$$

This can be easily seen as follows. Keep in mind that B_0 is labeled by s_r . Let $D \in \mathcal{H}(u, v)$. If $B_0 \notin D$, then $D \in S_1$. If $B_0 \in D$, then word(D) is obtained from word $(D \setminus \{B_0\})$ by appending the letter s_r at the end, and thus we have $\operatorname{Hecke}(D) = \operatorname{Hecke}(D \setminus \{B_0\}) * s_r$, and therefore either $\operatorname{Hecke}(D \setminus \{B_0\}) = u$ or $\operatorname{Hecke}(D \setminus \{B_0\}) = u'$. Hence either $D \in S_2$ or $D \in S_3$. Conversely, any $D \in S_1 \cup S_2 \cup S_3$ belongs to $\mathcal{H}(u, v)$, since $u * s_r = u' * s_r = u$. By the above claim and in view of (3.3), we obtain that

$$\mathfrak{G}_{u}(y_{v};y) = \sum_{D \in S_{1} \cup S_{2} \cup S_{3}} (-1)^{|D|-\ell(u)} \operatorname{wt}(D) = \sum_{D \in \mathcal{H}(u,v)} (-1)^{|D|-\ell(u)} \operatorname{wt}(D).$$

Case 2. s_r is not a descent of u. Let $D \in \mathcal{H}(u, v)$. We claim that $B_0 \notin D$. Suppose otherwise that $B_0 \in D$. Consider $D' = D \setminus \{B_0\}$. If s_r is a descent of $\operatorname{Hecke}(D')$, then $\operatorname{Hecke}(D) = \operatorname{Hecke}(D')$, while if s_r is not a descent of $\operatorname{Hecke}(D')$, then $\operatorname{Hecke}(D) = \operatorname{Hecke}(D') s_r$. In both cases, s_r is a descent of $u = \operatorname{Hecke}(D)$, leading to a contradiction. Therefore, we see that $\mathcal{H}(u, v) = \{\phi(D) \mid D \in \mathcal{H}(u, v')\}$. By Lemma 3.2 and by induction hypothesis,

$$\mathfrak{G}_{u}(y_{v};y) = \mathfrak{G}_{u}(y_{v'};y) = \sum_{D \in \mathcal{H}(u,v')} (-1)^{|D|-\ell(u)} \mathrm{wt}(D) = \sum_{D \in \mathcal{H}(u,v)} (-1)^{|D|-\ell(u)} \mathrm{wt}(D).$$

This completes the proof.

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