ON THE EXISTENCE OF DENSE SUBSTRUCTURES IN FINITE GROUPS

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ABSTRACT. Fix $k \ge 6$. We prove that any large enough finite group G contains k elements which span quadratically many triples of the form $(a, b, ab) \in S \times G$, given any dense set $S \subseteq G \times G$. The quadratic bound is asymptotically optimal. In particular, this provides an elementary proof of a special case of a conjecture of Brown, Erdős and Sós. We remark that the result was recently discovered independently by Nenadov, Sudakov and Tyomkyn.

1. INTRODUCTION

This note is devoted to the study of a special case of a long-standing conjecture of Brown, Erdős and Sós [1] in 1976, which was originally formulated as a hypergraph extremal problem, and was found equivalent to the following by Solymosi, see [4].

Conjecture 1 (Brown-Erdős-Sós [1]). Fix $k \ge 6$. For every c > 0, there exists a threshold N = N(c) such that if A is a finite quasigroup of order larger than N, then for every $S \subseteq A \times A$ with $|S| \ge c|A|^2$ where c > 0, there exists a set of k elements of A which spans at least k - 3 triples of the form $(a, b, ab) \in S \times A$.

This conjecture is proved true only when k = 6, by Ruzsa and Szemerédi [3] in 1978. The problem turns out to be delicate and remains unapproachable over decades in its full generality, for all $k \ge 7$.

It is desirable to look for subfamilies of quasigroups for which the conjecture is valid and a natural candidate is groups, where associativity holds. By exploiting this additional structure, the groundbreaking result of Solymosi [4] shows the validity of the conjecture when k = 7 for finite groups.

Using the regularity lemma, Solymosi and the author [5] extended the result of [4] to k = 11, 12 and $k \ge 15$. Surprisingly, instead of the conjectured lower bound k - 3, we found asymptotically 4k/3 triples spanned by a set of k elements. One is then led to the question: For finite groups, what is the right magnitude of the maximum number of triples spanned by k elements?

In this note we give an elementary proof that such lower bound is quadratic in k, matching the trivial upper bound k^2 . An immediate consequence is an alternate proof of conjecture 1 for finite groups when k is large.

Theorem 2. Let $k \ge 6$ be an integer, then there exists a threshold N = N(k) such that if G is a finite group of order larger than N, then for every $S \subseteq G \times G$ with $|S| \ge c|G|^2$ where c > 0, there exists a set of k elements of G which spans at least $\frac{c}{210}k^2$ triples (a, b, ab) from S, i.e. $(a, b) \in S$.

A stronger result, where the coefficient of k^2 is independent of c, was recently discovered independently by Nenadov, Sudakov, and Tyomkyn.

The rest of this note is dedicated to the proof of theorem 2. The main ingredient is the construction of explicit subsets whose arbitrary two-fold products are highly structured. This is demonstrated in section 2 for the model groups \mathbb{Z}_n and \mathbb{Z}_m^n , which are respectively large in the order and the exponent.

The case of general abelian groups can be readily reduced to these model groups, using the Fundamental Theorem of Finite Abelian Groups, which states that every finite abelian group is a direct product of cyclic groups. Finally, the argument can be carried over to arbitrary finite groups by considering cosets of the form ℓH , Hr and ℓHr , where H is a large abelian subgroup whose existence is guaranteed by a theorem of Pyber's [2]. This is the content of section 3.

2. Idea of the proof

Fix $k \ge 1$. Let G be a finite (abelian) group. For every element $x \in G$, we will construct a subset A_x of G with the following properties:

- (1) The set A_x has size k.
- (2) The set $A_x A_y := \{ab \in G : a \in A_x, b \in A_y\}$ has size at most 2k.
- (3) If $x \neq y$, then $A_x \neq A_y$.
- (4) Any $(a, b, ab) \in G^3$ is contained in k^2 many $B_{x,y}$'s, where

$$B_{x,y} := \{ (a, b, ab) \in G^3 : a \in A_x, b \in A_y \}.$$

Then, by pigeonhole principle, together with (3) and (4), there exists $B_{x,y}$ that contains at least $|S|k^2/|G|^2$ triples from S, for any $S \subseteq G \times G$. Note that the triples in $B_{x,y}$ are spanned by the elements $A_x \cup A_y \cup A_x A_y$, which has size at most 4k, by condition (1) and (2). The assumption that $|S| \ge c|G|^2$ implies that there is a set of 4k elements of G which spans at least ck^2 triples from S.

To fix the idea, we consider the model cases \mathbb{Z}_n and \mathbb{Z}_m^n .

2.1. When $G \cong \mathbb{Z}_n$ is a cyclic group. Fix $k \ge 1$ and let $n \ge 4k$. For $x \in \mathbb{Z}_n$, let

$$A_x = \{x + i \in \mathbb{Z}_n : 0 \le i \le k - 1\}$$

be a set of k elements of \mathbb{Z}_n . We check the other 3 conditions one by one. The set

$$A_x + A_y = \{a + b : a \in A_x, b \in A_y\} = \{x + y + j \in \mathbb{Z}_n : 0 \le j \le 2k - 2\}$$

has 2k - 1 elements. It is clear that different elements $x, y \in \mathbb{Z}_n$ yield different sets A_x, A_y . To see that the last condition holds, one can choose x from the set $\{a, a - 1, \ldots, a - (k - 1)\} \subseteq \mathbb{Z}_n$, and choose y from the set $\{b, b - 1, \ldots, b - (k - 1)\} \subseteq \mathbb{Z}_n$, given any $(a, b, a + b) \in \mathbb{Z}_n^3$.

2.2. When $G \cong \mathbb{Z}_m^n$, for some $m \ge 2$. Fix integers $\rho \ge 1, 1 \le t < m$ and let n be large such that $m^n \ge 4tm^{\rho}$. (We suppose for now that $k = tm^{\rho} - 1$.) For $x = (x_1, \ldots, x_n) \in \mathbb{Z}_m^n$, let

$$A_x = \{(z_1, \dots, z_{\rho}, x_{\rho+1} + i, x_{\rho+2}, \dots, x_n) \in \mathbb{Z}_m^n : z_i \in \mathbb{Z}_m, 0 \le i \le t-1\} \setminus \{x\}$$

be a set of $tm^{\rho} - 1$ elements of \mathbb{Z}_m^n . The set

$$A_x + A_y = \{a + b : a \in A_x, b \in A_y\}$$

$$\subseteq \{(z_1, \dots, z_{\rho}, x_{\rho+1} + y_{\rho+1} + j, x_{\rho+2} + y_{\rho+2}, \dots, x_n + y_n) \in \mathbb{Z}_m^n : z_i \in \mathbb{Z}_m, 0 \le j \le 2t - 2\}$$

has size at most $(2t-1)m^{\rho} \leq 2(tm^{\rho}-1)$. Since $x \notin A_x$, it is easy to see that different elements $x, y \in \mathbb{Z}_m^n$ yield different sets A_x , A_y . Finally, given $(a, b, a + b) \in (\mathbb{Z}_m^n)^3$, the number of $B_{x,y}$ that contains (a, b, a + b)is $(tm^{\rho}-1)^2$. Indeed, one can choose x from the set

$$\{(z_1,\ldots,z_{\rho},a_{\rho+1}-i,a_{\rho+2},\ldots,a_n)\in\mathbb{Z}_m^n: z_i\in\mathbb{Z}_m, 0\leq i\leq t-1\}\setminus\{a\}$$

and choose y from the set

$$\{(z_1,\ldots,z_{\rho},b_{\rho+1}-i,b_{\rho+2},\ldots,b_n)\in\mathbb{Z}_m^n:z_i\in\mathbb{Z}_m,0\leq i\leq t-1\}\setminus\{b\}$$

3. Proof of theorem 2

Let us recall Pyber's theorem on the existence of large abelian subgroup of a finite group.

Theorem 3 (Pyber [2]). There exists $\mu > 0$ such that every finite group G contains an abelian subgroup H of order at least $e^{\mu}\sqrt{\log |G|}$.

For a fixed $k \ge 1$, let G be a finite group that has so large an order that contains an abelian subgroup H with

(1)
$$|H| \ge e^{\mu \sqrt{\log|G|}} \ge k^{3k \log k}.$$

By the Fundamental Theorem of Finite Abelian Groups, H is isomorphic to

$$\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_\tau},$$

where m_i 's are prime powers, with $|H| = \prod_{i=1}^{\tau} m_i$. Let $\phi : \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{\tau}} \to H$ be an isomorphism. By pigeonhole principle, there exist elements $\ell, r \in G$ such that

$$(2) \qquad \qquad |S \cap (\ell H \times Hr)| > c|H|^2.$$

Depending on the values m_i 's, we have 2 cases.

If $m_i > k$ for some *i*, reorder the cyclic groups if necessary, assume that $m_1 > k$. In this case, we set $\rho = 0$ and t = k. We show in this section that we can get a set of 4(k-1) < 4k elements of *G* that spans at least $c(k-1)^2 \ge \frac{c}{4}k^2$ triples from *S*.

If $m_i \leq k$ for all *i*, reorder the cyclic groups if necessary, assume that $m_1 = m_2 = \cdots = m_\lambda = m \geq 2$ is the most popular index among the m_i 's and it occurs λ times. In this case, we set integers $1 \leq t < m$ and $\rho \geq 1$ such that

(3)
$$tm^{\rho} \le k < (t+1)m^{\rho},$$

and we get a set of $4(tm^{\rho}-1) \leq 4k$ elements of G that spans at least $c(tm^{\rho}-1)^2 \geq \frac{c}{16}k^2$ triples from S. We note that in the second case, $\rho < \lambda$. By (1), we have

$$k^{3k \log k} \le |H| = \prod_{i=1}^{\tau} m_i \le k^{\tau},$$

which implies that $\tau \ge 3k \log k \ge 2k \log k / \log 2 \ge 2k \log k / \log m$. Since there are at most k possible distinct values of m_i 's, $\lambda \ge 2 \log k / \log m > \log k / \log m$. On the other hand, (3) implies that $k \ge m^{\rho}$, and so $\rho \le \log k / \log m$.

Now, with the above chosen ρ and t, we do the following. Let $\omega_1 = (1, 0, \ldots, 0) \in \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{\tau}}$, $\omega_2 = (0, 1, 0, \ldots, 0) \in \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{\tau}}$, and so on. For $x \in H$, define

$$A_{\ell x}^{(1)} := \{\ell x \phi(\omega_1)^{\sigma_1} \cdots \phi(\omega_\rho)^{\sigma_\rho} \phi(\omega_{\rho+1})^{\sigma} : 0 \le \sigma_i \le m-1 \text{ and } 0 \le \sigma \le t-1\} \setminus \{\ell x\} \subseteq \ell H,$$

and

$$A_{xr}^{(2)} := \{x\phi(\omega_1)^{\sigma_1} \cdots \phi(\omega_\rho)^{\sigma_\rho} \phi(\omega_{\rho+1})^{\sigma_r} : 0 \le \sigma_i \le m-1 \text{ and } 0 \le \sigma \le t-1\} \setminus \{xr\} \subseteq Hr.$$

We now check the 4 corresponding conditions stated in section 2 one by one, as 4 lemmata.

Lemma 4. For each $x \in H$, we have

$$|A_{\ell x}^{(1)}| = |A_{xr}^{(2)}| = tm^{\rho} - 1.$$

Proof. Since $\tau \ge \rho + 1$, we have $m_i = m$ for all $1 \le i \le \rho + 1$. Hence, the elements

$$\{\sigma_i \omega_i \in \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_\tau} : 1 \le i \le \rho + 1 \text{ and } 0 \le \sigma_i \le m - 1\}$$

are all distinct. This implies that the elements

$$\{\phi(\omega_i)^{\sigma_i} \in H : 1 \le i \le \rho + 1 \text{ and } 0 \le \sigma_i \le m - 1\}$$

are distinct as well. With t < m, each of the sets $A_{\ell x}^{(1)}$ and $A_{xr}^{(2)}$ has size $tm^{\rho} - 1$.

Lemma 5. For $x, y \in H$, we have

$$|A_{\ell x}^{(1)} A_{yr}^{(2)}| \le 2(tm^{\rho} - 1).$$

Proof. Recall that H is abelian. We write

$$\begin{aligned} A_{\ell x}^{(1)} A_{yr}^{(2)} \\ &\subseteq \{\ell x \phi(\omega_1)^{\sigma_1} \cdots \phi(\omega_\rho)^{\sigma_\rho} \phi(\omega_{\rho+1})^{\sigma} y \phi(\omega_1)^{\psi_1} \cdots \phi(\omega_\rho)^{\psi_\rho} \phi(\omega_{\rho+1})^{\psi} r : 0 \le \sigma_i, \psi_j \le m-1 \text{ and } 0 \le \sigma, \psi \le t-1 \} \\ &= \{\ell x \phi(\omega_1)^{\sigma_1 + \psi_1} \cdots \phi(\omega_\rho)^{\sigma_\rho} + \psi_\rho \phi(\omega_{\rho+1})^{\sigma+\psi} yr : 0 \le \sigma_i, \psi_j \le m-1 \text{ and } 0 \le \sigma, \psi \le t-1 \} \\ &= \{\ell x \phi(\omega_1)^{\sigma_1} \cdots \phi(\omega_\rho)^{\sigma_\rho} \phi(\omega_{\rho+1})^{\sigma} yr : 0 \le \sigma_i \le m-1 \text{ and } 0 \le \sigma \le 2t-2 \}, \end{aligned}$$

which shows that the set $A_{\ell x}^{(1)} A_{xr}^{(2)} \subseteq \ell Hr$ has size at most $(2t-1)m^{\rho} \leq 2(tm^{\rho}-1)$.

The lemma below guarantees that the sets $B_{\ell x,yr}$ are distinct for different pairs of $(x,y) \in H^2$, where

$$B_{\ell x,yr} := \{ (\ell a, br, \ell a br) \in \ell H \times Hr \times \ell Hr : \ell a \in A_{\ell x}^{(1)}, br \in A_{yr}^{(2)} \}$$

Lemma 6. (1) If $A_{\ell x}^{(1)} = A_{\ell y}^{(1)}$ for some $x, y \in H$, then x = y. (2) If $A_{xr}^{(2)} = A_{yr}^{(2)}$ for some $x, y \in H$, then x = y. *Proof.* Note that $\ell x \notin A_{\ell x}^{(1)}$. This allows us to recover x from the set $A_{\ell x}^{(1)}$. Consider $\phi^{-1}(\ell^{-1}A_{\ell x}^{(1)}) \subseteq \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_{\tau}}$, which is the same as

$$\{\phi^{-1}(x) + \sigma_1\omega_1 + \dots + \sigma_\rho\omega_\rho + \sigma\omega_{\rho+1} : 0 \le \sigma_i \le m-1 \text{ and } 0 \le \sigma \le t-1\} \setminus \{\phi^{-1}(x)\}.$$

Hence, the elements of the set $\phi^{-1}(\ell^{-1}A_{\ell x}^{(1)})$ as well as $\phi^{-1}(x)$ only differ in the first $\rho + 1$ coordinates. If we consider only the *i*-coordinate, where $1 \leq i \leq \rho$, every but one element appears $tm^{\rho-1}$ times. This exceptional element from \mathbb{Z}_{m_i} is the *i*-th coordinate of $\phi^{-1}(x)$. Now, consider the $(\rho + 1)$ -coordinate. One of the elements appears only $m^{\rho} - 1$ times. This is the $(\rho + 1)$ -coordinate of $\phi^{-1}(x)$.

The proof of the second statement is similar.

Lemma 7. Given a triple $(\ell a, br, \ell a br) \in \ell H \times Hr \times \ell Hr$. The number of $B_{\ell x, ur}$'s which contains $(\ell a, br, \ell a br)$ is $(tm^{\rho} - 1)^2$.

Proof. We need to choose $x, y \in H$ such that $\ell a \in A_{\ell x}^{(1)}$ and $br \in A_{yr}^{(2)}$. To have $\ell a \in A_{\ell x}^{(1)}$, it is equivalent to have

$$a \in \{x\phi(\omega_1)^{\sigma_1} \cdots \phi(\omega_\rho)^{\sigma_\rho} \phi(\omega_{\rho+1})^{\sigma} : 0 \le \sigma_i \le m-1 \text{ and } 0 \le \sigma \le t-1\} \setminus \{x\},\$$

which is in turn the same as choosing x from the $(tm^{\rho} - 1)$ -element set

$$\{a\phi(\omega_1)^{-\sigma_1}\cdots\phi(\omega_{\rho})^{-\sigma_{\rho}}\phi(\omega_{\rho+1})^{-\sigma}: 0 \le \sigma_i \le m-1 \text{ and } 0 \le \sigma \le t-1\} \setminus \{a\}$$

Similarly, one can choose y from the $(tm^{\rho} - 1)$ -element set

$$\{b\phi(\omega_1)^{-\sigma_1}\cdots\phi(\omega_{\rho})^{-\sigma_{\rho}}\phi(\omega_{\rho+1})^{-\sigma}: 0 \le \sigma_i \le m-1 \text{ and } 0 \le \sigma \le t-1\} \setminus \{b\}.$$

Hence, there are $(tm^{\rho}-1)^2$ such $B_{\ell x,yr}$'s containing the given triple $(\ell a, br, \ell abr) \in \ell H \times Hr \times \ell Hr$.

Finally, consider the pairs $((\ell a, br, \ell abr), B_{\ell x, yr})$, where $(\ell a, br, \ell abr) \in B_{\ell x, yr}$ are triples from S. By (2), the number of triples from $S \cap (\ell H \times Hr)$ is at least $c|H|^2$. Hence, by lemma 7, the number of pairs we consider is at least $c|H|^2(tm^{\rho}-1)^2$. There are $|H|^2$ different $B_{\ell x,yr}$'s by lemma 6. Using pigeonhole principle, there is a set $B_{\ell x,yr}$ which contains at least $c|H|^2(tm^{\rho}-1)^2/|H|^2 = c(tm^{\rho}-1)^2$ many triples from S. These $c(tm^{\rho}-1)^2$ triples are spanned by the set $A_{\ell x}^{(1)} \cup A_{yr}^{(2)} \cup A_{\ell x}^{(1)} A_{yr}^{(2)}$, which has at most

$$|A_{\ell x}^{(1)}| + |A_{yr}^{(2)}| + |A_{\ell x}^{(1)}A_{yr}^{(2)}| \le tm^{\rho} - 1 + tm^{\rho} - 1 + 2(tm^{\rho} - 1) = 4(tm^{\rho} - 1)$$

elements of G by lemma 4 and lemma 5, as desired.

Hence, we found 4k elements of G that span at least $\frac{c}{16}k^2$ triples from S. By adjusting the constants, theorem 2 is proved.

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