THE LEAST H-EIGENVALUE OF SIGNLESS LAPLACIAN OF NON-ODD-BIPARTITE HYPERGRAPHS

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Abstract. Let G be a connected non-odd-bipartite hypergraph with even uniformity. The least H-eigenvalue of the signless Laplacian tensor of G is simply called the least eigenvalue of G and the corresponding H-eigenvectors are called the first eigenvectors of G. In this paper we give some numerical and structural properties about the first eigenvectors of G which contains an odd-bipartite branch, and investigate how the least eigenvalue of G changes when an odd-bipartite branch attached at one vertex is relocated to another vertex. We characterize the hypergraph(s) whose least eigenvalue attains the minimum among a certain class of hypergraphs which contain a fixed non-odd-bipartite connected hypergraph. Finally we present some upper bounds of the least eigenvalue and prove that zero is the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs.

1. Introduction

Since Lim [13] and Qi [16] independently introduced the eigenvalues of tensors or hypermatrices in 2005, the spectral theory of tensors developed rapidly, especially the well-known Perron-Frobenius theorem of nonnegative matrices was generalized to nonnegative tensors [2, 6, 20, 21, 22]. The signless Laplacian tensors $\mathcal{Q}(G)$ [17] were introduced to investigating the structure of hypergraphs, just like signless Laplacian matrices to simple graphs. As $\mathcal{Q}(G)$ is nonnegative, by using Perron-Frobenius theorem, many results about its spectral radius are presented [9, 10, 12, 14, 23].

Let G be a k-uniform connected hypergraph. Shao et al. [18] prove that zero is an H-eigenvalue of $\mathcal{Q}(G)$ if and only if k is even and G is odd-bipartite. Some other equivalent conditions are summarized in [5]. Note that zero is an eigenvalue of $\mathcal{Q}(G)$ if and only if k is even and G is odd-colorable [5]. So, there exist odd-colorable but non-odd-bipartite hypergraphs [4, 15], for which zero is an N-eigenvalue. Hu and Qi [7] discuss the H-eigenvectors of zero eigenvalue of $\mathcal{Q}(G)$ related to the odd-bipartitions of G, and use N-eigenvectors of zero eigenvalue of $\mathcal{Q}(G)$ to discuss some kinds of partition of G, where an eigenvector is called *H-(or N-)eigenvector* if it can (or cannot) be scaled into a real vector.

Except the above work, the least H-eigenvalue of $\mathcal{Q}(G)$ receives little attention. In this paper, we focus on the least H-eigenvalue of $\mathcal{Q}(G)$. Qi [16] proved that each eigenvalue of $\mathcal{Q}(G)$ of a connected k-uniform hypergraph G has a nonnegative real part by using Gershgorin disks, which implies that the least H-eigenvalue of $\mathcal{Q}(G)$

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is at least zero, and is zero if and only if k is even and G is odd-bipartite. If k is even, then $\mathcal{Q}(G)$ are positive semi-definite [17], and its least H-eigenvalue is a solution of minimum problem over a real unit sphere; see Eq. (2.3). So, throughout of this paper, when discussing the least H-eigenvalue of $\mathcal{Q}(G)$, we always assume that G *is connected non-odd-bipartite with even uniformity* k. For convenience, the least H-eigenvalue of $\mathcal{Q}(G)$ is simply called the *least eigenvalue* of G and the corresponding H-eigenvectors are called the *first eigenvectors* of G.

In this paper we give some numerical and structural properties about the first eigenvectors of G which contains an odd-bipartite branch, and investigate how the least eigenvalue of G changes when an odd-bipartite branch attached at one vertex is relocated to another vertex. We characterize the hypergraph(s) whose least eigenvalue attains the minimum among a certain class of hypergraphs which contain a fixed non-odd-bipartite connected hypergraph. Finally we present some upper bounds of the least eigenvalue and prove that zero is the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs. The perturbation result on the least eigenvalue in this paper is a generalization of that on the least eigenvalue of the signless Laplacian matrix of a simple graph in [19].

2. Preliminaries

2.1. **Eigenvalues of tensors.** A real *tensor* (also called *hypermatrix*) $\mathcal{A} = (a_{i_1 i_2 \ldots i_k})$ of order k and dimension n refers to a multi-dimensional array with entries $a_{i_1 i_2 \ldots i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1, 2, \ldots, n\}$ and $j \in [k]$. Obviously, if $k = 2$, then $\mathcal{A}$ is a square matrix of dimension n. The tensor $\mathcal{A}$ is called *symmetric* if its entries are invariant under any permutation of their indices.

Given a vector $x \in \mathbb{C}^n$, $\mathcal{A}x^k \in \mathbb{C}$ and $\mathcal{A}x^{k-1} \in \mathbb{C}^n$, which are defined as follows:

$$\mathcal{A}x^k = \sum_{i_1, i_2, \ldots, i_k \in [n]} a_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$
$$(\mathcal{A}x^{k-1})_i = \sum_{i_2, \ldots, i_k \in [n]} a_{i i_2 \ldots i_k} x_{i_2} \cdots x_{i_k}, i \in [n].$$

Let $\mathcal{I} = (i_{i_1 i_2 \ldots i_k})$ be the *identity tensor* of order k and dimension n, that is, $i_{i_1 i_2 \ldots i_k} = 1$ if $i_1 = i_2 = \cdots = i_k \in [n]$ and $i_{i_1 i_2 \ldots i_k} = 0$ otherwise.

Definition 2.1 ([13, 16])**.** Let $\mathcal{A}$ be a real tensor of order k dimension n. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda\mathcal{I} - \mathcal{A})x = 0$, or equivalently $\mathcal{A}x^{k-1} = \lambda x^{[k-1]}$, has a solution $x \in \mathbb{C}^n \backslash \{0\}$, then λ is called an *eigenvalue* of $\mathcal{A}$ and x is an *eigenvector* of $\mathcal{A}$ associated with λ, where $x^{[k-1]} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})$.

In the above definition, (λ, x) is called an *eigenpair* of $\mathcal{A}$. If x is a real eigenvector of $\mathcal{A}$, surely the corresponding eigenvalue λ is real. In this case, λ is called an *H-eigenvalue* of $\mathcal{A}$. Denote by $\lambda_{\min}(\mathcal{A})$ the least H-eigenvalue of $\mathcal{A}$.

A real tensor $\mathcal{A}$ of even order k is called *positive semidefinite* (or *positive definite*) if for any $x \in \mathbb{R}^n \backslash \{0\}$, $\mathcal{A}x^k \geq 0$ (or $\mathcal{A}x^k > 0$).

Lemma 2.2 ([16], Theorem 5)**.** *Let $\mathcal{A}$ be a real symmetric tensor of order k and dimension n, where k is even. Then the following results hold.*

(1) *$\mathcal{A}$ always has H-eigenvalues, and $\mathcal{A}$ is positive definite (or positive semi-definite) if and only if its least H-eigenvalue is positive (or nonnegative).*

(2) $\lambda_{\min}(\mathcal{A}) = \min\{\mathcal{A}x^k : x \in \mathbb{R}^n, \|x\|_k = 1\}$, *where* $\|x\|_k = \left(\sum_{i=1}^n |x_i|^k\right)^{\frac{1}{k}}$. *Furthermore, x is an optimal solution of the above optimization if and only if it is an eigenvector of $\mathcal{A}$ associated with $\lambda_{\min}(\mathcal{A})$.*

2.2. **Uniform hypergraphs.** A *hypergraph* $G = (V(G), E(G))$ is a pair consisting of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $e_j \subseteq V(G)$ for each $j \in [m]$. If $|e_j| = k$ for all $j \in [m]$, then G is called a k*-uniform* hypergraph. The *degree* $d_G(v)$ or simply $d(v)$ of a vertex $v \in V(G)$ is defined as $d(v) = |\{e_j : v \in e_j\}|$. The *order* of G is the cardinality of $V(G)$, denoted by $\nu(G)$, and its *size* is the cardinality of $E(G)$, denoted by $\varepsilon(G)$. A *walk* in a G is a sequence of alternate vertices and edges: $v_0e_1v_1e_2\ldots e_lv_l$, where $v_i, v_{i+1} \in e_i$ for $i = 0, 1, \ldots, l-1$. A walk is called a *path* if all the vertices and edges appeared on the walk are distinct. A hypergraph G is called *connected* if any two vertices of G are connected by a walk or path.

If a hypergraph is both connected and acyclic, it is called a *hypertree*. The k*-th power* of a simple graph H, denoted by H^k, is obtained from H by replacing each edge (a 2-set) with a k-set by adding $(k-2)$ additional vertices [8]. The k-th power of a tree is called *power hypertree*, which is surely a k-uniform hypertree. In particular, the k-th power of a path P_m (respectively, a star S_m) (as a simple graph) with m edges is called a *hyperpath* (respectively, *hyperstar*), denote by P_m^k (respectively, S_m^k). In a k-th power hypertree T, an edge is called a *pendent edge* of T if it contains $k-1$ vertices of degree one, which are called the *pendent vertices* of T.

Lemma 2.3 ([1])**.** *Let G be a connected k-uniform hypergraph. Then G is a hypertree if and only if $\varepsilon(G) = \frac{\nu(G)-1}{k-1}$.*

The odd-bipartite hypergraphs was introduced by Hu and Qi [7], which is considered as a generalization of the ordinary bipartite graphs. The odd-bipartition is closely related to odd-traversal [15].

Definition 2.4 ([7])**.** Let k be even. A k-uniform hypergraph $G = (V, E)$ is called *odd-bipartite*, if there exists a bipartition $\{V_1, V_2\}$ of V such that each edge of G intersects V_1 (or V_2) in an odd number of vertices (such bipartition is called an *odd-bipartition* of G); otherwise, G is called *non-odd-bipartite*.

Let G be a k-uniform hypergraph on n vertices $v_1, v_2, \ldots, v_n$. The *adjacency tensor* of G [3] is defined as $\mathcal{A}(G) = (a_{i_1i_2\ldots i_k})$, an order k dimensional n tensor, where

$$a_{i_1i_2\ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(G)$ be a diagonal tensor of order k and dimension n, where $d_{i\ldots i} = d(v_i)$ for $i \in [n]$. The tensor $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ is called the *signless Laplacian tensor* of G [17]. Observe that the adjacency (signless Laplacian) tensor of a hypergraph is symmetric.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$. Then x can be considered as a function defined on the vertices of G, that is, each vertex v_i is mapped to $x_i =: x_{v_i}$. If x is an eigenvector of $\mathcal{Q}(G)$, then it defines on G naturally, i.e., x_v is the entry of x corresponding to v. If G_0 is a sub-hypergraph of G, denote by $x|_{G_0}$ the restriction of x on the vertices of G_0, or a subvector of x indexed by the vertices of G_0.

Denote by $E_G(v)$, or simply $E(v)$, the set of edges of G containing v. For a subset U of $V(G)$, denote $x^U := \Pi_{v\in U} x_u$, and $x_U^k := \sum_{v\in U} x_u^k$. Then we have

$$\mathcal{Q}(G)x^k = \sum_{e\in E(G)} (x_e^k + kx^e), \tag{2.1}$$

and for each $v \in V(G)$,

$$(\mathcal{Q}(G)x^{k-1})_v = d(v)x_v^{k-1} + \sum_{e\in E(v)} x^{e\setminus\{v\}}.$$

So the eigenvector equation $\mathcal{Q}(G)x^{k-1} = \lambda x^{[k-1]}$ is equivalent to that for each $v \in V(G)$,

$$(\lambda - d(v))x_v^{k-1} = \sum_{e\in E(v)} x^{e\setminus\{v\}}. \tag{2.2}$$

From Lemma 2.2(2), if k is even, then $\lambda_{\min}(G) := \lambda_{\min}(\mathcal{Q}(G))$ can be expressed as

$$\lambda_{\min}(G) = \min_{x\in\mathbb{R}^n, \|x\|_k=1} \sum_{e\in E(G)} (x_e^k + kx^e). \tag{2.3}$$

Note that if k is odd, the Eq. (2.3) does not hold. The reason is as follows. If G contains at least one edge, then by Perron-Frobenius theorem, the spectral radius $\rho(\mathcal{Q}(G))$ of $\mathcal{Q}(G)$ is positive associated with a unit nonnegative eigenvector x. Now

$$\lambda_{\min}(G) \le \mathcal{Q}(G)(-x)^k = -\mathcal{Q}(G)x^k = -\rho(\mathcal{Q}(G)) < 0,$$

a contradiction as $\lambda_{\min}(G) \ge 0$ (see [17, Theorem 3.1]).

Lemma 2.5. *Let G be a k-uniform hypergraph, and (λ, x) be an eigenpair of $\mathcal{Q}(G)$. If $E(u) = E(v)$ and $\lambda \ne d(u)$, then $x_u^k = x_v^k$.*

Proof. Consider the eigenvector equation of x at u and v respectively,

$$(\lambda - d(u))x_u^k = \sum_{e\in E(u)} x^e, \; (\lambda - d(v))x_v^k = \sum_{e\in E(v)} x^e.$$

As $E[u] = E[v]$, $d(u) = d(v)$ and $\sum_{e\in E(u)} x^e = \sum_{e\in E(v)} x^e$. The result follows. $\square$

Lemma 2.6 ([11])**.** *Let G be a k-uniform hypergraph with the minimum degree $\delta(G) > 0$, where k is even. Then $\lambda_{\min}(G) < \delta(G)$.*

3. Properties of the first eigenvectors

Let G_1, G_2 be two vertex-disjoint hypergraphs, and let $v_1 \in V(G_1), v_2 \in V(G_2)$. The *coalescence* of G_1, G_2 with respect to v_1, v_2, denoted by $G_1(v_1) \diamond G_2(v_2)$, is obtained from G_1, G_2 by identifying v_1 with v_2 and forming a new vertex u. The graph $G_1(v_1) \diamond G_2(v_2)$ is also written as $G_1(u) \diamond G_2(u)$. If a connected graph G can be expressed in the form $G = G_1(u) \diamond G_2(u)$, where G_1, G_2 are both nontrivial and connected, then G_1 is called a *branch* of G with *root* u. Clearly G_2 is also a branch of G with root u in the above definition.

We will give some properties of the first eigenvectors of a connected k-uniform G which contains an odd-bipartite branch. We stress that k is *even* in this and the following sections.

Lemma 3.1. *Let $G = G_0(u) \diamond H(u)$ be a connected k-uniform hypergraph, where H is odd-bipartite. Let x be a first eigenvector of G. Then the following results hold.*

(1) *$x^e \le 0$ for each $e \in E(H)$.*
(2) *If $x_u = 0$, then $\sum_{e\in E_{G_0}(u)} x^{e\setminus\{u\}} = 0$, and $x^{e\setminus\{u\}} = 0$ for each $e \in E_H(u)$.*
(3) *There exists a first eigenvector of G such that it is nonnegative on one part and nonpositive on the other part for any odd-bipartition of H.*

Proof. Let $\{U, W\}$ be an odd-bipartition of H, where $u \in U$. Without loss of generality, we assume that $\|x\|_k = 1$ and $x_u \ge 0$. Let $\tilde{x}$ be such that

$$\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0)\setminus\{u\}; \\ |x_v|, & \text{if } v \in U; \\ -|x_v|, & \text{if } v \in W. \end{cases}$$

Note that $\|\tilde{x}\|_k = \|x\|_k = 1$, and for each $e \in E(H)$,
(a) $\tilde{x}_e^k = x_e^k$.
(b) $\tilde{x}^e \le x^e$ with equality if and only if $x^e \le 0$.

We prove the assertion (1) by a contradiction. Suppose that there exists an edge $e \in E(H)$ such that $x^e > 0$. Then $\tilde{x}^e < x^e$. By (a), (b), and Eq. (2.3), we have

$$\lambda_{\min}(G) \le \mathcal{Q}(G)\tilde{x}^k < \mathcal{Q}(G)x^k = \lambda_{\min}(G),$$

a contradiction. So $x^e \le 0$ for each $e \in E(H)$, and $\tilde{x}$ is also a first eigenvector as $\mathcal{Q}(G)\tilde{x}^k = \mathcal{Q}(G)x^k$. The assertions (1) and (3) follow.

For the assertion (2), let $\bar{x}$ be such that

$$\overline{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0)\setminus\{u\}; \\ -|x_v|, & \text{if } v \in U; \\ |x_v|, & \text{if } v \in W. \end{cases}$$

By a similar discussion, $\bar{x}$ is also a first eigenvector of G. Note that $x_u = 0$ and consider the eigenvector equation Eq. (2.2) of $\tilde{x}$ and $\bar{x}$ at u, respectively.

$$(\lambda_{\min}(G) - d(u))\tilde{x}_u^{k-1} = 0 = \sum_{e\in E_{G_0}(u)} \tilde{x}^{e\setminus\{u\}} + \sum_{e\in E_H(u)} \tilde{x}^{e\setminus\{u\}},$$

$$\begin{aligned}(\lambda_{\min}(G) - d(u))\bar{x}_u^{k-1} = 0 \quad &= \sum_{e\in E_{G_0}(u)} \bar{x}^{e\setminus\{u\}} + \sum_{e\in E_H(u)} \bar{x}^{e\setminus\{u\}} \\ &= \sum_{e\in E_{G_0}(u)} \tilde{x}^{e\setminus\{u\}} - \sum_{e\in E_H(u)} \tilde{x}^{e\setminus\{u\}}.\end{aligned}$$

Thus $\sum_{e\in E_{G_0}(u)} x^{e\setminus\{u\}} = \sum_{e\in E_{G_0}(u)} \tilde{x}^{e\setminus\{u\}} = 0$ and $\sum_{e\in E_H(u)} \tilde{x}^{e\setminus\{u\}} = 0$. As $\tilde{x}^{e\setminus\{u\}} \le 0$ for each edge $e \in E_H(u)$, we have $\tilde{x}^{e\setminus\{u\}} = 0$ for each $e \in E_H(u)$. The assertion (2) follows by the definition of $\tilde{x}$. □

Lemma 3.2. *Let $G = G_0(u) \diamond H(u)$ be a connected non-odd-bipartite k-uniform hypergraph, where H is odd-bipartite. Then*

$$\lambda_{\min}(G_0) \ge \lambda_{\min}(G),$$

with equality if and only if for any first eigenvector y of G_0, $y_u = 0$ and $\tilde{y}$ is a first eigenvector of G, where $\tilde{y}$ is defined by

$$\tilde{y}_v = \begin{cases} y_v, & \text{if } v \in V(G_0); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that y is a first eigenvector of G_0, $\|y\|_k = 1$, and $y_u \geq 0$. Let $\{U, W\}$ be an odd-bipartition of H, where $u \in U$. Define $\bar{y}$ by

$$\bar{y}_v = \begin{cases} y_v, & \text{if } v \in V(G_0); \\ y_u, & \text{if } v \in U \backslash \{u\}; \\ -y_u, & \text{if } v \in W. \end{cases}$$

Then $\|\bar{y}\|_k = 1 + (\nu(H) - 1)y_u^k$, and

$$\begin{aligned} \mathcal{Q}(G)\bar{y}^k &= \sum_{e \in E(G)} (\bar{y}_e^k + k\bar{y}^e) \\ &= \sum_{e \in E(G_0)} (\bar{y}_e^k + k\bar{y}^e) + \sum_{e \in E(H)} (\bar{y}_e^k + k\bar{y}^e) \\ &= \mathcal{Q}(G_0)y^k + \sum_{e \in E(H)} (ky_u^k - ky_u^k) \\ &= \lambda_{\min}(G_0). \end{aligned}$$

By Eq. (2.3), we have

$$\lambda_{\min}(G) \leq \frac{\mathcal{Q}\bar{y}^k}{\|\bar{y}\|_k^k} = \frac{\lambda_{\min}(G_0)}{1 + (\nu(H) - 1)y_u^k} \leq \lambda_{\min}(G_0),$$

where the first equality holds if and only if $\bar{y}$ is also a first eigenvector of G, and the second equality holds if and only if $y_u = 0$ (Note that $\lambda_{\min}(G_0) > 0$ as G_0 is connected and non-odd-bipartite). The result now follows. □

Corollary 3.3. *Let $G = G_0(u) \diamond H(u)$ be a connected non-odd-bipartite k-uniform hypergraph, where H is odd-bipartite.*

(1) *If y is a first eigenvector of G_0 with $y_u \neq 0$, then*

$$\lambda_{\min}(G_0) > \lambda_{\min}(G).$$

(2) *If x is a first eigenvector of G such that $x_u = 0$ and $x|_{G_0} \neq 0$, then*

$$\lambda_{\min}(G_0) = \lambda_{\min}(G).$$

Proof. By Lemma 3.2, we can get the assertion (1) immediately. Let x be a first eigenvector of G as in (2). By Lemma 3.1(2), $\sum_{e \in E_{G_0}(u)} x^{e \backslash \{u\}} = 0$. Considering the eigenvector equation (2.2) of x at each vertex of $V(G_0)$, we have

$$\mathcal{Q}(G_0)(x|_{G_0})^{k-1} = \lambda_{\min}(G)(x|_{G_0})^{[k-1]}.$$

So $x|_{G_0}$ is an eigenvector of $\mathcal{Q}(G_0)$ associated with the eigenvalue $\lambda_{\min}(G)$. The result follows by Lemma 3.2. □

Lemma 3.4. *Let $G = G_0(u) \diamond H(u)$ be a connected non-odd-bipartite k-uniform hypergraph, where H is odd-bipartite. If x is a first eigenvector of G, then*

$$\beta_H(x) := d_H(u)x_u^k + \sum_{e \in E_H(u)} x^e \leq 0. \tag{3.1}$$

Furthermore, if $\beta_H(x) = 0$ and $x|_{G_0} \neq 0$, then $x_u = 0$ and $\lambda_{\min}(G_0) = \lambda_{\min}(G)$; or equivalently if $x_u \neq 0$, then $\beta_H(x) < 0$.

Proof. Let $\lambda := \lambda_{\min}(G)$. By Eq. (2.2), for each $v \in V(G_0) \backslash \{u\}$,

$$((\mathcal{Q}(G) - \lambda \mathcal{I})x^{k-1})_v = ((\mathcal{Q}(G_0) - \lambda \mathcal{I})(x|_{G_0})^{k-1})_v = 0. \tag{3.2}$$

For the vertex u,

$$\begin{aligned}
\lambda x_u^{k-1} = (\mathcal{Q}(G)x^{k-1})_u &= d_G(u)x_u^{k-1} + \sum_{e \in E_G(u)} x^{e \backslash \{u\}}, \\
&= d_{G_0}(u)x_u^{k-1} + \sum_{e \in E_{G_0}(u)} x^{e \backslash \{u\}} + d_H(u)x_u^{k-1} + \sum_{e \in E_H(u)} x^{e \backslash \{u\}} \\
&= (\mathcal{Q}(G_0)(x|_{G_0})^{k-1})_u + d_H(u)x_u^{k-1} + \sum_{e \in E_H(u)} x^{e \backslash \{u\}}.
\end{aligned}$$

So,

$$((\mathcal{Q}(G_0) - \lambda \mathcal{I})(x|_{G_0})^{k-1})_u = -\left(d_H(u)x_u^{k-1} + \sum_{e \in E_H(u)} x^{e \backslash \{u\}} \right). \tag{3.3}$$

By Lemma 3.2 and Lemma 2.2(1), $\mathcal{Q}(G_0) - \lambda \mathcal{I}$ is positive semidefinite. Then $(\mathcal{Q}(G_0) - \lambda \mathcal{I})y^k \geq 0$ for any real and nonzero y. So, by Eq. (3.2) and Eq. (3.3), we have

$$\begin{aligned}
0 \leq (\mathcal{Q}(G_0) - \lambda \mathcal{I})(x|_{G_0})^k &= (x|_{G_0})^\top ((\mathcal{Q}(G_0) - \lambda \mathcal{I})(x|_{G_0})^{k-1} \\
&= -x_u \left(d_H(u)x_u^{k-1} + \sum_{e \in E_H(u)} x^{e \backslash \{u\}} \right) \\
&= -\beta_H(x).
\end{aligned}$$

So we have $\beta_H(x) \leq 0$.

Suppose that $\beta_H = 0$ and $x|_{G_0} \neq 0$. If $x_u = 0$, by Corollary 3.3(2), $\lambda_{\min}(G_0) = \lambda_{\min}(G)$. If $x_u \neq 0$, then $d_H(u)x_u^{k-1} + \sum_{e \in E_H(u)} x^{e \backslash \{u\}} = 0$. By Eq. (3.2) and Eq. (3.3), $(\lambda_{\min}(G), x|_{G_0})$ is an eigenpair of $\mathcal{Q}(G_0)$, implying that $\lambda_{\min}(G) = \lambda_{\min}(G_0)$ by Lemma 3.2. However, $(x|_{G_0})_u = x_u \neq 0$, a contradiction to Corollary 3.3(1). □

Lemma 3.5. *Let $G = G_0(r) \diamond T(r)$ be a connected non-odd-bipartite k-uniform hypergraph, where T is a power hypertree. If x is a first eigenvector of G and $x_p \neq 0$ for some $p \in V(T)$, then $x_q \neq 0$ whenever q is a vertex of T such that p lies on the unique path from r to q.*

Proof. It suffices to consider three vertices u, v, w in a common edge $e \in E(T)$, where $d(u) \geq 2$, $d(v) = 1$, $d(w) \geq 2$, and u lies on the path from r to w. We will show $x_u \neq 0 \Rightarrow x_v \neq 0 \Rightarrow x_w \neq 0$. Write $G = \bar{G}_0(u) \diamond \bar{T}(u)$, where $\bar{G}_0$ contains G_0 as a sub-hypergraph, and $\bar{T}$ is a sub-hypergraph of T such that e is the only edge of $\bar{T}$ containing u. Suppose that $x_u \neq 0$. If $x_v = 0$, by Lemma 3.4,

$$\beta_{\bar{T}}(x) = x_u^k + x^e = x_u^k \leq 0,$$

a contradiction. So $x_v \neq 0$. If $x_w = 0$, then by Eq. (2.2), $x_v = 0$ as $\lambda_{\min}(G) < \delta(G) = 1$ by Lemma 2.6, also a contradiction. So $x_w \neq 0$. □

Lemma 3.6. *Let $G = G_0(r) \diamond T(r)$ be a connected non-odd-bipartite k-uniform hypergraph, where T is a power hypertree. If x is a first eigenvector of G and $x_r \neq 0$, then $|x_u| < |x_w|$ whenever u, w are two vertices of T such that u lies on the unique path from r to w, $d(u) \geq 2$ and $d(w) \geq 2$.*

Proof. By Lemma 3.5, $x_i \neq 0$ for any vertex $i \in V(T)$. It suffices to consider three vertices u, v, w in a common edge $e \in E(T)$, where $d(u) \geq 2$, $d(v) = 1$, $d(w) \geq 2$, and u lies on the path from r to w. We will show that $|x_u| < |x_w|$. By the eigenvector equation of x at v, noting that $0 < \lambda := \lambda_{\min}(G) < 1$, by Lemma 2.5 we have

$$(1-\lambda)|x_v|^k = |x^e| = |x_u||x_v|^{k-2}|x_w|,$$

which implies that

$$|x_v| = \left(\frac{|x_u||x_w|}{1-\lambda}\right)^{\frac{1}{2}}. \tag{3.4}$$

We can write $G = \bar{G}_0(w) \diamond \bar{T}(w)$, where $\bar{G}_0$ contains G_0 and the edge e as the only one containing w. Then by Lemma 3.4, noting $x_w \neq 0$,

$$\beta_{\bar{T}}(x) = (d_T(w)-1)x_w^k + \sum_{\tilde{e} \in E_T(w)\setminus\{e\}} x^{\tilde{e}} < 0.$$

By Lemma 3.1(1), we have

$$(d_T(w)-1)|x_w|^k < \sum_{\tilde{e} \in E_T(w)\setminus\{e\}} |x^{\tilde{e}}|. \tag{3.5}$$

Considering the eigenvector equation of x at w, by Lemma 2.5, Eq. (3.4) and Eq. (3.5), we have

$$\begin{aligned}
(d_T(w)-\lambda)|x_w|^k &= \sum_{\tilde{e} \in E_T(w)\setminus\{e\}} |x^{\tilde{e}}| + |x_u||x_v|^{k-2}|x_w| \\
&> (d_T(w)-1)|x_w|^k + |x_u|\left(\frac{|x_u||x_w|}{1-\lambda}\right)^{\frac{k}{2}-1}|x_w| \\
&= (d_T(w)-1)|x_w|^k + (1-\lambda)^{1-\frac{k}{2}}|x_u|^{\frac{k}{2}}|x_w|^{\frac{k}{2}}.
\end{aligned}$$

So

$$(1-\lambda)|x_w|^k > (1-\lambda)^{1-\frac{k}{2}}|x_u|^{\frac{k}{2}}|x_w|^{\frac{k}{2}},$$

and hence

$$|x_w|^{\frac{k}{2}} > (1-\lambda)^{\frac{k}{2}}|x_w|^{\frac{k}{2}} > |x_u|^{\frac{k}{2}}.$$

□

Denote by $G = G_0(r) \diamond P_m^k(r)$ the coalescence of G_0 and P_m^k by identifying one vertex of G_0 and one pendent vertex of P_m^k and forming a new vertex r.

Lemma 3.7. *Let $G = G_0(r) \diamond P_m^k(r)$ be a connected non-odd-bipartite k-uniform hypergraph, where P_m^k is a hyperpath with m edges. Starting from the root r, label edges of P_m^k as $e_m, e_{m-1}, \ldots, e_1$, and some vertices of those edges as*

$$r = 2m, 2m-1, 2m-2, \ldots, 2i, 2i-1, 2i-2, \ldots, 2, 1, 0, \tag{3.6}$$

where $\{2i, 2i-1, 2i-2)\} \subseteq e_i$ for $i \in [m]$, $d_{P_m}(2i) = 2$ and $d_{P_m}(2i-1) = 1$ for $i \in [m-1]$, $d_{P_m}(2m) = d_{P_m}(0) = 1$. If x is a first eigenvector of G and $x_r \neq 0$, Then

$$|x_{2i}| = f_i(\lambda_{\min}(G))^{\frac{2}{k}}|x_0|,\ i \in [m], \tag{3.7}$$

where $f_i(x)$ is defined recursively as $f_0(x) = 1$, $f_1(x) = (1-x)^{\frac{k}{2}}$,

$$f_{i+1}(x) = (2-x)(1-x)^{\frac{k}{2}-1}f_i(x) - f_{i-1}(x),\ i \in [m-1].$$

Furthermore, $0 < f_i(\lambda_{\min}(G)) < 1$*, and* $f_i(\lambda_{\min}(G))$ *is strictly decreasing in* i*.*

Proof. By Lemma 3.5, $x_v \neq 0$ for each $v \in V(P_m)$. Let $\lambda := \lambda_{\min}(G)$. Then $0 < \lambda < 1$ by Lemma 2.6. By Lemma 2.5 and Eq. (2.2),

$$|x_0| = |x_1|,\ |x_2| = (1-\lambda)|x_0|.$$

So, by Lemma 2.5, Lemma 3.1(1) and Eq. (3.4), considering the eigenvector equation of x at the vertex $2i$ $(1 \leq i \leq m-1)$, we have

$$\begin{aligned}(2-\lambda)|x_{2i}|^{k-1} &= |x_{2i-1}|^{k-2}|x_{2i-2}| + |x_{2i+1}|^{k-2}|x_{2i+2}| \\ &= \left(\frac{|x_{2i}x_{2i-2}|}{1-\lambda}\right)^{\frac{k}{2}-1}|x_{2i-2}| + \left(\frac{|x_{2i}x_{2i+2}|}{1-\lambda}\right)^{\frac{k}{2}-1}|x_{2i+2}| \\ &= (1-\lambda)^{1-\frac{k}{2}}|x_{2i}|^{\frac{k}{2}-1}\left(|x_{2i-2}|^{\frac{k}{2}} + |x_{2i+2}|^{\frac{k}{2}}\right).\end{aligned}$$

Thus, for $i = 1, \ldots, m-1$,

$$|x_{2i+2}|^{\frac{k}{2}} = (2-\lambda)(1-\lambda)^{\frac{k}{2}-1}|x_{2i}|^{\frac{k}{2}} - |x_{2i-2}|^{\frac{k}{2}}.$$

It is easy to verify that for $i \in [m]$,

$$|x_{2i}|^{\frac{k}{2}} = f_i(\lambda)|x_0|^{\frac{k}{2}}.$$

By Lemma 3.6, for $i \in [m]$, $0 < f_i(\lambda) < 1$, and $f_i(\lambda)$ is strictly decreasing in i. □

4. Perturbation of the least eigenvalues

We first give a perturbation result on the least eigenvalues under relocating an odd-bipartite branch. Let G_0, H be two vertex-disjoint hypergraphs, where v_1, v_2 are two distinct vertices of G_0, and u is a vertex of H (called the *root* of H). Let $G = G_1(v_2) \diamond H(u)$ and $\tilde{G} = G_1(v_1) \diamond H(u)$. We say that $\tilde{G}$ is obtained from G by *relocating* H rooted at u from v_2 to v_1; see Fig. 4.1.

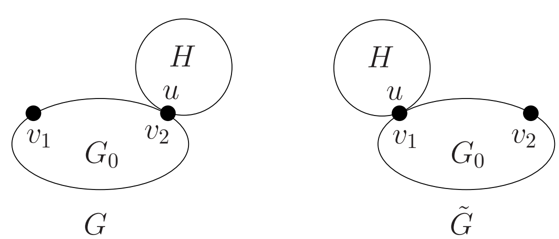

Figure 4.1. Relocating H from v_2 to v_1

Lemma 4.1. *Let* $G = G_0(v_2) \diamond H(u)$ *and* $\tilde{G} = G_0(v_1) \diamond H(u)$ *be connected non-odd-bipartite* k*-uniform hypergraphs, where* H *is odd-bipartite. If* x *is a first eigenvector of* G *such that* $|x_{v_1}| \geq |x_{v_2}|$*, then*

$$\lambda_{\min}(\tilde{G}) \leq \lambda_{\min}(G),$$

with equality if and only if $x_{v_1} = x_{v_2} = 0$*, and* $\tilde{x}$ *defined in (4.4) is a first eigenvector of* $\tilde{G}$*.*

Proof. Let x be a first eigenvector of G such that $\|x\|_k = 1$ and $x_{v_1} \geq 0$. We divide the discussion into three cases. Denote $\lambda := \lambda_{\min}(G)$.

Case 1: $x_{v_2} > 0$. Write $x_{v_1} = \delta x_{v_2}$, where $\delta \geq 1$. Define $\tilde{x}$ on $\tilde{G}$ by

$$\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0); \\ \delta x_v, & \text{if } v \in V(H)\backslash\{u\}. \end{cases} \tag{4.1}$$

Then $\|\tilde{x}\|_k^k = 1 + (\delta^k - 1)\sum_{v \in V(H)\backslash\{u\}} x_v^k$, and

$$\begin{aligned} \mathcal{Q}(\tilde{G})\tilde{x}^k &= \sum_{e \in E(\tilde{G})} (\tilde{x}_e^k + k\tilde{x}^e) \\ &= \mathcal{Q}(G)x^k + (\delta^k - 1)\sum_{e \in E(H)} (x_e^k + kx^e) \\ &= \lambda + (\delta^k - 1)\sum_{e \in E(H)} (x_e^k + kx^e). \end{aligned}$$

By the eigenvector equation of x at each vertex $v \in V(H)\backslash\{u\}$,

$$d(v)x_v^k + \sum_{e \in E_H(v)} x^e = \lambda x_v^k. \tag{4.2}$$

By the eigenvector equation of x at u,

$$d(u)x_u^k + \sum_{e \in E_G(u)} x^e = \alpha_{G_0}(x) + \beta_H(x) = \lambda x_u^k, \tag{4.3}$$

where $\alpha_{G_0}(x) := d_{G_0}(u)x_u^k + \sum_{e \in E_{G_0}(u)} x^e$. By Eq. (4.2) and Eq. (4.3), we have

$$\alpha_{G_0} + \sum_{e \in E(H)} (x_e^k + kx^e) = \lambda \sum_{v \in V(H)} x_v^k.$$

So

$$\begin{aligned} \sum_{e \in E(H)} (x_e^k + kx^e) &= \lambda \sum_{v \in V(H)} x_v^k - \alpha_{G_0}(x) \\ &= \lambda \sum_{v \in V(H)} x_v^k - (\lambda x_u^k - \beta_H(x)) \\ &= \lambda \sum_{v \in V(H)\backslash\{u\}} x_v^k + \beta_H(x). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{Q}(\tilde{G})\tilde{x}^k &= \lambda + (\delta^k - 1)\left(\lambda \sum_{v \in V(H)\backslash\{u\}} x_v^k + \beta_H(x)\right) \\ &= \lambda\left(1 + (\delta^k - 1)\sum_{v \in V(H)\backslash\{u\}} x_v^k\right) + (\delta^k - 1)\beta_H(x) \\ &= \lambda\|\tilde{x}\|_k^k + (\delta^k - 1)\beta_H(x). \end{aligned}$$

As $x_{v_2} \neq 0$, $\beta_H(x) < 0$ by Lemma 3.4,

$$\lambda_{\min}(\tilde{G}) \leq \frac{\mathcal{Q}(\tilde{G})\tilde{x}^k}{\|\tilde{x}\|_k^k} = \lambda + \frac{(\delta^k - 1)\beta_H(x)}{\|\tilde{x}\|_k^k} \leq \lambda = \lambda_{\min}(G),$$

where the first equality holds if and only if $\tilde{x}$ is a first eigenvector of $\tilde{G}$, and the second equality holds if and only if $\delta = 1$, i.e., $x_{v_1} = x_{v_2}$. By the eigenvector equations of x and $\tilde{x}$ at v_2 respectively, we will get $\beta_H(x) = 0$, a contradiction. So, in this case, $\lambda_{\min}(\tilde{G}) < \lambda_{\min}(G)$.

Case 2: $x_{v_2} = 0$. First assume $x_{v_1} = 0$. Define $\tilde{x}$ on $\tilde{G}$ by

$$\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0); \\ x_v, & \text{if } v \in V(H)\backslash\{u\}. \end{cases} \tag{4.4}$$

Then $\|\tilde{x}\|_k^k = 1$, and

$$\lambda_{\min}(\tilde{G}) \le \mathcal{Q}(\tilde{G})\tilde{x}^k = \mathcal{Q}(G)x^k = \lambda_{\min}(G),$$

with equality if and only if $\tilde{x}$ is a first eigenvector of $\tilde{G}$.

Now assume that $x_{v_1} > 0$. By Corollary 3.3(2) and its proof, $\lambda_{\min}(G) = \lambda_{\min}(G_0)$ as $x_u = 0$ and $x|_{G_0} \ne 0$; furthermore, $x|_{G_0}$ is a first eigenvector of G_0. By Corollary 3.3(1), $\lambda_{\min}(G_0) > \lambda_{\min}(\tilde{G})$ as $(x|_{G_0})_{v_1} \ne 0$, thinking of v_1 a coalescence vertex between G_0 and H in $\tilde{G}$. So $\lambda_{\min}(\tilde{G}) < \lambda_{\min}(G)$.

Case 3: $x_{v_2} < 0$. Write $x_{v_1} = -\delta x_{v_2}$, where $\delta \ge 1$. Define $\tilde{x}$ on $\tilde{G}$ by

$$\tilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(G_0); \\ -\delta x_v, & \text{if } v \in V(H)\backslash\{u\}. \end{cases} \tag{4.5}$$

By a similar discussion to Case 1 by replacing δ by $-\delta$, we also have $\lambda_{\min}(\tilde{G}) < \lambda_{\min}(G)$. □

Corollary 4.2. *Let G be a connected non-odd-bipartite k-uniform hypergraph, and $G_{s,t}$ be the hypergraph obtained by coalescing G with two hyperpaths P_s^k and P_t^k by identifying a pendent vertex of P_s^k and a pendent vertex of P_t^k both with a vertex u of G, where $s \ge t \ge 1$. If x is a first Q-eigenvector of $G_{s,t}$ and $x_u \ne 0$, then*

$$\lambda_{\min}(G_{s,t}) > \lambda_{\min}(G_{s+1,t-1}).$$

Proof. Using the method of labeling vertices as in Eq. (3.6), we label some of the vertices P_s^k as

$$u = 2s, 2s-1, 2s-2, \ldots, 2, 1, 0,$$

and label some of the vertices of P_t^k as

$$u = 2\bar{t}, 2\bar{t}-1, 2\bar{t}-2, \ldots, \bar{2}, \bar{1}, \bar{0}.$$

Then, by Lemma 3.7

$$|x_u| = f_s(\lambda)^{\frac{2}{k}}|x_0| = f_t(\lambda)^{\frac{2}{k}}|x_{\bar{0}}|,$$

where $\lambda := \lambda_{\min}(G_{s,t})$, and $f_i(x)$ is defined as in Lemma 3.7. As $s \ge t$, $0 < f_s(\lambda) \le f_t(\lambda)$ also by Lemma 3.7. So, combining the eigenvector equation on $\bar{0}$, $|x_0| \ge |x_{\bar{0}}| > |x_{\bar{2}}| > 0$. Now relocating the pendent edge of P_t^k rooted at $\bar{2}$ and attaching to the pendent vertex 0 of P_s^k, we arrive at the hypergraph $G_{s+1,t-1}$. The result follows by Lemma 4.1. □

Lemma 4.3. *Let G and $G_{s,t}$ be as defined in Corollary 4.2. Then*

$$\lambda_{\min}(G_{s,t}) \ge \lambda_{\min}(G_{s+t,0}).$$

Furthermore, if x is a first Q-eigenvector of $G_{s,t}$ and $x_u \ne 0$, then

$$\lambda_{\min}(G_{s,t}) > \lambda_{\min}(G_{s+t,0}).$$

Proof. Suppose that the labeling of some vertices of P_s^k and P_t^k is as in the proof of Corollary 4.2. Let x be a first eigenvector of $G_{s,t}$. If $x_u = 0$, then $|x_{\bar{0}}| \ge |x_u|$; otherwise $|x_{\bar{0}}| > |x_u|$ by Lemma 3.6. Relocating P_t^k rooted at u and attaching to the pendent vertex 0 of P_s^k, we arrive at the hypergraph $G_{s+t,0}$. The result follows by Lemma 4.1. □

A hypergraph is called a *minimizing hypergraph* in a certain class of hypergraphs if its least eigenvalue attains the minimum among all hypergraphs in the class. Denoted by $\mathcal{T}_m(G_0)$ the class of hypergraphs with each obtained from a fixed connected non-odd-bipartite hypergraph G_0 by attaching some hypertrees at some vertices of G_0 respectively (i.e. identifying a vertex of a hypertree with some vertex of G_0 each time) such that the number of its edges equals $\varepsilon(G_0)+m$. We will characterize the minimizing hypergraph(s) in $\mathcal{T}_m(G_0)$.

Theorem 4.4. *Let G_0 be a connected non-odd-bipartite k-uniform hypergraph. If G is a minimizing hypergraph in $\mathcal{T}_m(G_0)$, then $G = G_0(u) \diamond P_m(u)$ for some vertex u of G_0.*

Proof. Suppose that G is a minimizing hypergraph in $\mathcal{T}_m(G_0)$, and G has no the structure as desired in the theorem. We will get a contradiction by the following three cases.

Case 1: G contains hypertrees attached at two or more vertices of G_0. Let T_1, T_2 be two hypertrees attached at v_1, v_2 of G_0 respectively. Let x be a first eigenvector of G. Assume $|x_{v_1}| \ge |x_{v_2}|$. Relocating T_2 rooted at v_2 and attaching to v_1, we will get a hypergraph $\bar{G} \in \mathcal{T}_m(G_0)$ such that $\lambda_{\min}(\bar{G}) \le \lambda_{\min}(G)$ by Lemma 4.1. Repeating the above operation, we finally arrive at a hypergraph $G^{(1)}$ with only one hypertree $T^{(1)}$ attached at one vertex u_0 of G_0 such that $\lambda_{\min}(G^{(1)}) \le \lambda_{\min}(G)$.

Case 2: $T^{(1)}$ contains edges with three or more vertices of degree greater than one, i.e. $T^{(1)}$ is not a power hypertree. Let e be one of such edges containing u, v, w with $d(u), d(v), d(w)$ all greater than one. Let x be a first eigenvector of $G^{(1)}$, and assume that $|x_u| \ge |x_w|$. Relocating the hypertree rooted at w and attaching to u, we will get a hypergraph $\hat{G} \in \mathcal{T}_m(G_0)$ such that $\lambda_{\min}(\hat{G}) \le \lambda_{\min}(G^{(1)})$ by Lemma 4.1. Repeating the above operation on the edge e until e contains exactly 2 vertices of degree greater than one, and on each other edges like e, we finally arrive at a hypergraph $G^{(2)}$ such that the unique hypertree $T^{(2)}$ attached at u_0 is a power hypertree, and $\lambda_{\min}(G^{(2)}) \le \lambda_{\min}(G^{(1)})$.

Case 3: $T^{(2)}$ contains more than one pendent edges except the edge(s) containing u_0. Let x be a first eigenvector of $G^{(2)}$. We assert that $|x_{u_0}| = \max_{v \in V(G_0)} |x_v|$. Otherwise, there exists a vertex v_0 of G such that $|x_{v_0}| > |x_{u_0}|$. Relocating $T^{(2)}$ rooted at u_0 and attaching to v_0, we will get a hypergraph $\tilde{G} \in \mathcal{T}_m(G_0)$ such that $\lambda_{\min}(\tilde{G}) < \lambda_{\min}(G^{(2)})$ by Lemma 4.1. Then $\lambda_{\min}(\tilde{G}) < \lambda_{\min}(G)$, a contradiction to G being minimizing. We also assert that there exists one pendent vertex w_0 of $T^{(2)}$ such that $x_{w_0} \ne 0$. Otherwise by Lemma 3.5, $x|_{T^{(2)}} = 0$, in particular $x_{u_0} = 0$, and hence $x = 0$ by the first assertion, a contradiction.

Note that $T^{(2)}$ consists of $d_{T^{(2)}}(u_0)$ sub-hypertrees sharing a common vertex u_0. Let $T_1^{(2)}$ be the sub-hypertrees of $T^{(2)}$ attached at u_0 which contains w_0. If $d_{T^{(2)}}(u_0) = 1$, let p be the furthest vertex of degree greater 2 on the path starting from u_0 to w_0, and let T_p be the hypertree attached to p which contains no vertices of the path except p. Relocating T_p rooted at p and attaching to w_0, we will arrive at a hypergraph still in $\mathcal{T}_m(G_0)$ but with a smaller least eigenvalue by Lemma 3.6 and Lemma 4.1 regardless of x_p being zero or not, a contradiction. If $d_{T^{(2)}}(u_0) > 1$, let $T_2^{(2)}$ be the sub-hypertree of $T^{(2)}$ attached at u_0 which contains no w_0. Relocating $T_2^{(2)}$ from u_0 and attaching to w_0, we still arrive at a hypergraph in $\mathcal{T}_m(G_0)$ but with a smaller least eigenvalue, also a contradiction. The result now follows. $\square$

5. Least limit point of the least eigenvalues

In this section we will investigate the upper bounds of the least eigenvalues, from which we show that the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs is zero.

Lemma 5.1. *Let G be a non-odd-bipartite k-uniform hypergraph. Then G contains an odd-bipartite sub-hypergraph with at least $\frac{\varepsilon(G)}{2}$ edges.*

Proof. Let $T \subseteq V(G)$ be a random subset given by $\Pr[v \in T] = \frac{1}{2}$, these choices being mutually independent. Set $B = V(G)\backslash T$. Call an edge e odd-transversal if exactly the cardinality of $e \cap T$ is odd. Let X be the number of odd-transversal edges. We decompose

$$X = \sum_{e \in E(G)} X_e,$$

where X_e is the indicator random variable for e being odd-transversal, i.e, $X_e = 1$ if e is odd-transversal, and $X_e = 0$ otherwise. Then the expectation

$$E[X_e] = \Pr[X_e] = \sum_{i \text{ is odd}, i \in [k]} \binom{k}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{k-i} = \frac{1}{2}.$$

So $E(X) = \sum_{e \in E(G)} E(X_e) = \frac{\varepsilon(G)}{2}$. Thus $X \geq \frac{\varepsilon(G)}{2}$ for some choice of T, and the set of those odd-transversal edges forms an odd-bipartite sub-hypergraph. $\square$

Theorem 5.2. *Let $G = G_0(u) \diamond H(u)$ be a connected non-odd-bipartite k-uniform hypergraph, where H is odd-bipartite. Then*

$$\lambda_{\min}(G) \leq \frac{k\varepsilon(G_0)}{\nu(G)}.$$

Proof. By Lemma 5.1, there is a proper subset T of $V(G_0)$ such that the number of odd transversal edges of G_0 respect to T is at least $\frac{\varepsilon(G_0)}{2}$. Let $\{U, W\}$ be an odd-bipartition of H, where $u \in U$. Define x by

$$x_v = \begin{cases} 1, & \text{if } v \in T \cup U; \\ -1, & \text{otherwise.} \end{cases}$$

Then $\|x\|_k^k = \nu(G)$. We write $e \sim T$ (or $e \nsim T$) to denote that e is odd-transversal (or not odd-transversal) respect to T. By Eq. (2.3) and Lemma 5.1,

$$\begin{aligned}\lambda_{\min}(G) &\le \frac{\mathcal{Q}x^k}{\|x\|_k^k}\\ &= \frac{1}{\nu(G)}\left(\sum_{e\in E(G_0),e\nsim T}(x_e^k+kx^e)+\sum_{e\in E(G_0),e\sim T}(x_e^k+kx^e)+\sum_{e\in E(H)}(x_e^k+kx^e)\right)\\ &= \frac{1}{\nu(G)}\sum_{e\in E(G_0),e\nsim T}2k\\ &\le \frac{k\varepsilon(G_0)}{\nu(G)}.\end{aligned}$$

□

Theorem 5.3. *Let $G = G_0(u) \diamond H(u)$ be a connected non-odd-bipartite k-uniform hypergraph, where H is odd-bipartite. Then*

$$\lambda_{\min}(G) \le \frac{d_{G_0}(u)}{\nu(H)}.$$

Proof. Let $\{U, W\}$ be an odd-bipartition of H, where $u \in U$. Define x by

$$x_v = \begin{cases} 1, & \text{if } v \in U;\\ -1, & \text{if } v \in W;\\ 0, & \text{otherwise.}\end{cases}$$

Then $\|x\|_k^k = \nu(H)$, and

$$\begin{aligned}\lambda_{\min}(G) &\le \frac{\mathcal{Q}x^k}{\|x\|_k^k}\\ &= \frac{1}{\nu(H)}\left(\sum_{e\in E(G_0)}(x_e^k+kx^e)+\sum_{e\in E(H)}(x_e^k+kx^e)\right)\\ &= \frac{d_{G_0}(u)}{\nu(H)}.\end{aligned}$$

□

Remark 5.4. In Theorem 5.2, the upper bound

$$\frac{k\varepsilon(G_0)}{\nu(G)} = \frac{k\varepsilon(G_0)}{\nu(G_0)}\frac{\nu(G_0)}{\nu(G)} = d(G_0)\frac{\nu(G_0)}{\nu(G)},$$

where $d(G_0)$ is the average degree of the vertices of G_0. So, if fixing G_0, and letting H have enough vertices, then the bounds in Theorems 5.2 and 5.3 will be much smaller than $\delta(G)$, the upper bound in Lemma 2.6.

By Theorem 5.3 and Lemma 2.3, we get the following result immediately.

Corollary 5.5. *Let $G = G_0(u) \diamond T_m(u)$ be a connected non-odd-bipartite k-uniform hypergraph, where $T_m(u)$ is a hypertree with m edges. Then*

$$\lambda_{\min}(G) \le \frac{d_{G_0}(u)}{(k-1)m+1}.$$

By Lemma 3.2, $\lambda_{\min}(G_0(r) \diamond P_m^k(r))$ and $\lambda_{\min}(G_0(r) \diamond S_m^k(r))$ are both decreasing in m, which implies that they have limits. By Corollary 5.5, those two limits are both 0. As for a connected non-odd-bipartite hypergraph, its least eigenvalue is greater than 0. So we get the following result.

Corollary 5.6. *Zero is the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs.*

References

1. C. Berge, *Hypergraphs: Combinatorics of finite sets*, North-Holland, 1989.
2. K. C. Chang, K. Pearson, T. Zhang, Perron-Frobenius theorem for nonnegative tensors, *Commu. Math. Sci.*, **6**(2008), 507-520.
3. J. Cooper, A. Dutle, Spectra of uniform hypergraphs, *Linear Algebra Appl.*, **436**(9)(2012), 3268-3292.
4. Y.-Z. Fan, M. Khan, Y.-Y. Tan, The largest H-eigenvalue and spectral radius of Laplaican tensor of non-odd-bipartite generalized power hypergraphs, *Linear Algebra Appl.*, **504**(2016), 487-502.
5. Y.-Z. Fan, Y. Wang, Y.-H. Bao, J.-C. Wan, M. Li, Z. Zhu, Eigenvectors of Laplacian or signless Laplacian of hypergraphs associated with zero eigenvalue, Available at arXiv: 1807.00544v2.
6. S. Friedland, S. Gaubert, L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, *Linear Algebra Appl.*, **438**(2013), 738-749.
7. S. Hu, L. Qi, The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph, *Discrete Appl. Math.*, **169**(2014), 140-151.
8. S. Hu, L. Qi, J. Y. Shao, Cored hypergraphs, power hypergraphs and their Laplacian H-eigenvalues, *Linear Algebra Appl.*, **439** (2013), 2980-2998.
9. S. Hu, L. Qi, J. Xie, The largest Laplacian and signless Laplacian H-eigenvalues of a uniform hypergraph, *Linear Algebra Appl.*, **469**(2015), 1-27.
10. M. Khan, Y.-Z. Fan, On the spectral radius of a class of non-odd-bipartite even uniform hypergraphs, *Linear Algebra Appl.*, **480**(2015), 93-106.
11. M. Khan, Y.-Z. Fan, Y.-Y. Tan, The H-spectra of a class of generalized power hypergraphs, *Discrete Math.*, **339**(2016), 1682-1689.
12. H. Li, J.-Y. Shao, L. Qi, The extremal spectral radii of k-uniform supertrees, *J. Comb. Optim.*, **32**(2016), 741-764.
13. L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, *Proceedings of the 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 2005, pp. 129-132.
14. H. Lin, B. Mo, B. Zhou, W. Weng, Sharp bounds for ordinary and signless Laplacian spectral radii of uniform hypergraphs, *Appl. Math. Comput.*, **285**(2016), 217-227.
15. V. Nikiforov, Hypergraphs and hypermatrices with symmetric spectrum, *Linear Algebra Appl.*, **519**(2017), 1-18.
16. L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symbolic Comput.*, **40**(6)(2005), 1302-1324.
17. L. Qi, H^+-eigenvalues of Laplacian and signless Laplacian tensor, *Commu. Math. Sci.*, **12**(2014), 1045-1064.
18. J.-Y. Shao, H.-Y. Shan and B.-F. Wu, Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs, *Linear Multilinear Algebra*, **63**(2015), 2359-2372.
19. Y. Wang, Y.-Z. Fan, The least eigenvalue of signless Laplacian of graphs under perturbation, *Linear Algebra Appl.*, **436** (2012), 2084-2092.
20. Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, *SIAM J Matrix Anal. Appl.*, **31**(5)(2010), 2517-2530.
21. Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors II, *SIAM J Matrix Anal. Appl.*, **32**(4)(2011), 1236-1250.
22. Y. Yang, Q. Yang, On some properties of nonnegative weakly irreducible tensors, Available at arXiv: 1111.0713v2.

23. X. Yuan, L. Qi, J. Shao, The proof of a conjecture on largest Laplacian and signless Laplacian H-eigenvalues of uniform hypergraphs, *Linear Algebra Appl.*, **490**(2016), 18-30.
24. L. Zhang, L. Qi, G. Zhou, M-tensors and some applications, *SIAM J Matrix Anal. Appl.*, **35**(2)(2014), 437-452.

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