# Restricted permutations refined by number of crossings and nestings 

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#### Abstract

Let $\mathrm{st}=\left\{\mathrm{st}_{1}, \ldots, \mathrm{st}_{k}\right\}$ be a set of $k$ statistics on permutations with $k \geq 1$. We say that two given subset of permutations $T$ and $T^{\prime}$ are st-Wilf-equivalent if the joint distributions of all statistics in st over the sets of $T$-avoiding permutations $S_{n}(T)$ and $T^{\prime}$-avoiding permutations $S_{n}\left(T^{\prime}\right)$ are the same. The main purpose of this paper is the (cr,nes)-Wilfequivalence classes for all single patterns in $S_{3}$, where cr and nes denote respectively the statistics number of crossings and nestings. One of the main tools that we use is the bijection $\Theta: S_{n}(321) \rightarrow S_{n}(132)$ which was originally exhibited by Elizalde and Pak in [10]. They proved that the bijection $\Theta$ preserves the number of fixed points and excedances. Since the given formulation of $\Theta$ is not direct, we show that it can be defined directly by a recursive formula. Then, we prove that it also preserves the number of crossings. Due to the fact that the sets of non-nesting permutations and 321-avoiding permutations are the same, these properties of the bijection $\Theta$ leads to an unexpected result related to the q,p-Catalan numbers of Randrianarivony defined in [17].


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## 1 Introduction and main result

Let $E=\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ be a set of $n$ integers such that $e_{1}<e_{2}<\ldots<e_{n}$. A permutation $\sigma$ of $E$ is a bijection from $E$ to itself which can be written linearly as $\sigma=\sigma\left(e_{1}\right) \sigma\left(e_{2}\right) \ldots \sigma\left(e_{n}\right)$. We shall refer to $|\sigma|:=n$ as the length of $\sigma$. For any permutation $\sigma$ of $E$, the reduction of $\sigma$ is $\operatorname{red}(\sigma):=\tau \sigma \tau^{-1}$, where $\tau$ is the unique order preserving bijection from $E$ to $[n]:=$ $\{1,2, \ldots, n\}$. Example: $\sigma=43795$ is a permutation of $E=\{3,4,5,7,9\}$ (i.e. $\sigma(3)=4$, $\sigma(4)=3, \sigma(5)=7, \sigma(7)=9$ and $\sigma(9)=5)$ and we have $\operatorname{red}(\sigma)=21453$. We will denote by $S_{n}$ the set of all permutations of $[n]$.

The concept of crossing and nesting on permutations were introduced by A. de Médicis and X.G. Viennot [7] and several authors extended their study, eg [1, 6, 16, 17]. Recently, S. Burrill et al. [1] generalized the definition and introduced the notion of $k$-crossing and $k$-nesting. In this work, we are only interested on 2-crossing and 2-nesting that we simply call crossing and nesting. A crossing in a permutation $\sigma$ is a pair of indexes $(i, j)$ such that $i<j<\sigma(i)<\sigma(j)$ or $\sigma(i)<\sigma(j) \leq i<j$. A nesting of $\sigma$ is similarly defined as a pair $(i, j)$
such that $i<j<\sigma(j)<\sigma(i)$ or $\sigma(j)<\sigma(i) \leq i<j$. We denote respectively by $\operatorname{cr}(\sigma)$ and nes $(\sigma)$ the number of crossings and nestings of $\sigma$. For better understanding, one can use arc diagram representations. For $\pi=46298171035 \in S_{10}$ (see Fig. 1), we have $\operatorname{cr}(\pi)=8$ (upper crossings are $(1,2),(2,4),(2,5),(4,8)$ and lower crossings are $(3,9),(6,9),(6,10)$, $(9,10)$ ) and nes $(\pi)=4$ (the only upper nesting is $(4,5)$ and the three lower nestings are $(3,6),(7,9)$ and $(7,10))$.


Figure 1: Arc diagrams of $\pi=46298171035 \in S_{10}$.
It was known that the joint distribution of the statistics cr and nes over $S_{n}$ is symmetric (e.g. Corollary 2.3 [17] and Proposition 4 in [6]), i.e. $\sum_{\sigma \in S_{n}} x^{\operatorname{cr}(\sigma)} y^{\operatorname{nes}(\sigma)}=\sum_{\sigma \in S_{n}} x^{\operatorname{nes}(\sigma)} y^{\operatorname{cr}(\sigma)}$. Combining analytical and bijection methods, the continued fraction expansion of the ordinary generating function of this joint distribution was first computed by Randrianarivony [17]. He obtained the following identity

$$
\sum_{n \geq 0} \sum_{\sigma \in S_{n}} x^{\operatorname{cr}(\sigma)} y^{\operatorname{nes}(\sigma)} z^{n}=\frac{1}{1-\frac{[1]_{x, y} \cdot z}{1-\frac{[1]_{x, y} \cdot z}{1-\frac{[2]_{x, y} \cdot z}{1-\frac{[2]_{x, y} \cdot z}{1-\frac{[3]_{x, y} \cdot z}{1-\frac{[3]_{x, y} \cdot z}{\ddots}}}}}},}
$$

where $[n]_{x, y}=x^{n-1}+x^{n-2} y+\ldots+x y^{n-2}+y^{n-1}$ for any integer $n \geq 1$. We will denote respectively by $N C_{n}$ and $N N_{n}$ the set of all noncrossing and nonnesting permutations of [n]. It is well known in the literature that $\left|N C_{n}\right|=\left|N N_{n}\right|=C_{n}$, the $n$-th Catalan number $C_{n}=\frac{1}{n+1}\binom{2 n}{n}$.

Let us now define some other statistics on permutations. For that, we fix $\sigma \in S_{n}$. Say that index $i$ is a fixed point (resp excedance, descent) of $\sigma$ if $\sigma(i)=i($ resp $\sigma(i)>i, \sigma(i)>$ $\sigma(i+1)$ ). An inversion of $\sigma$ is a pair of indexes $(i, j)$ such that $i<j$ and $\sigma(i)>\sigma(j)$. We denote respectively by $\operatorname{fp}(\sigma), \operatorname{exc}(\sigma)$ and $\operatorname{inv}(\sigma)$ the number of fixed points, excedances and inversions of $\sigma$. Define the major index of $\sigma$, denoted by $\operatorname{maj}(\sigma)$, as the sum of all descents of $\sigma$.

Let $\sigma \in S_{n}$ and $\tau \in S_{k}$ with $k \leq n$. Say that the subsequence $\sigma\left(i_{1}\right) \sigma\left(i_{2}\right) \ldots \sigma\left(i_{k}\right)$, with $i_{1}<i_{2}<\ldots<i_{k}$, is an occurrence of $\tau$ if $\operatorname{red}\left[\sigma\left(i_{1}\right) \sigma\left(i_{2}\right) \ldots \sigma\left(i_{k}\right)\right]=\tau$. If no such subsequence exists in $\sigma$, we say that it avoids $\tau$ or it is $\tau$-avoiding. For example, the subsequences 2415 , 2418, 2416 and 2417 are the four occurrences of the pattern 2314 in $\pi=24135867 \in S_{8}$. We let the reader to verify that $\pi$ is 321 -avoiding. For any given subset of permutations $T$, we will denote by $S_{n}(T)$ the set of all permutations of $[n]$ that avoid all patterns in $T$ and $S(T)=\cup_{n} S_{n}(T)$. For the case of patterns of length 3, it is well known [14] that regardless of the pattern $\tau \in S_{3},\left|S_{n}(\tau)\right|=C_{n}$. Bijective proof of the fact that $\left|S_{n}(321)\right|=\left|S_{n}(132)\right|$ interested several authors. There exists several constructed bijections between $S_{n}(321)$ and
$S_{n}(132)$. Each of them has its own properties. See for example [4] and reference therein or [ $2,10,14]$. In this paper, we are particularly interested on the (fp,exc)-preserving bijection $\Theta: S_{n}(321) \rightarrow S_{n}(132)$ of Elizalde and Pak [10] that will help us for solving our problem.

Let $\left\{\mathrm{st}_{1}, \ldots, \mathrm{st}_{k}\right\}$ be a set of $k$ statistics on $S_{n}$ with $k \geq 1$. Two given subset of patterns $T$ and $T^{\prime}$ are $\left(\mathrm{st}_{1}, \ldots, \mathrm{st}_{k}\right)$-Wilf equivalent if the following identity holds

$$
\sum_{\sigma \in S_{n}(T)} x_{1}^{\mathrm{st}_{1}(\sigma)} \ldots x_{k}^{\mathrm{st}_{k}(\sigma)}=\sum_{\sigma \in S_{n}\left(T^{\prime}\right)} x_{1}^{\mathrm{st}_{1}(\sigma)} \ldots x_{k}^{\mathrm{st}_{k}(\sigma)}
$$

Our paper is naturally inspired from some previous known works. In [19], Robertson et al. focused on the fp-Wilf-equivalence classes. Elizalde [11] generalized the result of Robertson et al. and proposed the (fp,exc)-Wilf-equivalence classes. Recently, Dokos et al. [8] studied the Wilf-equivalence classes modulo inv and maj. We summarize in table 1 their known results for single pattern of length 3 .

| Statistic st | st-Wilf-equivalence classes | Reference |
| :---: | :---: | :---: |
| fp | $\{132,213,321\},\{231,312\}$ and $\{123\}$ | $[19]$ |
| (fp,exc) | $\{132,213,321\},\{231\},\{312\}$ and $\{123\}$ | $[11]$ |
| inv | $\{132,213\},\{231,312\},\{321\}$ and $\{123\}$ | $[8]$ |
| maj | $\{132,231\},\{213,312\},\{321\}$ and $\{123\}$ | $[8]$ |
| (fp,exc,inv) | $\{132,213\},\{231,312\},\{321\}$ and $\{123\}$ | $[11,8]$ |

Table 1: Some known results on fp,exc,inv,maj-Wilf-equivalence classes.
In this paper, we focus on the (cr,nes)-Wilf-equivalence classes for singleton patterns in $S_{3}$. Following the notation in [8], we denote by $[T]_{s t}$ the st-Wilf-equivalent class for any subset of patterns $T$ and set of statistics st. After these necessary preliminaries, we are now in a position to present the main result of this paper that can be stated as follows.
Theorem 1.1. For single patterns in $S_{3}$, the non singleton cr and nes-Wilf-equivalence classes are the following
i) $[132]_{\text {nes }}=\{132,213\}$ and $[231]_{\text {nes }}=\{231,312\}$,
ii) $[132]_{\mathrm{cr}}=\{132,213,321\}$,
iii) $[132]_{\text {cr,nes }}=\{132,213\}$.

When we combine this result with those of Elizalde and Dokos et al., we get a more generalized one (see Section 4). The connection with the result of Randrianarivony [17] is due to the fact that nonnesting permutations and 321-avoiding permutations are the same.

The rest of this paper is now organized as follow. In Section 2, we first recall the bijection $\Theta$ of Elizalde and Pak before presenting an interesting inductive formula for it. In Section 3, we establish the cr-preserving of the bijection $\Theta$. In Section 4, we provide the proof of Theorem 1.1 using the bijection $\Theta$ and some known trivial bijections on permutations namely reverse, complement and inverse. In Section 5, we discuss the unexpected connection to the $q, p$-Catalan numbers defined and interpreted by Randrianarivony [17]. In Section 6, we finally conclude this paper with two interesting remarks. The first one is the correspondence of decomposition between our combinatorial objects while the second one is the cr-preserving of the direct bijection $\Gamma$ of Robertson [20] which comes from a recent result of Saracino [21].

## 2 Review of Elizalde and Pak's bijection

As mentioned in introduction, Elizalde and Pak exhibited a bijection $\Theta$ from $S_{n}(321)$ to $S_{n}(132)$ which preserves the statistics fp and exc. Using Dyck paths as intermediate object, they defined the bijection $\Theta$ as a composition of two bijections. The first one is a bijection $\Psi$ from $S_{n}(321)$ to $\mathcal{D}_{n}$ (set of $n$-Dyck paths) and the second one is a bijection $\Phi$ from $S_{n}(132)$ to $\mathcal{D}_{n}$. So, we have $\Theta=\Phi^{-1} \circ \Psi$. In this section, after some reviews on Dyck paths and notions of tunnels introduced by Elizalde et al., we recall the two bijections $\Psi$ and $\Phi^{-1}$. Then, we exploit the bijection $\Theta$ and find a new description of its definition.

### 2.1 Tunnels on Dyck paths

A Dyck path of semi-length $n$, called also $n$-Dyck path, is a path in the first quadrant which starts from the origin $(0,0)$, ends at $(2 n, 0)$, and consists of $n$ up-steps $(1,1)$ and $n$ downsteps $(1,-1)$. Usually, we encode each up-step by a letter $u$ and each down-step by $d$. The resulting encoding of a Dyck path is called Dyck word. We denote by $\mathcal{D}_{n}$ the set of all $n$ Dyck paths. If $D=D_{1} \ldots D_{2 n} \in \mathcal{D}_{n}$, we have $|D|_{u}=|D|_{d}=n$ and $|D(k)|_{u} \geq|D(k)|_{d}$ for any initial sub-word $D(k)=D_{1} \ldots D_{k}$ of length $k$ of $D$, where $|w|_{a}$ denotes the number of occurrences of letter $a$ in a word $w$. In [9, 10], Elizalde et al. introduced the statistic number of tunnels on Dyck paths that they used to enumerate restricted permutations according to the number of fixed points and excedances. In fact, a tunnel of a Dyck path $D \in \mathcal{D}_{n}$ is an horizontal segment between two lattice points of $D$ that intersects $D$ only at these two points, and stays always below $D$. They distinguished tunnels according to the coordinates of their midpoints. Graphically, right and left tunnels of a Dyck paths are respectively those with their midpoints stay on the right, and on the left of the vertical line through the middle of the path $(x=n)$. Centered tunnels are those whose midpoints stay on the vertical line $x=n$. We denote respectively by $\operatorname{rt}(D), \operatorname{ct}(D)$ and $\operatorname{lt}(D)$ the number of right tunnels, centered tunnels and left tunnels. The Dyck path in Fig. 2 below has the following characteristics: $\operatorname{rt}(D)=3, \operatorname{ct}(D)=1$ and $\operatorname{lt}(D)=4$.


Figure 2: Counting tunnels of the Dyck path $D=u d u d u u u d d u d d u u d d$.
As seen in figure 2, each tunnel is a segment that goes from the beginning of an up-step $u$ to the end of a down-step $d$. As mentioned in [9], such tunnel is also in obvious one-to-one correspondence with decomposition of the Dyck word $D=A u B d C$, where $B$ and $A C$ are both Dyck paths.

### 2.2 The bijection $\Psi: S_{n}(321) \rightarrow \mathcal{D}_{n}$

The bijection $\Psi$ is essentially due to Knuth [14] and is a composition of two bijections that we describe here.

The first is the Robinson-Schensted-Knuth correspondence or simply RSK correspondence. It is a bijection between $S_{n}$ and pairs of standard Young tableaux of identical shape
$\lambda \vdash n$. This $R S K$ is based on the insertion algorithm known as $R S K$ algorithm (see [14, 23]). Let $\sigma \in S_{n}$ and $(P, Q)=R S K(\sigma)$. The tableau $P$ is known as the insertion tableau, and $Q$ the recording tableau. The insertion tableau $P$ is obtained by reading the permutation $\sigma$ from left to right and, at each step, inserting $\sigma(i)$ to the partial tableau obtained so far. Assume that $\left(P^{(i-1)}, Q^{(i-1)}\right)=\operatorname{RSK}(\sigma(1) \ldots \sigma(i-1))$. We obtain $\left(P^{(i)}, Q^{(i)}\right)$ by inserting $(i, \sigma(i))$ in $\left(P^{(i-1)}, Q^{(i-1)}\right)$, i.e. inserting $\sigma(i)$ in $P^{(i-1)}$ and $i$ in $Q^{(i-1)}$. Insertion follows the following rules: if $\sigma(i)$ is larger than all of elements on the first row of $P^{(i-1)}$, then place $\sigma(i)$ at the end of the first row of $P^{(i-1)}$. Otherwise, it takes the place of the leftmost element $x$ on the first row that is larger than $\sigma(i)$ (we say that $x$ is bumped by $\sigma(i)$ ) and then insert $x$ in the second row by the same way. The recording tableau $Q^{(i)}$ has the same shape as $P^{(i)}$ and is obtained by placing $i$ in the position of the square that was created at step $i$ on insertion of $\sigma(i)$. By this way, we get $(P, Q)=\left(P^{(n)}, Q^{(n)}\right)$. One of the known properties of $R S K$ is that the number of rows of $P$ (and as well $Q$ ) is equals to the length of the longest decreasing subsequence of $\sigma$. Consequently, $\sigma$ is 321-avoiding if and only if P has at most two rows. The duality is also among the famous properties of the $R S K$ correspondence. It says that $R S K\left(\sigma^{-1}\right)=(Q, P)$ if and only if $\operatorname{RSK}(\sigma)=(P, Q)$ (e.g. [14]). Below is an example of the construction of $(P, Q)$ from a 321-avoiding permutation $\pi=24135867$ (see Fig. 3).


Figure 3: Construction of $(P, Q)=\operatorname{RSK}(\pi)$, with $\pi=24135867$.
In [10], Elizalde and Pak presented a matching algorithm which matches some nonexcedance values with excedance values of a given permutation. We remark that when we change the output of this algorithm, we directly get the second rows of $P$ and $Q$. In fact, the modified matching algorithm that we present here matches non-excedances with excedance values.

## Matching algorithm:

INPUTS: excedances $e_{1}<\ldots<e_{k}$ and non-excedances $a_{1}<\ldots<a_{n-k}$ of $\sigma$
OUTPUT: List of matched excedances $\mathcal{M}$
BEGIN
Let $p:=1 ; q:=1$ and $\mathcal{M}:=\{ \}$
REPEAT UNTIL $p>k$ OR $q>n-k$.
IF $e_{p}>a_{q}$ THEN $q:=q+1$;
ELSE IF $\sigma\left(e_{p}\right)<\sigma\left(a_{q}\right)$ THEN $p:=p+1$;
$\operatorname{ELSE} \mathcal{M}:=\mathcal{M} \cup\left\{\left(\sigma\left(e_{p}\right), a_{q}\right)\right\} ; p:=p+1 ; q:=q+1 ;$
END

A given permutation $\sigma$ is bi-increasing if its excedance and non-excedance values are both increasing. That means $\sigma\left(i_{1}\right) \ldots \sigma\left(i_{k}\right)$ and $\sigma\left(j_{1}\right) \ldots \sigma\left(j_{n-k}\right)$ are both increasing subsequences, where $\left\{i_{1}, \ldots, i_{k}\right\}$ and $\left\{j_{1}, \ldots, j_{n-k}\right\}$ are respectively excedances and non-excedances of $\sigma$ in increasing order. By Reifegerste [18], all 321-avoiding permutations are bi-increasing. In this work, we sometime refer to this known property.

Let us consider $\sigma \in S_{n}(321)$ and let $\mathcal{M}=\left\{\left(E_{1}, a_{1}\right), \ldots,\left(E_{l}, a_{l}\right)\right\}$ be the output of the matching algorithm with $l \leq n$ and suppose that $(P, Q)=\operatorname{RSK}(\sigma)$. Since $\sigma$ is bi-increasing,
then $E_{1}, \ldots, E_{l}$ are the excedance values of $\sigma$ that are bumped respectively by the nonexcedance values $\sigma\left(a_{1}\right), \ldots, \sigma\left(a_{l}\right)$ when applying the $R S K$ algorithm. In other words, the second rows of $P$ and $Q$ are respectively $\left[E_{1}, \ldots, E_{l}\right]$ and $\left[a_{1}, \ldots, a_{l}\right]$. So we have the following remark.
Remark 2.1. The bumped excedance values by the $R S K$ algorithm and the matched excedance values by the matching algorithm on a 321 -avoiding permutation are the same.

For example, when we apply the matching algorithm with the permutation $\pi=$ $24135867 \in S_{8}(321)$ of the above example, we get as output $\mathcal{M}=\{(2,3),(4,4),(8,7)\}$. If $(P, Q)=R S K(\pi)$, the second rows of $P$ and $Q$ are respectively $[2,4,8]$ and $[3,4,7]$. So, we can therefore deduce the first rows of $P$ and $Q$ that are respectively $[1,3,5,6,7]$ and $[1,2,5,6,8]$.

The second correspondence is a simple transformation of the pair of standard Young tableaux $(P, Q)$, result of $R S K$, into Dyck path $D=\Psi(\sigma)$. The first half of the Dyck path $D$ is obtained from $P$ by adjoining, for $i$ from 1 to $n$, an up-step if $i$ is in the first row of $P$, and a down-step if $i$ is in the second row. The second half of $D$ is obtained from $Q$ by adjoining, for $j$ from $n$ down to 1 , an up-step if $j$ is in the second row of $Q$ and a down-step if $j$ is in the first row. If we denote respectively by $D^{(L)}$ and $D^{(R)}$ the left and right half sub-paths of $D$ produced respectively by the tableaux $P$ and $Q$, then we can write $\Psi(\sigma)=D^{(L)} D^{(R)}$.


Figure 4: The corresponding Dyck path from $(P, Q)=\operatorname{RSK}(24135867)$.
From this given definition of $\Psi$, we have the following obvious proposition.
Proposition 2.2. For any Dyck path $D$, we have $\left|D^{(L)}\right|_{d}=\left|D^{(R)}\right|_{u}$.
Proof. Let us consider a Dyck path $D$. There exists a permutation $\sigma \in S(321)$ such that $D=\Psi(\sigma)$. If $(P, Q)=R S K(\sigma)$, then $\left|D^{(L)}\right|_{d}$ and $\left|D^{(R)}\right|_{u}$ are respectively equal to the sizes of the second rows of $P$ and $Q$. So, $\left|D^{(L)}\right|_{d}$ and $\left|D^{(R)}\right|_{u}$ are equal.

### 2.3 The bijection $\Phi^{-1}: \mathcal{D}_{n} \rightarrow S_{n}(132)$

As mentioned in [10], the bijection $\Phi$ is essentially the same bijection between $S_{n}(132)$ and $\mathcal{D}_{n}$ given by Krattenthaler [13], up to reflection of the path over a vertical line. Here, we will present the bijection $\Phi^{-1}$ in a slightly different way.

Starting from a Dyck path $D \in \mathcal{D}_{n}$, we will construct the corresponding permutation by the following procedure. From left to right, number the up-steps of $D$ from $n$ down to 1 and the down-steps from 1 up to $n$. Then, we have $\Phi^{-1}(D)(n+1-i)=j$ if and only if tunnel from up-step numbered $n+1-i$ (i.e. the $i$-th up-step) is ending to down-step numbered $j$ (i.e. the $j$-th down-step). By this way, it is not hard to show that the map $\Phi^{-1}$ is a well defined biijection. See Fig. 5 for graphical illustration.

Let us consider $\sigma \in S_{n}$ and two integers $a$ and $b$ satisfying $1 \leq a \leq b \leq n$. We denote by $\sigma(a \ldots b):=\sigma(a) \sigma(a+1) \ldots \sigma(b)$ the contiguous subsequence of $\sigma$ from its $a$-th to $b$-th letter and for any operator $* \in\{<, \leq, \neq, \geq,>\}$ and a given number $x$, we can write $\sigma(a \ldots b) * x$ if only if $\sigma(i) * x$ for all $i \in[a ; b]$. Example: if we consider $\pi=6413275 \in S_{7}$, then we have $\pi(2 \ldots 5)=4132 \leq 4$. Our first observation of the bijection $\Phi$ described by the above procedure leads to the following proposition.


Figure 5: Corresponding 132-avoiding permutation from a Dyck path.
Proposition 2.3. Let us assume that $D \in \mathcal{D}_{n}$ and $j=\left|D^{(R)}\right|_{u}+1$. The permutation $\sigma=\Phi^{-1}(D)$ satisfies the following properties:
(i) if $j \geq 2$, then we have $\sigma^{-1}(1 \ldots j-1) \geq j \leq \sigma(1 \ldots j-1)$,
(ii) for all $i \geq j$, if $\sigma(i)>i$, then we have $\sigma^{-1}(i)<i$.

Proof. Let us consider $D \in \mathcal{D}_{n}$ and suppose that $j=\left|D^{(R)}\right|_{u}+1 \geq 2$. We range in the following table all of assigned numbers to up-steps and down-steps of $D$ for getting $\sigma=$ $\Phi^{-1}(D)$. When looking at the second column of Table 2, we get $\sigma(1 \ldots j-1) \geq j$. Similarly,

| Sub-path | $D^{(L)}$ | $D^{(R)}$ |
| :---: | :---: | :---: |
| For up-steps | $n, \ldots, j+1, j$ | $j-1, \ldots, 2,1$ |
| For down-steps | $1,2, \ldots, j-1$ | $j, j+1, \ldots, n$ |

Table 2: Assigned numbers to up-steps and down-steps of $D$.
when looking at the first column, we also get $\sigma^{-1}(1 \ldots j-1) \geq j$. So, we easily obtain the proof of the first property.

Now let us consider an integer $i$ such that that $\sigma(i)>i \geq j$. If we return to the Dyck path $D$, then the tunnel which matches the up-step numbered $i$ with the down-step numbered $\sigma(i)$ decomposes the Dyck path $D$ as $D=\ldots u B d \ldots$, where $u$ is the ( $n+1-i$ )-th up-step of $D, d$ is the $\sigma(i)$-th down-step of $D, B$ is a sub-Dyck path of $D$ which contains at least the down-steps numbered $j, j+1, \ldots, \sigma(i)-1$ and its up-steps are obviously numbered by numbers less than $i$. This implies that we must have $\sigma^{-1}(j \ldots \sigma(i)-1)<i$. Since $i \in$ $\{j, j+1, \ldots, \sigma(i)-1\}$, so we get $\sigma^{-1}(i)<i$. This ends the proof of the second property of our proposition.

Let us just end this section with the following obvious remark which implies that $\Phi^{-1}$ exchanges left and centered tunnels of a Dyck path and non-excedances of the corresponding 132-avoiding permutation.
Remark 2.4. Tunnel of Dyck path $D \in \mathcal{D}_{n}$ which matches the up-step numbered $n+1-i$ with the down-step numbered $j$ is left or centered if and only if $n+1-i \geq j=\Phi^{-1}(D)(n+$ $1-i$ ).

### 2.4 A direct formulation of $\Theta$

As seen in the previous sub-sections, the original definition of $\Theta$ uses Young tableaux and Dyck paths as auxiliary combinatorial objects. In this section, we propose a recursive formula for the bijection $\Theta$ which does not use these auxiliary objects. For that, we introduce some needed notations. Given a permutation $\sigma \in S_{n}$,

- $\sigma^{+a}$ denotes the obtained permutation from $\sigma$ by adding $a$ to each of its number. Example: $312^{+2}=534$.
- $\sigma^{a \rtimes b}$ denotes the obtained permutation from $\sigma$ by adding $b$ all of its numbers greater or equal to $a$. Example: $4132^{3 \times 2}=6152$.
- if $a$ and $b$ satisfy $1 \leq a, b \leq|\sigma|+1$, then $\sigma^{(a, b)}$ denotes the obtained permutation from $\sigma$ by inserting the number $b$ at the $a$-th position of $\sigma$. More precisely, we have $\sigma^{(a, b)}=\sigma^{b \rtimes 1}(1 \ldots a-1) \cdot b \cdot \sigma^{b \rtimes 1}(a \ldots|\sigma|)$. Example: $3142^{(2,3)}=43152$.
Proposition 2.5. If $\sigma \in S_{n}^{k}(321):=\left\{\sigma \in S_{n}(321) \mid \sigma(n)=k\right\}$ and $\pi=\operatorname{red}[\sigma(1) \ldots \sigma(n-1)]$ (i.e. $\sigma=\pi^{(n, k)}$ ), then we have $\Theta(\sigma)=\Theta(\pi)^{(n-k+j, j)}$, where $j-1$ is the number of matched excedance values less than $k$ of $\pi$. Furthermore, $n-k+j$ is the minimum of non-excedance of $\Theta(\sigma)$.

Proof. Let us assume that $\sigma \in S_{n}^{k}(321)$ and $\pi=\operatorname{red}[\sigma(1 \ldots n-1)]$. Since $\sigma=\pi^{(n, k)}$, the pair $(P, Q)=\operatorname{RSK}(\sigma)$ is obtained from $\left(P^{\prime}, Q^{\prime}\right)=\operatorname{RSK}(\sigma(1 \ldots n-1))$ by inserting $(n, k)$. Following the logical of insertion of $(n, k)$ in $\left(P^{\prime}, Q^{\prime}\right)$ to get $(P, Q)$, we observe that $\Psi(\sigma)$ can also be obtained from $\Psi(\pi)$. For better understanding, we reason graphically and we distinguish three cases according to the values of $k$. In all cases, we denote by ( $i-1$ ) (resp $(j-1)$ ) the position of the rightmost number less than $k$ in the first row (resp second row) of $P^{\prime}$. In other words, the number at the $i$-th (resp $j$-th) column of the first row (resp second row) of $P^{\prime}$ is greater than $k$ if it exists. Notice that $j-1$ can be interpreted as the number of bumped excedance values less than $k$ by the RSK algorithm and following Remark 2.1 it is also the number of the matched excedance values less than $k$ by the matching algorithm.

Let us first suppose that $k=n$. To get $(P, Q)$, we just add $n$ at the end of the first rows of $P^{\prime}$ and $Q^{\prime}$ (see Table 3). Consequently, when we translate $(P, Q)$ into Dyck path, we get $\Psi(\sigma)=\Psi(\pi)^{(L)} \cdot u d . \Psi(\pi)^{(R)}$, where $\Psi(\sigma)^{(L)} \Psi(\sigma)^{(R)}=\Psi(\sigma)$. So, when we number the steps of $\Psi(\sigma)$ to get $\Theta(\sigma)$ according to the procedure described in section 2.3, those of newly added up-step $u$ and down-step $d$ are respectively $n+1-i$ and $j$.


Table 3: Insertion of $(n, n)$.

Suppose now that $k<n$ but it is also greater than all numbers in the first row of $P^{\prime}$. In this case, when inserting $k$ in $P^{\prime}$, bumping does not occur but there exists a part $A$ in the second row of $P^{\prime}$ such that its elements are all greater than $k$. As we can see in Table 4, $\Psi(\pi)^{(L)}$ is ending with a sequence of down-steps produced by $A$. The corresponding Dyck path $\Psi(\sigma)$ of $(P, Q)$ can be obtained from $\Psi(\pi)$. Indeed, we have $\Psi(\sigma)^{(R)}=d . \Psi(\pi)^{(R)}$ and we obtain $\Psi(\sigma)^{(L)}$ from $\Psi(\pi)^{(L)}$ by inserting a new up-step (produced by $k$ ) just before the $j$-th down-step (produced by the minimum of $A$ ). So, when numbering the steps of $\Psi(\sigma)$ to get $\Theta(\sigma)$, that of the new added up-step is $n+1-i$ which is just before the down-step numbered $j$.


Table 4: Insertion of $(n, k)$ with $k<n$ and bumping does not occur.

The last case that we discuss here is the case where bumping occur, i.e. $k<n$ and there is at least an element of the first row of $P^{\prime}$ which is greater than $k$. Let us denote by $x$ the leftmost of such element. It is therefore at the $i$-th column of $P^{\prime}$. In the second row of $P^{\prime}$, we still denote by $A$ the part of elements greater than $k$ which begins at the $j$-th comlumn. Notice that any elements of $A$ is also less that $x$, i.e. $A \subseteq\{k+1, k+2, \ldots, x-1\}$. That is why, as we can see in Table 5, the path $\Psi(\pi)^{(L)}$ is ending with a sequence of down-steps produced by $A$ followed by a sequence of up-steps produced $x$ and the numbers in its right. When inserting $k$ in $P^{\prime}, k$ takes the place of $x$ (i.e. $x$ is bumped by $k$ ) and $x$ creates a new cell at the end of the second row. Similarly to the previous cases, the deduction of $\Psi(\sigma)$ from $\Psi(\pi)$ simply follows the logical of insertion of $(n, k)$ in $\left(P^{\prime}, Q^{\prime}\right)$ to get $(P, Q)$. In fact, we have $\Psi(\sigma)^{(R)}=u . \Psi(\pi)^{(R)}$ and $\Psi(\sigma)^{(R)}$ is obtained from $\Psi(\pi)^{(R)}$ by replacing the $i$-th up-step (the up-step produced by $x$ ) by a down-step because $x$ becomes an element of the second row, then inserting a new up-step (the up-step produced by $k$ ) just before the $j$-th down-step.


Table 5: Insertion of $(n, k)$ with $k<n$ and bumping occur.

In all the cases discussed above, when we look at the path $\Psi(\sigma)$, we can observe three things.

- Firstly, we have $\Theta(\sigma)(n+1-i)=j$.
- Secondly, when we remove the up-step number $n+1-i$ and the down-step number $j$ of $\Psi(\sigma)$, the remaining Dyck path is $\Psi(\pi)$. This also implies that when we remove $\Theta(\sigma)(n+1-i)$ (which is equal to $j$ ) in $\Theta(\sigma)$, the obtained reduction is none other than $\Theta(\pi)$. In other words, we have $\Theta(\sigma)=\Theta(\pi)^{(n+1-i, j)}$. Moreover, since $\sigma$ is biincreasing (see [18]), the number $i$ equals to the number of non-excedance values less than $k$ of $\sigma$ plus the number of non-bumped excedance values of $\sigma$ less than $k$. Consequently, since the number of bumped excedance values less than $k$ is $j-1$, then we
have $i=k-(j-1)=k+1-j$. This implies that $n+1-i=n-k+j$ and we finally get $\Theta(\sigma)=\Theta(\pi)^{(n-k+j, j)}$.
- Thirdly, since $n-k+j \geq j$ then $n-k+j$ is a non-excedance of $\Theta(\sigma)$. Moreover, knowing that $n-k+j$ is the minimum of the numbers assigned to up-steps of $\Psi(\sigma)^{(L)}$ in which associated tunnel is left or centered, it is also the minimum of the non-excedance of $\Theta(\sigma)$ (see Remark 2.4).

This ends the proof of Proposition 2.5.
The given property of the bijection $\Theta$ in Proposition 2.5 is really fundamental and allows us to define it directly without passing to Young tableaux and Dyck paths. So we have the following theorem.
Theorem 2.6. If $\sigma \in S_{n}(321)$ and $\pi=\operatorname{red}(\sigma(1 \ldots n-1))$, then we have

$$
\Theta(\sigma)= \begin{cases}\sigma & \text { if }|\sigma|=1 \\ \Theta(\pi)^{(n-\sigma(n)+j, j)} & \text { if }|\sigma|>1\end{cases}
$$

where $j-1$ is the number of matched excedance values less than $\sigma(n)$. To compute $\Theta^{-1}(\alpha)$ from a given permutation $\alpha \in S(132)$, we use the following relation

$$
\Theta^{-1}(\alpha)=\left\{\begin{array}{ll}
\alpha & \text { if }|\alpha|=1 \\
\Theta^{-1}(\beta)^{(|\alpha|,|\alpha|+\alpha(k)-k)} & \text { if }|\alpha|>1
\end{array},\right.
$$

where $k$ is the minimum of the non-excedance of $\alpha$ and $\beta^{(k, \alpha(k))}=\alpha$.
Notice that, the number of matched excedance values is obtained by applying the matching algorithm. Example: for $\sigma=4162735 \in S_{7}(321)$, we summarize in table 6 the computation of $\Theta(\sigma)$.

| $l$ | $\sigma_{l}=\operatorname{red}[\sigma(1) \ldots \sigma(l)]$ | $(l-\sigma(l)+j, j)$ | $\Theta\left(\sigma_{l-1}\right)^{(l-\sigma(l)+j, j)}$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | - | 1 |
| 2 | 21 | $(2,1)$ | 21 |
| 3 | 213 | $(2,2)$ | 321 |
| 4 | 3142 | $(3,1)$ | 4312 |
| 5 | 31425 | $(3,3)$ | 54312 |
| 6 | 415263 | $(4,1)$ | 654123 |
| 7 | $\sigma=4162735$ | $(4,2)$ | $\Theta(\sigma)=7652134$ |

Table 6: Recursive computation of $\Theta(4162735)$.
If $\alpha=7652134$, we have $\Theta^{-1}(\alpha)=\Theta^{-1}(654123)^{(7,5)}$ since $|\alpha|=7, k=4$ and $\alpha(k)=2$.

## 3 The (fp, exc, cr)-preserving of the bijection $\Theta$

In [10], Elizalde and Pak proved that the bijection $\Theta$ preserves the number of fixed points and excedances.
Theorem 3.1. [10] The bijection $\Theta$ preserves the number of fixed points and excedances. It means that, for any permutation $\sigma \in S(321)$, we have $(\mathrm{fp}, \mathrm{exc})(\Theta(\sigma))=(\mathrm{fp}, \mathrm{exc})(\sigma)$.

To prove this theorem, Elizalde and Pak showed that the bijections $\Psi$ and $\Phi$ exchange the statistics ( $\mathrm{fp}, \mathrm{exc}$ ) on 321 or 132-avoiding permutations and (ct,rt) on Dyck paths. More precisely, we have $\Theta:(\mathrm{fp}, \mathrm{exc}) \xrightarrow{\Psi}(\mathrm{ct}, \mathrm{rt}) \xrightarrow{\Phi^{-1}}(\mathrm{fp}, \mathrm{exc})$. Using the given recursive definition of $\Theta$ in section 2.4, we will prove by induction on $n$ that it also preserves the number of crossings. For that, we need some operations and notations to be defined.

Let us fix two permutations $\alpha$ and $\beta$. The direct sum of $\alpha$ and $\beta$ is $\alpha \oplus \beta:=\alpha \cdot \beta^{+|\alpha|}$. Example: $312 \oplus 231=312564$ since $231^{+3}=564$. Our aim is to define a new operation on $S(132)$ that we need to prove the cr-preserving of the bijection $\Theta$. For that, we need the following fundamental proposition.
Proposition 3.2. Assume that $\sigma \in S_{n}(132)$ and denote by $T(\sigma):=\left\{i \in[n] \mid \sigma^{-1}(i)>i<\sigma(i)\right\}$. We have the following properties.
(a) $T(\sigma)=\varnothing$ if and only if $\sigma(1)=1$.
(b) If $j \in T(\sigma)$, then we also have $i \in T(\sigma)$ for all $i \leq j$.
(c) We have $|T(\sigma)| \leq \frac{n}{2}$.
(d) If $D=\Phi(\sigma)$, then $|T(\sigma)|=\left|D^{(R)}\right|_{u}$.
(e) If $k=1+|T(\sigma)|$, then we have $\sigma^{(k, k)} \in S_{n+1}(132)$.

Proof. Let $\sigma \in S_{n}(132)$. Notice first that the property (a) is obvious since we have $\sigma(1)=1$ if and only if $\sigma=12 \ldots n$. Using the fact that $\sigma$ is 132 -avoiding, we can easily prove by contradiction that the property $(b)$ also hold. Suppose that $t=\max T(\sigma)$. According to the property (b), we have $t=|T(\sigma)|$ and so $\sigma^{-1}(1 \ldots t)>t<\sigma(1 \ldots t)$. This is possible only if we have $n-t \geq t$. This implies that $t \leq \frac{n}{2}$ and ends the proof of the property (c). The property ( $d$ ) comes from Proposition 2.3.
Suppose now that $k=1+|T(\sigma)|$. Like the property (b), the property $(e)$ can be proved easily by contradiction. Firstly, we have $\sigma^{(k, k)}=\sigma^{k \rtimes 1}(1 \ldots k-1) \cdot k \cdot \sigma^{k \rtimes 1}(k \ldots n)$ and according to the property $(b)$, we have $\sigma^{k \rtimes 1}(1 \ldots k-1)=(\sigma(1 \ldots k-1))^{+1}$ since $\sigma(1 \ldots k-1) \geq k$. Secondly, if $\sigma^{(k, k)} \notin S_{n+1}(132)$, then it is $k \cdot \sigma^{k \rtimes 1}(k \ldots n)$ which contains at least a 132-pattern because $\sigma^{k \rtimes 1}(1 \ldots k-1) . k$ is 132 -avoiding. Suppose that $k \sigma^{k \rtimes 1}\left(i_{1}\right) \sigma^{k \rtimes 1}\left(i_{2}\right)$ is a 132-pattern of $k \cdot \sigma^{k \rtimes 1}(k \ldots n)$ for $k<i_{1}<i_{2}$. Knowing the maximality of $t=k-1$, we have to examine two cases.

- If $\sigma^{-1}(k)>k$, then $\sigma(k) \sigma\left(i_{1}-1\right) \sigma\left(i_{2}-1\right)$ is a 132-pattern for $\sigma$ since $\sigma(k)<k$.
- If $\sigma^{-1}(k) \leq k$, then $k \sigma\left(i_{1}-1\right) \sigma\left(i_{2}-1\right)$ is a 132-pattern for $\sigma$.

This contradict the fact that $\sigma$ is 132-avoiding. Finally, we must have $\sigma^{(k, k)} \in S_{n+1}(132)$.
For any given permutation $\sigma \in S_{n}$ and an integer $p \geq 1$, we write $\sigma^{\left\{\left(a_{1}, b_{1}\right), \ldots,\left(a_{p}, b_{p}\right)\right\}}:=$ $\left(\ldots\left(\sigma^{\left(a_{1}, b_{1}\right)}\right) \ldots\right)^{\left(a_{p}, b_{p}\right)} \in S_{n+p}$, where $1 \leq a_{i}, b_{i} \leq n+i$ for all $i \in[p]$. Furthermore, if $\pi$ is a permutation of length $p$, then we also write $\sigma^{(a, \pi)}:=\sigma^{\{(a, \pi(1)),(a+1, \pi(2)), \ldots,(a+p-1, \pi(p))\}}$. Example: $3142^{(3,213)}=3142^{\{(3,2),(4,1),(5,3)\}}=41253^{\{(4,1),(5,3)\}}=523164^{(5,3)}=6241375$. Let $\alpha$ and $\beta$ be two 132-avoiding permutations. We define the direct product of $\alpha$ and $\beta$ as

$$
\alpha \otimes \beta:=\beta^{\left(k, \alpha^{+(k-1)}\right)}=\beta^{k \rtimes|\alpha|}(1 \ldots k-1) \cdot \alpha^{+(k-1)} \cdot \beta^{k \rtimes|\alpha|}(k \ldots|\beta|),
$$

where $k=1+|T(\beta)|$. Example: $312 \otimes 543612=543612^{(3,534)}=875346912$ since $k=3$, $312^{+2}=534$ and $543612^{3 \times 6}=876912$. We say that $\sigma$ is $\oplus$-irreducible if it cannot be written as the direct sum of two non-empty permutations. Otherwise, $\sigma$ is called $\oplus$-decomposable. Each
$\oplus$-decomposable permutation $\sigma$ can be written uniquely as the direct sum of $\oplus$-irreducible ones, called the $\oplus$-components of $\sigma$. Similarly, we also define the notions of $\otimes$-irreducible, $\otimes$ decomposable and $\otimes$-components but now over $S(132)$. We observe that $\operatorname{cr}\left(\sigma_{1} \oplus \sigma_{2}\right)=\operatorname{cr}\left(\sigma_{1}\right)+$ $\operatorname{cr}\left(\sigma_{2}\right)$ for all permutations $\sigma_{1}$ and $\sigma_{2}$. In the following proposition we prove similar result with the new operation $\otimes$ over $S(132)$.
Proposition 3.3. For all $\sigma_{1}$ and $\sigma_{2} \in S(132)$, we have $\operatorname{cr}\left(\sigma_{1} \otimes \sigma_{2}\right)=\operatorname{cr}\left(\sigma_{1}\right)+\operatorname{cr}\left(\sigma_{2}\right)$.
Proof. Suppose that $\sigma=\sigma_{1} \otimes \sigma_{2}$ with $\sigma_{1}, \sigma_{2} \in S(132)$. By definition, we have $\sigma\left(k \ldots k+\left|\sigma_{1}\right|-\right.$ $1)=\sigma_{1}^{+(k-1)}$ and $\sigma(1 \ldots k-1) \cdot \sigma\left(k+\left|\sigma_{1}\right| \ldots\left|\sigma_{1}\right|+\left|\sigma_{2}\right|\right)=\sigma_{2}^{k \times\left|\sigma_{1}\right|}$, where $k=1+\left|T\left(\sigma_{2}\right)\right|$. Since $\sigma_{1}^{+(k-1)}$ is a permutation of $\left\{k, \ldots,\left|\sigma_{1}\right|+k-1\right\}$, when referring to the arc diagrams of the product $\sigma$, no arrows in $\sigma$ go between an entry in $\sigma_{1}^{+(k-1)}$ and an entry in $\sigma_{2}^{k x\left|\sigma_{1}\right|}$. Consequently, we have $\operatorname{cr}(\sigma)=\operatorname{cr}\left(\sigma_{1}^{+(k-1)}\right)+\operatorname{cr}\left(\sigma_{2}^{k \rtimes\left|\sigma_{1}\right|}\right)$. So, the desired result follows from the fact that $\operatorname{cr}\left(\pi^{+a}\right)=\operatorname{cr}(\pi)$ and $\operatorname{cr}\left(\pi^{a \rtimes b}\right)=\operatorname{cr}(\pi)$ for any permutation $\pi$ and integers $a$ and $b$.

Proposition 3.4. For all $\sigma_{1}, \sigma_{2} \in S(321)$, we have $\Theta\left(\sigma_{1} \oplus \sigma_{2}\right)=\Theta\left(\sigma_{2}\right) \otimes \Theta\left(\sigma_{1}\right)$.
Proof. Let $\sigma=\sigma_{1} \oplus \sigma_{2}$ with $\sigma_{1} \in S_{n}(321)$ and $\sigma_{2} \in S_{m}(321)$. Denote first by $\left(P_{1}, Q_{1}\right)=$ $R S K\left(\sigma_{1}\right)$. To get $(P, Q)=R S K(\sigma)$, we have to insert $\sigma_{2}^{\prime}=\sigma_{2}^{+n}$ in $\left(P_{1}, Q_{1}\right)$. Since all of numbers in $\sigma_{2}^{\prime}$ are greater than all of inserted ones in $\left(P_{1}, Q_{1}\right)$, we get $(P, Q)=\left(P_{1} \cdot P_{2}, Q_{1} \cdot Q_{2}\right)$, where $\left(P_{2}, Q_{2}\right)=\operatorname{RSK}\left(\sigma_{2}^{\prime}\right)$ and the dots denote concatenations. So, the Dyck path produced by $(P, Q)$ is $D_{1}^{(L)}\left(D_{2}^{(L)} D_{2}^{(R)}\right) D_{1}^{(R)}=\Psi(\sigma)$, where $D_{1}^{(L)}, D_{1}^{(R)}, D_{2}^{(L)}$ and $D_{2}^{(R)}$ are respectively the sub-paths produced by $P_{1}, Q_{1}, P_{2}$ and $Q_{2}$. Moreover, we have $D_{1}^{(L)} D_{1}^{(R)}=\Psi\left(\sigma_{1}\right)$ and $D_{2}^{(L)} D_{2}^{(R)}=\Psi\left(\sigma_{2}\right)$.

Let $k=\left|D_{1}^{(R)}\right|_{u}+1$. We range in the following table all of assigned numbers to up-steps and down-steps of $\Psi(\sigma)$ for getting $\Theta(\sigma)$.

| Sub-path | $D_{1}^{(L)}$ | $D_{2}^{(L)} D_{2}^{(R)}$ | $D_{1}^{(R)}$ |
| :---: | :---: | :---: | :---: |
| For up-steps | $n+m, \ldots, k+m$ | $k+m-1, \ldots, k+1, k$ | $k-1, \ldots, 2,1$ |
| For down-steps | $1,2, \ldots, k-1$ | $k, k+1, \ldots, k+m-1$ | $k+m, \ldots, n+m$ |

Table 7: Assigned numbers to up-steps and down-steps of $\Psi(\sigma)$.

When we compute $\pi=\Phi^{-1}\left(D_{1}^{(L)}\left(D_{2}^{(L)} D_{2}^{(R)}\right) D_{1}^{(R)}\right)=\Theta(\sigma)$ by the described procedure in section 2.3, we get two subsequences to be discussed.

- The first one $\pi(k \ldots k+m-1)$ is the sequence produced by $D_{2}^{(L)} D_{2}^{(R)}$ and is a permutation of $\{k, k+1, \ldots k+m-1\}$. In other words, we have $\pi(k \ldots k+m-1)=\pi_{2}^{+(k-1)}$, where $\pi_{2}=\operatorname{red}[\pi(k \ldots k+m-1)]=\Phi^{-1}\left(D_{2}^{(L)} D_{2}^{(R)}\right)=\Theta\left(\sigma_{2}\right) \in S_{m}(132)$.
- The second one $\pi(1 \ldots k-1) \cdot \pi(k+m \ldots m+n)$ is the sequence produced by $D_{1}^{(L)} D_{1}^{(R)}$ and is a permutation of $[n+m]-\{k, k+1, \ldots k+m-1\}$. When looking at the first and third columns of table 6 , we get $\pi^{-1}(1 \ldots k-1) \geq k+m \leq \pi(1 \ldots k-1)$. Moreover, if $\pi_{1}=\operatorname{red}[\pi(1 \ldots k-1) \cdot \pi(k+m \ldots m+n)]$ then we have $\pi_{1}^{k \rtimes m}=\pi(1 \ldots k-1) \cdot \pi(k+$ $m \ldots m+n), \pi_{1}=\Phi^{-1}\left(D_{1}^{(L)} D_{1}^{(R)}\right)=\Theta\left(\sigma_{1}\right) \in S_{n}(132)$.

From these two points, we can write $\pi=\pi_{1}^{k \rtimes m}(1 \ldots k-1) \cdot \pi_{2}^{+(k-1)} \pi_{1}^{k \rtimes m}(k \ldots n)$. Furthermore, we have $k=1+\left|T\left(\pi_{1}\right)\right|$ since $\left|D_{1}^{(R)}\right|_{u}=\left|T\left(\pi_{1}\right)\right|$. Consequently, using Proposition 2.3, we have $\pi=\pi_{2} \otimes \pi_{1}=\Theta\left(\sigma_{2}\right) \otimes \Theta\left(\sigma_{1}\right)$. This completes the proof of Proposition 3.4.

It is obvious from Proposition 3.4 that $\sigma \in S(321)$ is $\oplus$-irreducible if and only if $\Theta(\sigma) \in$ $S(132)$ is $\otimes$-irreducible. Moreover, one can easily verify that $\oplus$ is an associative and stable operation on $S(321)$. As direct consequence of Proposition 3.4, the direct product $\otimes$ is also stable and associative on $S(132)$.

Let $\sigma$ be a 321-avoiding permutation. Now, let us adopt the following notations $A_{1}(\pi, a, b):=|\{b \leq i<a \mid \pi(i)<b\}|, A_{2}(\pi, a, b):=\left|\left\{b \leq i<a \mid a<\pi^{-1}(i)\right\}\right|$, $A_{3}(\pi, a, b):=\left|\left\{b \leq i<a \mid \pi^{-1}(i)<i<\pi(i)\right\}\right|$ and $A_{4}(\pi, a, b):=\mid\{b \leq i<a \mid \pi(i)<i<$ $\left.\pi^{-1}(i)\right\} \mid$ for any permutation $\pi$ and two integers $a$ and $b$ satisfying $b \leq a \leq|\pi|+1$. We will prove the following lemma which has an important role for later demonstration.
Lemma 3.5. Let $\pi \in S_{n-1}$ and $a, b$ two non negative integers satisfying $b \leq a \leq n$. If $\sigma=\pi^{(a, b)}$, we have $\operatorname{cr}(\sigma)=\operatorname{cr}(\pi)+\operatorname{cr}(\pi, a, b)$, where $\operatorname{cr}(\pi, a, b)=A_{1}(\pi, a, b)+A_{2}(\pi, a, b)+$ $A_{3}(\pi, a, b)-A_{4}(\pi, a, b)$.
Proof. Let $\pi \in S_{n-1}$ and $\sigma=\pi^{(a, b)}$ such that $a$ and $b$ satisfy the condition of the lemma. The two subsequences $\sigma(1 \ldots b-1)$ and $\pi(1 \ldots b-1)$ are in order isomorphic. The same is true for $\sigma(a+1 \ldots n)$ and $\pi(a \ldots n-1)$. Furthermore, if $i \in\{b, \ldots a-1\}$, we have the following properties
(a) it is easy to verify that all crossings of $\pi$, except lower crossings of the form $\pi(i)<i<$ $\pi^{-1}(i)$, remain crossings for $\sigma$,
(b) the new lower arc $(a, b)$ crosses with any arc $(i, \pi(i))$ such that $\pi(i)<b \leq i$ and with each arc $\left(i, \pi^{-1}(i)\right)$ such that $i<a<\pi^{-1}(i)$. So $(i, a)$ or $\left(a, \pi^{-1}(i)\right)$ which is not a crossing of $\pi$ becomes one for $\sigma$,
(c) if $\pi^{-1}(i)<i<\pi(i)$, then $\left(\pi^{-1}(i), i\right)$ becomes a crossing of $\sigma$ since $\pi^{-1}(i)<i<$ $\sigma\left(\pi^{-1}(i)\right)=i+1<\sigma(i)=\pi(i)+1$.

From (a) we get the $\mathrm{cr}(\pi)-A_{4}(\pi, a, b)$, from (b) the $A_{1}(\pi, a, b)+A_{2}(\pi, a, b)$ and from (c) the $A_{3}(\pi, a, b)$. Together, that give the desired relation of Lemma 3.5.

Remark 3.6. Notice that if $b<x<a$, we have $\operatorname{cr}\left(\pi^{(x, x)}, a, b\right)=\operatorname{cr}(\pi, a-1, b)$. More generally, if $b<x_{1}<x_{2}<\ldots<x_{p}<a$, we have $\operatorname{cr}\left(\pi^{\left\{\left(x_{1}, x_{1}\right),\left(x_{2}, x_{2}\right), \ldots,\left(x_{p}, x_{p}\right)\right\}}, a, b\right)=\operatorname{cr}(\pi, a-p, b)$.
Lemma 3.7. Let $\sigma \in S_{n}^{k}$ (321). For all integer $i$ such that $k \leq i<n$, either $(\sigma(i)<k$ and $\left.\sigma^{-1}(i)<i\right)$ or $\sigma^{-1}(i)<i<\sigma(i)$.

Proof. Let $\sigma \in S_{n}^{k}(321)$ and $i$ an integer such that $k \leq i<n$. It is easy to show by contradiction that if $i$ is an excedance of $\sigma$ then $\sigma^{-1}(i)<i$ and if $i$ is a non-excedance of $\sigma$ then $\sigma(i)<k$ and $\sigma^{-1}(i)<i$.
Lemma 3.8. Let $\sigma$ be an $\oplus$-irreducible permutation in $S_{n}^{k}(321)$ and let $\pi=\operatorname{red}[\sigma(1 \ldots n-1)]$, ie. $\sigma=\pi^{(n, k)}$. We have $\operatorname{cr}(\Theta(\pi), n-k+j, j)=\operatorname{cr}(\pi, n, k)$, where $j-1$ is the number of matched excedance values less than $k$. Furthermore, we have the following properties
(i) If $\pi$ is $\oplus$-irreducible, then $\operatorname{cr}(\pi, n, k)=n-k$.
(ii) If $\pi$ is not $\oplus$-irreducible, then there exists $l>1$ such that $\alpha=\pi(1 \ldots n-l-1)$ is $\oplus$ irreducible and $\operatorname{cr}(\pi, n, k)=\operatorname{cr}(\alpha, n-l, k)=n-l-k$.

Proof. Let $\sigma$ be an $\oplus$-irreducible permutation in $S_{n}^{k}(321)$ and let $\pi=\operatorname{red}[\sigma(1 \ldots n-1)]$. We have two cases to be considered.
(i) $\pi$ is $\oplus$-irreducible: Using Lemma 3.7, we obtain $\operatorname{cr}(\pi, n, k)=A_{1}(\pi, n, k)+$ $A_{3}(\pi, n, k)=n-k$ since $A_{2}(\pi, n, k)=A_{4}(\pi, n, k)=0$. Since $\Theta(\pi)$ is also $\otimes$-irreducible and $n-k+j$ is the minimum of non excedance of $\Theta(\sigma)=\Theta(\pi)^{(n-k+j, j)}$, we must have $\left(\Theta(\pi)^{-1}(i) \geq n-k+j\right.$ and $\left.\Theta(\pi)(i)>i\right)$ or $\Theta(\pi)^{-1}(i)<i<\Theta(\pi)(i)$ for $j \leq i<n-k+j$. That implies $\operatorname{cr}(\Theta(\pi), n-k+j, j)=A_{2}(\Theta(\pi), n-k+j, j)+A_{3}(\Theta(\pi), n-k+j, j)=$ $n-k+j-j=n-k$ since $A_{1}(\Theta(\pi), n-k+j, j)=A_{4}(\Theta(\pi), n-k+j, j)=0$. Hence, we get $\operatorname{cr}(\Theta(\pi), n-k+j, j)=\operatorname{cr}(\pi, n, k)=n-k$. This ends the proof of the (i) of Lemma 3.8.
(ii) $\pi$ is $\oplus$-decomposable: On one hand, there exists an integer $m>1$ such that $\pi=\pi_{1} \oplus \pi_{2} \oplus \ldots \oplus \pi_{m}$. It is easy to show that $\left|\pi_{i}\right|=1$ for all $i \neq 1$ and $\left|\pi_{1}\right| \geq k$. Consequently, if we denote by $\alpha=\pi_{1}$ and $n-1-l=|\alpha|$ the length of $\alpha$, then we have $\pi=$ $\alpha \oplus 12 \ldots l=\alpha^{\{(n-l, n-l) \ldots,(n-1, n-1)\}}$. So, from Remark 3.6, we get $\operatorname{cr}(\pi, n, k)=\operatorname{cr}(\alpha, n-l, k)$. Since $\alpha$ is $\oplus$-irreducible, we get $\operatorname{cr}(\pi, n, k)=\operatorname{cr}(\alpha, n-l, k)=n-l-k$. On the other hand, we have $\Theta(\pi)=12 \ldots l \otimes \Theta(\alpha)=\Theta(\alpha)^{\left(i, 12 \ldots l^{+(i-1)}\right)}$, where $i=1+|T(\Theta(\alpha))|$. Furthermore, since $n-k+j$ is the minimum of the non-excedances of $\Theta(\sigma)=\Theta(\pi)^{(n-k+j, j)}$, we must have $j<i<i+l-1<n-k+j$. Otherwise, $\Theta(\sigma)$ may not be $\otimes$-irreducible or not 132-avoiding. Consequently, using again Remark 3.6, we get $\operatorname{cr}(\Theta(\pi), n-k+j, j)=\operatorname{cr}(\Theta(\alpha), n-l-k+$ $j, j)$. According to the i) of this lemma and knowing that $\alpha$ is $\oplus$-irreducible, we obtain $\operatorname{cr}(\Theta(\pi), n-k+j, j)=\operatorname{cr}(\Theta(\alpha), n-l-k+j, j)=\operatorname{cr}(\alpha, n-l, k)=n-l-k$. This also ends the proof of the (ii) of Lemma 3.8.

Now we can prove the cr-preserving of the bijection $\Theta$ that is the main object of this section.
Theorem 3.9. For all $\sigma \in S_{n}(321)$, we have $\operatorname{cr}(\Theta(\sigma))=\operatorname{cr}(\sigma)$.
Proof. Combining Lemma 3.5 with Lemma 3.8, we can proceed by induction of $n$. Theorem 3.9 is obvious for $n=1,2,3$. We assume that Theorem 3.9 holds for $k<n$ and let us consider $\sigma \in S_{n}(321)$.

Suppose first that $\sigma$ is $\oplus$-decomposable. We can decompose it as a direct sum $\sigma=$ $\oplus_{i=1}^{l} \sigma_{i}$ of $\oplus$-irreducible ones. From Proposition 3.4, we get $\Theta(\sigma)=\otimes_{i=1}^{l} \Theta\left(\sigma_{l+1-i}\right)$ such that $\Theta\left(\sigma_{i}\right)$ are all $\otimes$-irreducible. Applying the induction hypothesis, we get $\operatorname{cr}(\Theta(\sigma))=$ $\sum_{k=1}^{l} \operatorname{cr}\left(\Theta\left(\sigma_{k}\right)\right)=\sum_{k=1}^{l} \operatorname{cr}\left(\sigma_{k}\right)=\operatorname{cr}(\sigma)$.

Suppose now that $\sigma$ is $\oplus$-irreducible. Let $\pi=\operatorname{red}[\sigma(1 \ldots n-1)] \in S_{n-1}(321)$ and $\Theta(\sigma)=\Theta(\pi)^{(n-k+j, j)}$, where $j-1$ is the number of matched excedance values less than $\sigma(n)$. When we apply the induction hypothesis with Lemma 3.8, we get $\operatorname{cr}(\Theta(\sigma))=$ $\operatorname{cr}(\Theta(\pi))+\operatorname{cr}(\Theta(\pi), n-\sigma(n)+j, j)=\operatorname{cr}(\pi)+\operatorname{cr}(\pi, n, \sigma(n))=\operatorname{cr}(\sigma)$. This ends the proof of Theorem 3.9.

Combining this result with those of Elizalde and Pak (see Theorem 3.1), we get the following one.
Theorem 3.10. The bijection $\Theta$ is ( $\mathrm{fp}, \mathrm{exc}, \mathrm{cr}$ )-preserving, i.e. for all $\sigma \in S_{n}(321)$, we have $(\mathrm{fp}, \mathrm{exc}, \mathrm{cr})(\Theta(\sigma))=(\mathrm{fp}, \mathrm{exc}, \mathrm{cr})(\sigma)$.

As illustration example to end this section, we draw in Fig. 6 the arc diagrams for $\pi=$ 4162735 and $7652134=\Theta(\pi)$. Observe that we have $(\mathrm{fp}, \mathrm{exc}, \mathrm{cr})(\Theta(\pi))=(\mathrm{fp}, \mathrm{exc}, \mathrm{cr})(\pi)=$ $(0,3,5)$.


Figure 6: Arc diagrams of $\pi=4162735$ and $7652134=\Theta(\pi)$.

## 4 Wilf-equivalence classes modulo cr and nes

In this section, we will prove our main result (Theorem 1.1) using the bijection $\Theta$ and some trivial involutions on permutations. These involutions are, the reverse $r$, the complement $c$, and the inverse $i$ such that for any permutation $\sigma$ of $[n]$,
(a) the reverse of $\sigma$ is $r(\sigma)=\sigma(n) \sigma(n-1) \ldots \sigma(1)$. In other word, $r(\sigma)(j)=\sigma(n+1-j)$ for all $j$.
(b) the complement of $\sigma$ is $c(\sigma)=(n+1-\sigma(1))(n+1-\sigma(2)) \ldots(n+1-\sigma(n))$. It means that $c(\sigma)(j)=n+1-\sigma(j)$ for all $j$.
(c) the inverse of $\sigma$ is $i(\sigma)$, a permutation such that $i(\sigma)(j)=k$ if and only if $\sigma(k)=j$. We usually denote $\sigma^{-1}=i(\sigma)$.

To simplify writing, we write $f g:=f \circ g$ for all $f$ and $g$ in $\{r, c, i\}$. Let $\sigma$ be a permutation in $S_{n}$. From the above definitions, the reverse-complement of $\sigma$ is $r c(\sigma)$ such that $r c(\sigma)(n+1-$ $i)=n+1-\sigma(i)$ for all $i$ and the reverse-complement-inverse of $\sigma$ is $r c i(\sigma)$ such that $r c i(\sigma)(n+$ $1-\sigma(i))=n+1-i$ for all $i$. For example, if $\pi=41532$, we have $r(\pi)=23514, c(\pi)=$ 25134, $\pi^{-1}=25413, r c(\sigma)=43152$ and $r c i(\pi)=35214$. Notice that for any composition $f$ of $r, c$ and $i$ and for any subset of permutations $T$, we have the following equivalence

$$
\sigma \in S_{n}(T) \Longleftrightarrow f(\sigma) \in S_{n}(f(T))
$$

This equivalence is well known in the literature, see for example [22, 24]. As in [11] and [8], we need this equivalence to prove most of our results.
Lemma 4.1. The bijection rc preserves the number of nestings.
Proof. Let $\sigma$ be a permutation in $S_{n}$. Using definition of the bijection $r c$, Lemma 4.1 holds from the following facts.

- $(i, j)$ is an upper nesting of $\sigma \Longleftrightarrow(n+1-j, n+1-i)$ is a lower nesting of $r c(\sigma)$ that does not involve a fixed point,
- $(i, j)$ is a lower nesting of the kind $\sigma(j)<\sigma(i)<i<j$ of $\sigma \Longleftrightarrow(n+1-j, n+1-i)$ is an upper nesting of $r c(\sigma)$,
- The number of upper arcs and the number of lower arcs that embrace a fixed point (as a loop) are always equal. In other words, we have $|\{j<i \mid \sigma(j)>i\}|=\mid\{j>i \mid$ $\sigma(j)<i\} \mid$ if $i$ is a fixed point of $\sigma$.

Observe that $r c$ exchanges lower and upper arcs of $\sigma$, except loops.
Notice that $r c$ does not preserve the number of crossings. Take as example $\pi=312$ and $r c(\pi)=231$, while $1=\operatorname{cr}(312) \neq \operatorname{cr}(231)=0$.
Lemma 4.2. The bijection rci preserves the number of crossings and nestings.

Proof. Let $\sigma$ be a permutation of $[n]$. By definition, we have $\operatorname{rci}(\sigma)(n+1-\sigma(i))=n+1-i$ for all $i \in[n]$. So, the following equivalences immediately hold:

$$
\begin{aligned}
i<j<\sigma(i)<\sigma(j) & \Leftrightarrow n+1-\sigma(j)<n+1-\sigma(i)<n+1-j<n+1-i \\
& \Leftrightarrow n+1-\sigma(j)<n+1-\sigma(i)<r c i(\sigma)(n+1-\sigma(j))<r c i(\sigma)(n+1-\sigma(i)) .
\end{aligned}
$$

This means that $(i, j)$ is an upper-crossing of $\sigma$ if and only if $(n+1-\sigma(j), n+1-\sigma(i))$ is an upper-crossing of $r c i(\sigma)$. Similarly, we can easily show the following equivalences

- $(i, j)$ is a crossing of $\sigma \Longleftrightarrow(n+1-\sigma(j), n+1-\sigma(i))$ is a crossing of $r c i(\sigma)$;
- $(i, j)$ is a nesting of $\sigma \Longleftrightarrow(n+1-\sigma(i), n+1-\sigma(j))$ is a nesting of $r c i(\sigma)$.

Lemma 4.2 follows from these properties.
Here is an immediate consequence of these two previous lemmas.
Corollary 4.3. For any subset of patterns $T$, the following statements are true
i) $T$ and $r c(T)$ are nes-Wilf-equivalent,
ii) $T$ and $r c i(T)$ are (cr, nes)-Wilf-equivalent.

Having all the necessary tools, we are now able to prove the following theorem which is equivalent to Theorem 1.1.
Theorem 4.4. For patterns of length 3, we have the following statements.
i) The nes-Wilf equivalence classes are: $\{123\},\{321\},\{213,132\}$ and $\{231,312\}$,
ii) The cr-Wilf equivalence classes are: $\{123\},\{312\},\{231\}$ and $\{132,213,321\}$,
iii) The (cr, nes)-Wilf equivalence classes are: $\{123\},\{321\},\{312\},\{231\}$ and $\{132,213\}$.

Proof. Knowing that $213=r c i(132)$ and $312=r c(231)$, from Corollary 4.3, the patterns 213 and 132 are of the same cr, nes and (cr, nes)-Wilf equivalence classes and the patterns 231 and 312 are of the same nes-Wilf equivalence classes. Moreover, following Theorem 3.9 and due to the cr-preserving of the bijection $\Theta$ of Elizalde and Pak, the patterns 321 and 132 are also of the same cr-Wilf-equivalence class. To complete the proof, we can observe Table 8 containing $\operatorname{Nes}_{n}(\tau ; x)=\sum_{\sigma \in S_{n}(\tau)} x^{\operatorname{nes}(\sigma)}$ and $C r_{n}(\tau ; x)=\sum_{\sigma \in S_{n}(\tau)} x^{\operatorname{cr}(\sigma)}$ for $n=4$ and $\tau \in S_{3}$ obtained by simple computation.

| Pattern $\tau$ | $\mathrm{Cr}_{4}(\tau ; x)$ | $\operatorname{Nes}_{4}(\tau ; x)$ |
| :---: | :---: | :---: |
| 123 | $7+6 x+x^{2}$ | $4+8 x+2 x^{2}$ |
| 132,213 | $8+4 x+2 x^{2}$ | $7+5 x+2 x^{2}$ |
| 321 | $8+4 x+2 x^{2}$ | 14 |
| 231 | $8+5 x+x^{2}$ | $8+5 x+x^{2}$ |
| 312 | $13+x$ | $8+5 x+x^{2}$ |

Table 8: $\mathrm{Cr}_{4}(\tau ; x)$ and $\operatorname{Nes}_{4}(\tau ; x)$ for $\tau \in S_{3}$.

So, when we combine our result with those of Elizalde and Dokos et al., we get the following one.
Theorem 4.5. For pattern in $S_{3}$, the non singleton Wilf-equivalence classes are
i) $[132]_{\mathrm{fp}, \mathrm{exc}, \text { inv,cr,nes }}=\{132,213\}$,
ii) $[231]_{\text {fp,inv }, \text { nes }}=\{231,312\}$,
iii) $[132]_{\mathrm{fp}, \mathrm{exc}, \mathrm{cr}}=\{132,213,321\}$.

Proof. For i) and ii), it is easy to show that the bijection rci preserves all statistics in \{fp, exc, inv, cr, nes\} and the bijection $r c$ preserves all statistics in $\{\mathrm{fp}$, inv, nes $\}$. Especially for iii), we use the (fp, exc, cr)-preserving of the bijection $\Theta$.

## 5 Connection to the $q, p$-Catalan number of Randrianarivony

In this section, we present an unexpected result on the joint distribution of the statistics exc and cr (see Theorem 5.3). The connection with the q,p-Catalan number of Randrianarivony is due to the following lemma about the characterization of nonnesting permutations in terms of avoiding permutations.
Lemma 5.1. For all integer $n$, we have $N N_{n}=S_{n}(321)$.
Proof. It is clear that a given permutation $\sigma$ is nonnesting if and only if it is bi-increasing. Following Reifegerste [18], bi-increasing permutations and 321-avoiding permutations are the same.

Randrianarivony [17] defined a $q, p$-Catalan numbers $C_{n}(q, p)$ through the relation

$$
\begin{equation*}
C_{n}(q, p)=C_{n-1}(q, p)+q \sum_{k=0}^{n-2} p^{k} C_{k}(q, p) C_{n-1-k}(q, p) \tag{5.1}
\end{equation*}
$$

with $C_{0}(q, p)=C_{1}(q, p)=1$ and he purposed some combinatorial interpretations of $C_{n}(q, p)$ in terms of noncrossing and nonnesting permutations. According to Lemma 5.1, one of its proved results can be stated as follow.
Theorem 5.2. [17] For all integer $n \geq 0$, we have $\sum_{\sigma \in S_{n}(321)} q^{\operatorname{exc}(\sigma)} p^{\operatorname{cr}(\sigma)}=C_{n}(q, p)$.
Proof. See [17].
Notice that the iii) of Theorem 4.5 is equivalent to the following identities.

$$
\begin{equation*}
\sum_{\sigma \in S_{n}(213)} x^{\mathrm{fp}(\sigma)} q^{\operatorname{exc}(\sigma)} p^{\operatorname{cr}(\sigma)}=\sum_{\sigma \in S_{n}(132)} x^{\mathrm{fp}(\sigma)} q^{\operatorname{exc}(\sigma)} p^{\operatorname{cr}(\sigma)}=\sum_{\sigma \in S_{n}(321)} x^{\mathrm{fp}(\sigma)} q^{\operatorname{exc}(\sigma)} p^{\operatorname{cr}(\sigma)} \tag{5.2}
\end{equation*}
$$

The case $\mathrm{p}=1$ of (5.2) was treated in $[9,10,11,12]$. With Theorem 5.2 , the case $x=1$ leads to the following one.
Theorem 5.3. For all non negative integer $n$ and for all $\tau \in\{213,132,321\}$, we have

$$
\sum_{\sigma \in S_{n}(\tau)} q^{\operatorname{exc}(\sigma)} p^{\operatorname{cr}(\sigma)}=C_{n}(q, p)
$$

Corollary 5.4. For any $\tau \in\{213,132,321\}$, the continued fraction expansion of $\sum_{\sigma \in S(\tau)} q^{\operatorname{exc}(\sigma)} p^{\operatorname{cr}(\sigma)} z^{|\sigma|}$ is

$$
\begin{equation*}
\frac{1}{1-\frac{z}{1-\frac{q z}{1-\frac{p z}{1-\frac{q p z}{1-\frac{p^{2} z}{1-\frac{q p^{2} z}{\ddots}}}}}} . . . .} \tag{5.3}
\end{equation*}
$$

Proof. Let us denote by $C(q, p, z):=\sum_{\sigma \in S(\tau)} q^{\operatorname{exc}(\sigma)} p^{\operatorname{cr}(\sigma)} z^{|\sigma|}$ for $\tau \in\{213,132,321\}$. According to Theorem 5.3, we have $C(q, p, z)=\sum_{n \geq 0} C_{n}(q, p) z^{n}$. So, using recurrence (5.1), we get the following identity which leads to (5.3)

$$
C(q, p, z)=\frac{1}{1-\frac{z}{1-q z C(q, p, p z)}} .
$$

Let us end this section with an interesting remark on the recursion formula for the polynomial distribution of the statistic inv over the set $S_{n}(321)$. If we denote by $I_{n}(q)=$ $\sum_{\sigma \in S_{n}(321)} q^{\operatorname{inv}(\sigma)}$, we have the following recurrence formula which was first conjectured in [8]

$$
\begin{equation*}
I_{n}(q)=I_{n-1}(q)+\sum_{k=0}^{n-2} q^{k+1} I_{k}(q) I_{n-1-k}(q) . \tag{5.4}
\end{equation*}
$$

Using other objects like 2-Motzkin paths and polyominoes, this recursion was later proved by Cheng et al. in [5]. Based on the scanning-elements algorithm, Mansour and Shattuck [15] also provided another proof. We observe that (5.4) can be obtained from (5.1) by setting $p=q$, i.e. $I_{n}(q)=C_{n}(q, q)$. Indeed, it was proved in [7, 17] that $\operatorname{inv}(\sigma)=2 \operatorname{nes}(\sigma)+\operatorname{cr}(\sigma)+$ $\operatorname{exc}(\sigma)$ for all permutation $\sigma$. So, if $\sigma \in S_{n}(321)$, then we have $\operatorname{inv}(\sigma)=\operatorname{cr}(\sigma)+\operatorname{exc}(\sigma)$ and we get $I_{n}(q)=C_{n}(q, q)$. The continued fraction expansion of the generating function of $I_{n}(q)$ presented in [15] (Theorem 1) is also obtained from (5.3) by setting $p=q$.

## 6 Concluding remarks

We conclude this paper with two remarks. The first one is about the decomposition of Dyck path involving centered multitunnel and the second one is about the direct bijection $\Gamma$ : $S_{n}(321) \rightarrow S_{n}(132)$ defined by A. Robertson [20].

According to the original definition of Elizalde and Deutsch, let us remember what a multitunnel is. A multitunnel of a Dyck paths $D$ is a concatenation of tunnels in which each tunnel starts at the point where the previous one ends. Centered multitunnels are those whose midpoints stay on the vertical line $x=n$. For example, the Dyck path $D=$
ududuuuddudduudd in Fig. 2 has three centered multitunnels. As mentioned in [9], each centered multitunnel is in obvious one-to-one correspondence with decomposition of the Dyck word $D=A B C$ where $B$ and $A C$ are Dyck paths, $B$ is the section that runs along the entire multitunnel, $A$ and $C$ have the same length. So, an operation © defined by $D_{1} \odot$ $D_{2}=D_{1}^{(L)} D_{2} D_{1}^{(R)}$ is well defined and stable on Dyck paths. Inspiring from the proof of Proposition 3.4, we remark that there is a correspondence between the three operations $\oplus$, $\bigcirc$ and $\otimes$.
Remark 6.1. For all $\sigma_{1}, \sigma_{2} \in S(321)$, we have $\sigma_{2} \oplus \sigma_{2} \longmapsto \Psi\left(\sigma_{1}\right) \odot \Psi\left(\sigma_{2}\right)$ and for any Dyck paths $D_{1}$ and $D_{2}$, we also have $D_{1} \odot D_{2} \longmapsto \Phi^{-1}\left(D_{2}\right) \otimes \Phi^{-1}\left(D_{1}\right)$.

In terms of statistic, the number of $\oplus$-components of $\sigma$, the number of centered multitunnels of $\Psi(\sigma)$ and the number of $\otimes$-components of $\Theta(\sigma)$ are the same for any $\sigma \in S(321)$. We illustrates this correspondence by an example in Figure 7.


Figure 7: Example of correspondence between $\oplus$, ๑ and $\otimes$.
Let us recall the direct bijection $\Gamma: S_{n}(321) \rightarrow S_{n}(132)$ defined by A. Robertson. Let $\sigma(i) \sigma(j) \sigma(k)$ and $\sigma(x) \sigma(y) \sigma(z)$ be two occurrences of pattern 132 in a permutation $\sigma$. Say that $\sigma(i) \sigma(j) \sigma(k)$ is smaller than $\sigma(x) \sigma(y) \sigma(z)$ if $(i, j, k) \prec(x, y, z)$, where $\prec$ denote the lexicographic ordering of triples of positive integers. Let $\mathcal{M}$ be an operation that creates the permutation $\mathcal{M} \sigma$ from $\sigma$ by converting the smallest occurrence of 132-pattern into 321pattern. It is clear that $\mathcal{M} \sigma=\sigma$ if $\sigma$ is 132 -avoiding. We denote by $\mathcal{M}^{j} \sigma=\mathcal{M}^{j-1} \sigma, j \geq 1$. It is known that, for every $\sigma \in S_{n}, \mathcal{M}^{j} \sigma$ must be 132 -avoiding for some finite $j$. So, we can define the bijection $\Gamma$ as follow

$$
\Gamma(\sigma)= \begin{cases}\sigma & \text { if } \sigma \in S_{n}(132) \\ \mathcal{M}^{r} \sigma & \text { if } \sigma \notin S_{n}(132)\end{cases}
$$

where $r$ is the smallest positive integer such that $\mathcal{M}^{r} \sigma$ is 132-avoiding.
For example, if $\pi=4162735 \in S_{7}(321)$, then we have $\mathcal{M} \pi=6152734, \mathcal{M}^{2} \pi=6521734$, $\mathcal{M}^{3} \pi=6571324$ and $\mathcal{M}^{4} \pi=6573214=\Gamma(\pi)$. We remark that the following result holds and seems to be interesting.
Theorem 6.2. The bijection $\Gamma$ is also (fp,exc,cr)-preserving.
Indeed, Bloom and Saracino [2, 3] proved that the bijection $\Gamma$ is (fp,exc)-preserving. Recently, Saracino [21] showed how the bijections $\Theta$ and $\Gamma$ are related each other by a simple relation. He proved the following theorem.
Theorem 6.3. [21] We have $\Gamma(\sigma)=\Theta \circ \operatorname{rci}(\sigma)$ for any $\sigma \in S_{n}(321)$, where rci is the reverse-complement-inverse.
Since the bijections $\Theta$ and $r c i$ are both cr-preserving, so do the bijection $\Gamma$. Below is a graphical illustration example to close this paper. For $\pi=4162735$ we have $\operatorname{cr}(\Gamma(\pi))=\operatorname{cr}(\pi)=5$.


Figure 8: Arc diagrams of $\pi=4162735$ and $6573214=\Gamma(\pi)$.

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