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The geometry connectivity of hypergraphs

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Abstract

Let $\mathcal{G}$ be a k-uniform hypergraph, $\mathcal{L}_{\mathcal{G}}$ be its Laplacian tensor. And $\beta(\mathcal{G})$ denotes the maximum number of linearly independent nonnegative eigenvectors of $\mathcal{L}_{\mathcal{G}}$ corresponding to the eigenvalue 0. In this paper, $\beta(\mathcal{G})$ is called the geometry connectivity of $\mathcal{G}$. We show that the number of connected components of $\mathcal{G}$ equals the geometry connectivity $\beta(\mathcal{G})$.

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1. Introduction

In 1973, Fiedler pointed that a graph G is connected if and only if the second smallest eigenvalue of Laplacian matrix L_G is more than zero, this eigenvalue is called *algebraic connectivity* of graph G, denoted by $\alpha(G)$ [1]. Usually, this conclusion is called the Fiedler Theorem.

Theorem 1.1. *[1] A graph G is connected if and only if $\alpha(G) > 0$.*

The Fiedler Theorem give the tight relation to the fundamental graph property and eigenvalues of graphs, it attracts much attention and huge literatures followed. In 1975, Fiedler further studied the algebraic connectivity of graph G in [2]. He showed that the eigenvector corresponding to $\alpha(G)$ induces partitions of the vertices of G that are natural connected clusters [3, 4]. This property is important and efficient for partitioning of graphs. After the publication of [2], the eigenvector corresponding to $\alpha(G)$ has been adopted by computer scientists and used in algorithmic partitioning applications, see [5, 6].

Further, the Fiedler Theorem can be generalized as follows.

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Theorem 1.2. *[7, 8] The number of connected components of a graph G equals the algebraic multiplicity of Laplacian eigenvalue 0 of G.*

The connectivity and the number of connected components hypergraphs are important topics as well. But for the k-uniform hypergraph $\mathcal{G}$ ($k \geqslant 3$), from the Example 2.6, we can see that the Theorem 1.1 and Theorem 1.2 can't be generalized to hypergraphs directly. In [9, 10, 11], the authors characterized the connectivity of hypergraphs by subtensors of the Laplacian tensor.

In this paper, we study the connectivity and the number of connected components of k-uniform hypergraphs in terms of eigenvectors of Laplacian tensor. We give the concept of the (Z-)geometry connectivity of k-uniform hypergraph $\mathcal{G}$ as follows.

Definition 1.3. *The (Z-)geometry connectivity of k-uniform hypergraph $\mathcal{G}$, denoted by $\beta(\mathcal{G})$ ($\beta_Z(\mathcal{G})$), is defined as the maximum number of linearly independent nonnegative (Z-)eigenvectors of Laplacian tensor corresponding to the (Z-)eigenvalue 0.*

We show that the number of connected components of a k-uniform hypergraph $\mathcal{G}$ is the (Z-)geometry connectivity $\beta(\mathcal{G})$ ($\beta_Z(\mathcal{G})$).

2. Preliminaries

In this section, we introduce some concepts and lemmas.

For a positive integer n, denote $[n] = \{1, 2, \ldots, n\}$. Let $\mathbb{R}^{[k,n]}$ and $\mathbb{R}_+^{[k,n]}$ denote the set of k-order n-dimensional real tensors and nonnegative tensors, respectively. When $k = 2$, the $\mathbb{R}^{[2,n]}$ (resp. $\mathbb{R}_+^{[2,n]}$) is the set of all $n \times n$ real (resp. nonnegative) matrices. Let $\mathbb{C}^n$, $\mathbb{R}^n$ and $\mathbb{R}_+^n$ denote the set of n-dimensional complex vectors, real vectors and nonnegative vectors, respectively.

A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_k}) \in \mathbb{R}^{[k,n]}$ is called *symmetric* if its each entry $a_{i_1 i_2 \cdots i_k}$ is invariant under any permutation of $i_1, i_2, \ldots, i_k$.

A tensor $\mathcal{I} = (\delta_{i_1 \cdots i_k}) \in \mathbb{R}^{[k,n]}$ is called the *identity tensor* if whose entry $\delta_{i \cdots i} = 1$ for all i and zero otherwise.

For $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$, it is associated to a directed graph $G(\mathcal{A}) = (V(\mathcal{A}), E(\mathcal{A}))$, where $V(\mathcal{A}) = \{1, 2, \ldots, n\}$ and $E(\mathcal{A}) = \{(i, j) : a_{i i_2 \cdots i_m} > 0, j \in \{i_2, \ldots, i_m\}\}$. A nonnegative tensor $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is called *weakly irreducible* if the associated directed graph $G(\mathcal{A})$ is strongly connected (see [12, 13]).

In 2005, the eigenvalue of tensors was proposed by Qi [14] and Lim [15], independently. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. If there exist $\lambda \in \mathbb{C}$ and a nonzero vector

$x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{2.1}$$

then λ is called an *eigenvalue* of $\mathcal{A}$ and x is called an *eigenvector* of $\mathcal{A}$ corresponding to λ, where $\mathcal{A}x^{m-1} \in \mathbb{C}^n$, $(\mathcal{A}x^{m-1})_i = \sum\limits_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}$, $i \in [n]$, and $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^{\mathrm{T}}$. The $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda$ is an eigenvalue of $\mathcal{A}\}$ is called the *spectral radius* of $\mathcal{A}$. If there exist $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x \quad \text{and} \quad x^{\mathrm{T}}x = 1, \tag{2.2}$$

then λ is called a *Z-eigenvalue* of $\mathcal{A}$ and x is called a *Z-eigenvector* of $\mathcal{A}$ corresponding to λ. Since the work of Qi [14] and Lim [15], the research on eigenvalues of tensors and its applications has attracted much attention (see [16, 17, 18]).

In [12, 19, 20, 21], Perron-Frobenius theory of tensors were established. Next, we introduce some results on Perron-Frobenius theory of tensors that we used in this paper.

Lemma 2.1. *[20] Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. If some eigenvalue of $\mathcal{A}$ has a positive eigenvector corresponding to it, then this eigenvalue must be $\rho(\mathcal{A})$.*

Lemma 2.2. *[12, 21] Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. Then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ and there exists a unique positive eigenvector corresponding to $\rho(\mathcal{A})$ up to a multiplicative constant.*

Lemma 2.3. *[21] Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. Suppose x is an eigenvector corresponding to $\rho(\mathcal{A})$. Then x contains no zero elements.*

Let a hypergraph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, where $V(\mathcal{G}) = \{1, 2, \ldots, n\}$ and $E(\mathcal{G}) = \{e_1, e_2, \ldots, e_m\}$ are the vertex set and edge set of $\mathcal{G}$, respectively. If each edge of $\mathcal{G}$ contains k vertices, then $\mathcal{G}$ is called a *k-uniform hypergraph*. Clearly, 2-uniform hypergraphs are exactly the ordinary graphs. The degree of a vertex i of $\mathcal{G}$ is denoted by d_i, where $d_i = |\{e_j : i \in e_j, j = 1, \ldots, m\}|$, $i \in [n]$. If all vertices of $\mathcal{G}$ have the same degree, then $\mathcal{G}$ is called *regular*. The *adjacency tensor* [22] of k-uniform hypergraph $\mathcal{G}$, denoted by $\mathcal{A}_{\mathcal{G}}$, is a k-order n-dimensional nonnegative symmetric tensor with entries

$$a_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(\mathcal{G}); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}_{\mathcal{G}} = \mathcal{D}_{\mathcal{G}} - \mathcal{A}_{\mathcal{G}}$ be the *Laplacian tensor* of $\mathcal{G}$ [9], where $\mathcal{D}_{\mathcal{G}}$ is a diagonal tensor, whose diagonal entries are $d_1, \ldots, d_n$, respectively.

A *path* P in a k-uniform hypergraph $\mathcal{G}$ is defined to be an alternating sequence of vertices and edges $v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$, where $v_0, \ldots, v_l$ (resp. $e_1, \ldots, e_l$) are distinct vertices (resp. edges) of $\mathcal{G}$ and $v_{i-1}, v_i \in e_i$ for $i = 1, \ldots, l$. If there exists a path starting at u and terminating at v for all $u, v \in V(\mathcal{G})$, then $\mathcal{G}$ is called *connected*. Let X be a subset of $V(\mathcal{G})$, $\mathcal{G}(X)$ denote the sub-hypergraph of $\mathcal{G}$ induced by X. If $\mathcal{G}(X)$ is connected and there isn't the paths starting at the vertices in X and terminating at vertices in $V \backslash X$, then $\mathcal{G}(X)$ is called a *connected component* of $\mathcal{G}$.

Lemma 2.4. *[13] Let $\mathcal{G}$ be a k-uniform hypergraph. Then $\mathcal{G}$ is connected if and only if adjacency tensor $\mathcal{A}_{\mathcal{G}}$ is weakly irreducible.*

In the following, we give an example to show that the Theorem 1.1 and Theorem 1.2 can't be generalized to hypergraphs directly.

Example 2.5. *The Figure 1 is a 4-uniform hypergraph $\mathcal{G}$, by Theorem 4.3 in [22], we get that the eigenvalues of $\mathcal{A}_{\mathcal{G}}$ are $0, -1, 1, -i, i$ and the corresponding algebraic multiplicity are 36,16,16,16,16, respectively, where $i^2 = -1$.*

Since $\mathcal{L}_{\mathcal{G}} = \mathcal{D} - \mathcal{A} = \mathcal{I} - \mathcal{A}$, by the definition of eigenvalue of tensors, we have the eigenvalues of $\mathcal{L}_{\mathcal{G}}$ are $1, 2, 0, 1+i, 1-i$ and the corresponding algebraic multiplicity are 36,16,16,16,16, respectively.

Obviously, the number of connected component of $\mathcal{G}$ is 1. But the algebraic multiplicity of Laplacian eigenvalue 0 is 16. Thus, the Theorem 1.1 and Theorem 1.2 can't be generalized to hypergraphs directly.

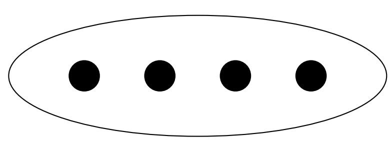

Figure 1: 4-uniform hypergraph $\mathcal{G}$

3. Main results

In this section, we show that the number of connected components of a k-uniform hypergraph $\mathcal{G}$ equals the (Z-)geometry connectivity $\beta(\mathcal{G})$ ($\beta_Z(\mathcal{G})$). Before we show the main results, we first give the following result.

Lemma 3.1. *Let $\mathcal{G}$ be a connected k-uniform hypergraph. Then $\beta(\mathcal{G}) = 1$.*

Proof. Let $\tilde{\mathcal{L}_\mathcal{G}} = \nabla\mathcal{I} - \mathcal{L}_\mathcal{G}$, where ∇ is the maximum degree of $\mathcal{G}$. It's easy to check that $(0, \mathbf{e})$ is an eigenpair of $\mathcal{L}_\mathcal{G}$, where $\mathbf{e} = (1, 1, \ldots, 1)^\mathrm{T} \in \mathbb{R}^n$. Then $(\nabla, \mathbf{e})$ is an eigenpair of $\tilde{\mathcal{L}_\mathcal{G}}$.

Since $\mathcal{G}$ is connected, from Lemma 2.4, $\mathcal{A}_\mathcal{G}$ is weakly irreducible. So we have $\tilde{\mathcal{L}_\mathcal{G}} = \nabla\mathcal{I} - \mathcal{L}_\mathcal{G} = \nabla\mathcal{I} - \mathcal{D}_\mathcal{G} + \mathcal{A}_\mathcal{G}$ is a nonnegative weakly irreducible tensor. Then, by Lemma 2.1, we get that ∇ is the spectral radius of $\tilde{\mathcal{L}_\mathcal{G}}$.

From Lemma 2.2 and Lemma 2.3, we get that the nonnegative eigenvector of a nonnegative weakly irreducible tensor corresponding to spectral radius is unique up to a multiplicative constant. So, $\mathbf{e}$ is a unique nonnegative eigenvector corresponding to spectral radius ∇ of $\tilde{\mathcal{L}_\mathcal{G}}$ up to a multiplicative constant.

Let $(0, x)$ is an eigenpair of $\mathcal{L}_\mathcal{G}$. Obviously,

$$\tilde{\mathcal{L}_\mathcal{G}} x^{k-1} = (\nabla\mathcal{I} - \mathcal{L}_\mathcal{G})x^{k-1} = \nabla x^{[k-1]} - \mathcal{L}_\mathcal{G} x^{k-1} = \nabla x^{[k-1]}.$$

So (∇, x) is an eigenpair of $\tilde{\mathcal{L}_\mathcal{G}}$. Hence, $\mathbf{e}$ is a unique nonnegative eigenvector corresponding to eigenvalue 0 of $\mathcal{L}_\mathcal{G}$ up to a multiplicative constant, i.e., $\beta(\mathcal{G}) = 1$. $\square$

Let $\mathcal{A} = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{[k,n]}$, S be a subset of $[n]$, and $\mathcal{A}[S] = (a_{i_1 \cdots i_k})$ denote a k-order $|S|$-dimensional *subtensor* of $\mathcal{A}$, where $i_1, i_2, \ldots, i_k \in S$.

Theorem 3.2. *Let $\mathcal{G}$ be a k-uniform hypergraph. Then the number of connected components of $\mathcal{G}$ is the geometry connectivity $\beta(\mathcal{G})$.*

Proof. Denote $\mathcal{L}_\mathcal{G}$ the Laplacian tensors of $\mathcal{G}$ and $\mathbf{e} = (1, 1, \ldots, 1)^\mathrm{T} \in \mathbb{R}^n$. Let $\mathcal{G}(V_1), \ldots, \mathcal{G}(V_r)$ be the connected components of $\mathcal{G}$. Then subtensor $\mathcal{L}_\mathcal{G}[V_i]$ of $\mathcal{L}_\mathcal{G}$ is the Laplacian tensors of sub-hypergraphs $\mathcal{G}(V_i)$, $i = 1, \ldots, r$. Then we have

$$\mathcal{L}_\mathcal{G}[V_i](\mathbf{e}[V_i])^{k-1} = 0, \ i = 1, \ldots, r. \tag{3.1}$$

Thus, $\mathbf{e}[V_i]$ is a nonnegative eigenvector corresponding to eigenvalue 0 of $\mathcal{L}_\mathcal{G}[V_i]$, $i = 1, \ldots, r$.

Let $\mathbf{e}_{V_i} = (x_j^{(i)}) \in \mathbb{R}^n$, its entry $x_j^{(i)} = 1$ if $j \in V_i$ and zero otherwise, $i = 1, \ldots, r$. Then by (3.1), we have

$$\mathcal{L}_\mathcal{G}(\mathbf{e}_{V_i})^{k-1} = 0, \ i = 1, \ldots, r,$$

i.e., $(0, \mathbf{e}_{V_1}), \ldots, (0, \mathbf{e}_{V_r})$ are eigenpairs of $\mathcal{L}_\mathcal{G}$. And since $V_i \bigcap V_j = \varnothing$, $i, j = 1, 2, \ldots, r$, $i \neq j$. We get that $\mathbf{e}_{V_1}, \ldots, \mathbf{e}_{V_r}$ is linearly independent eigenvectors corresponding to 0 of $\mathcal{L}_\mathcal{G}$. Next, we prove $\mathbf{e}_{V_1}, \ldots, \mathbf{e}_{V_r}$ is a maximal linearly independent group of nonnegative eigenvectors corresponding to eigenvalue 0 of $\mathcal{L}_\mathcal{G}$, i.e., $\beta(\mathcal{G}) = r$.

For any nonnegative eigenvector $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ corresponding to eigenvalue 0 of $\mathcal{L}_{\mathcal{G}}$, we have $\mathcal{L}_{\mathcal{G}} x^{k-1} = 0$ and $x \in \mathbb{R}^n_+$. Let $x_{V_i} = (x_j^{(i)}) \in \mathbb{R}^n$, its entry $x_j^{(i)} = x_j$ if $j \in V_i$ and zero otherwise, $i = 1, \ldots, r$. So $x = x_{V_1} + \cdots + x_{V_r}$. Since $\mathcal{G}(V_1), \ldots, \mathcal{G}(V_r)$ are the connected components of $\mathcal{G}$, we get

$$\mathcal{L}_{\mathcal{G}}(x_{V_i})^{k-1} = 0, \ i = 1, \ldots, r.$$

Then,

$$\mathcal{L}_{\mathcal{G}}[V_i](x[V_i])^{k-1} = 0, \ i = 1, \ldots, r.$$

So, $x[V_i]$ is a nonnegative eigenvector corresponding to eigenvalue 0 of $\mathcal{L}_{\mathcal{G}}[V_i]$ if $x[V_i] \neq 0$, $i \in [r]$. From Lemma 3.1, we know that the number of maximal linearly independent nonnegative eigenvectors of $\mathcal{G}(V_i)$ corresponding to eigenvalue 0 of $\mathcal{L}_{\mathcal{G}}[V_i]$ is 1. Thus, $x[V_i] = c_i \mathbf{e}[V_i]$, i.e., $x_{V_i} = c_i \mathbf{e}_{V_i}$, c_i is a constant, $i = 1, \ldots, r$. Therefore, x is a linear combination of $\mathbf{e}_{V_1}, \ldots, \mathbf{e}_{V_r}$. So $\mathbf{e}_{V_1}, \ldots, \mathbf{e}_{V_r}$ is a maximal linearly independent group of nonnegative eigenvectors corresponding to eigenvalue 0 of $\mathcal{L}_{\mathcal{G}}$, i.e., $\beta(\mathcal{G}) = r$. □

By Theorem 3.2, we can get the following result directly.

Theorem 3.3. *A k-uniform hypergraph $\mathcal{G}$ is connected if and only if $\beta(\mathcal{G}) = 1$.*

Obviously, the maximum numbers of linearly independent vectors of sets $\{x : \mathcal{L}_{\mathcal{G}} x^{k-1} = 0 \text{ and } x \in \mathbb{R}^n_+\}$ and $\{x : \mathcal{L}_{\mathcal{G}} x^{k-1} = 0, x^{\mathrm{T}} x = 1 \text{ and } x \in \mathbb{R}^n_+\}$ are equal, i.e., $\beta(\mathcal{G}) = \beta_Z(\mathcal{G})$. By Theorem 3.2, we have the following result.

Theorem 3.4. *Let $\mathcal{G}$ be a k-uniform hypergraph. Then the number of connected components of $\mathcal{G}$ is the Z-geometry connectivity $\beta_Z(\mathcal{G})$.*

Let $\mathcal{G}$ be a k-uniform hypergraph and $\mathcal{A}_{\mathcal{G}}$ be its adjacency tensor. Let $\beta_\rho(\mathcal{G})$ denote the maximum number of linearly independent nonnegative eigenvectors of $\mathcal{A}_{\mathcal{G}}$ corresponding to spectral radius $\rho(\mathcal{A}_{\mathcal{G}})$. We have the following conclusion.

Theorem 3.5. *Let $\mathcal{G}$ be a k-uniform d-regular hypergraph. Then the number of connected components of $\mathcal{G}$ is $\beta_\rho(\mathcal{G})$.*

Proof. It's easy to check that $(d, \mathbf{e})$ is an eigenpair of $\mathcal{A}_{\mathcal{G}}$, where $\mathbf{e} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. From Lemma 2.1, d is the spectral radius of $\mathcal{A}_{\mathcal{G}}$.

Since $\mathcal{G}$ is a d-regular hypergraph, we have its Laplacian tensor is $\mathcal{L}_{\mathcal{G}} = d\mathcal{I} - \mathcal{A}_{\mathcal{G}}$. Thus, (λ, x) is an eigenpair of $\mathcal{A}_{\mathcal{G}}$ if and only if $(d - \lambda, x)$ is an eigenpair of $\mathcal{L}_{\mathcal{G}}$.

So, (d, x) is an eigenpair of $\mathcal{A}_{\mathcal{G}}$ if and only if $(0, x)$ is an eigenpair of $\mathcal{L}_{\mathcal{G}}$. Thus, $\beta_\rho(\mathcal{G}) = \beta(\mathcal{G})$.

By Theorem 3.2, the statement holds. □

Remark 3.6. *Let $\mathcal{G}$ is an ordinary graph with r connected components. Since Laplacian matrix $\mathcal{L}_{\mathcal{G}}$ is symmetric, the algebraic multiplicity and the geometry multiplicity of Laplacian eigenvalue 0 are equal, are both r. It's easy to know that we can choose r nonnegative eigenvectors as the basis of characteristic subspace of Laplacian eigenvalue 0. Therefore, when $\mathcal{G}$ is an ordinary graph, Theorem 3.2 is the Theorem 1.2, and Theorem 3.3 is the Fiedler's result Theorem 1.1, respectively.*

Similarly, when $\mathcal{G}$ is a d-regular ordinary graph with r connected components, we can choose r nonnegative eigenvectors as the basis of characteristic subspace of spectral radius. Thus, by Theorem 3.5, we get that the result of regular graphs "the number of connected components of $\mathcal{G}$ is equal to the algebraic multiplicity of spectral radius of the adjacency matrix [7, 8]".

When $\mathcal{G}$ is a k-uniform hypergraph, the maximum number of linearly independent nonnegative eigenvectors and the maximum number of linearly independent eigenvectors corresponding to 0 of $\mathcal{L}_{\mathcal{G}}$ isn't equal. For example, for 4-uniform hypergragh $\mathcal{G}$ in Figure 1, it's easy to check that $(1,1,1,1)^{\mathrm{T}}$, $(1,1,-1,-1)^{\mathrm{T}}$, $(1,-1,1,-1)^{\mathrm{T}}$ and $(1,-1,-1,1)^{\mathrm{T}}$ are the linearly independent eigenvectors of $\mathcal{L}_{\mathcal{G}}$ corresponding to 0. Thus, the geometry connectivity $\beta(\mathcal{G})$ isn't equal to the maximum number of linearly independent eigenvectors corresponding to 0 of $\mathcal{L}_{\mathcal{G}}$.

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