Some Criteria for a Signed Graph to Have Full Rank

S. Akbari^{*}, A. Ghafari, K. Kazemian, M. Nahvi

Department of Mathematical Sciences Sharif University of Technology, Tehran, Iran

Abstract

A weighted graph G^{ω} consists of a simple graph G with a weight ω , which is a mapping, $\omega: E(G) \to \mathbb{Z} \setminus \{0\}$. A signed graph is a graph whose edges are labeled with -1 or 1. In this paper, we characterize graphs which have a sign such that their signed adjacency matrix has full rank, and graphs which have a weight such that their weighted adjacency matrix does not have full rank. We show that for any arbitrary simple graph G, there is a sign σ so that G^{σ} has full rank if and only if G has a $\{1, 2\}$ -factor. We also show that for a graph G, there is a weight ω so that G^{ω} does not have full rank if and only if G has at least two $\{1, 2\}$ -factors.

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1 Introduction

Throughout this paper, by a graph we mean a simple, undirected and finite graph. Let G be a graph. We denote the edge set and the vertex set of G by E(G) and V(G), respectively. By order and size of G, we mean the number of vertices and the number of edges of G, respectively. The adjacency matrix of a simple graph G is denoted by $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. We denote the complete graph of order n by K_n . A $\{1,2\}$ -factor of a graph G is a spanning subgraph of G which is a disjoint union of copies of K_2 and cycles. For a $\{1,2\}$ -factor H of G, the number of cycles of H is denoted by c(H). The number of $\{1,2\}$ -factors of a graph G is denoted by t(G). The perrank of a graph G of order n is defined to be the order of its largest subgraph which is a disjoint union of copies of K_2 and cycles, and we say G has full perrank if perrank(G) = n. For a graph G, a zero-sum flow is an assignment of non-zero real numbers to the edges of G such that the total sum of the assignments of all edges incident with any vertex is zero. For a positive integer k, a zero-sum k-flow of G is a zero-sum flow of G using the numbers $\{\pm 1, \ldots, \pm (k-1)\}$.

We call a matrix *integral* if all of its entries are integers. A set X of n entries of an $n \times n$ matrix A is called a *transversal*, if X contains exactly one entry of each row and each column of A. A transversal is called a *non-zero transversal* if all its entries are non-zero. The identity matrix is denoted by I. Also, j_n is an $n \times 1$ matrix with all entries 1.

A weighted graph G^{ω} consists of a simple graph G with a weight ω , which is a mapping, $\omega: E(G) \to \mathbb{Z} \setminus \{0\}$. A signed graph G^{σ} is a weighted graph where $\sigma: E(G) \to \{-1, 1\}$. The weighted

^{*} Email addresses:
s_akbari@sharif.edu, ghafaribaghestani_a@mehr.sharif.edu, kazemian_kimia@mehr.sharif.edu, nahvi_mina@mehr.sharif.edu

adjacency matrix of the weighted graph G^{ω} is denoted by $A(G^{\omega}) = [a_{ij}^{\omega}]$, where $a_{ij}^{\omega} = \omega(v_i v_j)$ if v_i and v_j are adjacent vertices, and $a_{ij}^{\omega} = 0$, otherwise. The rank of a weighted graph is defined to be the rank of its weighted adjacency matrix. A bidirected graph G is a graph such that each edge is composed of two directed half edges. Function $f : E(G) \to \mathbb{Z} \setminus \{0\}$ is a nowhere-zero \mathbb{Z} -flow of G if for every vertex v of G we have $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$, where $E^+(v)$ (resp. $E^-(v)$) is the set of all edges with tails (resp. heads) at v. For a positive integer k, a nowhere-zero k-flow of G is a nowhere-zero \mathbb{Z} -flow of G using the numbers $\{\pm 1, \ldots, \pm (k-1)\}$. For a graph G, where $E(G) = \{e_1, \ldots, e_m\}$ and $V(G) = \{v_1, \ldots, v_n\}$, we define $M_G(x_1, \ldots, x_m) = [m_{ij}]$ to be an $n \times n$ matrix, where $m_{ij} = \begin{cases} x_k & \text{If } e_k = v_i v_j \\ 0 & \text{otherwise} \end{cases}$. Now, define $f_G(x_1, \ldots, x_m) = \det(M_G(x_1, \ldots, x_m))$.

In order to establish our results, first we need the following well-known theorem, which has many applications in algebraic combinatorics.

Theorem A. [3] Let F be an arbitrary field and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree deg(f) of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is non-zero. Then, if S_1, \ldots, S_n are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.

The following remark is a necessary tool in proving our results.

Remark 1.1. For a graph G, each non-zero transversal in A(G) corresponds to a $\{1,2\}$ -factor of G, and each $\{1,2\}$ -factor H of G corresponds to $2^{c(H)}$ non-zero transversals in M.

In this paper, we prove the following theorems:

Theorem. Let G be a graph. Then there exists a sign σ for G so that G^{σ} has full rank if and only if G has full perrank.

Theorem. Let G be a graph. Then there exists a weight ω for G so that G^{ω} does not have full rank if and only if $t(G) \geq 2$.

2 Signed Graphs Which Have Full Rank

Our first result is the following theorem.

Theorem 2.1. Let G be a graph. Then there exists a sign σ for G so that G^{σ} has full rank if and only if G has full perrank.

Proof. First, assume that G has full perrank. Let m = |E(G)|, n = |V(G)|. Suppose that $\overline{f}(x_1, \ldots, x_m)$ is the polynomial obtained by replacing x_i^2 by 1 in $f_G(x_1, \ldots, x_m)$, for all $i, 1 \le i \le m$. Clearly, $deg_{x_i}\overline{f} \le 1$ for all $i, 1 \le i \le m$.

Let U be the set of all $\{1,2\}$ -factors of G. We have $U \neq \emptyset$. For each $H \in U$, we define a(H) as the number of K_2 components of H. We choose F_1 in U so that $a(F_1) = \max_{H \in U}(a(H))$. Note that F_1 does not contain any even cycles. So F_1 is a disjoint union of $a(F_1)$ copies of K_2 , and $c(F_1)$ odd cycles. Assume that all edges in the cycles of F_1 are e_1, \ldots, e_k . There are $2^{c(F_1)}$ non-zero transversals in M corresponding to F_1 , and all terms in $\overline{f}(x_1, \ldots, x_m)$ created by these non-zero transversals are $(-1)^{a(F_1)}x_1 \ldots x_k$. Let X be a non-zero transversal in M which makes the term $ax_1 \ldots x_k$ in $\overline{f}(x_1, \ldots, x_m)$ for some $a \in \mathbb{R}$ and does not correspond to F_1 . Let F_2 be the $\{1, 2\}$ -factor of G associated with X. The edges in the cycles of F_2 are e_1, \ldots, e_k . Therefore $a(F_2) = a(F_1)$, which is maximum. So F_2 has no even cycles and $a = (-1)^{a(F_1)}$. So $x_1 \ldots x_k$ has a non-zero coefficient in $\overline{f}(x_1, \ldots, x_m)$. Thus $\overline{f} \neq 0$ and it has a term $c \prod_{i=1}^m x_i^{t_i}$ with the maximum degree $\sum_{i=1}^m t_i$, where $t_i \in \{0, 1\}$ for each i, $1 \leq i \leq m$. Let $S_i = \{-1, 1\}$ for each $i, 1 \leq i \leq m$, so $|S_i| \geq 2$. By Theorem A there exists $(s_1, \ldots, s_m) \in S_1 \times \cdots \times S_m$ so that $\overline{f}(s_1, \ldots, s_m) \neq 0$. By defining σ as a sign assigning the same sign as s_i to e_i for each $i, 1 \leq i \leq m$, one can see that $det(A(G^{\sigma})) \neq 0$.

Now, assume that there exists such a sign for G. It is clear that $A(G^{\sigma})$ has a non-zero transversal, and the associated edges with this transversal, regardless of their signs, form a $\{1, 2\}$ -factor of G.

Corollary 2.2. Let G be a graph. Then $\max_{\sigma}(rank(G^{\sigma})) = perrank(G)$.

In the sequel we propose two following problems.

Problem 1. Let G be a graph. Determine $\min(rank(G^{\sigma}))$.

Problem 2. Find an efficient algorithm that can lead us to the desirable sign in Theorem 2.1.

3 Weighted Graphs Which Do Not Have Full Rank

In order to establish our next result, first we need the following lemma and theorems.

Lemma A. [2] Let G be a 2-edge connected bipartite graph. Then G has a zero-sum 6-flow.

Theorem B. [2] Suppose G is not a bipartite graph. Then G has a zero-sum flow if and only if for any edge e of G, $G \setminus \{e\}$ has no bipartite component.

Theorem C. [4] Every bidirected graph with a nowhere-zero \mathbb{Z} -flow has a nowhere-zero 12-flow.

According to [1], if we orient all edges of a simple graph in a way that all edges adjacent to each vertex v belong to $E^+(v)$, then a nowhere-zero bidirected flow corresponds to a zero-sum flow. Therefore, the following corollary is a result of Theorem C.

Corollary A. Every graph with a zero-sum flow has a zero-sum 12-flow.

Now, we can prove the following theorem.

Theorem 3.1. Let G be a graph. Then there exists a weight ω for G so that G^{ω} does not have full rank if and only if $t(G) \geq 2$.

Proof. Define $X = \{H | t(H) \ge 2 \text{ and for any } \omega, rank(A(H^{\omega})) = |V(H)|\}$. By contradiction assume that $X \ne \emptyset$. Let $n = \min_{H \in X} |V(H)|$ and $m = \min_{H \in X, |V(H)|=n} |E(H)|$ and $G \in X$ be a graph of order n and size m. Let $E(G) = \{e_1, \ldots, e_m\}$. Obviously, G is connected. For all $a_1, \ldots, a_m \in \mathbb{Z} \setminus \{0\}$,

we have $f_G(a_1, \ldots, a_m) \neq 0$. For each $i, 1 \leq i \leq m$, we can write $f_G = x_i^2 g_{G_i} + x_i h_{G_i} + l_{G_i}$, where g_{G_i} , h_{G_i} and l_{G_i} are polynomials in variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$. One can see that g_{G_i} is the zero polynomial if and only if e_i belongs to no K_2 component of any $\{1, 2\}$ -factor of G, h_{G_i} is the zero polynomial if and only if e_i belongs to no cycle of any $\{1, 2\}$ -factor of G, and l_{G_i} is the zero polynomial if and only if e_i belongs to all $\{1, 2\}$ -factor of G.

If there exists an edge e_i of G belonging to no $\{1,2\}$ -factor of G, then $t(G \setminus \{e_i\}) \ge 2$ and for each weight ω we have $det(A((G \setminus \{e_i\})^{\omega})) \ne 0$, which is a contradiction. So each edge of G is contained in at least one $\{1,2\}$ -factor. Moreover, if there exists an edge $e_i = uv$ which belongs to no cycle of any $\{1,2\}$ -factor of G but belongs to all $\{1,2\}$ -factors of G, then we have $t(G \setminus \{u,v\}) \ge 2$ and for each weight ω , $det(A((G \setminus \{u,v\})^{\omega})) \ne 0$, a contradiction. Furthermore we show that if e_i appears in a cycle of a $\{1,2\}$ -factor, then neither of the polynomials g_{G_i} and l_{G_i} is the zero polynomial. By contradiction assume that $g_{G_i}l_{G_i} \equiv 0$. If $g_{G_i} \equiv 0$ and $l_{G_i} \not\equiv 0$, then according to Theorem A there are $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in \mathbb{Z} \setminus \{0\}$ so that $h_{G_i}l_{G_i}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \ne 0$. It can be seen that $f_G(a_1, \ldots, a_m) = 0$, where $a_i = -\frac{l_{G_i}}{h_{G_i}}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)$. Note that for each j,

 $1 \leq j \leq m, a_j$ is a non-zero rational number. Since f_G is a homogeneous polynimial, it has a root in $(\mathbb{Z}\setminus\{0\})^m$, a contradiction. If $l_{G_i} \equiv 0$ and $g_{G_i} \not\equiv 0$, then the same argument leads to a contradiction. If $g_{G_i} = l_{G_i} \equiv 0$ and $e_i, e_{j_1}, \ldots, e_{j_p}$ are all edges of a cycle C of a $\{1, 2\}$ -factor of G, then for each $k, 1 \leq k \leq p$, we have $h_{G_i} = x_{j_k}p_k + q_k$, where p_k and q_k are polynomials in variables $\{x_j\}_{j\in I}$, where $I = \{1, \ldots, n\}\setminus\{i, j_k\}$. Obviously, $p_k \not\equiv 0$. If $q_k \not\equiv 0$, then using Theorem A one can see that h_{G_i} has a root in $(\mathbb{Z}\setminus\{0\})^{m-1}$, a contradiction. Therefore, one can see that $h_{G_i} = x_{j_1} \cdots x_{j_p}h_1$, where h_1 is a polynomial in variables $\{x_j\}_{j\in J}$, where $J = \{1, \ldots, m\}\setminus\{i, j_1, \ldots, j_p\}$. So C is a subgraph of every $\{1, 2\}$ -factor of G. Now, by considering the graph $G\setminus V(C)$ and noting that $t(G\setminus V(C)) \geq 2$, we obtain a contradiction. So we have $g_{G_i}, l_{G_i} \not\equiv 0$ for all $i, 1 \leq i \leq m$. So, for each edge e_i of G, there exists a $\{1, 2\}$ -factor of G containing e_i in a K_2 component.

If G has a zero-sum flow, then $M_G(a_1, \ldots, a_m)j_n = 0$, and as a result $f_G(a_1, \ldots, a_m) = 0$, where for each $i, 1 \le i \le m, a_i$ is the non-zero integer assigned to the edge e_i in the flow, a contradiction. Thus, assume that G has no zero-sum flow. Now, we have two cases:

- 1. The graph G is not bipartite. According to Theorem B, $G \setminus \{e_i\}$ has a bipartite component for some $i, 1 \le i \le m$. We have two cases:
 - (a) The edge e_i is not a cut edge. The graph $G \setminus \{e_i\}$ is a bipartite graph, say $G \setminus \{e_i\} = (X, Y)$, where the vertices adjacent to e_i belong to X. There is a $\{1, 2\}$ -factor of G having e_i in a K_2 component, so |X| 2 = |Y|. Also, there exists a $\{1, 2\}$ -factor of G not having e_i , therefore we have |X| = |Y|, a contradiction.
 - (b) Now, assume that e_i is a cut edge. The graph $G \setminus \{e_i\}$ has two components H and F, where F = (X, Y) is bipartite.
- 2. Now, suppose that G is bipartite. According to Lemma A, G has a cut edge e_i . We denote the bipartite connected components of the graph $G \setminus \{e_i\}$ by H and F = (X, Y).

In both Cases 1b and 2, there exists a $\{1, 2\}$ -factor of G having e_i in a K_2 component. Therefore, $F \setminus u$ has a $\{1, 2\}$ -factor, where $u \in X$ is the vertex in F adjacent to e_i , so we have |X| = |Y| + 1. On the other hand, there exists a $\{1, 2\}$ -factor of G which does not contain e_i . Hence, F has a $\{1, 2\}$ -factor. So we have |X| = |Y|, a contradiction.

Now, let G^w be a weighted graph which has full rank. Let $E(G) = \{e_1, \ldots, e_m\}$. By contradiction assume that t(G) < 2. Then f_G has one monomial and therefore for some $i, 1 \le i \le m$, $w(e_i) = 0$, a contradiction.

Remark 3.2. Let G be a graph with $t(G) \ge 2$. According to Lemma A, if G is bipartite, then there exists a weight ω : $E(G) \to \{\pm 1, \ldots, \pm 5\}$ such that G^{ω} does not have full rank. If G is not bipartite, then according to Corollary A, there exists a weight ω : $E(G) \to \{\pm 1, \ldots, \pm 11\}$ such that G^{ω} does not have full rank.

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References

- S. Akbari, A. Daemi, O. Hatami, A. Javanmard, A. Mehrabian, Zero-sum flows in regular graphs, Graphs and Combinatorics 26 (5) (2010) 603-615.
- [2] S. Akbari, S. Ghareghani, G.B. Khosrovshahi, A Mahmoody, On zero-sum 6-flows of graphs, Linear Algebra and its Applications 430 (11-12) (2009) 3047-3052.
- [3] N. Alon, Combinatorial nullstellensatz, Combinatorics, Probability and Computing 8(1-2) (1999) 7-29.
- [4] M. DeVos, Flows on bidirected graphs, preprint, 2013. Available at arXiv:1310.8406 [math.CO]