# Some Criteria for a Signed Graph to Have Full Rank 

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#### Abstract

A weighted graph $G^{\omega}$ consists of a simple graph $G$ with a weight $\omega$, which is a mapping, $\omega: E(G) \rightarrow \mathbb{Z} \backslash\{0\}$. A signed graph is a graph whose edges are labeled with -1 or 1 . In this paper, we characterize graphs which have a sign such that their signed adjacency matrix has full rank, and graphs which have a weight such that their weighted adjacency matrix does not have full rank. We show that for any arbitrary simple graph $G$, there is a sign $\sigma$ so that $G^{\sigma}$ has full rank if and only if $G$ has a $\{1,2\}$-factor. We also show that for a graph $G$, there is a weight $\omega$ so that $G^{\omega}$ does not have full rank if and only if $G$ has at least two $\{1,2\}$-factors.


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## 1 Introduction

Throughout this paper, by a graph we mean a simple, undirected and finite graph. Let $G$ be a graph. We denote the edge set and the vertex set of $G$ by $E(G)$ and $V(G)$, respectively. By order and size of $G$, we mean the number of vertices and the number of edges of $G$, respectively. The adjacency matrix of a simple graph $G$ is denoted by $A(G)=\left[a_{i j}\right]$, where $a_{i j}=1$ if $v_{i}$ and $v_{j}$ are adjacent, and $a_{i j}=0$ otherwise. We denote the complete graph of order $n$ by $K_{n}$. A $\{1,2\}$-factor of a graph $G$ is a spanning subgraph of $G$ which is a disjoint union of copies of $K_{2}$ and cycles. For a $\{1,2\}$-factor $H$ of $G$, the number of cycles of $H$ is denoted by $c(H)$. The number of $\{1,2\}$-factors of a graph $G$ is denoted by $t(G)$. The perrank of a graph $G$ of order $n$ is defined to be the order of its largest subgraph which is a disjoint union of copies of $K_{2}$ and cycles, and we say $G$ has full perrank if $\operatorname{perrank}(G)=n$. For a graph $G$, a zero-sum flow is an assignment of non-zero real numbers to the edges of $G$ such that the total sum of the assignments of all edges incident with any vertex is zero. For a positive integer $k$, a zero-sum $k$-flow of $G$ is a zero-sum flow of $G$ using the numbers $\{ \pm 1, \ldots, \pm(k-1)\}$.
We call a matrix integral if all of its entries are integers. A set $X$ of $n$ entries of an $n \times n$ matrix $A$ is called a transversal, if $X$ contains exactly one entry of each row and each column of $A$. A transversal is called a non-zero transversal if all its entries are non-zero. The identity matrix is denoted by $I$. Also, $j_{n}$ is an $n \times 1$ matrix with all entries 1 .
A weighted graph $G^{\omega}$ consists of a simple graph $G$ with a weight $\omega$, which is a mapping, $\omega: E(G) \rightarrow \mathbb{Z} \backslash\{0\}$. A signed graph $G^{\sigma}$ is a weighted graph where $\sigma: E(G) \rightarrow\{-1,1\}$. The weighted

[^0]adjacency matrix of the weighted graph $G^{\omega}$ is denoted by $A\left(G^{\omega}\right)=\left[a_{i j}^{\omega}\right]$, where $a_{i j}^{\omega}=\omega\left(v_{i} v_{j}\right)$ if $v_{i}$ and $v_{j}$ are adjacent vertices, and $a_{i j}^{\omega}=0$, otherwise. The rank of a weighted graph is defined to be the rank of its weighted adjacency matrix. A bidirected graph $G$ is a graph such that each edge is composed of two directed half edges. Function $f: E(G) \rightarrow \mathbb{Z} \backslash\{0\}$ is a nowhere-zero $\mathbb{Z}$ flow of $G$ if for every vertex $v$ of $G$ we have $\sum_{e \in E^{+}(v)} f(e)=\sum_{e \in E^{-}(v)} f(e)$, where $E^{+}(v)$ (resp. $\left.E^{-}(v)\right)$ is the set of all edges with tails (resp. heads) at $v$. For a positive integer $k$, a nowhere-zero $k$-flow of $G$ is a nowhere-zero $\mathbb{Z}$-flow of $G$ using the numbers $\{ \pm 1, \ldots, \pm(k-1)\}$. For a graph $G$, where $E(G)=\left\{e_{1}, \ldots, e_{m}\right\}$ and $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$, we define $M_{G}\left(x_{1}, \ldots, x_{m}\right)=\left[m_{i j}\right]$ to be an $n \times n$ matrix, where $m_{i j}=\left\{\begin{array}{ll}x_{k} & \text { If } e_{k}=v_{i} v_{j} \\ 0 & \text { otherwise }\end{array}\right.$. Now, define $f_{G}\left(x_{1}, \ldots, x_{m}\right)=\operatorname{det}\left(M_{G}\left(x_{1}, \ldots, x_{m}\right)\right)$. In order to establish our results, first we need the following well-known theorem, which has many applications in algebraic combinatorics.

Theorem A. 3] Let $F$ be an arbitrary field and let $f=f\left(x_{1}, \ldots, x_{n}\right)$ be a polynomial in $F\left[x_{1}, \ldots, x_{n}\right]$. Suppose the degree $\operatorname{deg}(f)$ of $f$ is $\sum_{i=1}^{n} t_{i}$, where each $t_{i}$ is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_{i}^{t_{i}}$ in $f$ is non-zero. Then, if $S_{1}, \ldots, S_{n}$ are subsets of $F$ with $\left|S_{i}\right|>t_{i}$, there are $s_{1} \in S_{1}, s_{2} \in S_{2}, \ldots, s_{n} \in S_{n}$ so that $f\left(s_{1}, \ldots, s_{n}\right) \neq 0$.

The following remark is a necessary tool in proving our results.

Remark 1.1. For a graph $G$, each non-zero transversal in $A(G)$ corresponds to a $\{1,2\}$-factor of $G$, and each $\{1,2\}$-factor $H$ of $G$ corresponds to $2^{c(H)}$ non-zero transversals in $M$.

In this paper, we prove the following theorems:
Theorem. Let $G$ be a graph. Then there exists a sign $\sigma$ for $G$ so that $G^{\sigma}$ has full rank if and only if $G$ has full perrank.

Theorem. Let $G$ be a graph. Then there exists a weight $\omega$ for $G$ so that $G^{\omega}$ does not have full rank if and only if $t(G) \geq 2$.

## 2 Signed Graphs Which Have Full Rank

Our first result is the following theorem.

Theorem 2.1. Let $G$ be a graph. Then there exists a sign $\sigma$ for $G$ so that $G^{\sigma}$ has full rank if and only if $G$ has full perrank.
 $\bar{f}\left(x_{1}, \ldots, x_{m}\right)$ is the polynomial obtained by replacing $x_{i}^{2}$ by 1 in $f_{G}\left(x_{1}, \ldots, x_{m}\right)$, for all $i, 1 \leq i \leq$ $m$. Clearly, $\operatorname{deg}_{x_{i}} \bar{f} \leq 1$ for all $i, 1 \leq i \leq m$.
Let $U$ be the set of all $\{1,2\}$-factors of $G$. We have $U \neq \varnothing$. For each $H \in U$, we define $a(H)$ as the number of $K_{2}$ components of $H$. We choose $F_{1}$ in $U$ so that $a\left(F_{1}\right)=\max _{H \in U}(a(H))$. Note that $F_{1}$ does not contain any even cycles. So $F_{1}$ is a disjoint union of $a\left(F_{1}\right)$ copies of $K_{2}$, and $c\left(F_{1}\right)$ odd cycles. Assume that all edges in the cycles of $F_{1}$ are $e_{1}, \ldots, e_{k}$. There are $2^{c\left(F_{1}\right)}$ non-zero transversals in $M$ corresponding to $F_{1}$, and all terms in $\bar{f}\left(x_{1}, \ldots, x_{m}\right)$ created by these non-zero transversals are $(-1)^{a\left(F_{1}\right)} x_{1} \ldots x_{k}$.

Let $X$ be a non-zero transversal in $M$ which makes the term $a x_{1} \ldots x_{k}$ in $\bar{f}\left(x_{1}, \ldots, x_{m}\right)$ for some $a \in \mathbb{R}$ and does not correspond to $F_{1}$. Let $F_{2}$ be the $\{1,2\}$-factor of $G$ associated with $X$. The edges in the cycles of $F_{2}$ are $e_{1}, \ldots, e_{k}$. Therefore $a\left(F_{2}\right)=a\left(F_{1}\right)$, which is maximum. So $F_{2}$ has no even cycles and $a=(-1)^{a\left(F_{1}\right)}$. So $x_{1} \ldots x_{k}$ has a non-zero coefficient in $\bar{f}\left(x_{1}, \ldots, x_{m}\right)$. Thus $\bar{f} \not \equiv 0$ and it has a term $c \prod_{i=1}^{m} x_{i}^{t_{i}}$ with the maximum degree $\sum_{i=1}^{m} t_{i}$, where $t_{i} \in\{0,1\}$ for each $i$, $1 \leq i \leq m$. Let $S_{i}=\{-1,1\}$ for each $i, 1 \leq i \leq m$, so $\left|S_{i}\right| \geq 2$. By Theorem A there exists $\left(s_{1}, \ldots, s_{m}\right) \in S_{1} \times \cdots \times S_{m}$ so that $\bar{f}\left(s_{1}, \ldots, s_{m}\right) \neq 0$. By defining $\sigma$ as a sign assigning the same $\operatorname{sign}$ as $s_{i}$ to $e_{i}$ for each $i, 1 \leq i \leq m$, one can see that $\operatorname{det}\left(A\left(G^{\sigma}\right)\right) \neq 0$.

Now, assume that there exists such a sign for $G$. It is clear that $A\left(G^{\sigma}\right)$ has a non-zero transversal, and the associated edges with this transversal, regardless of their signs, form a $\{1,2\}$-factor of $G$.

Corollary 2.2. Let $G$ be a graph. Then $\max _{\sigma}\left(\operatorname{rank}\left(G^{\sigma}\right)\right)=\operatorname{perrank}(G)$.
In the sequel we propose two following problems
Problem 1. Let $G$ be a graph. Determine $\min _{\sigma}\left(\operatorname{rank}\left(G^{\sigma}\right)\right)$.
Problem 2. Find an efficient algorithm that can lead us to the desirable sign in Theorem 2.1.

## 3 Weighted Graphs Which Do Not Have Full Rank

In order to establish our next result, first we need the following lemma and theorems.

Lemma A. 2 Let $G$ be a 2-edge connected bipartite graph. Then $G$ has a zero-sum 6-flow.

Theorem B. [2] Suppose $G$ is not a bipartite graph. Then $G$ has a zero-sum flow if and only if for any edge e of $G, G \backslash\{e\}$ has no bipartite component.

Theorem C. [4] Every bidirected graph with a nowhere-zero $\mathbb{Z}$-flow has a nowhere-zero 12-flow.

According to [1], if we orient all edges of a simple graph in a way that all edges adjacent to each vertex $v$ belong to $E^{+}(v)$, then a nowhere-zero bidirected flow corresponds to a zero-sum flow. Therefore, the following corollary is a result of Theorem C.

Corollary A. Every graph with a zero-sum flow has a zero-sum 12-flow.

Now, we can prove the following theorem.

Theorem 3.1. Let $G$ be a graph. Then there exists a weight $\omega$ for $G$ so that $G^{\omega}$ does not have full rank if and only if $t(G) \geq 2$.

Proof. Define $X=\left\{H \mid t(H) \geq 2\right.$ and for any $\omega$, $\left.\operatorname{rank}\left(A\left(H^{\omega}\right)\right)=|V(H)|\right\}$. By contradiction assume that $X \neq \varnothing$. Let $n=\min _{H \in X}|V(H)|$ and $m=\min _{H \in X,|V(H)|=n}|E(H)|$ and $G \in X$ be a graph of order $n$ and size $m$. Let $E(G)=\left\{e_{1}, \ldots, e_{m}\right\}$. Obviously, $G$ is connected. For all $a_{1}, \ldots, a_{m} \in \mathbb{Z} \backslash\{0\}$,
we have $f_{G}\left(a_{1}, \ldots, a_{m}\right) \neq 0$. For each $i, 1 \leq i \leq m$, we can write $f_{G}=x_{i}^{2} g_{G_{i}}+x_{i} h_{G_{i}}+l_{G_{i}}$, where $g_{G_{i}}, h_{G_{i}}$ and $l_{G_{i}}$ are polynomials in variables $x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m}$. One can see that $g_{G_{i}}$ is the zero polynomial if and only if $e_{i}$ belongs to no $K_{2}$ component of any $\{1,2\}$-factor of $G, h_{G_{i}}$ is the zero polynomial if and only if $e_{i}$ belongs to no cycle of any $\{1,2\}$-factor of $G$, and $l_{G_{i}}$ is the zero polynomial if and only if $e_{i}$ belongs to all $\{1,2\}$-factors of $G$.
If there exists an edge $e_{i}$ of $G$ belonging to no $\{1,2\}$-factor of $G$, then $t\left(G \backslash\left\{e_{i}\right\}\right) \geq 2$ and for each weight $\omega$ we have $\operatorname{det}\left(A\left(\left(G \backslash\left\{e_{i}\right\}\right)^{\omega}\right)\right) \neq 0$, which is a contradiction. So each edge of $G$ is contained in at least one $\{1,2\}$-factor. Moreover, if there exists an edge $e_{i}=u v$ which belongs to no cycle of any $\{1,2\}$-factor of $G$ but belongs to all $\{1,2\}$-factors of $G$, then we have $t(G \backslash\{u, v\}) \geq 2$ and for each weight $\omega, \operatorname{det}\left(A\left((G \backslash\{u, v\})^{\omega}\right)\right) \neq 0$, a contradiction. Furthermore we show that if $e_{i}$ appears in a cycle of a $\{1,2\}$-factor, then neither of the polynomials $g_{G_{i}}$ and $l_{G_{i}}$ is the zero polynomial. By contradiction assume that $g_{G_{i}} l_{G_{i}} \equiv 0$. If $g_{G_{i}} \equiv 0$ and $l_{G_{i}} \not \equiv 0$, then according to Theorem A there are $a_{1}, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m} \in \mathbb{Z} \backslash\{0\}$ so that $h_{G_{i}} l_{G_{i}}\left(a_{1}, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m}\right) \neq 0$. It can be seen that $f_{G}\left(a_{1}, \ldots, a_{m}\right)=0$, where $a_{i}=-\frac{l_{G_{i}}}{h_{G_{i}}}\left(a_{1}, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m}\right)$. Note that for each $j$, $1 \leq j \leq m, a_{j}$ is a non-zero rational number. Since $f_{G}$ is a homogeneous polynimial, it has a root in $(\mathbb{Z} \backslash\{0\})^{m}$, a contradiction. If $l_{G_{i}} \equiv 0$ and $g_{G_{i}} \not \equiv 0$, then the same argument leads to a contradiction. If $g_{G_{i}}=l_{G_{i}} \equiv 0$ and $e_{i}, e_{j_{1}}, \ldots, e_{j_{p}}$ are all edges of a cycle $C$ of a $\{1,2\}$-factor of $G$, then for each $k, 1 \leq k \leq p$, we have $h_{G_{i}}=x_{j_{k}} p_{k}+q_{k}$, where $p_{k}$ and $q_{k}$ are polynomials in variables $\left\{x_{j}\right\}_{j \in I}$, where $I=\{1, \ldots, n\} \backslash\left\{i, j_{k}\right\}$. Obviously, $p_{k} \not \equiv 0$. If $q_{k} \not \equiv 0$, then using Theorem A one can see that $h_{G_{i}}$ has a root in $(\mathbb{Z} \backslash\{0\})^{m-1}$, a contradiction. Therefore, one can see that $h_{G_{i}}=x_{j_{1}} \cdots x_{j_{p}} h_{1}$, where $h_{1}$ is a polynomial in variables $\left\{x_{j}\right\}_{j \in J}$, where $J=\{1, \ldots, m\} \backslash\left\{i, j_{1}, \ldots, j_{p}\right\}$. So $C$ is a subgraph of every $\{1,2\}$-factor of $G$. Now, by considering the graph $G \backslash V(C)$ and noting that $t(G \backslash V(C)) \geq 2$, we obtain a contradiction. So we have $g_{G_{i}}, l_{G_{i}} \not \equiv 0$ for all $i, 1 \leq i \leq m$. So, for each edge $e_{i}$ of $G$, there exists a $\{1,2\}$-factor of $G$ not containing $e_{i}$, and also there exists a $\{1,2\}$-factor of $G$ containing $e_{i}$ in a $K_{2}$ component.
If $G$ has a zero-sum flow, then $M_{G}\left(a_{1}, \ldots, a_{m}\right) j_{n}=0$, and as a result $f_{G}\left(a_{1}, \ldots, a_{m}\right)=0$, where for each $i, 1 \leq i \leq m, a_{i}$ is the non-zero integer assigned to the edge $e_{i}$ in the flow, a contradiction. Thus, assume that $G$ has no zero-sum flow. Now, we have two cases:

1. The graph $G$ is not bipartite. According to Theorem $\mathrm{B}, G \backslash\left\{e_{i}\right\}$ has a bipartite component for some $i, 1 \leq i \leq m$. We have two cases:
(a) The edge $e_{i}$ is not a cut edge. The graph $G \backslash\left\{e_{i}\right\}$ is a bipartite graph, say $G \backslash\left\{e_{i}\right\}=$ $(X, Y)$, where the vertices adjacent to $e_{i}$ belong to $X$. There is a $\{1,2\}$-factor of $G$ having $e_{i}$ in a $K_{2}$ component, so $|X|-2=|Y|$. Also, there exists a $\{1,2\}$-factor of $G$ not having $e_{i}$, therefore we have $|X|=|Y|$, a contradiction.
(b) Now, assume that $e_{i}$ is a cut edge. The graph $G \backslash\left\{e_{i}\right\}$ has two components $H$ and $F$, where $F=(X, Y)$ is bipartite.
2. Now, suppose that $G$ is bipartite. According to Lemma A, $G$ has a cut edge $e_{i}$. We denote the bipartite connected components of the graph $G \backslash\left\{e_{i}\right\}$ by $H$ and $F=(X, Y)$.

In both Cases 1b and 2, there exists a $\{1,2\}$-factor of $G$ having $e_{i}$ in a $K_{2}$ component. Therefore, $F \backslash u$ has a $\{1,2\}$-factor, where $u \in X$ is the vertex in $F$ adjacent to $e_{i}$, so we have $|X|=|Y|+1$. On the other hand, there exists a $\{1,2\}$-factor of $G$ which does not contain $e_{i}$. Hence, $F$ has a $\{1,2\}$-factor. So we have $|X|=|Y|$, a contradiction.

Now, let $G^{w}$ be a weighted graph which has full rank. Let $E(G)=\left\{e_{1}, \ldots, e_{m}\right\}$. By contradiction assume that $t(G)<2$. Then $f_{G}$ has one monomial and therefore for some $i, 1 \leq i \leq m$, $w\left(e_{i}\right)=0$, a contradiction.

Remark 3.2. Let $G$ be a graph with $t(G) \geq 2$. According to Lemma $A$, if $G$ is bipartite, then there exists a weight $\omega: E(G) \rightarrow\{ \pm 1, \ldots, \pm 5\}$ such that $G^{\omega}$ does not have full rank. If $G$ is not bipartite, then according to Corollary A, there exists a weight $\omega: E(G) \rightarrow\{ \pm 1, \ldots, \pm 11\}$ such that $G^{\omega}$ does not have full rank.

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