

Vertex-monochromatic connectivity of strong digraphs

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Abstract

A vertex coloring of a strong digraph D is a *strong vertex-monochromatic connection coloring (SVMC-coloring)* if for every pair u, v of vertices in D there exists an (u, v) -monochromatic path having all the internal vertices of the same color. Let $smc_v(D)$ denote the maximum number of colors used in an SVMC-coloring of a digraph D . In this paper we determine the value of $smc_v(D)$ for the line digraph of a digraph. We also we give conditions to find the exact value of $smc_v(T)$, where T is a tournament.

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1 Introduction

Caro and Yuster [3] introduced the concept of monochromatic connection of an edge colored graph. An edge-coloring of a graph G is a *monochromatic-*

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connecting coloring if there exists a monochromatic path between any two vertices of G . The study of monochromatic connecting colorings arises from the rainbow connecting coloring problem, in which rainbow paths are considered (a path is said to be *rainbow* if no two edges of them are colored the same). The monochromatic connection problem has also been studied in oriented graphs [5]. An arc-coloring of a digraph D is a *strongly monochromatic connecting coloring* (SMC-coloring, for short) if for every pair u, v of vertices in D there exists a directed (u, v) -monochromatic path and a directed (v, u) -monochromatic path. The *strong monochromatic connection number* of a strong digraph D , denoted by $smc(D)$, is the maximum number of colors used in an SMC-coloring of D . Concerning the strong monochromatic connection number of an oriented graph the following result was proved in [5].

Theorem 1 *Let D be a strongly connected oriented graph of size m , and let $\Omega(D)$ be the minimum size of a strongly connected spanning subdigraph of D . Then*

$$smc(D) = m - \Omega(D) + 1.$$

Cai, Li and Wu [8] defined the vertex-version of the monochromatic connecting coloring concept. A path in a vertex colored graph is *vertex-monochromatic* if its internal vertices are colored the same. A vertex-coloring of a graph is a *vertex-monochromatic connecting coloring* (VMC-coloring) if there is a vertex-monochromatic path joining any two vertices of the graph. This concept also can be extended to digraphs. A directed path in a vertex colored digraph is *vertex-monochromatic* if its internal vertices are colored the same. A vertex-coloring of a digraph is a *strongly vertex-monochromatic connecting coloring* (SVMC-coloring) if for every pair u and v of vertices in D there exists a directed (u, v) -vertex-monochromatic path and a directed (v, u) -vertex-monochromatic path. The *monochromatic vertex-connecting number* of a strong digraph D , denoted by $smc_v(D)$, is the maximum number of colors that can be used in a strongly vertex-monochromatic connecting coloring of D .

For an overview of the monochromatic and rainbow connection subjects we refer the reader to [4,6,7].

In this work we study the SVMC-colorings of strong digraphs. The paper is organized as follows. In section 2 some basic definitions and notations are given. In section 3 lower and upper bounds for $smc_v(D)$ are presented. In section 4 we focus on the family of line digraphs. Finally, in section 5 we study the strong vertex-monochromatic connection number of strongly connected tournaments.

2 Definitions and Notation

All the digraphs considered in this work are simple; that is, digraphs with no parallel arcs, nor loops are considered. If (u, v) is an arc of D , then we use either uv or $u \rightarrow v$ denote it. Two vertices u and v of a digraph are *adjacent* if $u \rightarrow v$ or $v \rightarrow u$. All walks, paths and cycles are to be considered directed. A digraph is *connected* if its underlying graph is connected. A digraph D is *unilateral* if, for every pair $u, v \in V(D)$, either u is reachable from v , or v is reachable from u (or both). A p -*cycle* is a cycle of length p . The minimum integer p for which D has a p -cycle is the *girth of D* and it is denoted by $g(D)$. A digraph D is *strongly connected* or *strong* if for every pair of vertices $u, v \subseteq V(D)$, the vertex u is reachable from v and the vertex v is reachable from u . Given a strong digraph D , we use $\Omega(D)$ to denote the minimum size of a strongly connected spanning subdigraph of D . Let u, v be two vertices of D . We say that u *dominates* v , or v *is dominated by* u , if $v \in N^+(u)$. A set of vertices $S \subset V(D)$ is a *dominating set* if each vertex $v \in V(D) \setminus S$ is dominated by at least one vertex in S . A set of vertices S is an *absorbing set* if for each vertex $v \in V(D) \setminus S$ there exists a vertex $u \in S$ such that $v \in N^-(u)$. An orientation of a complete graph is a *tournament*. A subdigraph H is said to be *absorbing subdigraph* (*dominating subdigraph*) if the set $V(H)$ is an absorbing set (*dominating set*) of D .

Let $D = (V, A)$ be a strong digraph. The *subdigraph induced* by a set of vertices S is denoted by $D[S]$. Given a positive integer p , let $[p] = \{1, 2, \dots, p\}$. A vertex p -*coloring* of D is a surjective function $\Gamma : V \rightarrow [p]$. For each “color” $i \in [p]$ the set of vertices $\Gamma^{-1}(i)$ will be called the *chromatic class* (of color i), and if $|\Gamma^{-1}(i)| = 1$, the color i and the chromatic class $\Gamma^{-1}(i)$ will be called *singular*. A subdigraph H of D will be called *monochromatic* if $A(H)$ is contained in a chromatic class. A p -coloring Γ of D is an *optimal coloring* if $p = smc(D)$ and Γ is an SMC-coloring of D . For general concepts we may refer the reader to [1,2].

3 Bounds for $smc_v(D)$

In this section upper and lower bounds for the strong vertex-monochromatic connection number of a digraph D are given.

The next proposition is the digraph version of the bounds obtained [8] for the monochromatic vertex-connection number of a graph.

Proposition 1 *Let D be a strong digraph of order n and diameter d . Then*

- i) $smc_v(D) = n$ if and only if $d \leq 2$.
- ii) If $d \geq 3$ then $smc_v(D) \leq n - d + 2$.

Proof.

- i) Let $\Gamma : V(D) \rightarrow [n]$ be an SVMC-coloring of D . Let u, v be two vertices in D such that $d(u, v) = d$. Let P be a (u, v) -vertex-monochromatic path. Since Γ is an SVMC-coloring of D and all the vertices of D have a different color it follows that the length of P is at most two. Therefore $d(u, v) \leq 2$ and the result follows. If $d \leq 2$, the coloring that assigns to every vertex a different color is an SVMC-coloring of D .
- ii) Let u and v be two vertices in D such that $d(u, v) = d$. Let P be a vertex-monochromatic path connecting u and v . Observe that there are at least $d - 2$ vertices in P using the same color, therefore $smc_v(D) \leq n - (d - 2)$ and the result follows.

■

The following example shows that the upper bound of item *ii*) of the above theorem is tight. Let D be the digraph with vertex set $V(D) = \{v_1, v_2, \dots, v_n\}$ and arc set $A(D) = \{v_i v_1, v_2 v_i : i = 3, 4, \dots, n\} \cup \{v_i v_2\}$. Observe that $\Omega_v(D) = 3$, $diam(D) = 3$ and $smc_v(D) = n - diam(D) + 2 = n - 1 > n - \Omega_v(D) + 1$.

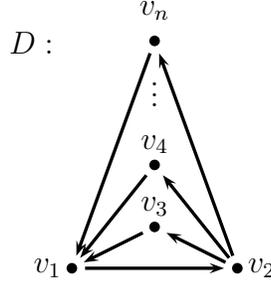


Fig. 1. $smc_v(D) = n - diam(D) + 2$.

Theorem 2 Let D be a strong digraph and let Γ be an SVMC-coloring of D . Let S be the set of singular chromatic classes of Γ and let D^* be the digraph induced by $V(D) \setminus S$.

- i) If for every vertex v of D there exists a vertex x such that $d(v, x) \geq 3$, then $V(D^*)$ is a total absorbing set of D .
- ii) If for every vertex v of D there exists a vertex x such that $d(x, v) \geq 3$, then $V(D^*)$ is a total dominating set of D .
- iii) If $g(D) \geq 5$, then D^* is strong, absorbing and dominating set of D .

Proof. i) Let v be a vertex of D . Let $x \in V(D)$ such that $d(v, x) \geq 3$.

Since Γ is an SVMC-coloring there exists a (v, x) -vertex-monochromatic path $P = (v, x_1, \dots, x_r, x)$ with $r \geq 2$, such that the color of the internal vertices of P belongs to a non-singular class, implying that D^* is a total absorbing set.

ii) Let v be a vertex of D and let $x \in V(D)$ such that $d(x, v) \geq 3$. Let $P = (x, x_1, \dots, x_r, v)$, $r \geq 2$, be a (x, v) -vertex-monochromatic path. Since the color of the internal vertices of P are non-singular, then the set $V(D^*)$ is a total dominating set of D .

iii) Absorbing and dominating properties follows from items i) and ii). Let $u, v \in V(D^*)$ and suppose that there is no (u, v) -path in D^* . Since Γ is an SVMC-coloring, there exists an (u, v) -vertex-monochromatic path P of length 2 connecting and a (v, u) -vertex-monochromatic path P' of length at least 3 (because $g(D) \geq 5$). Suppose that $P' = (u, x_1, \dots, x_\ell, v)$. Note that the color used in the internal vertices of P' is a non-singular color. Therefore $x_i \in V(D^*)$ for $i = 1, 2, \dots, \ell$. Since Γ is an SVMC-coloring of D and $g(D) \geq 5$, it follows that the (x_i, x_{i-1}) -vertex-monochromatic paths are totally contained in D^* . Hence D^* is strongly connected. ■

Let D be a strong digraph and let H be an absorbent, dominant and strong subdigraph of D . By coloring the vertices of H with one single color and the remaining vertices with distinct colors, an SVMC-coloring of D with $n - |V(H)| + 1$ colors is obtained. Let $\Omega_v(D)$ denote the minimum order of an absorbent, dominant and strong subdigraph of D . Therefore

$$smc_v(D) \geq n - \Omega_v(D) + 1. \quad (1)$$

Theorem 3 *Let D be a strong digraph of order n and girth $g(D) \geq 4$. Let Γ be an SVMC-coloring of D that uses $smc_v(D)$ colors. If ℓ is the minimum cardinality of a non-singular chromatic class of Γ , then*

$$n - \Omega_v(D) + 1 \leq smc_v(D) \leq n - \Omega_v(D) + \frac{\Omega_v(D)}{\ell}.$$

Proof. The left hand of the inequality is a consequence of (1). Let Γ be an SVMC-coloring of D that uses $smc_v(D)$ colors. Let a_i denote the number of the chromatic classes with cardinality i . Observe that $a_1 + a_2 + \dots + a_r = smc_v(D)$, where r is the cardinality of the largest chromatic class. Let ℓ be the minimum cardinality of a non-singular chromatic class. Hence

$$n = a_1 + \sum_{i=\ell}^r ia_i \geq \ell(a_1 + \sum_{i=\ell}^r ia_i) - (\ell - 1)a_1 = \ell smc_v(D) - (\ell - 1)a_1.$$

Therefore,

$$smc_v(D) \leq \frac{n + (\ell - 1)a_1}{\ell}.$$

Let D^* be the subdigraph of D induced by the non-singular classes of Γ . By Theorem ?? the set $V(D^*)$ is strong, absorbing and dominating. Then $a_1 + \Omega_v(D) \leq n$. Hence

$$smc_v(D) \leq \frac{n + (\ell - 1)a_1}{\ell} \leq \frac{n + (\ell - 1)(n - \Omega_v(D))}{\ell} = n - \Omega_v(D) + \frac{\Omega_v(D)}{\ell}.$$

■

Corollary 4 *Let D be a strong digraph of order n . Then*

$$n - \Omega_v(D) + 1 \leq smc_v(D) \leq n - \frac{\Omega_v(D)}{2}.$$

4 Line digraphs

Recall that the line digraph $L(D)$ of a digraph $D = (V, A)$ has A for its vertex set and (e, f) is an arc in $L(D)$ whenever the arcs e and f in D have a vertex in common which is the head of e and the tail of f . A digraph D is called a *line digraph* if there exists a digraph H such that $L(H)$ is isomorphic to D . In this section we determine the value of $smc_v(D)$ for a line digraph D .

Proposition 2 *Let D be a strong digraph and let H be a spanning and strong subdigraph of D . If $L(H)$ is the subdigraph of $L(D)$ induced by the arcs of H , then $L(H)$ is a strong, absorbing and dominating subdigraph of $L(D)$.*

Proof. Let D be a strong digraph and let H be a spanning strong subdigraph of D . Since H is strong it follows that $L(H)$ is a strong subdigraph of $L(D)$. Furthermore, since H is an spanning and strong subdigraph of D for every vertex $e = (u, v)$ of $L(D)$ there are two vertices f_1 and f_2 of $L(H)$ such that $f_1 = (w_1, u)$ and $f_2 = (v, w_2)$. Therefore f_1 dominates the vertex e and f_2 absorbs the vertex e . ■

Let H be the line digraph a of digraph D . Let $\Gamma : V(H) \rightarrow [k]$ be an SVMC-coloring of H . Notice that the coloring Γ induces a coloring Γ' of the arcs in D . Let $\Gamma' : A(D) \rightarrow [k]$ the coloring that assigns to each arc e in D the color $\Gamma(e)$ of the vertex $e \in V(H)$.

Let D be a strong digraph. An ordered pair (u, v) of vertices of D is said to be a *bad pair* of D if $N^+(u) = \{v\}$ and $N^-(v) = \{u\}$. Observe that if (u, v) is a bad pair then uv is an arc of D and the pair (v, u) is not a bad pair.

Lemma 1 *Let D be a strong digraph and let H be the line digraph of D . Let Γ be an SVMC-coloring of H and let Γ' be the arc coloring of D that assigns to each arc $e \in A(D)$ the color $\Gamma(e)$ of vertex $e \in V(H)$. Given two vertices u and v in D there exists an (v, u) -monochromatic path in D if one of the following conditions holds.*

- i) The ordered pair (u, v) is not a bad pair.*
- ii) The ordered pair (u, v) is a bad pair and there exists an arc vw in D such that (v, w) is not a bad pair.*
- iii) The ordered pair (u, v) is a bad pair and there exists an arc wu in D such that (w, u) is not a bad pair.*
- iv) If the previous cases do not happen and D is different from C_3 .*

Proof. Let D be a strong digraph and let $H = L(D)$. Let u, v be two vertices of D . Let Γ be an SVMC-coloring of H and let Γ' be the arc coloring of D induced by Γ .

- i) Suppose that (u, v) is not a bad pair. Assume that there exists a vertex $w \in N^-(v)$ such that $w \neq u$. Since D is strong there exists a vertex $w_1 \in N^+(u)$ (it may happen that $w_1 = v$). Since Γ is an SVMC-coloring of H , there exists an vertex-monochromatic path $P = (wv, vv_1, \dots, v_{l-1}v_l, v_lu, uw_1)$ connecting the vertices wv and uw_1 of H . The path P induces a monochromatic path connecting the vertices v and u in D . If there exists a vertex $w \in N^+(u)$ such that $w \neq v$. Since H is strong there exists a vertex $w_1 \in N^-(v)$ and there is a vertex-monochromatic path P connecting the vertices w_1v and uw which induces a monochromatic path connecting the vertices v and u in D .*
- ii) Suppose that (u, v) is a bad pair and there exists a vertex $w \in N^+(v)$ such that (v, w) is not a bad pair. By the above item there is a (w, v) -monochromatic path P of the same color of the arc uv . If (w, u) is not a bad pair then there exists a (u, w) -monochromatic path P' containing the arc uv (because (u, v) is a bad pair). The union of P and P' contains a (v, u) -monochromatic path. If (w, u) is a bad pair, then $N^+(w) = \{u\}$ and $N^-(u) = \{w\}$. Since (v, w) is not a bad pair there exists a vertex z such that $z \in N^-(w) \setminus \{v\}$ or $z \in N^+(v) \setminus \{w\}$. Note that $(v, z), (u, z)$ and (z, u) are not bad pairs because $N^+(v) \neq \{z\}$ and v and u and z are not adjacent. Since (v, z) and (z, u) are not bad pairs there is a (z, v) -monochromatic path P and a (u, z) -monochromatic path P' of the same color because both paths contains the arc uv . The union of P and P' contains a (v, u) -monochromatic path.*
- iii) Suppose that the ordered pair (u, v) is a bad pair and there exists an arc*

wu in D such that (w, u) is not a bad pair. By item *i*) there exists a (u, w) -monochromatic path P that uses the arc wu and therefore of the same color of wu . If (v, w) is not a bad pair, then by item *i*) there exists a (w, v) -monochromatic P' containing the arc wv and therefore of the same color of P . The union of P' and P contains a (v, u) -monochromatic path in D . Continue assuming that (v, w) is a bad pair. Therefore there is a vertex such that either $z \in N^+(w)$ or $z \in N^-(u)$. Observe that $(z, v), (v, z)$ and (w, z) are not bad pairs. Hence there exists (v, z) -monochromatic path a (z, v) -monochromatic path and a (z, w) -monochromatic path in D . The union of these paths contains a (v, u) -monochromatic path.

iv) If D is isomorphic to C_3 , then H is also isomorphic to C_3 . Since $smc_v(C_3) = 3$ (see item *i*) of Proposition 1). The coloring Γ induces a coloring Γ' of the arcs in D with three colors that is not an SMC-coloring of D .

■

Theorem 5 *Let D be a strong directed graph different from the cycle of length 3. Then*

$$smc_v(L(D)) = smc(D).$$

Proof. Let D be a strong digraph of size m . By Theorem 1 it follows that $smc(D) = m - \Omega(D) + 1$. Let H be a strong and spanning subdigraph of D of size $\Omega(D)$. By Proposition 2 it follows that $L(H)$ is a strong, absorbing and dominating subdigraph of $L(D)$. By (1) it follows

$$smc_v(L(D)) \geq |V(L(D))| - |V(H)| + 1 = m - \Omega(D) + 1 = smc(D).$$

Observe that if D is different from C_3 , then every pair of vertices of D satisfies one of the items of Lemma 1. Hence, if Γ is an SVMC-coloring of $L(D)$ it follows that the coloring Γ' of D induced Γ is an SMC-coloring of D and therefore $smc_v(L(D)) \leq smc(D)$, and the result follows. ■

5 Monochromatic vertex-connecting number of tournaments

In this section a condition on $\Omega_v(T)$ of a strong tournament T is given in order to find the exact value of $smc_v(T)$.

Theorem 6 *Let T be a strong tournament of diameter $d \geq 6$. If $\Omega(T) \leq 2d - 6$, then*

$$smc_v(T) = n - \Omega_v(T) + 1.$$

Proof. Let Γ be an SVMC-coloring of T and let u, v be two vertices of T such that $d(u, v) = d \geq 6$. Let $P = (u, x_1, x_2, \dots, x_s, v)$ be a (u, v) -vertex-monochromatic path. Since P is a (u, v) -path of T , it follows that $s \geq d - 1$.

Observe that the subdigraph induced by $\{x_1, x_2, \dots, x_s\}$ is strong. Let H be the biggest strong subdigraph of T containing the set $\{x_1, x_2, \dots, x_s\}$ such that all the vertices of H are colored the same. We claim that $V(H)$ is an absorbing and dominating set of T .

Claim 1. $V(H)$ is an absorbing set of T . Suppose that there exists a vertex $w \in V(T) \setminus V(H)$ such that $x \rightarrow w$ for every vertex $x \in V(H)$. Since Γ is an SVMC-coloring of T there exists a (w, v) -vertex-monochromatic path $P' = (w, y_1, y_2, \dots, y_r, v)$. Note that $(u, x_1, w, y_1, \dots, y_r, v)$ is a (u, v) -path, hence $r \geq d - 3$. If the color of the internal vertices of P' is different to the color of the vertices in H , then

$$smc_v(D) \leq n - |V(H)| - (|V(P')| - 2) + 2 \leq n - (d - 1) - (d - 3) + 2 = n - 2d + 6.$$

Combining the above inequality with (1) it follows that

$$n - (2d - 6) + 1 \leq n - \Omega_v(T) + 1 \leq smc_v(T) \leq n - 2d + 6,$$

giving a contradiction. Hence, the color of the internal vertices of P is equal to the color of the vertices of H . Observe that $x_s \rightarrow y_1$, otherwise (u, x_1, w, y_1, x_s, v) is a (u, v) -path of length 5 contradicting that $d(u, v) = d \geq 6$. Furthermore, for every $y_i \in V(P')$, $i = 2, \dots, s$, it follows that $x \rightarrow y_i$. If $y_i \rightarrow x$ for some $i = 2, \dots, s$, the subdigraph induced by $V(H) \cup \{y_1, y_2, \dots, y_s\}$ would be a strong subdigraph of T bigger than H , contradicting the election of H . Therefore $y \notin V(H)$ for every $y \in V(P')$ and

$$smc_v(D) \leq n - |V(H)| - (|V(P')| - 2) + 1 \leq n - (d - 1) - (d - 3) + 1 = n - 2d + 5,$$

and using (1), a contradiction is obtained.

Claim 2. $V(H)$ is a dominating set of T . Suppose that there exists a vertex $w \in V(T) \setminus V(H)$ such that $w \rightarrow x$ for every $x \in V(H)$. Let $P' = (u, y_1, y_2, \dots, y_r, w)$ be a (u, w) -vertex-monochromatic path. Since $(u, y_1, y_2, \dots, w, x_s, v)$ is a (u, v) -path, it follows that $s \geq d - 3$. If the color of the internal vertices of P' is different to the color of the vertices of H , using a similar reasoning as in the proof of *Claim 1* a contradiction is obtained. Hence, the color of the internal vertices of P is equal to the color of the vertices of H . Observe that $x_s \rightarrow y_1$, otherwise (u, y_1, x_s, v) is a (u, v) -path of length 4, a contradiction. Moreover, for every $y_i \in V(P')$, $i = 2, \dots, s$, it follows that $x \rightarrow y_i$, for every $x \in V(H)$. If not, the digraph induced by $V(H) \cup \{y_1, y_2, \dots, y_i\}$ is a strong subdigraph of T bigger than H , giving a contradiction. Therefore $y \notin V(H)$ for every $y \in V(P')$ and using a reasoning analogous to the proof of *Claim 1* the result is followed.

Hence H is an absorbent, dominant and strong subdigraph of T . Since Γ is an optimal SVMC-coloring of T that assign the same color to every vertex in H , it follows that $smc_v(T) = n - \Omega_v(T) + 1$ and the result follows. ■

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