An odd $[1, b]$-factor in regular graphs from eigenvalues

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Abstract

An odd $[1, b]$-factor of a graph G is a spanning subgraph H such that for each vertex $v \in V(G)$, $d_H(v)$ is odd and $1 \le d_H(v) \le b$. Let $\lambda_3(G)$ be the third largest eigenvalue of the adjacency matrix of G. For positive integers $r \ge 3$ and even n, Lu, Wu, and Yang [10] proved a lower bound for $\lambda_3(G)$ in an n-vertex r-regular graph G to gurantee the existence of an odd $[1, b]$-factor in G. In this paper, we improve the bound; it is sharp for every r.

Keywords: Odd $[1, b]$-factor, eigenvalues

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1 Introduction

In this paper we deal only with finite and undirected graphs without loops or multiple edges. The *adjacency matrix* $A(G)$ of G is the n-by-n matrix in which entry $a_{i,j}$ is 1 or 0 according to whether v_i and v_j are adjacent or not, where $V(G) = \{v_1, \dots, v_n\}$. The *eigenvalues* of G are the eigenvalues of its adjacency matrix $A(G)$. Let $\lambda_1(G), \dots, \lambda_n(G)$ be its eigenvalues in nonincreasing order. Note that the spectral radius of G, written $\rho(G)$ equals $\lambda_1(G)$.

The degree of a vertex v in $V(G)$, written $d_G(v)$, is the number of vertices adjacent to v. An *odd* (or *even*) $[a, b]$-*factor* of a graph G is a spanning subgraph H of G such that for each vertex $v \in V(G)$, $d_H(v)$ is odd (or even) and $a \le d_H(v) \le b$; an $[a, a]$-factor is called the a-*factor*. For a positive integer r, a graph is r-*regular* if every vertex has the same degree r. Note that $\lambda_1(G) = r$ if G is r-regular. Many researchers proved the conditions for a graph

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to have an a-factor, or (even or odd) $[a,b]$-factor. (See [2, 9, 11, 12]) Brouwer and Haemers started to investiage the relations between eigenvalues and the existence of 1-factor.

In fact, they [5] proved that if G is an r-regular graph without an 1-factor, then

$$\lambda_3(G) > \begin{cases} r-1+\frac{3}{r+1} & \text{if } r \text{ is even,} \\ r-1+\frac{3}{r+2} & \text{if } r \text{ is odd} \end{cases}$$

by using Tuttes 1-Factor Theorem [13], which is a special case of Berge-Tutte Formula [3]. Cioabă, Gregory, and Haemers [6] improved their bound and in fact proved that if G is an r-regular graph without an 1-factor, then

$$\lambda_3(G) \geq \begin{cases} \theta = 2.85577... & \text{if } r = 3, \\ \frac{1}{2}(r-2+\sqrt{r^2+12}) & \text{if } r \geq 4 \text{ is even,} \\ \frac{1}{2}(r-3+\sqrt{(r+1)^2+16}) & \text{if } r \geq 5 \text{ is odd,} \end{cases}$$

where θ is the largest root of $x^3 - x^2 - 6x + 2 = 0$. More generally, O and Cioabă [7] determined connections between the eigenvalues of a t-edge connected r-regular graph and its matching number when $1 \leq t \leq r-2$. In 2010, Lu, Wu, and Yang [10] proved that if an r-regular graph G with even number of vertices has no odd $[1,b]$-factor, then

$$\lambda_3(G) > \begin{cases} r - \frac{\lceil \frac{r}{b} \rceil - 2}{r+1} + \frac{1}{(r+1)(r+2)} & \text{if } r \text{ is even and } \lceil \frac{r}{b} \rceil \text{ is even,} \\ r - \frac{\lceil \frac{r}{b} \rceil - 1}{r+1} + \frac{1}{(r+1)(r+2)} & \text{if } r \text{ is even and } \lceil \frac{r}{b} \rceil \text{ is odd,} \\ r - \frac{\lceil \frac{r}{b} \rceil - 1}{r+1} + \frac{1}{(r+2)^2} & \text{if } r \text{ is odd and } \lceil \frac{r}{b} \rceil \text{ is even,} \\ r - \frac{\lceil \frac{r}{b} \rceil - 2}{r+1} + \frac{1}{(r+2)^2} & \text{if } r \text{ is odd and } \lceil \frac{r}{b} \rceil \text{ is odd.} \end{cases}$$

To prove the above bounds in the paper [10], they used Amahashi's result.

Theorem 1.1. [1] *Let G be a graph and let b be a positive odd integer. Then G contains an odd $[1,b]$-factor if and only if for every subset $S \subseteq V(G)$, $o(G-S) \leq b|S|$, where $o(H)$ is the number of odd components in a graph H.*

Thoerem 1.1 guarantees that if there is no odd $[1,b]$-factor in an r-regular graph, then there exists a subset $S \in V(G)$ such that $o(G-S) > b|S|$. By counting the number of edges between S and $G-S$, we can show that $G-S$ has at least three odd components Q_1, Q_2, Q_3 such that $|[V(Q_i), S]| \leq r-1$ (see the proof of Theorem [10] or Theorem 3.2). Then they found lower bounds for the largest eigenvalue in a graph in the family $\mathcal{F}_{r,b}$, where $\mathcal{F}_{r,b}$ is a family of such a possible component depending on r and b, and those bounds are appeared above.

In this paper, we improve their bound and in fact prove that if G is an n-vertex r-regular graph without an odd $[1,b]$-factor, then

$$\lambda_3(G) \geq \rho(r,b),$$

where

$$\rho(r,b)=\begin{cases}\frac{r-2+\sqrt{(r+2)^2-4(\lceil\frac{r}{b}\rceil-2)}}{2} & \text{if both } r \text{ and } \lceil\frac{r}{b}\rceil \text{ are even,}\\ \frac{r-2+\sqrt{(r+2)^2-4(\lceil\frac{r}{b}\rceil-1)}}{2} & \text{if } r \text{ is even and } \lceil\frac{r}{b}\rceil \text{ is odd,}\\ \frac{r-3+\sqrt{(r+3)^2-4(\lceil\frac{r}{b}\rceil-2)}}{2} & \text{if both } r \text{ and } \lceil\frac{r}{b}\rceil \text{ are odd,}\\ \frac{r-3+\sqrt{(r+3)^2-4(\lceil\frac{r}{b}\rceil-1)}}{2} & \text{if } r \text{ is odd and } \lceil\frac{r}{b}\rceil \text{ is even.}\end{cases}$$

The bounds that we found are sharp in a sense that there exists a graph H in $\mathcal{F}_{r,b}$ such that $\lambda_1(H)=\rho(r,b)$.

For undefined terms, see West [14] or Godsil and Royle [8].

2 Construction

Suppose that $\varepsilon=\begin{cases}2 & \text{if } r \text{ and } \lceil\frac{r}{b}\rceil \text{ has same parity}\\ 1 & \text{otherwise}\end{cases}$ and $\eta=\lceil\frac{r}{b}\rceil-\varepsilon$. In this section, we provide graphs $H_{r,\eta}$ such that $\lambda_1(H_{r,\eta})=\rho(r,b)$. These graphs show that the bounds in Theorem 3.2 are sharp.

Now, we define the graph $H_{r,\eta}$ as follows:

$$H_{r,\eta}=\begin{cases}\mathrm{K}_{r+1-\eta}\vee\overline{\frac{\eta}{2}\mathrm{K}_2} & \text{if } r \text{ is even,}\\ \overline{\mathrm{C}_\eta}\vee\overline{\frac{r+2-\eta}{2}\mathrm{K}_2} & \text{if } r \text{ is odd.}\end{cases}$$

To compute the spectral radius of $H_{r,\eta}$, the notion of equitable partition of a vertex set in a graph is used. Consider a partition $V(G)=V_1\cup\cdots\cup V_s$ of the vertex set of a graph G into s non-empty subsets. For $1\le i,j\le s$, let $q_{i,j}$ denote the average number of neighbours in V_j of the vertices in V_i. The quotient matrix of this partition is the $s\times s$ matrix whose (i,j)-th entry equals $q_{i,j}$. The eigenvalues of the quotient matrix interlace the eigenvalues of G. This partition is *equitable* if for each $1\le i,j\le s$, any vertex $v\in V_i$ has exactly $q_{i,j}$ neighbours in V_j. In this case, the eigenvalues of the quotient matrix are eigenvalues of G and the spectral radius of the quotient matrix equals the spectral radius of G (see [4],[8] for more details).

Theorem 2.1. *For $r\ge 3$ and $b\ge 1$, we have $\lambda_1(H_{r,\eta})=\rho(r,b)$.*

Proof. We prove this theorem only in the case when r is odd because the proof of the other case is similar.

Consider the vertex partition $\{V(\overline{\mathrm{C}_\eta}),V(\overline{\frac{r+2-\eta}{2}\mathrm{K}_2})\}$ of $H_{r,\eta}$. The quotient matrix of the vertex partitions equals

$$Q=\begin{pmatrix}\eta-3 & r+2-\eta\\ \eta & r-\eta\end{pmatrix}$$

The characteristic polynomail of Q is

$$p(x)=(x-\eta+3)(x-r+\eta)-(r+2-\eta)\eta.$$

Since the vertex partition is equitable, the largest root of the graph $H_{r,\eta}$ equals the largest root of the polynomial, which is $\lambda_1(Q) = \frac{r-3+\sqrt{(r+3)^2-4\eta}}{2}$. $\square$

3 Main results

In this section, we prove an upper bound for $\lambda_3(G)$ in an r-regular graph G with even number of vertices to guarantee the existence of an odd $[1, b]$-factor by using Theorem 1.1 and Theorem 3.1.

Theorem 3.1. [4, 8] *If H is an induced subgraph of a graph G, then $\lambda_i(H) \leq \lambda_i(G)$ for all $i \in \{1, \dots, |V(H)|\}$.*

Theorem 3.2. *Let $r \geq 3$, and b be a positive odd integer less than r. If $\lambda_3(G)$ of an r-regular graph G with even number of vertices is smaller than $\rho(r, b)$, then G has an odd $[1, b]$-factor.*

Proof. We prove the contrapositive. Assume that an r-regular graph G with even number of vertices has no odd $[1, b]$-factor. By Theorem 1.1, there exists a vertex subset $S \subseteq V(G)$ such that $o(G - S) > b|S|$. Note that since $|V(G)|$ is even, b is odd, and $o(G - S) \equiv |S|$ (mod 2), we have $o(G - S) \geq b|S| + 2$. Let $G_1, \dots, G_q$ be the odd components of $G - S$, where $q = o(G - S)$.

Claim 1. There are at least three odd components, say G_1, G_2, G_3, such that $|[V(G_i), S]| < \lceil \frac{r}{b} \rceil$ for all $i \in \{1, 2, 3\}$.

Assume to the contrary that there are at most two such odd components in $G - S$. Since G is r-regular, we have

$$r|S| \geq \sum_{i=1}^{q} |[V(G_i), S]| \geq \lceil \frac{r}{b} \rceil (q - 2) + 2 \geq \lceil \frac{r}{b} \rceil b|S| + 2 \geq r|S| + 2,$$

which is a contradiction.

By Theorem 3.1, we have

$$\lambda_3(G) \geq \lambda_3(G_1 \cup G_2 \cup G_3) \geq \min_{i \in \{1,2,3\}} \lambda_1(G_i). \tag{1}$$

Now, we prove that if H is an odd component of $G - S$ such that $|[V(H), S]| < \lceil \frac{r}{b} \rceil$, then $\lambda_1(H) \geq \rho(r, b)$.

Claim 2. If H is an odd components of $G - S$ such that $|[V(H), S]| < \lceil \frac{r}{b} \rceil$ and if $\lambda_1(H) \leq \lambda_1(H')$ for all odd components H' in $G - S$ such that $|[V(H'), S]| < \lceil \frac{r}{b} \rceil$, then we have $|V(H)| = \begin{cases} r + 2 \text{ if } r \text{ is odd,} \\ r + 1 \text{ if } r \text{ is even} \end{cases}$, and $2|E(H)| = \begin{cases} r(r + 2) - \eta \text{ if } r \text{ is odd,} \\ r(r + 1) - \eta \text{ if } r \text{ is even.} \end{cases}$

Let $x = \begin{cases} 1 \text{ if } r \text{ is odd,} \\ 0 \text{ if } r \text{ is even.} \end{cases}$ Since $|[V(H), S]| < \lceil \frac{r}{b} \rceil < r$ and G is r-regular, we have $|V(H)| \geq r + 1 + x$ since H has an odd number of vertices. If $|V(H)| > r + 1 + x$, then we have $|V(H)| \geq r + 3 + x$ since H has an odd number of vertices. Thus it suffices to show $\rho(r, b) < \lambda_1(H)$ if $|V(H)| \geq r + 3 + x$. By using the fact that $\lambda_1(G) \geq \frac{2|E(G)|}{|V(G)|}$ for any graph G, we have

$$\lambda_1(H) > \frac{r|V(H)| - \eta}{|V(H)|} \geq \frac{r(r + 3 + x) - \eta}{r + 3 + x} > \frac{r - 2 - x + \sqrt{(r + 2 + x)^2 - 4\eta}}{2}.$$

Now, we prove this theorem by considering two cases depending on the parity of r.

Case 1. r is even. By Claim 2, assume that H is an odd component of $G - S$ such that $|[V(H), S]| < \lceil \frac{r}{b} \rceil$, $|V(H)| = r + 1$, and $2|E(H)| = r(r + 1) - \eta$. Then there are at least $r + 1 - \eta$ vertices of degree r. Let V_1 be a set of vertices with degree r such that $|V_1| = r + 1 - \eta$, and let V_2 be the remaining vertices in $V(H)$. Then the quotient matrix of the vertex partition $\{V_1, V_2\}$ of H equals

$$\begin{pmatrix} r - \eta & \eta \\ r + 1 - \eta & \eta - 2 \end{pmatrix}$$

whose characteristic polynomial is $p(x) = (x - r + \eta)(x - \eta + 2) - \eta(r + 1 - \eta)$. Since the largest root of $p(x)$ equals $\rho(r, b)$, we have $\lambda_1(H) \geq \rho(r, b)$.

Case 2. r is odd. By Claim 2, assume that H is an odd component of $G - S$ such that $|[V(H), S]| < \lceil \frac{r}{b} \rceil$, $|V(H)| = r + 2$, and $2|E(H)| = r(r + 2) - \eta$. Then there are at least $r+2-\eta$ vertices of degree r. Let V_1 be a set of vertices with degree r such that $|V_1| = r+2-\eta$, and let V_2 be the remaining vertices in $V(H)$. Suppose that there are m_{12} edges between V_1 and V_2. Note that $(r + 2 - \eta)(\eta - 1) \leq m_{12} \leq (r + 2 - \eta)\eta$. Then the quotient matrix of the vertex partion $\{V_1, V_2\}$ of H equals

$$\begin{pmatrix} r - \frac{m_{12}}{r+2-\eta} & \frac{m_{12}}{r+2-\eta} \\ \frac{m_{12}}{\eta} & r - 1 - \frac{m_{12}}{\eta} \end{pmatrix}$$

whose characteristic polynomial is $q(x) = (x - r + \frac{m_{12}}{r+2-\eta})(x - r + 1 + \frac{m_{12}}{\eta}) - \frac{m_{12}^2}{(r+2-\eta)\eta}$.

Note that since $(r + 2 - \eta)(\eta - 1) \leq m_{12} \leq (r + 2 - \eta)\eta$, m_{12} can be expressed $m_{12} = (r + 2 - \eta)\eta - t$, where $0 \leq t \leq r + 2 - \eta$. Thus we have

$$q(x) = x^2 - (r - 3 + \frac{t(r + 2)}{(r + 2 - \eta)\eta})x - 3r + \eta - \frac{t}{r + 2 - \eta} + \frac{tr(r + 2)}{(r + 2 - \eta)\eta}$$

$$= x^2 - (r - 3)x - 3r + \eta - \frac{t(r + 2)}{(r + 2 - \eta)\eta}x - \frac{t}{r + 2 - \eta} + \frac{tr(r + 2)}{(r + 2 - \eta)\eta}.$$

Note that $q(\rho(r,b)) = -\frac{t(r+2)}{(r+2-\eta)\eta}(\rho(r,b) + \frac{\eta}{r+2} - r) \le 0$, since $\eta \ge 1$ and $0 \le t \le r+2-\eta$. $\square$

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