Discrete Mathematics 343 (2020) 111908

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Compatible spanning circuits in edge-colored graphs[☆]

Zhiwei Guo [a,b], Binlong Li [a,c], Xueliang Li [d], Shenggui Zhang [a,c,*]

[a] School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, PR China
[b] Faculty of EEMCS, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands
[c] Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, PR China
[d] Center for Combinatorics, Nankai University, Tianjin 300071, PR China

ARTICLE INFO

Article history:
Received 18 October 2018
Received in revised form 26 February 2020
Accepted 7 March 2020
Available online 19 March 2020

Keywords:
Edge-colored graph
Spanning eulerian subgraph
Supereulerian graph
Compatible spanning circuit
Ore-type degree condition

ABSTRACT

A spanning circuit in a graph is defined as a closed trail visiting each vertex of the graph. A compatible spanning circuit in an edge-colored graph refers to a spanning circuit in which each pair of edges traversed consecutively along the spanning circuit has distinct colors. As two extreme cases, sufficient conditions for the existence of compatible Hamilton cycles and compatible Euler tours have been obtained in previous literature. In this paper, we first establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible integer k, in edge-colored complete graphs and complete equipartition r-partite graphs. We also provide sufficient conditions for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times in edge-colored graphs satisfying Ore-type degree conditions.

1. Introduction

In this paper we consider only finite undirected graphs without loops or multiple edges. For terminology and notations not defined here, we refer the reader to Bondy and Murty [5].

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and edges of G, respectively. We write $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a vertex v of G, we denote by $E_G(v)$ the set of edges of G incident to v. The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is defined to be the cardinality of $|E_G(v)|$. In particular, we write $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$. We use $N_i(G)$ and $O(G)$ to denote the set of vertices of G of degree i and the vertices of G of odd degree, respectively. For two vertices u, v of G, a *uv-path* of G refers to a path of G connecting u and v, and the *distance* between u and v in G, denoted by $dist_G(u, v)$, is defined as the length of a shortest uv-path of G (if it exists). If the graph G is understood, we will denote $E_G(v)$, $d_G(v)$ and $dist_G(u, v)$ by $E(v)$, $d(v)$ and $dist(u, v)$, respectively.

A *spanning circuit* in a graph G is defined as a closed trail that visits (contains) each vertex of G. A *Hamilton cycle* of G can be regarded as a spanning circuit that visits each vertex of G exactly once; an *Euler tour* of G can be regarded as a spanning circuit that traverses each edge of G. Hence, a spanning circuit can be considered as one common relaxation between a Hamilton cycle and an Euler tour. A graph is said to be *hamiltonian* if it contains a Hamilton cycle, and *eulerian* if it admits an Euler tour. It is well-known that determining whether a graph is hamiltonian is NP-complete, and a lot of sufficient conditions for the existence of Hamilton cycles have been found. There is also a well-known characterization of eulerian

☆ Supported by NSFC, PR China (Nos. 11671320, 11601429 and U1803263), CSC, PR China (No. 201806290049) and the Fundamental Research Funds for the Central Universities, PR China (No. 3102019ghjd003).

* Corresponding author.
E-mail address: sgzhang@nwpu.edu.cn (S. Zhang).

https://doi.org/10.1016/j.disc.2020.111908

graphs that states a connected graph G is eulerian if and only if the degree of each vertex of G is even (see [5]). Moreover, Fleury obtained a polynomial-time algorithm for finding an Euler tour in an arbitrary eulerian graph (see [5]). A *spanning eulerian subgraph* of a graph G refers to an eulerian spanning subgraph of G. Clearly, each spanning circuit of a graph G corresponds to a spanning eulerian subgraph of G. A graph is said to be *supereulerian* if it contains a spanning eulerian subgraph (spanning circuit). Pulleyblank [17] proved that determining whether a graph is supereulerian is NP-complete. For more details on supereulerian graphs, see Catlin's excellent survey [6] and its supplement [13].

An *edge-coloring* of a graph G is defined as a mapping $c : E(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. An *edge-colored graph* refers to a graph with a fixed edge-coloring. Two edges of a graph are said to be *consecutive* with respect to a trail if they are traversed consecutively along the trail. A *compatible spanning circuit* in an edge-colored graph is defined as a spanning circuit in which any two consecutive edges have distinct colors. An edge-colored graph is said to be *properly colored* if each pair of adjacent edges of the graph has distinct colors. Thus, a compatible Hamilton cycle is properly colored, and a properly colored spanning circuit is compatible. Conversely, a compatible spanning circuit is not necessarily properly colored. Thus, a compatible spanning circuit can be viewed as a generalization of a properly colored spanning circuit. There are already some quite interesting results on compatible walks that are closely related to compatible spanning circuits. Compatible walks are of interest as generalizations of walks in undirected and directed graphs, as well as properly colored paths and cycles in edge-colored graphs. For more details on compatible walks, we refer the reader to Chapter 16 of [3] and recent papers [10,11,15]. Moreover, compatible spanning circuits are very useful in graph theory applications, for example, in genetic and molecular biology [16,20,21], in the design of printed circuit and wiring boards [22], and in channel assignment of wireless networks [1,18].

Let G be an edge-colored graph. For an edge e of G, we use $c(e)$ to denote the color of e. Denote by $C(G)$ the set of colors appearing on the edges of G, and let $d^i_G(v)$ be the cardinality of the set $\{e \in E_G(v) \mid c(e) = i\}$ for a vertex $v \in V(G)$ and a color $i \in C(G)$. Set $\Delta^{mon}_G(v) = \max\{d^i_G(v) \mid i \in C(G)\}$ for a vertex $v \in V(G)$, and set $\Delta^{mon}(G) = \max\{\Delta^{mon}_G(v) \mid v \in V(G)\}$; these two parameters are called the *maximum monochromatic degree* of a vertex v of G and the *maximum monochromatic degree* of an edge-colored graph G, respectively. When no confusion occurs, we will use the notation $\Delta^{mon}(v)$ instead of $\Delta^{mon}_G(v)$.

Let K^c_n denote an edge-colored complete graph on n vertices. Daykin [8] asked whether there exists a constant μ such that every K^c_n with $\Delta^{mon}(K^c_n) \le \mu n$ contains a compatible Hamilton cycle. This question was answered independently by Bollobás and Erdős [4] with $\Delta^{mon}(K^c_n) < n/69$, and Chen and Daykin [7] with $\Delta^{mon}(K^c_n) \le n/17$. Moreover, Bollobás and Erdős [4] proposed the following conjecture.

Conjecture 1 (*Bollobás and Erdős [4]*). *Let K^c_n be an edge-colored complete graph on n vertices. If $\Delta^{mon}(K^c_n) < \lfloor n/2 \rfloor$, then K^c_n contains a compatible Hamilton cycle.*

Afterward, Shearer [19] showed that $\Delta^{mon}(K^c_n) < n/7$ is sufficient. Alon and Gutin [2] showed that $\Delta^{mon}(K^c_n) \le (1 - 1/\sqrt{2} - o(1))n$ is sufficient. Recently, Lo [15] proved that Conjecture 1 is true asymptotically.

On the other hand, Kotzig [12] gave a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs, as follows.

Theorem 1 (*Kotzig [12]*). *Let G be an edge-colored eulerian graph. Then a compatible Euler tour exists if and only if $\Delta^{mon}(v) \le d(v)/2$ for each vertex v of G.*

From Theorem 1, we can obtain the following corollary.

Corollary 1. *Let K^c_n be an edge-colored complete graph on n vertices, where $n \ge 3$. If $\Delta^{mon}(K^c_n) \le (n-1)/2$, then K^c_n contains a compatible spanning circuit visiting each vertex exactly $\lfloor (n-1)/2 \rfloor$ times.*

Proof. Let $G = K^c_n$. First we suppose that n is odd. Since $d_G(v) = n - 1$ for every vertex $v \in V(G)$, G is an eulerian graph. If $\Delta^{mon}(G) \le (n-1)/2 = d_G(v)/2$, then the conclusion holds by Theorem 1.

Suppose now that n is even. Let $H = G - M$, where M is an arbitrary prefect matching of G. It follows that $d_H(v) = n - 2$ for every vertex $v \in V(G)$. Thus H is a spanning eulerian subgraph of G. If $\Delta^{mon}(G) \le (n-1)/2$, then $\Delta^{mon}(H) \le \Delta^{mon}(G) \le \lfloor (n-1)/2 \rfloor = d_H(v)/2$. Hence, the conclusion holds by Theorem 1. □

The corollary above implies that an edge-colored complete graph K^c_n ($n \ge 4$) satisfying $\Delta^{mon}(K^c_n) < \lfloor n/2 \rfloor$ (i.e., the condition stated in Conjecture 1) contains a compatible spanning circuit visiting each vertex exactly $\lfloor (n-1)/2 \rfloor$ times. Lo [15] proved that an edge-colored complete graph K^c_n with $\Delta^{mon}(K^c_n) < \lfloor n/2 \rfloor$ contains a compatible Hamilton cycle asymptotically. Compared to a compatible Hamilton cycle, we intuitively feel that a weaker condition might imply that K^c_n contains a compatible spanning circuit. However, from the following construction given by Fujita and Magnant [10], we can show that the condition $\Delta^{mon}(K^c_n) < \lfloor n/2 \rfloor$ to guarantee the existence of compatible spanning circuits (even if with no any restriction on the number of times that it visits each vertex) in K^c_n is best possible for n even.

Let K_{2m} be a complete graph on $2m$ ($m \ge 2$) vertices, and let u be one of the vertices of K_{2m}. We label the remaining vertices with $v_1, v_2, \ldots, v_{2m-1}$, respectively and color the edge uv_i with color i for each v_i, where $1 \le i \le 2m - 1$. Let $H = K_{2m} - u$, and we decompose H into $m - 1$ Hamilton cycles (see [14]). Also, we arbitrarily orient these Hamilton cycles such that they become directed cycles. We color the edge v_iv_j with color j if the arc $\overrightarrow{v_iv_j}$ is an arc of one of these Hamilton cycles. This provides an edge-coloring of K_{2m}. The complete graph K_{2m} satisfies $\Delta^{mon}(K_{2m}) = \lfloor 2m/2 \rfloor = m$, but it contains no compatible spanning circuit, because the vertex u cannot be visited compatibly.

Recall that a compatible Hamilton cycle can be regarded as a compatible spanning circuit visiting each vertex exactly once. Some conditions for the existence of compatible Hamilton cycles and compatible spanning circuits visiting each vertex exactly $\lfloor (n-1)/2 \rfloor$ times in edge-colored complete graphs on n vertices have been considered. It is natural to ask the following problem.

Problem 1. Under what conditions does an edge-colored graph on n vertices contain a compatible spanning circuit visiting each vertex exactly k times for a given integer k with $1 \le k \le \lfloor (n-1)/2 \rfloor$?

In this paper, we first prove the following two theorems, which give some partial answers to the question of Problem 1. We consider Problem 1 in edge-colored complete graphs, as follows.

Theorem 2. *Let K_n^c be an edge-colored complete graph on n vertices, where $n \ge 3$, and let k be an integer such that $1 \le k \le \lfloor (n-1)/2 \rfloor$. If $\Delta^{mon}(K_n^c) \le k$, then K_n^c contains a compatible spanning circuit visiting each vertex exactly k times.*

We postpone the proofs of all our results in order not to interrupt the flow of the narrative. We also consider Problem 1 in edge-colored complete equipartition r-partite graphs, as follows.

Theorem 3. *Let $(K_n^r)^c$ be an edge-colored complete equipartition r-partite graph on rn vertices, and let k be an integer such that $1 \le k \le \lfloor n(r-1)/2 \rfloor$. If $\Delta^{mon}((K_n^r)^c) \le k$, then $(K_n^r)^c$ contains a compatible spanning circuit visiting each vertex exactly k times.*

Motivated by Conjecture 1 and Corollary 1, we intuitively feel that it is possible that an edge-colored complete graph on n vertices satisfying the condition stated in Conjecture 1 contains a compatible spanning circuit visiting each vertex exactly k times for any given integer k with $1 < k < \lfloor (n-1)/2 \rfloor$. Thus, we pose the following problem.

Problem 2. Let K_n^c be an edge-colored complete graph on n vertices with $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$. Is it true that K_n^c contains a compatible spanning circuit visiting each vertex exactly k times for every integer k with $1 < k < \lfloor (n-1)/2 \rfloor$?

In this paper, we also provide some sufficient conditions for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times in edge-colored graphs satisfying Ore-type degree conditions, as shown in the following results.

Theorem 4. *Let G be an edge-colored connected graph on n $(n \ge 3)$ vertices such that $d(u) + d(v) \ge n$ for every pair of vertices u, v of G with $dist(u, v) = 2$. If $\Delta^{mon}(v) \le (d(v)-1)/2$ for each vertex v with $d(v) \ge 3$, and $\Delta^{mon}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times.*

From Theorem 4, we can obtain the following corollaries.

Corollary 2. *Let G be an edge-colored graph on n $(n \ge 3)$ vertices such that $d(u) + d(v) \ge n$ for every pair of nonadjacent vertices u, v of G. If $\Delta^{mon}(v) \le (d(v)-1)/2$ for each vertex v with $d(v) \ge 3$, and $\Delta^{mon}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times.*

Corollary 3. *Let G be an edge-colored graph on n $(n \ge 3)$ vertices such that $d(v) \ge n/2$ for every vertex v of G. If $\Delta^{mon}(v) \le (d(v)-1)/2$ for each vertex v with $d(v) \ge 3$, and $\Delta^{mon}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex at least $\lfloor (n-2)/4 \rfloor$ times.*

We further prove the following theorem.

Theorem 5. *Let G be an edge-colored 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \ge n/2$ for every pair of nonadjacent vertices u, v of G. If $\Delta^{mon}(v) \le (d(v)-1)/2$ for each vertex v with $d(v) \ge 3$, and $\Delta^{mon}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times.*

Remark 1. The following example shows that the bounds on $\Delta^{mon}(v)$ in Theorems 4 and 5 are tight for a vertex v of odd degree.

Example 1. Let m be an odd integer such that $m \ge 3$, and let G be a balanced complete bipartite graph on $2m$ vertices with partite sets X and Y. Let $X' \subset X$ and $Y' \subset Y$ with $|X'| = |Y'| = (m+1)/2$, and let x be a vertex in $X \setminus X'$. We color all the edges between X' and Y' with color i, and we color all the edges between x and Y' with color j $(j \ne i)$. Finally, we color the remaining edges of G with pairwise distinct new colors.

One can check that the graph G in Example 1 satisfies the property requested in Theorems 4 and 5, respectively, and it satisfies $\Delta^{mon}(x) = (m+1)/2 = (d(x)-1)/2 + 1$ for the above mentioned vertex x. However, the graph G contains no compatible spanning circuit visiting every vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times.

In the next section, we present the key ingredients for our proofs of the above results that are postponed to Section 3.

2. Preliminaries

In this section, we give some basic results which will be used in the later proofs of the main theorems that are postponed to Section 3.

Theorem 6 (*Laskar and Auerback [14]*). *Let G be a complete graph on n vertices, where $n \geq 3$. Then G can be decomposed into $(n-1)/2$ edge-disjoint Hamilton cycles for n odd, and G can be decomposed into $(n-2)/2$ edge-disjoint Hamilton cycles and one perfect matching for n even.*

Theorem 7 (*Laskar and Auerback [14]*). *Let G be a complete equipartition r-partite graph on rn vertices. Then G can be decomposed into $n(r-1)/2$ edge-disjoint Hamilton cycles for even $n(r-1)$, and G can be decomposed into $(n(r-1)-1)/2$ edge-disjoint Hamilton cycles and one perfect matching for odd $n(r-1)$.*

We list the following two key lemmas on supereulerian graphs that are essential for our proofs of Theorems 4 and 5, whose proofs will be given in Section 4.

Lemma 1. *Let G be a connected graph on n $(n \geq 3)$ vertices such that $d(u) + d(v) \geq n$ for every pair of vertices u, v of G with $dist(u, v) = 2$. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H.*

Lemma 2. *Let G be a 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of nonadjacent vertices u, v of G. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H.*

3. Proofs of the main theorems

Proof of Theorem 2. Let K_n^c be an edge-colored complete graph on n vertices, where $n \geq 3$. For any given integer k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, there exist k edge-disjoint Hamilton cycles in K_n^c by Theorem 6. Let H be a $2k$-regular spanning subgraph of K_n^c consisting of k edge-disjoint Hamilton cycles of K_n^c.

If $\Delta^{mon}(K_n^c) \leq k$, then $\Delta^{mon}(H) \leq \Delta^{mon}(K_n^c) \leq k$. Since $d_H(v) = 2k$ for every vertex v of H, there exists a compatible Euler tour in H by Theorem 1. Therefore, the edge-colored complete graph K_n^c contains a compatible spanning circuit visiting each vertex exactly k times. This completes the proof. $\square$

Proof of Theorem 3. Let $(K_n^r)^c$ be an edge-colored complete equipartition r-partite graph on rn vertices. For any given integer k with $1 \leq k \leq \lfloor n(r-1)/2 \rfloor$, there exist k edge-disjoint Hamilton cycles in $(K_n^r)^c$ by Theorem 7. Let H be a spanning subgraph of $(K_n^r)^c$ consisting of k edge-disjoint Hamilton cycles of $(K_n^r)^c$.

If $\Delta^{mon}((K_n^r)^c) \leq k$, then $\Delta^{mon}(H) \leq \Delta^{mon}((K_n^r)^c) \leq k$. Since $d_H(v) = 2k$ for every vertex v of H, there exists a compatible Euler tour in H by Theorem 1. Therefore, the edge-colored complete equipartition r-partite graph $(K_n^r)^c$ contains a compatible spanning circuit visiting each vertex exactly k times. This completes the proof. $\square$

Proof of Theorem 4. The proof is based on Lemma 1, which will be proved in Section 4. Let G be an edge-colored connected graph on n $(n \geq 3)$ vertices such that $d(u) + d(v) \geq n$ for every pair of vertices u, v of G with $dist(u, v) = 2$. By Lemma 1, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree, and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G with $d_G(v) \geq 3$, then $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each choice of the vertex v of H. Since H is a spanning eulerian subgraph of G, we have $d_H(v) = d_G(v) = 2$ for each vertex v of G with $d_G(v) = 2$. Thus $\Delta_H^{mon}(v) = 1 = d_H(v)/2$ for each of vertex v of G with $d_G(v) = 2$. Based on the argument above, there exists a compatible Euler tour in H by Theorem 1. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

Proof of Theorem 5. The proof is based on Lemma 2, which will be proved in Section 4. Let G be an edge-colored 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of nonadjacent vertices u, v of G. By Lemma 2, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree, and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G with $d_G(v) \geq 3$, then $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each choice of the vertex v of H. Since H is a spanning eulerian subgraph of G, we have $d_H(v) = d_G(v) = 2$ for each vertex v of G with $d_G(v) = 2$. Thus $\Delta_H^{mon}(v) = 1 = d_H(v)/2$ for each of vertex v of G with $d_G(v) = 2$. Based on the argument above, there exists a compatible Euler tour in H by Theorem 1. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

It remains to provide the proofs of Lemmas 1 and 2. This is done in the next section.

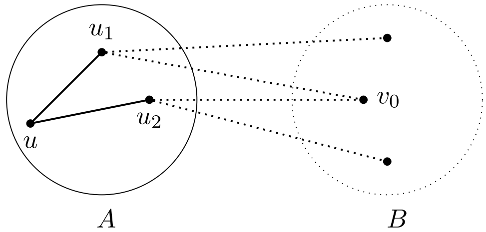


Fig. 1. The graph illustrating Case 1.1.

4. Proofs of Lemmas 1 and 2

Before proceeding with our proofs, we first introduce some additional terminology and notations, and give some auxiliary results which will be used in the later proofs of Lemmas 1 and 2.

A *linear forest* of a graph G is defined as a forest of G in which every component is a path. For a vertex v and a subgraph A of a graph G, denote by $N_G(v, A)$ the set of neighbors of v in G contained in A. For two disjoint subgraphs A and B of a graph G, let $N_G(B, A) = \bigcup_{v\in V(B)} N_G(v, A)$ and $E_G(B, A) = \{uv \in E(G) \mid u \in V(B), v \in V(A)\}$. We write $d_G(v, A) = |N_G(v, A)|$ and $e_G(B, A) = |E_G(B, A)|$. For two disjoint subsets S_1 and S_2 of $V(G)$, we can similarly define $N_G(S_1, S_2)$ and $E_G(S_1, S_2)$. In particular, we use $N_G(v, S_2)$ and $E_G(v, S_2)$ instead of $N_G(\{v\}, S_2)$ and $E_G(\{v\}, S_2)$, respectively. For a subset S of $V(G)$, we use $G[S]$ to denote the subgraph of G induced by S.

Lemma 3. *Let G be a hamiltonian graph on n vertices. Then there exists a linear forest F of G such that $N_1(F) = O(G)$ and $e(F) \le n/2$.*

Proof. If $O(G) = \emptyset$, then an empty linear forest (without edges) of G is a desired linear forest. Now we assume that $O(G) \ne \emptyset$. Let C be a Hamilton cycle of the graph G with a given orientation $\vec{C}$. Set $O(G) = \{v_1, v_2, \ldots, v_k\}$. For convenience, we assume that these vertices $v_1, v_2, \ldots, v_k$ appear in this order along $\vec{C}$. We use $\vec{C}[u, v]$ to denote the (directed) uv-path along $\vec{C}$. Let $F_1 = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i-1}, v_{2i}]$ and $F_2 = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i}, v_{2i+1}]$, where $v_{k+1} = v_1$. Clearly, F_i is a linear forest of G with $N_1(F_i) = O(G)$ for $i = 1, 2$. Since $e(F_1) + e(F_2) = n$, either $e(F_1) \le n/2$ or $e(F_2) \le n/2$. □

The following theorem due to Fan [9] is well known.

Theorem 8 (*Fan [9]*). *Let G be a 2-connected graph on n ($n \ge 3$) vertices. If $\max\{d(u), d(v)\} \ge n/2$ for every pair of vertices u, v of G with $dist(u, v) = 2$, then G is hamiltonian.*

We now have all the ingredients for our proofs of Lemmas 1 and 2.

4.1. Proof of Lemma 1

Let G be a connected graph on n ($n \ge 3$) vertices such that $d(u) + d(v) \ge n$ for every pair of vertices u, v with $dist(u, v) = 2$. It is not difficult to see that G is 2-connected. It follows that G is a hamiltonian graph by Theorem 8. By Lemma 3, there exists a linear forest F of G with $N_1(F) = O(G)$ and $e(F) \le n/2$. Let $G' = G - E(F)$. Thus $d_{G'}(v) \ge d_G(v) - 2$ and is even for each vertex $v \in V(G)$. If G' is connected, then it is a desired spanning eulerian subgraph of G. Next, we assume that G' is disconnected. We divide the remaining part of the proof into two cases that might occur.

We first consider the case that $G[V(A)]$ is not a complete graph for some component A of G' with $v(A) \le n/2$.

Case 1. There exists a component A of G' with $v(A) \le n/2$ such that $G[V(A)]$ is not a complete graph.

Let $B = G' - A$. Recall that $O(A) = \emptyset$ and A is not a complete graph. We can see that $v(A) \ge 4$. Thus, there exist two vertices $u_1, u_2 \in V(A)$ with $dist_G(u_1, u_2) = 2$. It follows that $n \le d_G(u_1) + d_G(u_2) \le v(A) - 2 + 2 + v(A) - 2 + 2 = 2v(A)$. Recall that $v(A) \le n/2$. We have $v(A) = v(B) = n/2$, $d_G(u_1, A) = d_G(u_2, A) = v(A) - 2$ and $d_G(u_1, B) = d_G(u_2, B) = 2$. Hence, $N_G(u_1, A) = N_G(u_2, A) = V(A) \setminus \{u_1, u_2\}$.

Depending on whether or not $N_G(u_1, B) \cap N_G(u_2, B) = \emptyset$, we divide Case 1 into two subcases (i.e., Cases 1.1 and 1.2).

Case 1.1. $N_G(u_1, B) \cap N_G(u_2, B) \ne \emptyset$ (see Fig. 1).

Let $v_0 \in N_G(u_1, B) \cap N_G(u_2, B)$. It is easy to see that $N_G(u_1, B) \cap N_G(u_2, B) = \{v_0\}$, otherwise there would be a cycle in F. Recall that $N_G(u_1, A) = V(A) \setminus \{u_1, u_2\}$. Let u be an arbitrary vertex in $V(A) \setminus \{u_1, u_2\}$. Since $u_1u \in E(G)$ and $v_0u \notin E(G)$, we have $dist_G(u, v_0) = 2$. It follows that $n \le d_G(u) + d_G(v_0) \le v(A) - 1 + 2 + d_G(v_0, B) + 2$, implying that $d_G(v_0, B) \ge v(B) - 3$.

If $d_G(v_0, B) = v(B) - 3$, then $d_G(u, B) = 2$ for every $u \in V(A)$. We have $e(F) \ge 2(n/2) > n/2$, a contradiction. If $d_G(v_0, B) = v(B) - 2$, then $d_G(u, B) \ge 1$ for every $u \in V(A)$. Recall that $d_G(u_1, B) = d_G(u_2, B) = 2$. We have $e(F) \ge 4 + (n/2 - 2) > n/2$, a contradiction. Now we assume that $d_G(v_0, B) = v(B) - 1$. Thus v_0 is adjacent to all vertices in $V(B) \setminus \{v_0\}$. Let v be an arbitrary vertex in $V(B) \setminus \{v_0\}$. Thus, there exists a vertex $u_i \in \{u_1, u_2\}$ such that

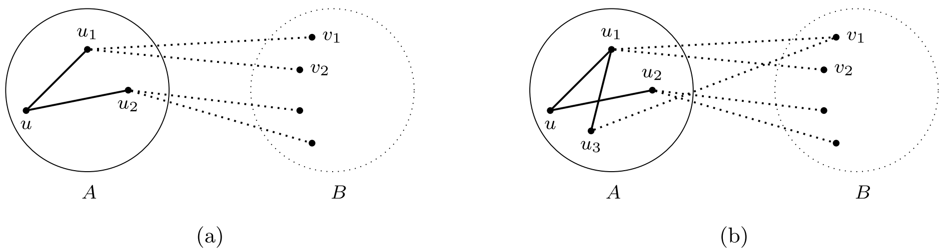


Fig. 2. The graphs illustrating Case 1.2.

$dist_G(u_i, v) = 2$. It follows that $n \le d_G(u_i) + d_G(v) \le v(A) - 2 + 2 + v(B) - 1 + d_G(v, A)$, implying that $d_G(v, A) \ge 1$. Recall that $d_G(v_0, A) = 2$. We have $e(F) \ge 2 + (n/2 - 1) > n/2$, a contradiction.

Case 1.2. $N_G(u_1, B) \cap N_G(u_2, B) = \emptyset$.

Let $N_G(u_1, B) = \{v_1, v_2\}$. First we suppose that $N_G(v_1, A) = \{u_1\}$ (see Fig. 2(a)). Recall that $N_G(u_1, A) = V(A) \setminus \{u_1, u_2\}$. Let u be an arbitrary vertex in $V(A) \setminus \{u_1, u_2\}$. We have $dist_G(u, v_1) = 2$. It follows that $n \le d_G(u) + d_G(v_1) \le v(A) - 1 + d_G(u, B) + v(B) - 1 + 1$, implying that $d_G(u, B) \ge 1$. Recall that $d_G(u_1, B) = d_G(u_2, B) = 2$. We have $e(F) \ge 4 + (n/2 - 2) > n/2$, a contradiction.

Suppose now that $N_G(v_1, A) = \{u_1, u_3\}$, where $u_3 \in V(A) \setminus \{u_1, u_2\}$ (see Fig. 2(b)). Let u be an arbitrary vertex in $V(A) \setminus \{u_1, u_2, u_3\}$. We have $dist_G(u, v_1) = 2$. It follows that $n \le d_G(u) + d_G(v_1) \le v(A) - 1 + 2 + d_G(v_1, B) + 2$, implying that $d_G(v_1, B) \ge v(B) - 3$.

If $d_G(v_1, B) = v(B) - 3$, then $d_G(u, B) = 2$ for every $u \in V(A) \setminus \{u_3\}$. Recall that $d_G(u_3, B) \ge 1$. We have $e(F) \ge 1 + 2(n/2 - 1) > n/2$, a contradiction. If $d_G(v_1, B) = v(B) - 2$, then $d_G(u, B) \ge 1$ for every $u \in V(A)$. Recall that $d_G(u_1, B) = d_G(u_2, B) = 2$. We have $e(F) \ge 4 + (n/2 - 2) > n/2$, a contradiction. Now we assume that $d_G(v_1, B) = v(B) - 1$. Let v be an arbitrary vertex in $V(B) \setminus \{v_1, v_2\}$. Thus, we have $dist_G(u_1, v) = 2$. It follows that $n \le d_G(u_1) + d_G(v) \le v(A) - 2 + 2 + v(B) - 1 + d_G(v, A)$, implying that $d_G(v, A) \ge 1$. Recall that $d_G(v_1, A) = 2$ and $d_G(v_2, A) \ge 1$. We have $e(F) \ge 2 + (n/2 - 1) > n/2$, a contradiction.

Now, we consider another case that $G[V(A)]$ is complete for every component A of G' with $v(A) \le n/2$.

Case 2. For every component A of G' with $v(A) \le n/2$, $G[V(A)]$ is a complete graph.

Depending on whether or not G' has an isolated vertex, we divide Case 2 into two subcases (i.e., Cases 2.1 and 2.2).

Case 2.1. G' has at least one isolated vertex.

We prove some claims in order to deal with Case 2.1.

Claim 1. *G' has exactly one isolated vertex.*

Proof. Let S be the set of isolated vertices of G'. Set $B = G' - S$ and $S' = N_G(S, B)$. If $V(B) \setminus S' = \emptyset$ (i.e., $V(G) = S \cup S'$), then $|S'| \le e_G(S, S')$ and $e(G[S]) < |S|$ (otherwise there would be a cycle in F). If $|S| \ge |S'|$, then $e(F) \ge 2|S| - e(G[S]) > |S| \ge n/2$, a contradiction; if $|S'| > |S|$, then $e(F) \ge e_G(S, S') > n/2$, also a contradiction. Therefore, we conclude that $V(B) \setminus S' \ne \emptyset$. Thus there exist two vertices $u_1 \in S$, $v_1 \in V(B) \setminus S'$ such that $dist_G(u_1, v_1) = 2$. It follows that $n \le d_G(u_1) + d_G(v_1) \le 2 + v(B) - 1$, implying that $v(B) \ge n - 1$ and $|S| \le 1$. Recall that G' has at least one isolated vertex. Hence, $|S| = 1$. □

Let u_1 be the isolated vertex of G'.

Claim 2. *$B = G' - u_1$ is connected.*

Proof. Suppose, to the contrary, that B is disconnected. Let B_1 be a component of B such that $N_G(u_1, B_1) \ne \emptyset$. Note that G' has no component with 2 vertices. Thus, every component of B has at least 3 vertices. We can see that $3 \le v(B_1) \le n - 4$, and there is a vertex $v \in V(B_1)$ with $dist_G(u_1, v) = 2$. Thus, we have $n \le d_G(u_1) + d_G(v) \le 2 + v(B_1) - 1 + 2 \le n - 1$, a contradiction. □

Clearly, we have $d_G(u_1) = 2$. Let $N_G(u_1, B) = \{v_1, v_2\}$. If $v_1v_2 \in E(G)$, then $v_1v_2 \in E(G')$, otherwise $v_1u_1v_2v_1$ would be a cycle in F. Thus $H = G' + \{u_1v_1, u_1v_2\} - \{v_1v_2\}$ is a desired spanning eulerian subgraph of G. Next, we assume that $v_1v_2 \notin E(G)$.

We claim that $N_G(v_1, B) = N_G(v_2, B)$. Suppose, to the contrary, that there exists a vertex $v_0 \in N_G(v_1, B) \setminus N_G(v_2, B)$. Since $dist_G(u_1, v_0) = 2$, we have $n \le d_G(u_1) + d_G(v_0) \le 2 + n - 1 - 2 < n$, a contradiction. Thus as we claimed, $N_G(v_1, B) = N_G(v_2, B)$.

Let $R = N_G(v_1, B) = N_G(v_2, B)$ and $S = N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$. Since $dist_G(v_1, v_2) = 2$, we have $|R| \ge (n - 2)/2$. If there exists a vertex $v \in S$ such that $E_G(v) \cap E(F) = \emptyset$, then let $H = G' + \{u_1v_1, u_1v_2\} - \{v_1v, v_2v\}$. Since $dist_G(u_1, v) = 2$ and $d_G(u_1) = 2$, we have $d_G(v) \ge n - 2$. It follows that $N_{G'}(v_1, B) \cap N_{G'}(v, B) \ne \emptyset$. Thus H is a desired spanning eulerian subgraph of G. Next, we assume that $E_G(v) \cap E(F) \ne \emptyset$ for every $v \in S$.

Claim 3. *There exist two vertices $v_3, v_4 \in S$ such that $v_3v_4 \in E(F)$.*

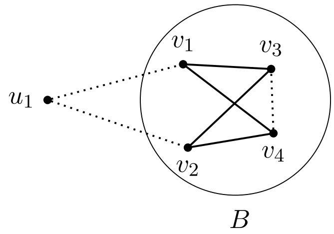


Fig. 3. The graph illustrating Case 2.1.

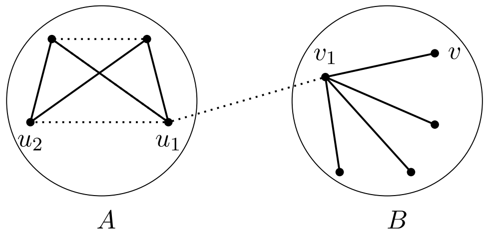


Fig. 4. The graph illustrating Case 2.2.1.

Proof. First we suppose that $S = R$. Let $F' = F - \{u_1v_1, u_1v_2\}$. We have $e(F') \le n/2 - 2$. Recall that $|S| = |R| \ge n/2 - 1$. Hence, there must exist two vertices $v_3, v_4 \in S$ such that $v_3v_4 \in E(F')$.

Suppose now that $|R \setminus S| = 1$. Let $v_5 \in R \setminus S$. Note that it is impossible that $\{v_1v_5, v_2v_5\} \subset E(F)$, otherwise $u_1v_1v_5v_2u_1$ would be a cycle in F. Without loss of generality, we suppose that $v_1v_5 \in E(F)$. Let $F' = F - \{u_1v_1, u_1v_2, v_1v_5\}$. We have $e(F') \le n/2 - 3$. Recall that $|S| = |R| - 1 \ge n/2 - 2$. Hence, there must exist two vertices $v_3, v_4 \in S$ such that $v_3v_4 \in E(F')$.

Finally, we suppose that $|R \setminus S| = 2$. Set $\{v_1v_5, v_2v_6\} \subset E(F)$. Let $F' = F - \{u_1v_1, u_1v_2, v_1v_5, v_2v_6\}$. We have $e(F') \le n/2 - 4$. Recall that $|S| = |R| - 2 \ge n/2 - 3$. Hence, there must exist two vertices $v_3, v_4 \in S$ such that $v_3v_4 \in E(F')$. $\square$

By Claim 3, $H = G' + \{u_1v_1, u_1v_2, v_3v_4\} - \{v_1v_3, v_2v_4\}$ is a desired spanning eulerian subgraph of G (see Fig. 3).

Now, we consider another subcase of Case 2 that G' has no isolated vertex.

Case 2.2. G' has no isolated vertex.

Claim 4. *G' has exactly two components.*

Proof. Let A be a component of G' with $v(A) \le n/2$ and B be another component of G' such that $N_G(A, B) \ne \emptyset$. Recall that every component of G' has at least 3 vertices. We can see that there exist two vertices $u \in V(A)$, $v \in V(B)$ such that $dist_G(u, v) = 2$. It follows that $n \le d_G(u) + d_G(v) \le v(A) - 1 + 2 + v(B) - 1 + 2$, implying that $v(A) + v(B) \ge n - 2$. If G' has the third component C, then $v(C) \le 2$, a contradiction. $\square$

Let A, B be two components of G'. We assume that $v(A) \le v(B)$, and if $v(A) = v(B)$ then $e(A) \le e(B)$. Depending on whether or not A is complete, we divide Case 2.2 into two subcases (i.e., Cases 2.2.1 and 2.2.2).

Case 2.2.1. A is not a complete graph (see Fig. 4).

Claim 5. *$A = G[V(A)] - M$ for some perfect matching M of $G[V(A)]$.*

Proof. Recall that $G' = G - E(F)$ and $G[V(A)]$ is a complete graph. Note that A is not a complete graph. If $d_G(u, A) - d_{G'}(u, A) \equiv 0 \ (mod\ 2)$ for each $u \in V(A)$, then $E(F) \cap E(G[V(A)])$ contains a cycle, a contradiction. Thus, there exists a vertex u of A such that $d_{G'}(u, A) = d_G(u, A) - 1$. Since every vertex of G' has an even degree, we have $d_{G'}(u, A) = d_G(u, A) - 1$ for each $u \in V(A)$. $\square$

Clearly there exists an edge $u_1v_1 \in E(F)$ with $u_1 \in V(A)$ and $v_1 \in V(B)$ (see Fig. 4). Let $u_1u_2 \in M$ (by Claim 5). Thus $dist_G(u_2, v_1) = 2$, otherwise $u_1u_2v_1u_1$ would be a cycle in F. It follows that $n \le d_G(u_2) + d_G(v_1) \le v(A) - 1 + 1 + d_G(v_1, B) + 2$, implying that $d_G(v_1, B) \ge v(B) - 2$. Let v be an arbitrary vertex in $N_G(v_1, B)$. Thus, we have $dist_G(u_1, v) = 2$. It follows that $n \le d_G(u_1) + d_G(v) \le v(A) - 1 + 1 + v(B) - 1 + d_G(v, A)$, implying that $d_G(v, A) \ge 1$. By Claim 5, we have $v(A) \ge 4$ and $e(M) \ge 2$. Thus $e(F) \ge v(B) - 2 + 1 + 2 > n/2$, a contradiction.

Now, we consider another subcase of Case 2.2 that A is a complete graph.

Case 2.2.2. A is a complete graph.

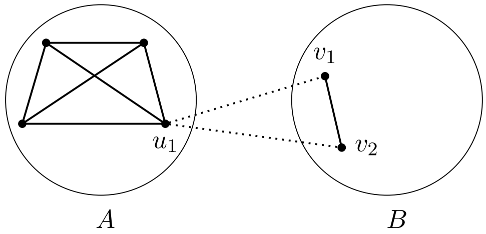
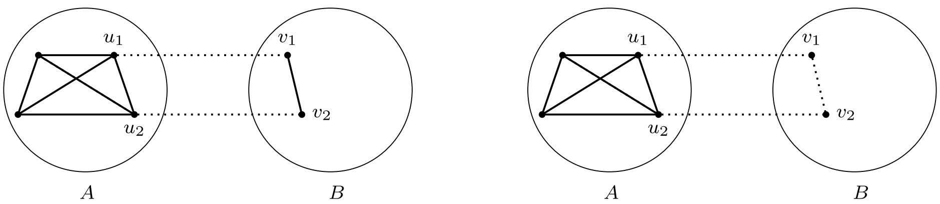


Fig. 5. The graph illustrating the case of $N_G(v_1, A) \cap N_G(v_2, A) \neq \emptyset$.



(a) (b)

Fig. 6. The graphs illustrating the case of $N_G(v_1, A) \cap N_G(v_2, A) = \emptyset$.

Let $N_G(A, B) = S$. Since G is 2-connected, we have $|S| \geq 2$. First we suppose that there exist two vertices v_1, v_2 of S such that $v_1v_2 \in E(G)$. If $N_G(v_1, A) \cap N_G(v_2, A) \neq \emptyset$, saying $u_1 \in N_G(v_1, A) \cap N_G(v_2, A)$, then $v_1v_2 \in E(G')$, otherwise $v_1u_1v_2v_1$ would be a cycle in F. Thus $H = G' + \{u_1v_1, u_1v_2\} - \{v_1v_2\}$ is a desired spanning eulerian subgraph of G (see Fig. 5).

Now let us consider the case that $N_G(v_1, A) \cap N_G(v_2, A) = \emptyset$. Let $u_1 \in N_G(v_1, A)$ and $u_2 \in N_G(v_2, A)$. We define a graph H such that

$$H = \begin{cases} G' + \{u_1v_1, u_2v_2\} - \{u_1u_2, v_1v_2\}, & \text{if } v_1v_2 \notin E(F); \\ G' + \{u_1v_1, u_2v_2, v_1v_2\} - \{u_1u_2\}, & \text{otherwise}. \end{cases}$$

Thus H is a desired spanning eulerian subgraph of G (see Fig. 6(a) and (b)).

Suppose now that S is an independent set in G. If $d_G(v, A) = 2$ for some $v \in S$, letting $N_G(v, A) = \{u_1, u_2\}$, then $H = G' + \{u_1v, u_2v\} - \{u_1u_2\}$ is a desired spanning eulerian subgraph of G. Next, we assume that $d_G(v, A) = 1$ for every $v \in S$. Clearly there exist two vertices $u \in V(A)$ and $v \in S$ such that $dist_G(u, v) = 2$. It follows that $n \leq d_G(u) + d_G(v) \leq v(A) - 1 + 2 + v(B) - |S| + 1$, implying that $|S| \leq 2$. Recall that $|S| \geq 2$. We have $|S| = 2$. Let u be an arbitrary vertex in A. Clearly there is a vertex $v \in V(B)$ with $dist_G(u, v) = 2$. It follows that $n \leq d_G(u) + d_G(v) \leq v(A) - 1 + d_G(u, B) + v(B) - 1 = n + d_G(u, B) - 2$, implying that $d_G(u, B) \geq 2$. Thus $N_G(u, B) = S$ for every $u \in A$, and F contains a cycle, a contradiction. This completes the proof. $\square$

4.2. Proof of Lemma 2

Let G be a 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of nonadjacent vertices u, v of G. The assertion is trivial if $n \leq 4$. Therefore, we assume that $n \geq 5$. It follows that G is a hamiltonian graph by Theorem 8. By Lemma 3, there exists a linear forest F of G with $N_1(F) = O(G)$ and $e(F) \leq n/2$. Let $G' = G - E(F)$. Thus $d_{G'}(v) \geq d_G(v) - 2$ and is even for each vertex $v \in V(G)$. If G' is connected, then it is a desired spanning eulerian subgraph of G. Next, we assume that G' is disconnected. We divide the remaining part of the proof into two cases that might occur.

We first consider the case that $G[V(A)]$ is complete for some component A of G' with $v(A) \leq n/2$.

Case 1. There exists a component A of G' with $v(A) \leq n/2$ such that $G[V(A)]$ is not a complete graph.

Let $B = G' - A$. Recall that $O(A) = \emptyset$ and A is not a complete graph. We can see that $v(A) \geq 4$. Thus, there exist two nonadjacent vertices $u_1, u_2 \in V(A)$. Without loss of generality, we suppose that $d_G(u_1) \geq n/2$. It follows that $n/2 \leq d_G(u_1) \leq v(A) - 2 + 2 = v(A)$. Recall that $v(A) \leq n/2$. Therefore, we have $v(A) = v(B) = n/2$, $d_G(u_1, A) = v(A) - 2$ and $d_G(u_1, B) = 2$. Hence, $N_G(u_1, A) = V(A) \setminus \{u_1, u_2\}$.

We claim that B is connected. Suppose, to the contrary, that B is disconnected. Note that $n \geq 8$ by the fact that $v(A) \geq 4$. Since $v(B) = n/2$, there exists a component B_1 of B with $v(B_1) \leq n/4$. We have $d_G(v) \leq n/4 - 1 + 2 < n/2$ for every $v \in V(B_1)$. Let u be an arbitrary vertex in A. If $u \in N_G(B_1, A)$, then $d_G(u, B) \geq 1$. If $u \notin N_G(B_1, A)$, then $n/2 \leq d_G(u) \leq v(A) - 1 + d_G(u, B)$, implying that $d_G(u, B) \geq 1$. Recall that $d_G(u_1, B) = 2$. We have $e(F) \geq 2 + (n/2 - 1) > n/2$, a contradiction. Thus as we claimed, B is connected.

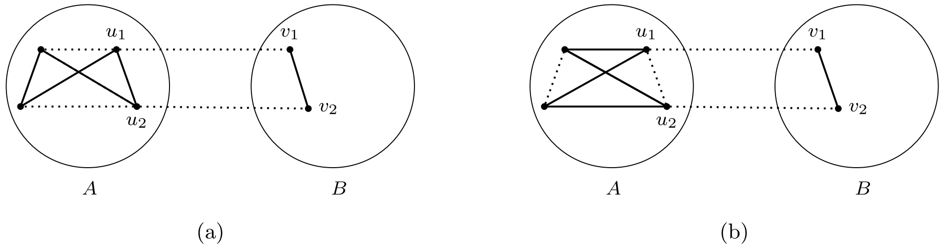


Fig. 7. The graphs illustrating the case of $v_1v_2 \in E(G')$.

Recall that $d_G(u_1, B) = 2$. Let $N_G(u_1, B) = \{v_1, v_2\}$. If $v_1v_2 \in E(G)$, then $v_1v_2 \in E(G')$, otherwise $v_1u_1v_2v_1$ would be a cycle in F. Thus $H = G' + \{u_1v_1, u_1v_2\} - \{v_1v_2\}$ is a desired spanning eulerian subgraph of G. Next, we assume that $v_1v_2 \notin E(G)$.

We claim that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$. Suppose, to the contrary, that $d_G(v) < n/2$ for some vertex $v \in V(B)$. Let u be an arbitrary vertex in A. If $u \in N_G(v, A)$, then $d_G(u, B) \geq 1$. If $u \notin N_G(v, A)$, then $n/2 \leq d_G(u) \leq v(A) - 1 + d_G(u, B)$, implying that $d_G(u, B) \geq 1$. Recall that $d_G(u_1, B) = 2$. We have $e(F) \geq 2 + (n/2 - 1) > n/2$, a contradiction. Thus as we claimed, $d_G(v) \geq n/2$ for every vertex $v \in V(B)$.

Since $v(B) = n/2$, $d_G(v, A) \geq 1$ for every $v \in V(B)$. Moreover, since $v_1v_2 \notin E(G)$, we have $d_G(v_1, A) = d_G(v_2, A) = 2$. Hence, $e(F) \geq 4 + (n/2 - 2) > n/2$, a contradiction.

Now, we consider another case that $G[V(A)]$ is complete for every component A of G' with $v(A) \leq n/2$.

Case 2. For every component A of G' with $v(A) \leq n/2$, $G[V(A)]$ is a complete graph.

Depending on whether or not every component A of G' with $v(A) \leq n/2$ is complete, we divide Case 2 into two subcases (i.e., Cases 2.1 and 2.2).

Case 2.1. There exists a component A of G' with $v(A) \leq n/2$ that is not a complete graph.

Claim 1. *$A = G[V(A)] - M$ for some perfect matching M of $G[V(A)]$.*

Proof. Recall that $G' = G - E(F)$ and $G[V(A)]$ is a complete graph. Hence, $G[V(A)]$ is regular. Note that A is not a complete graph. If $d_G(u, A) - d_{G'}(u, A) \equiv 0 \ (mod\ 2)$ for each $u \in V(A)$, then $E(F) \cap E(G[V(A)])$ contains a cycle, a contradiction. Thus, there exists a vertex u of A such that $d_{G'}(u, A) = d_G(u, A) - 1$. Since $G[V(A)]$ is regular and every vertex of G' has an even degree, we have $d_{G'}(u, A) = d_G(u, A) - 1$ for each $u \in V(A)$. □

Let $B = G' - A$. If $v(A) = n/2$, then there exists a vertex $u \in V(A)$ with $d_G(u, B) = 0$ (otherwise we would have $e(F) \geq n/4 + n/2 > n/2$ by Claim 1). Let v be an arbitrary vertex in $V(B)$. Since $uv \notin E(G)$ and $d_G(u) \leq v(A) - 1 < n/2$, we have $n/2 \leq d_G(v) \leq v(B) - 1 + d_G(v, A)$, implying that $d_G(v, A) \geq 1$. Thus, we have $e(F) \geq n/4 + n/2 > n/2$, a contradiction. Therefore, we conclude that $v(A) < n/2$.

By Claim 1, we have $v(A) \geq 4$ and $d_G(u) \leq v(A) - 1 + 1 = v(A) < n/2$ for every vertex $u \in V(A)$. For each vertex $v \in V(B)$, there exists a vertex $u \in V(A)$ such that $uv \notin E(G)$. This implies that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$.

If B is disconnected, then there exists a component B_1 of B with $v(B_1) \leq v(B)/2$. Recall that $v(A) \geq 4$. We have $d_G(v) \leq v(B_1) - 1 + 2 \leq v(B)/2 + 1 = (n - v(A))/2 + 1 < n/2$ for every $v \in V(B_1)$, a contradiction. Therefore, we conclude that B is connected.

Let $N_G(A, B) = S$. Since G is 2-connected, we have $|S| \geq 2$. For two vertices v_1, v_2 of S, let $u_1 \in N_G(v_1, A)$ and $u_2 \in N_G(v_2, A)$ (note that $u_1 \neq u_2$ by Claim 1). We define a graph $H_1(v_1, v_2)$ such that

$$H_1(v_1, v_2) = \begin{cases} G' + \{u_1v_1, u_2v_2\} - \{u_1u_2\}, & \text{if } u_1u_2 \notin M; \\ G' + \{u_1v_1, u_2v_2, u_1u_2\}, & \text{otherwise}. \end{cases}$$

If there exist two vertices v_1, v_2 of S such that $v_1v_2 \in E(G')$, then $H = H_1(v_1, v_2) - \{v_1v_2\}$ is a desired spanning eulerian subgraph of G (see Fig. 7(a) and (b)). Next, we assume that S is an independent set in G'.

Claim 2. *For every two vertices v_1, v_2 of S, one of the following holds*:
(1) there exists a vertex $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $E_G(v_3) \cap E(F) = \emptyset$;
(2) there exist two vertices $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ and $v_4 \in N_{G'}(v_1, B) \cup N_{G'}(v_2, B)$ such that $v_3v_4 \in E(F)$; or
(3) there exist two vertices $v_3, v_4 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $v_3v_5, v_4v_5 \in E(F)$ for some vertex $v_5 \in V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.

Proof. Let v_1, v_2 be two vertices of S, and $S_0 = N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$. Let $S_1 = N_{G'}(v_1, B) \setminus S_0$, $S_2 = N_{G'}(v_2, B) \setminus S_0$ and $S_3 = V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.

If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3) \cap E(F) = \emptyset$, then (1) holds. Now we assume that $E_G(v) \cap E(F) \neq \emptyset$ for every $v \in S_0$. If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3, S_3) \cap E(F) = \emptyset$, then (2) holds. Next, we assume that

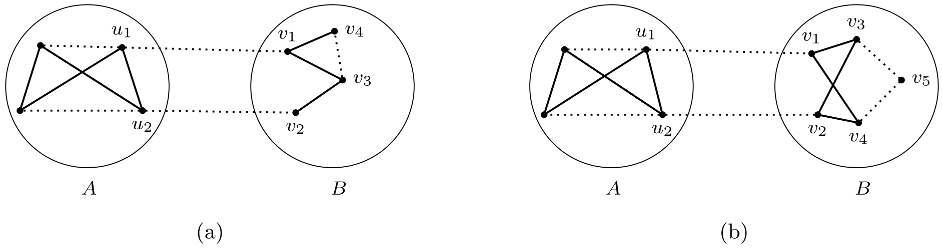


Fig. 8. The graphs illustrating the case that S is an independent set of G'.

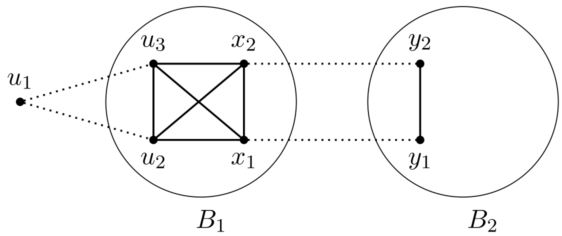


Fig. 9. The graph illustrating the case that B is disconnected.

$E_G(v, S_3) \cap E(F) \neq \emptyset$ for every $v \in S_0$. Recall that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$. We have $n/2 \leq d_G(v_1) \leq |S_1| + |S_0| + 2$, and $n/2 \leq d_G(v_2) \leq |S_2| + |S_0| + 2$. It follows that $|S_1| + |S_0| + |S_2| + |S_0| \geq n - 4$. However, we have $|S_1| + |S_0| + |S_2| + |S_3| \leq n - 6$ by the fact of $v(A) \geq 4$. Therefore, we have $|S_3| < |S_0|$. Thus (3) holds. □

Let v_1, v_2 be two vertices of S. First we suppose that (1) of Claim 2 holds. Recall that $d_G(v_i) \geq n/2$ for $i = 1, 2, 3$. One can check that $N_{G'}(v_1, B) \cap N_{G'}(v_3, B) \neq \emptyset$. Thus $H = H_1(v_1, v_2) - \{v_1v_3, v_2v_3\}$ is a desired spanning eulerian subgraph of G.

Suppose now that (2) of Claim 2 holds. Without loss of generality, we suppose that $v_4 \in N_{G'}(v_1, B)$. Thus $H = H_1(v_1, v_2) + \{v_3v_4\} - \{v_1v_4, v_2v_3\}$ is a desired spanning eulerian subgraph of G (see Fig. 8(a)).

Finally, we suppose that (3) of Claim 2 holds. Thus $H = H_1(v_1, v_2) + \{v_3v_5, v_4v_5\} - \{v_1v_3, v_2v_4\}$ is a desired spanning eulerian subgraph of G (see Fig. 8(b)).

Now, we consider another subcase of Case 2 that every component A of G' with $v(A) \leq n/2$ is complete.

Case 2.2. Every component A of G' with $v(A) \leq n/2$ is a complete graph.

Depending on whether or not G' has an isolated vertex, we divide Case 2.2 into two subcases (i.e., Cases 2.2.1 and 2.2.2).

Case 2.2.1. G' has at least one isolated vertex.

Let u_1 be an isolated vertex of G', and $B = G' - u_1$. Clearly, we have $d_G(u_1) = 2$. Let $N_G(u_1, B) = \{u_2, u_3\}$. First we suppose that $u_2u_3 \in E(G)$. It follows that $u_2u_3 \in E(G')$, otherwise $u_1u_2u_3u_1$ would be a cycle in F. If B is connected, then $H = G' + \{u_1u_2, u_1u_3\} - \{u_2u_3\}$ is a desired spanning eulerian subgraph of G. Next, we assume that B is disconnected.

We claim that B has exactly two components. Suppose, to the contrary, that B has at least three components. Note that neither u_2 nor u_3 is an isolated vertex of B. We conclude that there is no isolated vertex in B, otherwise there would be an isolated vertex u of B such that $u_1u \notin E(G)$ and $\max\{d(u_1), d(u)\} \leq 2 < n/2$ (recall that $n \geq 5$), a contradiction. Since every vertex of G' has an even degree, there exists a connected component B_1 of B with $3 \leq v(B_1) \leq n/2 - 2$. We have $d_G(v) \leq n/2 - 3 + 2 < n/2$ for every $v \in V(B_1)$. Therefore, there is a vertex $v \in V(B_1)$ such that $u_1v \notin E(G)$ and $\max\{d_G(u_1), d_G(v)\} < n/2$, a contradiction. Thus as we claimed, B has exactly two components.

Let B_1, B_2 be the two components of B such that $3 \leq v(B_1) \leq v(B_2)$. Note that there exists a vertex $v \in V(B_1)$ with $u_1v \notin E(G)$, and $d_G(u_1) < n/2$. It follows that $d_G(v) \geq n/2$ and hence $v(B_1) \geq n/2 - 1$. Therefore, we have $v(B_2) \leq n - 1 - (n/2 - 1) = n/2$. We conclude that B_1 and B_2 are both complete graphs. Without loss of generality, we suppose that u_2 and u_3 are in B_1. Since G is 2-connected, there exist two nonadjacent edges $x_1y_1, x_2y_2 \in E(F)$ with $x_1, x_2 \in V(B_1)$ and $y_1, y_2 \in V(B_2)$. If $\{x_1, x_2\} \cap \{u_2, u_3\} = \emptyset$, then $H = G' + \{u_1u_2, u_1u_3, x_1y_1, x_2y_2\} - \{u_2u_3, x_1x_2, y_1y_2\}$ is a desired spanning eulerian subgraph of G (see Fig. 9). Next, we assume that $\{x_1, x_2\} \cap \{u_2, u_3\} \neq \emptyset$. If $|\{x_1, x_2\} \cap \{u_2, u_3\}| = 1$, without loss of generality suppose that $x_1 = u_2$, then $H = G' + \{u_1u_2, u_1u_3, x_1y_1, x_2y_2\} - \{u_3x_2, y_1y_2\}$ is a desired spanning eulerian subgraph of G (see also Fig. 9). If $\{x_1, x_2\} = \{u_2, u_3\}$, setting $x_1 = u_2$ and $x_2 = u_3$, then $H = G' + \{u_1u_2, u_1u_3, x_1y_1, x_2y_2\} - \{y_1y_2\}$ is a desired spanning eulerian subgraph of G (see also Fig. 9).

Suppose now that $u_2u_3 \notin E(G)$. Without loss of generality, we suppose that $d_G(u_3) \geq n/2$. Let $G^* = G - \{u_1, u_2\}$. Let u_4 be a fixed vertex of $N_G(u_2, G^*)$. Let S be a subset of $V(G^*)$ such that

$$S = \begin{cases} O(G), & \text{if } d_G(u_2) \equiv 0 \ (mod\ 2); \\ O(G - \{u_2u_4\}), & \text{otherwise.} \end{cases}$$

Let v be an arbitrary vertex in $V(G^*) \setminus \{u_3\}$. Since $u_1v \notin E(G)$, we have $d_G(v) \geq n/2$. Thus $d_{G^*}(v) \geq (n-2)/2$. Recall that $d_G(u_3) \geq n/2$. Hence, we have $d_{G^*}(v) \geq (n-2)/2 = v(G^*)/2$ for every vertex $v \in V(G^*)$. By Dirac's theorem, G^* has a Hamilton cycle C. We give a orientation $\vec{C}$ on C. Let $S = \{v_1, v_2, \ldots, v_k\}$, where these vertices appear in this order along $\vec{C}$. Clearly, k is even. Let $F_1^* = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i-1}, v_{2i}]$ and $F_2^* = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i}, v_{2i+1}]$, where $v_{k+1} = v_1$. Thus F_i^* is a linear forest of G^* with $N_1(F_i^*) = S$ for $i = 1, 2$. Therefore, there exists a linear forest F^* of G^* such that $N_1(F^*) = S$, and $u_4 \notin N_2(F^*)$.

Now set

$$G'' = \begin{cases} G - E(F^*), & \text{if } d_G(u_2) \equiv 0 \ (mod\ 2); \\ G - (E(F^*) \cup \{u_2u_4\}), & \text{otherwise.} \end{cases}$$

We can see that $d_{G''}(v) \geq d_G(v) - 2$ and is even for each vertex $v \in V(G)$. If G'' is connected, then it is a desired spanning eulerian subgraph of G. Next, we assume that G'' is disconnected.

If G'' has an isolated vertex v, then $v \notin \{u_1, u_2\}$ (since $u_1u_2 \in E(G'')$). Recall that $n \geq 5$. We have $d_G(v) = 2 < n/2$, a contradiction. Therefore, we conclude that G'' has no isolated vertex. We prove another claim.

Claim 3. *G'' has exactly two components.*

Proof. Suppose, to the contrary, that G'' has at least three components. Thus, there exists a component B_1 of G'' with $v(B_1) \leq n/3$. Since every vertex of G'' has an even degree, we have $v(B_1) \geq 3$, implying that $n \geq 9$. Let v be an arbitrary vertex in $V(B_1) \setminus \{u_1, u_2\}$. It follows that $n/2 \leq d_G(v) \leq n/3 - 1 + 2 < n/2$, a contradiction. □

Now let B_1, B_2 be the two components of G'' such that $\{u_1, u_2\} \subseteq V(B_2)$, and let $B_2^* = B_2 - \{u_1, u_2\}$. If $v(B_1) < n/2 - 1$, then for every $v \in V(B_1)$, $d_G(v) \leq v(B_1) - 1 + 2 < n/2$, a contradiction. If $v(B_1) > n/2$, then $v(B_2^*) < n/2 - 2$. For every $v \in V(B_2^*)$, we have $d_{G^*}(v) \leq v(B_2^*) - 1 + 2 < n/2 - 1$, a contradiction. Therefore, we conclude that $n/2 - 1 \leq v(B_1) \leq n/2$, and $n/2 - 2 \leq v(B_2^*) \leq n/2 - 1$.

We claim that B_1 is complete. Suppose, to the contrary, that B_1 is not complete. Let v be an arbitrary vertex in $V(B_1)$. Recall that every component A of G' with $v(A) \leq n/2$ is a complete graph, and every vertex of B_1 has an even degree in G''. It follows that $d_G(v, B_1) \leq v(B_1) - 2$. Thus we have $n/2 \leq d_G(v) \leq v(B_1) - 2 + d_G(v, B_2) \leq n/2$. This implies that $v(B_1) = n/2$ and $d_G(v, B_2) = 2$. It follows that $e(F^*) \geq n$ and F^* contains a cycle, a contradiction. Thus as we claimed, B_1 is complete.

If there is a vertex $v \in V(B_2^*)$ such that $d_{G^*}(v, B_1) = 2$, letting $N_{G^*}(v, B_1) = \{v', v''\}$, then $H = G'' + \{vv', vv''\} - \{v'v''\}$ is a desired spanning eulerian subgraph of G. Next, we assume that $d_{G^*}(v, B_1) \leq 1$ for every $v \in V(B_2^*)$. If B_2^* is not complete, then there is a vertex $v \in V(B_2^*)$ with $d_{G^*}(v) \leq v(B_2^*) - 2 + 1 < n/2 - 1$, a contradiction. Therefore, we conclude that B_2^* is complete.

If $v(B_2^*) = 1$, then B_2 is a triangle and $V(B_2) = \{u_1, u_2, u_3\}$, contradicting the fact $u_2u_3 \notin E(G)$. Therefore, we assume that $v(B_2^*) \geq 2$. For every vertex $v \in V(B_2^*)$, $n/2 - 1 \leq d_{G^*}(v) \leq v(B_2^*) - 1 + d_{G^*}(v, B_1) \leq n/2 - 2 + d_{G^*}(v, B_1)$. This implies that $d_{G^*}(v, B_1) \geq 1$ for every $v \in V(B_2^*)$. Let $v_1, v_2 \in V(B_2^*)$ and $v_i' \in N_{G^*}(v_i, B_1)$ for $i = 1, 2$. We define a graph H such that

$$H = \begin{cases} G'' + \{v_1'v_1, v_1'v_2\} - \{v_1v_2\}, & \text{if } v_1' = v_2'; \\ G'' + \{v_1'v_1, v_2'v_2\} - \{v_1v_2, v_1'v_2'\}, & \text{otherwise.} \end{cases}$$

Thus H is a desired spanning eulerian subgraph of G.

Now, we consider another subcase of Case 2.2 that G' has no isolated vertex.

Case 2.2.2. G' has no isolated vertex.

We prove a number of claims in order to deal with Case 2.2.2.

Claim 4. *G' has exactly two components.*

Proof. Suppose, to the contrary, that G' has at least three components. Let A, B be the smallest two components of G', and $C = G' - (A \cup B)$. Since every vertex of G' has an even degree, we have $3 \leq v(A) \leq n/3$, implying that $n \geq 9$. Thus $v(A) \leq v(B) \leq (n - v(A))/2$. Let v be an arbitrary vertex of B. Then there exists a vertex $u \in V(A)$ such that $uv \notin E(G)$ and $d_G(u) \leq n/3 - 1 + 2 < n/2$. It follows that $n/2 \leq d_G(v) \leq v(B) - 1 + 2 \leq (n - v(A))/2 + 1 \leq (n-3)/2 + 1 < n/2$, a contradiction. □

Let A, B be the two components of G' such that $v(A) \leq v(B)$. If $v(A) = n/2$, then $v(B) = n/2$. Thus A and B are both complete graphs. Since G is 2-connected, there exist two nonadjacent edges $u_1v_1, u_2v_2 \in E(F)$ with $u_1, u_2 \in V(A)$ and $v_1, v_2 \in V(B)$. Thus $H = G' + \{u_1v_1, u_2v_2\} - \{u_1u_2, v_1v_2\}$ is a desired spanning eulerian subgraph of G (see Fig. 10(*b*)).

Now we assume that $3 \leq v(A) < n/2$, implying that $n \geq 7$. Recall that A is a complete graph. If there is a vertex $v \in V(B)$ such that $d_G(v, A) = 2$, letting $N_G(v, A) = \{u_1, u_2\}$, then $H = G' + \{u_1v, u_2v\} - \{u_1u_2\}$ is a desired spanning eulerian subgraph of G. Next, we assume that $d_G(v, A) \leq 1$ for every $v \in V(B)$.

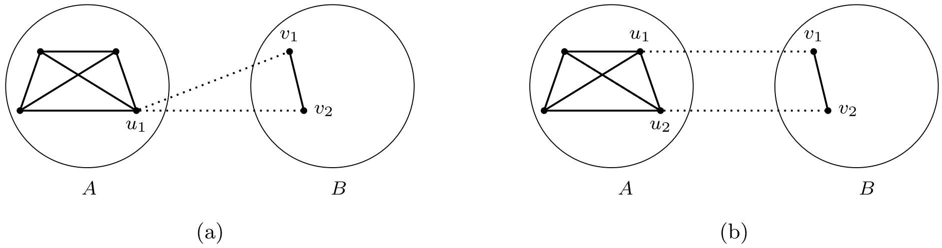


Fig. 10. The graphs illustrating the case of $v_1v_2 \in E(G')$.

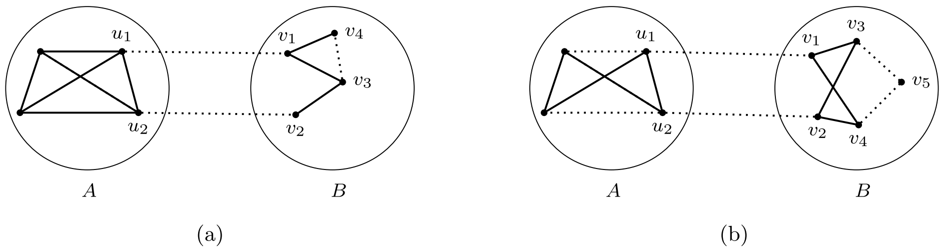


Fig. 11. The graphs illustrating the case that S is an independent set in G'.

Let $N_G(A, B) = S$. Since G is 2-connected, we have $|S| \geq 2$. Let v_1, v_2 be two vertices of S with $u_1 \in N_G(v_1, A)$ and $u_2 \in N_G(v_2, A)$. We define a graph $H_2(v_1, v_2)$ such that

$$H_2(v_1, v_2) = \begin{cases} G' + \{u_1v_1, u_1v_2\}, & \text{if } u_1 = u_2; \\ G' + \{u_1v_1, u_2v_2\} - \{u_1u_2\}, & \text{otherwise.} \end{cases}$$

If there exist two vertices v_1, v_2 of S such that $v_1v_2 \in E(G')$, then $H = H_2(v_1, v_2) - \{v_1v_2\}$ is a desired spanning eulerian subgraph of G (see Fig. 10(a) and (b)). Next, let us consider the case that S is an independent set in G'.

Claim 5. *$d_G(v) \geq n/2$ for every vertex $v \in V(B)$.*

Proof. First we suppose that $(n-2)/2 \leq v(A) \leq (n-1)/2$. If there exists a vertex $u \in V(A)$ with $d_G(u, B) = 0$, then $d_G(u) < n/2$. This implies that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$. If $d_G(u, B) \geq 1$ for every $u \in V(A)$, then there exists at most one vertex $u_1 \in V(A)$ such that $d_G(u, B) = 1$ for every $u \in V(A) \setminus \{u_1\}$ (otherwise $e(F) > n/2$). This implies that $d_G(u) < n/2$ for every $u \in V(A) \setminus \{u_1\}$. For every vertex $v \in V(B)$, there exists a vertex $u \in V(A)$ such that $uv \notin E(G)$ and $d_G(u) < n/2$ (recall that $v(A) \geq 3$ and $d_G(v, A) \leq 1$). Thus $d_G(v) \geq n/2$ for every vertex $v \in V(B)$.

Suppose now that $3 \leq v(A) \leq (n-3)/2$. It follows that $d_G(u) < n/2$ for every vertex $u \in V(A)$. For every vertex $v \in V(B)$, there exists a vertex $u \in V(A)$ such that $uv \notin E(G)$ and $d_G(u) < n/2$. Thus $d(v) \geq n/2$ for every vertex $v \in V(B)$. $\square$

Claim 6. *For every two vertices v_1, v_2 of S, one of the following holds:*
(1) there exists a vertex $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $E_G(v_3) \cap E(F) = \emptyset$;
(2) there exist two vertices $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ and $v_4 \in N_{G'}(v_1, B) \cup N_{G'}(v_2, B)$ such that $v_3v_4 \in E(F)$; or
(3) there exist two vertices $v_3, v_4 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $v_3v_5, v_4v_5 \in E(F)$ for some vertex $v_5 \in V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.

Proof. Let v_1, v_2 be two vertices of S, and $S_0 = N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$. Let $S_1 = N_{G'}(v_1, B) \setminus S_0$, $S_2 = N_{G'}(v_2, B) \setminus S_0$ and $S_3 = V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.

If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3) \cap E(F) = \emptyset$, then (1) holds. Now we assume that $E_G(v) \cap E(F) \neq \emptyset$ for every $v \in S_0$. If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3, S_3) \cap E(F) = \emptyset$, then (2) holds. Next, we assume that $E_G(v, S_3) \cap E(F) \neq \emptyset$ for every $v \in S_0$. By Claim 5, we have $n/2 \leq d_G(v_1) \leq |S_1| + |S_0| + 2$, and $n/2 \leq d_G(v_2) \leq |S_2| + |S_0| + 2$. It follows that $|S_1| + |S_0| + |S_2| + |S_0| \geq n - 4$. However, we have $|S_1| + |S_0| + |S_2| + |S_3| \leq n - 5$ by the fact of $v(A) \geq 4$. Therefore, we have $|S_3| < |S_0|$. Thus (3) holds. $\square$

Let v_1, v_2 be two vertices of S. First we suppose that (1) of Claim 6 holds. Recall that $d_G(v_i) \geq n/2$ for $i = 1, 2, 3$. One can check that $N_{G'}(v_1, B) \cap N_{G'}(v_3, B) \neq \emptyset$. Thus $H = H_2(v_1, v_2) - \{v_1v_3, v_2v_3\}$ is a desired spanning eulerian subgraph of G.

Suppose now that (2) of Claim 6 holds. Without loss of generality, we suppose that $v_4 \in N_{G'}(v_1, B)$. Thus $H = H_2(v_1, v_2) + \{v_3v_4\} - \{v_1v_4, v_2v_3\}$ is a desired spanning eulerian subgraph of G (see Fig. 11(a)).

Finally, we suppose that (3) of Claim 6 holds. Thus $H = H_2(v_1, v_2) + \{v_3v_5, v_4v_5\} - \{v_1v_3, v_2v_4\}$ is a desired spanning eulerian subgraph of G (see Fig. 11(b)). This completes the proof. $\square$

5. Conclusions and final remarks

In this work, we first considered the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible integer k, in some specified edge-colored graphs. We also considered the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times in some edge-colored graphs satisfying Ore-type degree conditions.

The proofs of Theorems 2 and 3 are based on a decomposition of a regular graph into edge-disjoint Hamilton cycles (and one perfect matching if it is odd regular). Motivated by the proof technique of the two theorems, we can prove a similar conclusion for any edge-colored regular graph G admitting a decomposition of G into edge-disjoint Hamilton cycles (and one perfect matching if G is odd regular).

Finally, we give an open problem on compatible spanning circuits in edge-colored graphs.

Problem 3. Let G be an edge-colored 2-connected graph on n vertices satisfying Fan's condition (see [9]), i.e., $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v of G with $dist(u, v) = 2$. Can G contain a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v)-1)/2 \rfloor$ times? If so, under what conditions does G contain such a compatible spanning circuit?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are particularly grateful to the referees for their helpful comments and suggestions that considerably improved the paper.

References

[1] S.K. Ahuja, Algorithms for Routing and Channel Assignment in Wireless Infrastructure Networks (Ph.D. thesis), Univ. Arizona, 2010.
[2] N. Alon, G. Gutin, Properly colored Hamilton cycles in edge-colored complete graphs, Random Structures Algorithms 11 (1997) 179–186.
[3] J. Bang-Jensen, G. Gutin, Digraghs: Theory, Algorithms and Applications, second ed., Springer-Verlag, London, 2009.
[4] B. Bollobás, P. Erdős, Alternating hamiltonian cycles, Israel J. Math. 23 (1976) 126–131.
[5] J.A. Bondy, U.S.R. Murty, Graph Theory, in: Graduate Texts in Mathematics, Vol. 244, Springer, New York, 2008.
[6] P.A. Catlin, Supereulerian graphs: a survey, J. Graph Theory 16 (1992) 177–196.
[7] C.C. Chen, D.E. Daykin, Graphs with Hamiltonian cycles having adjacent lines different colors, J. Combin. Theory Ser. B 21 (1976) 135–139.
[8] D.E. Daykin, Graphs with cycles having adjacent lines of different colors, J. Combin. Theory Ser. B 20 (1976) 149–152.
[9] G.H. Fan, New sufficient conditions for cycles in graphs, J. Combin. Theory Ser. B 37 (1984) 221–227.
[10] S. Fujita, C. Magnant, Properly colored paths and cycles, Discrete Appl. Math. 159 (2011) 1391–1397.
[11] G. Gutin, M. Jones, B. Sheng, M. Wahlstrom, A. Yeo, Chinese Postman problem on edge-colored multigraphs, Discrete Appl. Math. 217 (2017) 196–202.
[12] A. Kotzig, Moves without forbidden transitions in a graph, Mat. Časopis Sloven Akad. Vied 18 (1968) 76–80.
[13] H.-J. Lai, Y. Shao, H. Yan, An update on supereulerian graphs, WSEAS Trans. Math. 12 (2013) 926–940.
[14] R. Laskar, B. Auerback, On decomposition of r-partite graphs into edge-disjoint Hamilton circuits, Discrete Math. 14 (1976) 265–268.
[15] A. Lo, Properly coloured Hamiltonian cycles in edge-coloured complete graphs, Combinatorica 36 (2016) 471–492.
[16] P.A. Pevzner, Computational Molecular Biology: An Algorithmic Approach, MIT Press, Cambridge, MA, 2000.
[17] W.R. Pulleyblank, A note on graphs spanned by Eulerian graphs, J. Graph Theory 3 (1979) 309–310.
[18] S. Sankararaman, A. Efrat, S. Ramasubramanian, On channel discontinuity-constraint routing in wireless networks, Ad Hoc Netw. 134 (2014) 153–169.
[19] J. Shearer, A property of the colored complete graph, Discrete Math. 25 (1979) 175–178.
[20] M. Szachniuk, M.C. De Cola, G. Felici, The orderly colored longest path problem-a survey of applications and new algorithms, RAIRO-Oper. Res. 48 (2014) 25–51.
[21] M. Szachniuk, M. Popenda, R.W. Adamiak, J. Blazewicz, An assignment walk through 3D NMR spectrum, in: Proc. 2009 IEEE Symposium on Computational Intelligence in Bioinformatics and Computational Biology, 2009, pp. 215–219.
[22] I.L. Tseng, H.W. Chen, C.I. Lee, Obstacle-aware longest path routing with parallel MILP solvers, in: Proc WCECS-ICCS, Vol. 2, 2010, pp. 827–831.